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A.V. Porubov

ON FORMATION OF THE ROGUE WAVES AND HOLES IN OCEAN

Abstract. Two-dimensional nonlinear models are developed to account for abnormal surface ocean waves generation and propagation. These models are based on the Kadomtsev-Petviashvili equation, the 2D Benjamin-Ono equation and the 2D Gardner equation. Possible mechanisms of the waves formation suggested are the resonant interaction between semi-plane waves or waves with curved fronts or the transverse instability of a plane surface wave.

1. Introduction

This paper intends to consider some models describing sea waves of abnormally high amplitude, or the so-called rogue (or freak) waves [1, 2]. Observations of the rogue waves and various accidents with the ships in ocean caused by their attacks require a development of the theory of the rogue waves and understanding the mechanisms of the rogue waves formation. What is really dangerous that the rogue waves suddenly affect the ships even in the absence of a storm, and the crew cannot see these waves far from the vessel. Despite numerous works done by now [1, 2, 3, 4], many features of the waves remain unclear.

As a rule, rogue waves are considered as elevation free surface waves [2]. However, recently abnormal waves with negative amplitude were observed in the ocean [5, 6]. Certainly these wave are even more dangerous for a vessel than the elevation rogue waves, since their detection is unlikely either by eyes nor by a locator. The amplitude of the rogue wave may exceed 10 meters, hence, more likely, it is nonlinear wave. There is no common point of view what kind of wave is it, long or short [2]. Estimations done on the basis of some observations of both the elevation waves [7] and the deep holes [8] allow us to consider the rogue wave as a long non-linear wave.

In this paper, main attention is paid to the mathematical models. Their detailed physical justification may be found in our recent papers [7, 8]. Among the model equations employed are the Kadomtsev-Petviashvili (KP) equation, the 2D Benjamin-Ono equation and the 2D Gardner equation. The localization of an initial wave is suggested as a possible mechanism of the rogue wave generation. Localization of the wave is accompanied by an increase in its amplitude. In the two-dimensional case, localization may happen both along the direction of the wave propagation (plane localized wave) and in the plane where the wave evolves (2D localized wave). It is found that the former case may be described by exact solitary wave solutions, while the latter case requires a study of the transverse instability and numerical simulations. The conditions are obtained that establish the parameters of the incident waves and/or the ocean stratification required for the rogue wave or hole generation.

2. Long wave modelling of rogue waves

2.1. Kadomtsev-Petviashvili equation

The simplest model implies that the ocean is an inviscid liquid layer of permanent depth H with free deformable surface. Assume the plane $z = 0$ of the Cartesian coordinates coincides with undisturbed free surface of the layer, hence fluid occupies the region $-H < z < \eta$, $\eta(x, y, t)$ is a free surface disturbance. Let us denote velocity components along axes x, y, z by $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ respectively. Let t is time.

As usual, it is convenient to introduce the velocity potential, $u = \Phi_x$, $v = \Phi_y$ and $w = \Phi_z$. The basic equations and the boundary conditions may be found in [9]. The scales are introduced as follows: L (typical wave size) for x, Y for y, and H for z , $L/\sqrt{gH}$ for t, B for η, and $BL\sqrt{gH}/H$ for Φ. The small parameter of the problem, ε, is chosen according to estimations of the observed elevation rogue waves [7],

$$\varepsilon = B/H = H^2/L^2,$$

The simplified governing equation may be obtained if weak transverse variations are assumed, $L/Y = O(H/L) = O(\sqrt{\varepsilon})$. Introducing the phase variable $\theta = x - t$ and the slow time $\tau = \varepsilon\, t$, one can obtain from the basic equations [7] that the function $\eta(\theta, y, \tau)$ satisfies the equation

$$(2\eta_\tau + 3\eta\eta_\theta + 1/3\ \eta_{\theta\theta\theta})_\theta + \eta_{yy} = 0, \tag{2.1}$$

that is nothing but the Kadomtsev-Petviashvili (KP) equation [10, 11].

2.2. 2D Benjamin-Ono equation

As a rule, freak waves are considered as elevation free surface waves. However, there exist similar waves with deep troughs or surface holes that were observed in various places [5, 6]. It is important that these waves satisfy the relationship $A/H = O(H/L)$, hence, the KP equation is invalid in this case since its applicability requires another ratio between A/H and H/L, $A/H = O(H^2/L^2)$. We consider the model containing semi-unbounded inviscid and incompressible air layer interacting with a finite size layer of inviscid and incompressible water by the normal stresses. The wind in the atmosphere is taken into account in order to check its influence on the 2D localization of the wave on the water surface.

