Rend. Sem. Mat. Univ. Pol. Torino - Vol. 65, 2 (2007)
ISASUT Intensive Seminar

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# THERMOELASTIC STRESS ANALYSIS FOR LINEAR THERMOELASTIC BODIES

**Abstract.** The thermoelastic stress analysis for linear thermoelastic bodies is developed as a mathematical support to the infra-red radiometric method (SPATE) applied to structures under cyclic loading conditions.

## 1. Introduction

The use of high-tech materials for structural applications has shown in recent years that the study of mechanical behaviour of such materials is often inadequate. Studies on the interaction between mechanical and thermal effects in solid bodies have therefore received considerable attention.

Thermographic stress analysis has been adopted as a particularly convenient mean of experimental stress analysis based on the thermoelastic effect. The thermoelastic stress analysis tecnique is based upon the use of the SPATE (Stress Pattern Analysis by the measurement of Thermal Emission) equipment for the radiometric monitoring of the temperature changes induced by cyclic loading in the elastic range [1], [2], [9], [10], [11].

In [8] a theoretical analysis of the thermoelastic effect has been developed in order to provide a mathematical model as a support to SPATE and the results of tests carried out on concrete and mortar [1] are reported as experimental evidences of the theory described.

The aim of this paper is to generalize the results obtained in [8] : the intrinsic formulation of the linear theory of thermoelasticity is adopted [3] and the linear relations given in [8] between the variation of temperature and the variation of stress are obtained after suitable assumptions.

In Section 2, within the linear theory of elasticity for an isotropic continuum body, we deduce from the First Law of Thermodynamics a differential equation which gives a relation between stress and temperature.

In Section 3 we integrate this equation and we remark that if the principal stress components are two and three the solution depends on the first and the second invariant of the stress while the second invariant vanishes in the case of one component of the principal stress.

In Section 4 we show that if we linearize the equations obtained in Section 3, we get, at least for the case of dimension one, the same results of [9].

*This work was supported by the Italian M.U.R.S.T. research project “Modelli matematici per la scienza dei materiali” (Cofin 2002).

In Section 5 we apply the results obtained in Section 3 to the SPATE model considering sinusoidal principal components of stress.

## 2. The mathematical equations

Let us consider a continuum body as defined in [4] and let us assume the bounded regular region of space occupied by the body in a fixed reference configuration be closed with respect to mass transfer and open with respect to energy transfer. The body can be considered a thermodynamical closed system.

According to [3] we recall that the local form of the First Law of Thermodynamics for the system considered is:

$$(1) \qquad \rho\dot{e} = \mathbf{S}\cdot\dot{\mathbf{F}} - \mathrm{div}\mathbf{q} + \rho r,$$

where $e$ is the internal energy per unit mass, $\mathbf{q}$ is the heat flux vector per unit surface area and unit time, $r$ is the heat supply per unit mass and unit time, $\rho$ is the mass density, $\mathbf{S}$ and $\mathbf{F}$ are respectively the first Piola-Kirchhoff stress tensor and the deformation gradient.

If $\eta$ is the entropy per unit mass and $\vartheta$ the absolute temperature, we introduce the free energy per unit mass [3]

$$(2) \qquad \psi = e - \eta\vartheta.$$

If we assume that the body is elastic, the Second Law of Thermodynamics implies the following restrictions:

$$(3) \qquad \psi = \hat{\psi}(\mathbf{F},\vartheta), \qquad \mathbf{S} = \hat{\mathbf{S}}(\mathbf{F},\vartheta), \qquad \eta = \hat{\eta}(\mathbf{F},\vartheta)$$

$$(4) \qquad \hat{\mathbf{S}}(\mathbf{F},\vartheta) = \rho\partial_{\mathbf{F}}\hat{\psi}(\mathbf{F},\vartheta), \qquad \hat{\eta}(\mathbf{F},\vartheta) = -\partial_\vartheta\hat{\psi}(\mathbf{F},\vartheta).$$

