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A. Casasso*

NONLINEAR WAVES IN PLANE COSSERAT SOLIDS

Abstract. As it is well known, the propagation of nonlinear strain waves may be governed by a nonlinear dispersive-dissipative equation. In this work we study models in one or two dimensions. The final purpose is to apply the analytic technique developed by Samsonov [7] to dynamic equations arising in the theory of microstructured solids. To do this we search to reduce the 2 or 4 Euler-Lagrange equations of the model to 1 or 2 p.d.e. each depending only on one field variable.

1. Introduction

Nonlinear wave dynamics in dissipative solids has been discussed in two recent books by A. Porubov [6] and A. Samsonov [7], with the same goal of obtaining and exploiting physically and mathematically meaningful results related to the propagation of solitary waves mostly in complex wave guides. In particular in [7] it is presented a method of reduction of the dynamic p.d.e. to a second order Lie equation, hence to the Abel equation, but only for the 1D-case.

Now, we want to apply the same passages even for the 2D-case. This is possible if, after, we have reduced the 4 Lagrange equations to a couple of partial differential equations each depending only on one field variable. This reduction is the main purpose of this work, and it is used in one-dimensional or bi-dimensional models.

We want to remark that in our model the non linearity is due to a strain energy density which depends on the deformation variables, both macro and micro, the dissipation is introduced trough a linear combination of strain velocities.

2. Method of reduction

The aim of this work is to find the travelling wave solution (TW) of the initial p.d.e., that depends only upon the phase variable $z = \pm Vt$ and describes the wave propagation along the x-axis in time t and velocity V. The process used can be resumed in two principal steps. In the first we reduce the Euler-Lagrange equations to one or two p.d.e. equations each depending only on one field variable. To do this we choose a suitable form of the potential energy W, we calculate the Euler-Lagrange equations (two for the one-dimensional case and four for the two-dimensional case) and then we reduce this system to the wanted partial differential equation using the method of the "slaving principle", (for a general treatment of this principle, see, for instance, [4]).

The second step requires to distinguish the one- from the two-dimensional case. For the one-dimensional case we introduce the phase variable $z = x + Vt$, where V is

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the wave's velocity. We consider the vector

$$\mathbf{r} = \mathbf{r}(x, t) = u(x, t)\mathbf{i}$$

for the macrostructure and the vector

$$\mathbf{d} = \mathbf{d}(x, t) = \varphi(x, t)\mathbf{i}$$

for the microstructure. So we transform the function u(x,t) in a function u(z) that depends only on the variable z:

$$u(x, t) = u(z)$$

For the two-dimensional case, the procedure is a bit more difficult. To simplify it, we suppose that the components of vectors **r** and of vector **d** depend only on one direction, such that we have:

$$\mathbf{r}(x, y, t) = u(x, t)\mathbf{i} + v(y, t)\mathbf{j}$$

and

$$\mathbf{d}(x, y, t) = \varphi(x, t)\mathbf{i} + \chi(y, t)\mathbf{j}.$$

Hence two phase variables z and z'

$$z = x \pm Vt, \quad z' = y \pm V't$$

and two new functions $u(z)$ and $v(z')$ are introduced:

$$u(x, t) = u(z), \quad v(y, t) = v(z')$$

Thereafter, we reduce the starting partial diferential equation to an ordinary differential equation using the funtion $u(z)$ for the one-dimensional case and $u(z)$ and $v(z')$ for the two-dimensional case. Then we reduce this ordinary differential equation to a second order Lie equation, we pass to an Abel equation, a first order equation, and, last step, we reduce, if possible, the Abel equation to the Weierstrass equation and we integrate it to find the solution called "soliton".