Assume that the width of the water layer is H and it is bounded by the rigid bottom from below. The Cartesian coordinates (x, y, z) are put so as the plane $z = 0$ coincides with undisturbed free surface of the liquid layer. Let $\eta(x, y, t)$ is a disturbance of the water surface. Then the water occupies the region $-H < z < \eta$, while the air occupies $\eta < z < \infty$. Let us denote constant density of the water by ρ, the velocity components along the directions x, y, z by $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ respectively. Similar notations for the air are ρ', $u'(x, y, z, t)$, $v'(x, y, z, t)$ and $w'(x, y, z, t)$. It is convenient to use in the equations and in the boundary conditions the potentials Φ, Φ' defined by $u = \Phi_x$, $v = \Phi_y$ and $w = \Phi_z$, $u' = \Phi'_x$, $v' = \Phi'_y$ and $w' = \Phi'_z$. Suppose the wind in the air has the velocity U' directed along the x-axis.

Then the basic equations are written as usual [9] while the boundary conditions are:

$$\Phi_z = 0 \text{ at } z = -H, \tag{2.2}$$

$$\Phi' \to 0 \text{ at } z \to \infty, \tag{2.3}$$

and for $z = \eta$ we get

$$\rho(\Phi_t + 1/2(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + g\,\eta) = \rho'(\Phi'_t + U'\Phi'_x + 1/2(\Phi_x'^2 + \Phi_y'^2 + \Phi_z'^2) + g\,\eta), \tag{2.4}$$

$$\eta_t + \Phi_x\eta_x + \Phi_y\eta_y = \Phi_z. \tag{2.5}$$

$$\eta_t + \Phi'_x\eta_x + \Phi'_y\eta_y = \Phi'_z. \tag{2.6}$$

The following scales are used: L- for X, Y- for y, H-for z in the water and L - for z in the air, $L/\sqrt{gH}$-for t, A-for η and $AL\sqrt{gH}/H$- for Φ, Φ'. The small parameter ε is introduced as

$$\varepsilon = A/H = H/L.$$

It is assumed additionally that $L/Y = \sqrt{\varepsilon}$.

The governing equation is obtained following the well-known procedure [12, 13]. Finally, the phase variable $\theta = x - \sqrt{1-\sigma}t$ and the slow time $\tau = \varepsilon\sqrt{1-\sigma}\,t$ are introduced, $\sigma = \rho'/\rho$, and the equation for the function η is obtained,

$$\left(2\eta_\tau + 3\eta\eta_\theta + b\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta_{\theta'\theta'}\,d\theta'}{\theta' - \theta}\right)_\theta + \eta_{yy} = 0. \tag{2.7}$$

where

$$b = \sigma\left(1 - \frac{U'}{\sqrt{gH(1-\sigma)}}\right)^2.$$

Equation (2.7) is reduced to the Benjamin-Ono (BO) equation in the one-dimensional case, hence it may be called the two-dimensional generalization of the BO equation or the 2DBO equation.

2.3. 2D Gardner equation

Another model where the ratio $A/H = O(H/L)$ may be realized is the two-layer fluid model. It is known that it may be employed to account for the stratification in the ocean. Let the upper finite width layer has density ρ' and thickness h_0, while the lower one has density $\rho > \rho'$ and thickness H. Assume that both the interface and the free surface of the upper layer are deformable while an influence of the atmosphere is negligibly small. In the one-dimensional case, long nonlinear waves were considered early in [14]. It was shown there that both surface and internal waves are described by the Korteweg-de Vries (KdV) equations if $A/H = O(H^2/L^2)$. However, the coefficient at the quadratic nonlinear term in the KdV equation may be small itself at certain relationship between widths and densities. In this case the balance between nonlinearity and dispersion required for propagation of localized waves, is realized for $A/H = O(H/L)$, just the ratio observed for the deep holes [8].

Basic equations have the form similar to that used in the previous subsection (we denote now by $'$ variables in the upper liquid layer) with the exception of the wind, now $U' = 0$, and of the boundary conditions to be imposed at the upper free surface $z = h_0 + h(x, y, t)$,

$$\Phi'_t + 1/2(\Phi'^2_x + \Phi'^2_y + \Phi'^2_z) + g\,h = 0,$$

$$h_t + \Phi'_x h_x + \Phi'_y h_y = \Phi'_z. \tag{2.8}$$

Like in the previous subsection, weak transverse variations are considered with $L/Y = \sqrt{\varepsilon}$, where ε is defined in the previous subsection, A- typical amplitude of both the surface and internal waves. Now $\eta(x, y, t)$ is the interface disturbance, other notations are $\sigma = \rho'/\rho$, $m = h_0/H$. The scales are the same as in the previous subsection but for the scale for z in the upper layer now chosen equal to H. Like in the one-dimensional case [14], we obtain that the coefficient at the quadratic nonlinear term is equal to zero for $m = m^*$,

$$m^* = s_+ + s_- + \frac{1}{3}(3 - 3\sigma - \sigma^2),$$

where

$$s_\pm = \Big(\frac{\sigma}{54}[27(1+\sigma) + 2\sigma^2(9 - 18\sigma - 9\sigma^2 - \sigma^3)] \pm \frac{\sigma(1+2\sigma)}{6}\sqrt{\frac{(1-\sigma)(27+5\sigma)}{3}}\Big)^{1/3},$$