If we differentiate (2) and $(3)_1$ with respect to time, by means of (4), from (1) we get:

$$(5) \qquad \rho\eta\dot{\vartheta} = -\mathrm{div}\mathbf{q} + \rho r.$$

Defining the finite strain tensor $\mathbf{D}$ by [3], [4]:

$$(6) \qquad \mathbf{D} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right),$$

where $\mathbf{1}$ is the unit tensor, the restrictions (3) and (4) are substituted by:

$$(7) \qquad \psi = \tilde{\psi}(\mathbf{D},\vartheta), \qquad \mathbf{S} = \mathbf{F}\tilde{\mathbf{S}}(\mathbf{D},\vartheta), \qquad \eta = \tilde{\eta}(\mathbf{D},\vartheta)$$

and

$$(8) \qquad \tilde{\mathbf{S}}(\mathbf{D},\vartheta) = \rho\partial_{\mathbf{F}}\tilde{\psi}(\mathbf{D},\vartheta), \qquad \tilde{\eta}(\mathbf{D},\vartheta) = -\partial_\vartheta\tilde{\psi}(\mathbf{D},\vartheta),$$

as a consequence of material frame indifference [3], [4].

By differentiating $(8)_2$ with respect to time and introducing the specific heat at constant deformation [3]:

$$c = \vartheta\,\partial_\vartheta\,\tilde{\eta}(\mathbf{D}, \vartheta), \tag{9}$$

from (5) we get:

$$-\vartheta\,\partial_\vartheta\,\tilde{\mathbf{S}} \cdot \dot{\mathbf{D}} + \rho c \dot{\vartheta} = -\mathrm{div}\mathbf{q} + \rho r. \tag{10}$$

Now we assume that the gradient displacement $\nabla\mathbf{u}$ and its rate of change $\nabla\dot{\mathbf{u}}$ are small and that the body is subjected to small increment of temperature.

Nevertheless the stress tensor $\tilde{\mathbf{S}}$ depends on the temperature through the material functions which appear in the constitutive equations.

Therefore, introducing the infinitesimal strain tensor [3], [4]:

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{11}$$

if

$$\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}, \tag{12}$$

from (6) we get:

$$\mathbf{D} = \mathbf{E} + \frac{1}{2}\nabla\mathbf{u}^T\nabla\mathbf{u}. \tag{13}$$

If we assume that the body is isotropic, then the constitutive equation is [3]:

$$\mathbf{S} = 2\mu\mathbf{E} + [\lambda\mathrm{tr}\mathbf{E} - \beta(\vartheta - \vartheta_0)]\mathbf{1}. \tag{14}$$

In the above equation $\lambda$ and $\mu$ are the Lamé moduli, while $\beta$ is related to the coefficient of linear thermal expansion $\alpha$ by [11]:

$$\beta = (3\lambda + 2\mu)\alpha \tag{15}$$

From the experimental data we deduce that $\lambda$ and $\mu$ are functions of the temperature $\vartheta$ ([6], [11]).

Moreover from (13) and the assumption of linear approximation for the gradient displacement we can obtain:

$$\dot{\mathbf{D}} \cong \dot{\mathbf{E}} \tag{16}$$

Therefore by substituting (14) in (10), with the use of (16) we get the following equation:

$$\rho c\dot{\vartheta} = -\mathrm{div}\mathbf{q} + \rho r + \vartheta\,\{2\partial_\vartheta\,\mu\mathbf{E} + [\partial_\vartheta\,\lambda\mathrm{tr}\mathbf{E} - \partial_\vartheta\,\beta(\vartheta - \vartheta_0) - \beta]\mathbf{1}\}\,\dot{\mathbf{E}} \tag{17}$$

Let us assume that the thermodynamical process is adiabatic without internal heat sources, that is [3]:

$$-\mathrm{div}\mathbf{q} + \rho r = 0 \tag{18}$$

From experimental results and from the analysis of the order of magnitude we can deduce that $\partial_\vartheta \beta(\vartheta - \vartheta_0)$ is negligible with respect to $\beta$, while $\partial_\vartheta \mu \mathbf{E}$ and $\partial_\vartheta \lambda \mathrm{tr}\mathbf{E}$ are relevant, therefore, with the use of (18), equation (17) can be written as:

$$\rho c \frac{\dot{\vartheta}}{\vartheta} = 2\partial_\vartheta \mu \mathbf{E} \cdot \dot{\mathbf{E}} + [\partial_\vartheta \lambda \mathrm{tr}\mathbf{E} - \beta]\, \mathrm{tr}\dot{\mathbf{E}}. \tag{19}$$

Equation (19) represents the thermoelastic coupling between strain and temperature.