3. One-dimensional case

We consider now an example of model in one dimension. We deal with the vector **r** and **d** defined above and we choose the following form for the kinetic energy K and for the strain energy W:

$$K = \frac{1}{2}\left[\rho\dot{u}^2 + I\dot{\varphi}^2\right]$$

$$W = \frac{1}{2}\alpha u_x^2 + \frac{1}{6}\beta u_x^3 - A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{6}D\varphi_x^3$$

With the above mentioned formulas, the Euler-Lagrange equations which in general read

$$\begin{cases} \rho u_{tt} = \left(\dfrac{\partial W}{\partial u_x}\right)_x - \dfrac{\partial W}{\partial u} \\ \\ I\varphi_{tt} = \left(\dfrac{\partial W}{\partial \varphi_x}\right)_x - \dfrac{\partial W}{\partial \varphi} \end{cases}$$

become

$$(1) \qquad \begin{cases} \rho u_{tt} = \alpha u_{xx} + \beta u_x u_{xx} - A\varphi_x \\ \\ I\varphi_{tt} = C\varphi xx + D\varphi_x\varphi_{xx} + Au_x - B\varphi \end{cases}$$

To reduce this system to one partial differential equation depending only on the funtion u, we introduce the dimensionless form of the variables u, x, t and two new parameters:

$$U = \frac{u}{U_0},\ X = \frac{x}{L},\ T = \frac{c_0 t}{L},\ \delta = \left(\frac{l}{L}\right)^2,\ \epsilon = \left(\frac{U_0}{L}\right)$$

where L is the wave's length, l is the size of the microstructure. We suppose also that I, C and D verify the following equalities

$$I = \rho l^2 I^*,\ C = l^2 C^*,\ D = l^2 D^*$$

Now we use the slaving principle. It means that φ is determined in terms of U_x using a power expansion: $\varphi = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + ...$ The dimensionless form for equation $(1)_2$ yiels this expression for φ:

$$(2) \qquad \varphi = \epsilon\frac{A}{B}U_X + \frac{\delta}{B}\left(C^*\varphi_{XX} - \alpha I^*\varphi_{TT} + \frac{D^*}{L}\varphi_X\varphi_{XX}\right)$$

We evaluate φ_0 and φ_1 in terms of U and its partial derivatives obtaining:

$$\varphi_0 = \epsilon\frac{A}{B}U_X \qquad \varphi_1 = \epsilon\frac{A}{B^2}\left(C^*U_{XXX} - \alpha I^*U_{XTT} + \frac{D^*A\epsilon}{BL}U_{XX}U_{XXX}\right)$$

Inserting them into the governing equation $(1)_2$ in its dimensionless form, we get finally the single differential equation for U:

$$U_{TT} = U_{XX} + \frac{\beta\epsilon}{\alpha}\left(U_X^2\right)_X - \frac{A^2}{\alpha B}U_{XX} - \frac{\delta A^2}{\alpha B^2}C^*U_{XXXX} + \\ + \frac{\delta A^2}{\alpha B^2}\left[\alpha I^*U_{XXTT} - \frac{D^*A\epsilon}{BL}\left(U_{XX}U_{XXX}\right)_X\right]$$

To apply the second step in a simple way we suppose $D = 0$ in the previous model. In this case the microstrutured part of the strain energy depends only on φ and φ_x^2, and W is written as

$$W = \frac{1}{2}\alpha u_x^2 + \frac{1}{6}\beta u_x^3 - A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2$$

Let us introduce three positive dimensionless parameters:

- $\epsilon := V << 1$ accounting for the elastic strain;
- $\delta := \frac{l^2}{L^2} << 1$ characterizing the ratio between the microstructure size and the wave lenght;
- $\gamma := \frac{d}{l}$ characterizing the influence of the dissipation.