Assume that $m = m^* + \varepsilon m_1 + ...$, $v = v^* + \varepsilon v_1 + ...$ In this case, standard asymptotic procedure allows us to reduce basic equations to the governing equation for the free surface disturbance:

$$(h_\tau + a\,h_\theta^2 + c h_\theta^3 + b h_{\theta\theta\theta})_\theta + d h_{yy} = 0, \tag{2.9}$$

where $\theta = x - vt$, $\tau = \varepsilon^2 t$,

$$v^{*2} = \frac{1}{2}(1 + m^* - \sqrt{(1+m^*)^2 - 4(1-\sigma)m^*}),$$

$$a = \frac{3m_1[(2m^*(\sigma-2)+3-5\sigma)v^{*2}+(1-\sigma)(2m^{*2}+3m^*\sigma-1)]}{2v^*[(1+m^*)v^{*2}-2(1-\sigma)m^*](1+m^*-2v^{*2})},$$

$$b = \frac{v^*[(1+m^*)\{1-(1-3\sigma)m^*+m^{*2}\}v^{*2}-(1-\sigma)m^*(1+3m^*\sigma+m^{*2})]}{6[(1+m^*)v^{*2}-2(1-\sigma)m^*]},$$

$$c = \frac{m^*(1-\sigma)(4m^{*2}+m^*-1+4m^*\sigma)-4v^{*2}(m^{*2}(1-\sigma)-m^*(1-2\sigma)+1)}{v^*[(1+m^*)v^{*2}-2(1-\sigma)m^*]},$$

$$d = \frac{v^*}{2},$$

It is easy to check that $b > 0$, c is always negative, while the sign of a is defined by the sign of m_1. Equation (2.9) is nothing but a two-dimensional generalization of the Gardner equation or the 2D Gardner equation.

3. Mechanisms of the rogue waves formation

3.1. Resonant waves interaction

It is known that plane solitary wave solution of the KP equation(2.1),

$$\eta = \frac{4k^2}{3}\cosh^{-2} k(\theta + mZ - \frac{3m^2+2k^2}{6}\tau), \tag{3.1}$$

is stable to transverse disturbances, and exact two-dimensional localized travelling wave solution requires an opposite sign at η_{ZZ} or at $\eta_{\theta\theta\theta\theta}$ [10, 11]. In our case it is unlikely. Hence we cannot anticipate an appearance of 2D localized wave from the single solitary wave. However, the KP equation possesses a two-solitary wave solution [10, 15],

$$\eta = \frac{4}{3}\frac{\partial^2}{\partial\theta^2}\log(F),\; F = 1+\exp(\xi_1)+\exp(\xi_2)+\exp(A_{12}+\xi_1+\xi_2),$$

$$\xi_i = k_i(\theta + m_i Z - \frac{3m_i^2+k_i^2}{6}\tau),\; \exp A_{12} = \frac{(k_1-k_2)^2-(m_1-m_2)^2}{(k_1+k_2)^2-(m_1-m_2)^2} \tag{3.2}$$

It contains a hump in the area of the waves interaction. The hump moves keeping its shape and velocity, and its maximum amplitude may be up to four times higher than the amplitude of each interacting solitary wave. Detailed analysis of this solution and its possible application to the rogue waves description is done in [4]. Now we only mention that this solution describes propagation but not a formation of the localized structure, the last should be presented in the initial condition.

Recently we developed the idea of the use of the waves interaction. However, we consider another solution of the KP equation [7] that does not describe the waves interaction at $t = 0$ in contrast to the exact two-solitary wave solution suggested in [4]. Numerical solution describes a formation of a localized high wave (or a stem) only when the angle between the incident crested waves lies within a certain interval. Our estimations demonstrate rather fast formation of the stem that may be a reason why the rogue wave appears suddenly for the crew of a vessel. An increase in amplitude is up to four times. Much higher increase is achieved, up to 14 (!) times when the incident waves with curved fronts interact [7]. In the last case, the 2D localized wave is unstable and exists for a short time period. However, fast formation of a high wave *near* the ship gives rise enough time to it to affect the ship before an instability destroys the wave.