Let us now deduce from (19) a similar expression in terms of stress. Because of the isotropy of the material we can assume that $\mu > 0$, $3\lambda + 2\mu > 0$ [5]. Therefore the constitutive equation (14) can be inverted in

$$\mathbf{E} = \frac{1}{2\mu}\mathbf{S} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{tr}\mathbf{S}\,\mathbf{1} + \alpha(\vartheta - \vartheta_0)\mathbf{1}. \tag{20}$$

Let us now recall the well known relations between the Lamé moduli $\lambda,\ \mu$ and the Poisson's ratio $\nu$ and the Young's modulus $E$:

$$\begin{cases} \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \\[2ex] \mu = \dfrac{E}{2(1+\nu)} \end{cases} \tag{21}$$

By means of (21) we get $\partial_\vartheta \lambda$ and $\partial_\vartheta \mu$ in terms of $\partial_\vartheta E$ and $\partial_\vartheta \nu$, then with the use of (20) and the assumptions that both $(\vartheta - \vartheta_0)$ and its time-derivative are negligible, from (19) we deduce:

$$\frac{\dot{\vartheta}}{\vartheta} = -\left[\frac{\alpha}{\rho c} - \Gamma_1 \mathrm{tr}\mathbf{S}\right]\mathrm{tr}\dot{\mathbf{S}} + \Gamma_2 \mathbf{S} \cdot \dot{\mathbf{S}} \tag{22}$$

where

$$\begin{cases} \Gamma_1 = \dfrac{1}{\rho c}\left[-\dfrac{\nu}{E^2}\partial_\vartheta E + \dfrac{1}{E}\partial_\vartheta \nu\right] \\[2ex] \Gamma_2 = \dfrac{1}{\rho c}\left[\dfrac{1+\nu}{E^2}\partial_\vartheta E - \dfrac{1}{E}\partial_\vartheta \nu\right] \end{cases} \tag{23}$$

Let us now set:

$$\begin{cases} \sigma = I_1 = \mathrm{tr}\mathbf{S} \\ \dot{\sigma} = \mathrm{tr}\dot{\mathbf{S}} \\ \sigma_i = (\mathbf{S})_i \\ \dot{\sigma}_i = (\dot{\mathbf{S}})_i \end{cases} \qquad \text{with } i = 1, 2, 3 \tag{24}$$

where $(\mathbf{S})_i$ are the principal stresses with respect to an orthonormal basis $(\mathbf{e}_i)$. If we recall that $\mathbf{S} \cdot \dot{\mathbf{S}} = \sum_{i=1}^{3} (\mathbf{S})_i (\dot{\mathbf{S}})_i$, with the use of the (24), the relation (22) becomes:

$$\frac{\dot{\vartheta}}{\vartheta} = -\left[\frac{\alpha}{\rho c} - \Gamma_1 \sigma\right] \dot{\sigma} + \Gamma_2 \sum_{i=1}^{3} \sigma_i \dot{\sigma}_i \tag{25}$$

## 3. Stress-temperature relations

In order to get from (25) a relation between temperature and stress we must suppose that:

i) the mass density $\rho$ is constant during the thermodynamical process;

ii) the partial derivatives with respect to temperature $\vartheta$ of the Poisson's ratio $\nu$ and of the Young's modulus $E$ are constant with respect to time and temperature.

It follows therefore that $\Gamma_1$ and $\Gamma_2$ given by (23) are constant. If we introduce the "thermoelastic constant"

$$k = \frac{\alpha}{\rho c} \tag{26}$$

and we denote with $\tau$ the second invariant of the stress tensor:

$$\tau = I_2 = \sigma_1 \sigma_2 + \sigma_1 \sigma_3 + \sigma_2 \sigma_3, \tag{27}$$

under the assumption that $\vartheta = \vartheta_0$, $\sigma = \sigma_0$ and $\tau = \tau_0$ for $t = 0$, the integration of (25) with respect to time gives:

$$\ln \frac{\vartheta}{\vartheta_0} = -k(\sigma - \sigma_0) + \frac{1}{2}(\Gamma_1 + \Gamma_2)\left(\sigma^2 - \sigma_0^2\right) - \Gamma_2(\tau - \tau_0). \tag{28}$$