We assume the dissipation is weak and we introduce the functions $\nu = u_x$. Then the governing nonlinear p.d.e. for the macrostrain $\nu(x, t)$ is:

$$\nu_{tt} - \nu_{xx} - \epsilon\alpha_1(\nu^2)_{xx} - \gamma\alpha_2\nu_{xxt} + \delta(\alpha_3\nu_{xxxx} - \alpha_4\nu_{xxtt}) + \gamma\delta(\alpha_5\nu_{xxxxt} + \alpha_6\nu_{xxttt}) = 0$$

where $\alpha_1, ..., \alpha_6$ are given in [2]. When $\epsilon = O(\delta)$ nonlinearity and dispersion are in balance. If in addition $\gamma = 0$ we have the non-dissipative case governed by the double dispersive equation:

$$\nu_{tt} - \nu_{xx} - \epsilon\left[\alpha_1(\nu^2)_{xx} - \alpha_3\nu_{xxxx} + \alpha_4\nu_{xxtt}\right] = 0$$

Using the function $\nu(z) = \nu(x, t)$ and the boundary conditions $\frac{\partial^k \nu}{\partial z^k} \to 0$ for $|z| \to \infty$, $k = 0, 1, 2, 3$ we obtain the Abel equation:

$$v' = \frac{(V^2 - 1)}{\epsilon\alpha_3 - \epsilon V^2\alpha_4}\nu v^3 - \frac{\epsilon\alpha_1}{\epsilon\alpha_3 - \epsilon V^2\alpha_4}\nu^2 v^3$$

and the Weierstrass equation.

$$(\nu')^2 = \frac{(V^2 - 1)}{\epsilon\alpha_3 - \epsilon V^2\alpha_4}\nu^2 - \frac{2\epsilon\alpha_1}{\epsilon\alpha_3 - \epsilon V^2\alpha_4}\frac{\nu^3}{3}$$

The exact bell-shaped travelling solitary wave solution arises as a result of balance between nonlinear and dispersive terms and it is given by:

$$\nu(z) = \frac{3(V^2 - 1)}{2\epsilon\alpha_1}sech^2\left[2\sqrt{\frac{\epsilon\alpha_3 - \epsilon V^2\alpha_4}{V^2 - 1}}(z - c)\right]$$

4. 2D-case

Now we try to do the same thing for a two-dimensional model. In this case, we suppose that the vectors **r** and **d** are written in this form

$$\mathbf{r} = \mathbf{r}(x, y, t) = u(x, t)\mathbf{i} + v(y, t)\mathbf{j}$$

$$\mathbf{d} = \mathbf{d}(x, y, t) = \varphi(x, t)\mathbf{i} + \chi(y, t)\mathbf{j}$$

We can see that in this case u and φ depend only on the direction **i** and v and χ depend on the direction **j**. We choose the strain energy in terms of $u_x,\ v_y,\ \varphi,\ \chi,\ \varphi_x,\ \chi_y$ as follows:

$$W = \frac{1}{2}Au_x^2 + \frac{1}{2}Bv_y^2 + \frac{1}{2}C\varphi^2 + \frac{1}{2}D\chi^2 + \frac{1}{2}E\varphi_x^2 + \frac{1}{2}F\chi_y^2 + \frac{1}{2}Gu_x^2\varphi + \frac{1}{2}Hv_y^2\chi$$

Then the Euler-Lagrange equations become:

$$\left\{\begin{aligned} &\rho u_{tt} = Au_{xx} + Gu_{xx}\varphi + Gu_x\varphi_x \\ &\rho v_{tt} = Bv_{yy} + Hv_{yy}\chi + Hv_y\chi_y \\ &I\varphi_{tt} = E\varphi_{xx} - C\varphi - \frac{1}{2}Gu_x^2 \\ &I\chi_{tt} = F\chi_{yy} - D\chi - \frac{1}{2}Hv_y^2 \end{aligned}\right. \tag{3}$$

To obtain two partial differential equations, each one depending only on one field variable, we couple equation $(3)_1$ to $(3)_3$ and equation $(3)_2$ to $(3)_4$.