The well-known exact solitary wave solution solution $\eta = \eta_0$ of the BO equation may be written as

$$\eta_0 = \frac{4b\sqrt{k}}{3\,\left(k\xi^2 + 1\right)}, \tag{3.3}$$

where $\xi = (\theta - 0.5b\sqrt{k}\tau)$. It accounts for a moving elevation plane wave in the two-dimensional case. It is always stable to transverse disturbances for the positive sign of b [8, 13, 16]. One can see that the wind cannot affect the sign of b. The sign of the amplitude is defined by the sign of b, and holes are not described by this solution.

Hence the 2D localization elevation wave may only arise due to the plane waves interaction. This process was studied numerically in [17] where it was found a similarity with the KP case. Again a stem appears due to the waves interaction and its amplitude depends upon the angle between the incident plane waves. However, now the highest amplitude may be eight times larger than the amplitudes of the incident waves. The case of the curved initial waves interaction is not considered by now.

A similar scenario may be realized for the known solitary wave solution $h = h_s$ of the Gardner equation [14]:

$$h_s = \frac{3b\,k^2}{a(B_1\cosh(k\xi) + 1)}, \tag{3.4}$$

where

$$B_1 = \sqrt{1 + \frac{9bck^2}{2a^2}},\ \xi = \theta - bk^2\,\tau.$$

In the two-dimensional case, Eq. (3.4) accounts for the propagation of a plane solitary wave. In contrast to the exact solution of the 2DBO equation now the amplitude may be of either sign. The hump propagation is described for positive values of a, while the wave with a trough propagates for $a < 0$. The solution has an interesting feature for negative c: tendency to the extensive trough shape at $k \to \sqrt{-2a^2/(9b\,c)}$. The amplitude of the wave tends to the limiting value equal to $-2a/3c$. This solution is stable to transverse disturbances [8]. Similar to the KP and 2DBO equations, numerical solution reveals a localization with a great increase in amplitude of the wave as a result of semi-plane or curved incident waves interaction [18]. The results look very similar to those of the KP equation. However, no stem is observed for small negative values of the coefficient at the cubic non-linear term. Possible reason is in the fail of reality of B_1 that happens in the exact solution (3.4) as soon as negative c decreases.

3.2. Transverse instability of plane solitary waves

The Gardner equation (2.9) possesses an extra solution $h = h_p$ that may be employed for explanation of the holes. It appears due to the balance between cubic nonlinearity and dispersion and may be written through the Jacobi elliptic function,

$$h_p = \sqrt{-\frac{2b}{c}}\, k\, \kappa\, \mathrm{sn}(k\xi, \kappa) - \frac{a}{3c} \tag{3.5}$$

where $\xi = \theta - s\tau$, κ, $0 < \kappa < 1$, is the Jacobi function modulus, while the velocity is $s = -bk^2(1+\kappa^2) - a^2/(3c)$. Its minimum is larger that the maximum for negative a. Moreover, the troughs in the solution became extensive as $\kappa \to 1$, and they are separated by the extensive areas of moderate elevation. In the two-dimensional case, these troughs are similar to the shape of the solution (3.4) with the exception of the absence of the limiting amplitude [8].

The transverse instability of the solution (3.5) is studied similar to that of the solitary wave solutions [8]. One can check that the wave (3.5) is always unstable. This may result in an appearance of a periodical train of the waves modulated in the transverse direction, hence the sequence of two-dimensional localized waves. Then another mechanism of the rogue waves and holes formation may be suggested based on the 2D localization due to the transverse instability. It is to be noted the exact solution of the KP equation that may be obtained when the sign at dispersive or transverse η_{yy} term in Eq.(2.1)is negative [19]. This solution describes a transverse modulation of an initial plane solitary wave which is accompanied by an increase in amplitude. However, this increase is not very high, 1.5-2 times. Much more increase may be achieved if a 2D input is used. It was found [20] that growth up to 12 times happens if the input is not so smooth as the 2D Gaussian distribution. The process of the formation of the stable 2D localized wave is fast that may be used to explain sudden appearance of the rogue wave near the ship.

4. Conclusions

The solutions of the model equations considered reflect main features of the abnormally high or deep waves: their fast but rare appearance. The first feature as well as the growth of the amplitude depend upon the shape of the incident waves and the angle between them. The second feature is caused by the strict restrictions required for the existence of the solution, in particular, m should be near the special value m^* for the solution of the Gardner equation. It is found that the conditions yielding 2D localization are governed by physical factors through the signs of the coefficients of the equations. Thus the sign of a separates an appearance of the elevation wave or the hole wave, that is defined by the sign of m_1 or by the ocean stratification. This dependence upon the physical factors is important for a prediction of the rogue waves and holes in the sea.

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Alexey PORUBOV, A.F.Ioffe Physical Technical Institute of the Russian Academy of Sciences,
Polytekhnicheskaya st. 26, 194021, Saint-Petersburg, RUSSIA
e-mail: porubov@math.ioffe.ru