It is interesting to remark that in the two-dimensional case that is when the principal stresses are:

$$\sigma_1 \neq 0, \qquad \sigma_2 \neq 0, \qquad \sigma_3 = 0 \tag{29}$$

equation (28) still holds assuming $i = 1, 2$ in (24) and replacing (27) by

$$\tau = I_2 = \sigma_1 \sigma_2,$$

while in the one-dimensional case in which

$$\sigma_1 \neq 0, \qquad \sigma_2 = \sigma_3 = 0 \qquad \text{and } \sigma_1 = \sigma \tag{30}$$

the second invariant of the stress tensor vanishes and (28) is replaced by:

$$\ln \frac{\vartheta}{\vartheta_0} = -k(\sigma - \sigma_0) + \frac{1}{2}(\Gamma_1 + \Gamma_2)\left(\sigma^2 - \sigma_0^2\right). \tag{31}$$

Let us remark that the solution of (25) shows a non linear dependence on the first invariant of the stress in all the cases (see (28) and (31)) while only in two and three-dimensional cases a linear dependence on the second invariant appears.

## 4. The linear case

In classical theory [2], [7] from the generalised heat conduction equation, by assuming that straining occurs adiabatically with no conduction of heat and with no heat supply, the Kelvin formula has been deduced:

$$\frac{\Delta\vartheta}{\vartheta_0} = -k\Delta\sigma \tag{32}$$

with $k$ thermoelastic constant defined in (26).

We shall prove that, from (28) and (31) of the previous section, after suitable linearizations, it is possible to get linear relations between the variation of temperature and the variation of stress which are comparable to (32).

Infact, let us define the average values of the principal stresses, of the stress invariants and of the temperature by the following linear relations:

$$\begin{gathered} \sigma_{im} = \frac{1}{2}(\sigma_i + \sigma_{i0}), \quad \sigma_{i0} = \sigma_i \quad \text{for } t = 0,\ i = 1, 2, 3 \\ \sigma_m = \frac{1}{2}(\sigma + \sigma_0), \quad \tau_m = \frac{1}{2}(\tau + \tau_0), \quad \vartheta_m = \frac{1}{2}(\vartheta + \vartheta_0) \end{gathered} \tag{33}$$

We assume moreover that the principal stresses, the stress invariants and the temperature are related to their variations by:

$$\begin{gathered} \sigma_i = \sigma_{im} + \Delta\sigma_i, \quad i = 1, 2, 3 \\ \sigma = \sigma_m + \Delta\sigma, \quad \tau = \tau_m + \Delta\tau, \quad \vartheta = \vartheta_m + \Delta\vartheta \end{gathered} \tag{34}$$

From (33) with the use of (34) we get:

$$\begin{gathered} \sigma_i - \sigma_{i0} = 2\Delta\sigma_i \quad i = 1, 2, 3 \\ \sigma - \sigma_0 = 2\Delta\sigma, \quad \tau - \tau_0 = 2\Delta\tau, \quad \vartheta - \vartheta_0 = 2\Delta\vartheta \end{gathered} \tag{35}$$

By using the relations (33), (34) and (35) and linearizing the logarithm, as usually done if $|2\Delta\vartheta/\vartheta_0| < 1$:

$$\ln\left(\frac{\vartheta}{\vartheta_0}\right) = \ln\left(1 + \frac{2\Delta\vartheta}{\vartheta_0}\right) \approx \frac{2\Delta\vartheta}{\vartheta_0} \tag{36}$$

we obtain from (29) and (31) the linearized equations:

$$\begin{gathered} \frac{\Delta\vartheta}{\vartheta_0} = [-k + (\Gamma_1 + \Gamma_2)\,\sigma_m]\,\Delta\sigma - \Gamma_2\Delta\tau \\ \frac{\Delta\vartheta}{\vartheta_0} = [-k + (\Gamma_1 + \Gamma_2)\sigma_m]\,\Delta\sigma \end{gathered} \tag{37}$$

Let us remark that $(37)_2$ can be compared with the Kelvin's formula but the coefficient of $\Delta\sigma$ is the thermoelastic constant plus a coefficient depending on $\sigma_m$ and on an elasticity modulus [9]. We remark also that $(37)_1$ shows a dependence on the variation of the first and the second invariant of the stress and on the Poisson and Young moduli .