As in the one-dimensional case, we introduce the dimensionless variables and parameters

$$U = \frac{u}{U_0},\ V = \frac{v}{V_0},\ X = \frac{x}{L},\ Y = \frac{y}{L},\ T = \frac{c_0 t}{L},\ \delta = \left(\frac{l}{L}\right)^2, \epsilon = \left(\frac{U_0}{L}\right)$$

and we also suppose that I, E and F verify the following equalities:

$$I = \rho l^2 I^*,\ E = l^2 E^*,\ F = l^2 F^*$$

Now we must determine φ in terms of U_x and χ in terms of V_y. We expand them in powers of δ and we use the slaving principle in the same way as in the 1D-case. We get (from eqs. (3)):

$$\begin{cases} U_{TT} = U_{XX} + \dfrac{G}{A}\left(U_{XX}\varphi + U_X\varphi_X\right) \\ \varphi = -\dfrac{G\epsilon^2}{2C}U_X^2 + \dfrac{\delta}{C}(E^*\varphi_{XX} - AI^*\varphi_{TT}) \end{cases}$$

$$\begin{cases} V_{TT} = V_{YY} + \dfrac{H}{B}\left(V_{YY}\chi + V_Y\chi_Y\right) \\ \chi = -\dfrac{H\epsilon^2}{2D}V_Y^2 + \dfrac{\delta}{D}(F^*\chi_{YY} - BI^*\chi_{TT}) \end{cases}$$

We proceed as in the one-dimensional case, and finally we find a couple of partial differential equation depending on the function U and V. In conclusion we have trasformed the starting system of four Euler-Lagrange equations in the functions u, v, φ, χ in this system of two equations in the functions U and V:

$$\begin{cases} U_{TT} = U_{XX} - \dfrac{\epsilon^2G^2}{AC}\left\{\dfrac{1}{2}U_X^2U_{XX} + \dfrac{\delta}{C}U_{XX}\left[E^*\left(U_XU_{XX}\right)_X - AI^*\left(U_XU_{XT}\right)_T\right]\right. \\ \qquad\left. -U_{XX}^2 + \dfrac{\delta}{C}U_X\left[E^*\left(U_XU_{XX}\right)_{XX} - AI^*\left(U_XU_{XT}\right)_{XT}\right]\right\} \\ V_{TT} = V_{YY} - \dfrac{\epsilon^2H^2}{BD}\left\{\dfrac{1}{2}V_Y^2U_{YY} + \dfrac{\delta}{D}V_{YY}\left[F^*\left(V_YV_{YY}\right)_Y - BI^*\left(V_YV_{YT}\right)_T\right]\right. \\ \qquad\left. -V_{YY}^2 + \dfrac{\delta}{D}V_Y\left[F^*\left(V_YV_{YY}\right)_{YY} - BI^*\left(V_YV_{YT}\right)_{YT}\right]\right\} \end{cases}$$

Now we consider another two-dimensional model. The vectors **u** and **d** are the same as in the previous case while the strain energy W is slightly different: we add cubic terms for the derivatives of u, v, φ, χ:

$$\begin{aligned} W =& \frac{1}{2}A(u_x^2 + v_y^2) + \frac{1}{2}B(\varphi^2 + \chi^2) + \frac{1}{2}C(\varphi_x^2 + \chi_y^2) \\ &+\frac{1}{2}D(u_x^2\varphi + v_y^2\chi) + \frac{1}{6}E(u_x^3 + v_y^3) + \frac{1}{6}F(\varphi_x^3 + \chi_y^3) \end{aligned}$$

So, we obtain the Lagrange equations in this forms:

$$\begin{cases} \rho u_{tt} = Au_{xx} + Du_{xx}\varphi + Du_x\varphi_x + Eu_xu_{xx} \\ \rho v_{tt} = Av_{yy} + Dv_{yy}\chi + Dv_y\chi_y + Ev_yv_{yy} \\ I\varphi_{tt} = C\varphi_{xx} + F\varphi_x\varphi_{xx} - B\varphi - \dfrac{1}{2}Du_x^2 \\ I\chi_{tt} = C\chi_{yy} + F\chi_y\chi_{yy} - B\chi - \dfrac{1}{2}Dv_y^2 \end{cases} \tag{4}$$