## 5. SPATE model results

The SPATE model allows us to obtain the temperature variation related to the elastic deformation by spectroscopical experimental analysis.

The fundamental ipothesis of the SPATE model are the following:

1) the deformation must be adiabatic,

2) there must not exist other heat sources.

Now the relations obtained in Section 3 will be applied to SPATE model, analysing a cyclic loading [11] in which the temperature variation $(\vartheta - \vartheta_0)$ is small compared to the initial temperature $\vartheta_0$. We assume that the principal stress are:

$$\sigma_i = \sigma_{im} + a_i \sin(\omega t), \quad i = 1, 2, 3 \tag{38}$$

where $\sigma_{im}$, with $i = 1, 2, 3$, denote the average stresses and $a_i$, with $i = 1, 2, 3$, are arbitrary constants. Let us set:

$$\sigma_m = \sum_{i=1}^{3} \sigma_{im}, \qquad a = \sum_{i=1}^{3} a_i \tag{39}$$

The equation (28), with the use of (36), $(37)_1$, (38) and (39), becomes:

$$\frac{2\Delta\vartheta}{\vartheta_0} = A\sin(\omega t) + B[1 - \cos(2\omega t)] \tag{40}$$

where

$$\begin{cases} A = -ka + \Gamma_1 \sigma_m a + \Gamma_2 \displaystyle\sum_{i=1}^{3} \sigma_{im} a_i \\ B = \dfrac{1}{4}(\Gamma_1 + \Gamma_2)\, a^2 - \dfrac{1}{2}\Gamma_2 (a_1 a_2 + a_1 a_3 + a_2 a_3) \end{cases} \tag{41}$$

Formulas (40), (41) are the generalization of the results given in [11] and include the particular cases of dimension 1 and 2. Infact, if $i = 1, 2$ formulas (38)-(41) are valid replacing $(41)_2$ by

$$B = \frac{1}{4}(\Gamma_1 + \Gamma_2)\, a^2 - \frac{1}{2}\Gamma_2 a_1 a_2;$$

if $i = 1$ then

$$\sigma_1 = \sigma = \sigma_m + a\sin(\omega t), \quad \sigma_m = \sigma_{1m}, \quad a = a_1, \tag{42}$$

(39) and (40) still hold, with:

$$\begin{cases} A = -ka + (\Gamma_1 + \Gamma_2)\,\sigma_m a \\ B = \dfrac{1}{4}\,(\Gamma_1 + \Gamma_2)\,a^2 \end{cases} \tag{43}$$

Let us remark that, in the case of dimension 1, from (43) and (23) we deduce that the elastic behaviour of the material is described by the Young modulus, while in the cases of dimension 2 and 3 both Poisson and Young modulus are involved, as it turns out from (41) and (23). Moreover in case of dimension 1 it is possible to analyze the relation between the temperature variation and the stress variation. According to [1], let us replace in (42), (43) the constant a by the variation of stress $\Delta\sigma$ and let us assume in $(43)_1$ that the thermoelastic constant $k$ be relevant with respect to the other terms. Then from (40) we observe that according to the first addendum of the right member there is a loss of temperature for an increase of stress and an increase of temperature for a reduction of stress. The second addendum of the right member of (40) is relevant only for large variation of stress. In [1] a detailed discussion of tests performed on various specimens is reported and the conclusions are in agreement with our remarks.

## 6. Conclusions

The assumption of the dependence of the Poisson and Young moduli of elasticity from the temperature $\vartheta$ allows us to get a relation between the temperature variation and the stress variation. This relation presents coefficients which are not only the thermoelastic constant, as in the classical Kelvin's formula, but depend on the average stress, on the first and second invariant of the stress and on the modulus of elasticity. This result is supported by experimental results [11]. The application of our results to the SPATE model gives a solution which is the superposition of two cyclic functions with a phase difference of $\pi/2$ one with respect to the other. The relation between the temperature variation and the stress variation obtained can be compared with the experimental results given in [1].

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**AMS Subject Classification: 74A15, 74B05.**

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