The procedure is the same as in the previous case, and so, we omit all the passages. We can prove that, from equations $(4)_1$ and $(4)_3$ we obtain the following system

$$\begin{cases} U_{TT} = U_{XX} + \frac{D}{A}\left(U_{XX}\varphi + U_X\varphi_X\right) + \frac{E\epsilon}{A}U_X U_{XX} \\ \varphi = -\frac{D\epsilon^2}{2B}U_X^2 + \frac{\delta}{B}\left(C^*\varphi_{XX} - AI^*\varphi_{TT} + \frac{F^*}{L}\varphi_X\varphi_{XX}\right) \end{cases}$$

and from equation $(4)_2$ and $(4)_4$ the system

$$\begin{cases} V_{TT} = V_{YY} + \frac{D}{A}\left(V_{YY}\chi + V_Y\chi_Y\right) + \frac{E\epsilon}{A}V_Y V_{YY} \\ \chi = -\frac{D\epsilon^2}{2B}V_Y^2 + \frac{\delta}{B}\left(C^*\chi_{YY} - AI^*\chi_{TT} + \frac{F^*}{L}\chi_Y\chi_{YY}\right) \end{cases}$$

Now we write φ and χ in terms of U and V using the slaving principle. Our finally result is the reduction of the starting system (4) to this system:

$$\begin{cases} U_{TT} = U_{XX}\left(1 - \dfrac{3D^2\epsilon^2}{2AB}U_X^2\right) - \dfrac{\delta D^2\epsilon^2}{AB^2}\Big\{C^*\Big[U_{XX}\left(U_{XT}^2 + (U_{XT}^2)_X\right) + (U_X^2 U_{XTT})_X\Big] \\ \qquad - AI^*\Big[U_{XX}(U_X + U_{XT})_T + U_X(U_{XX} + U_{XXT})_T\Big] \\ \qquad - \dfrac{D\epsilon^2F^*}{BL}\Big[U_X U_{XX}(2U_{XX}^3 U_X^3 U_{XXXX}) + U_{XXX}(U_X^3 U_{XX})_X\Big]\Big\} + \dfrac{\epsilon E}{A}U_X U_{XX} \\ \\ V_{TT} = V_{YY}\left(1 - \dfrac{3D^2\epsilon^2}{2AB}V_Y^2\right) - \dfrac{\delta D^2\epsilon^2}{AB^2}\Big\{C^*\Big[V_{YY}\left(V_{YT}^2 + (V_{YT}^2)_Y\right) + (V_Y^2 U_{YTT})_Y\Big] \\ \qquad - AI^*\Big[V_{YY}(V_Y + V_{YT})_T + V_Y(V_{YY} + V_{YYT})_T\Big] \\ \qquad - \dfrac{D\epsilon^2F^*}{BL}\Big[V_Y V_{YY}(2V_{YY}^3 V_Y^3 V_{YYYY}) + V_{YYY}(V_Y^3 V_{YY})_Y\Big]\Big\} + \dfrac{\epsilon E}{A}V_Y V_{YY} \end{cases}$$

5. Conclusions

At this stage we have obtained the Abel equation, following the method presented by Samsonov [7], only for a one-dimensional model. The pourpose for future researches is to extend this work also to a two-dimensional model. In particular we want to reduce, if it is possible, the four Euler-Lagrange equations of a two-dimensional model, to a system of two Abel equations. Thereafter, our aim is to obtain a soliton solution.

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Alessia CASASSO, Department of Mathematics, University of Turin, Via Carlo Alberto 10, 10123, Turin, ITALY
e-mail: `alessia.casasso@unito.it`