Rend. Sem. Mat. Univ. Pol. Torino - Vol. 65, 2 (2007)
ISASUT Intensive Seminar

A.V. Porubov

ANALYTICAL SOLUTIONS AND UNSTEADY PROCESSES GOVERNED BY NON-LINEAR NON-INTEGRABLE EQUATIONS

Abstract. The benefit of finding particular exact and asymptotic solutions of non-integrable non-linear partial differential equations is considered. It is shown how explicit analytical solutions predict important features of the waves behavior even outside their formal applicability. In particular, an arbitrary initial pulse splits in a numerical solution into the train of localized waves each described by the analytical travelling wave solution. This happens both for the bell-shaped and kink-shaped localized waves. Also numerical simulations demonstrate an incident wave amplification/attenuation to a stable wave with the amplitude and velocity defined by the relationships obtained via an asymptotic solution. Physically reasonable equations are used to illustrate above mentioned statements.

1. Introduction

It would be nice to obtain an analytical solution of a governing non-linear equation. Most of the mathematical work in the realm of non-linear phenomena refers to integrable equations and their exact solutions. In this case rather general methods may be employed to obtain general solutions, see, e.g., [1, 2, 3]. Unfortunately, most of non-linear equations are non-integrable, and only particular solutions may be found.

Of special interest are the solutions that keep their shape on propagation. One of them is a bell-shaped solitary wave, see Fig. 1(a), that arises thanks to a balance between nonlinearity and dispersion. Another one is a shock wave or a kink-shaped wave, see Fig. 1(b), that usually appears due to a balance between nonlinearity and dissipation. One can find a lot of papers where particular exact solutions of these kinds are obtained. However, most of them do not consider an application of the solutions to the real physical problems. Indeed, exact travelling wave solutions of non-integrable equations are obtained as a rule. Hence, they require specific initial conditions. Moreover, some solutions do not contain free parameters, and special relationships between the equation coefficients are needed for their existence. Asymptotic solutions of non-linear equations are not travelling wave ones without fail. However, they usually describe a particular process, e.g., evolution of a single solitary wave [1]. Real physical problem requires a more general solution, in particular, evolution of an initial pulse of arbitrary shape. Usually such a problem may be solved only numerically. That is why people prefer to deal with numerical simulations. However, a solution of a non-linear equation is very sensitive to the values of the equation coefficients and to the initial conditions; this may be missed in a numerical modelling. Also numerical results may happen to be unusual, and its justification is needed. So, a natural question arises: may one employ particular analytical solutions to predict a behavior in the general problem when only numerical solution may be obtained? May the relationships between the coefficients

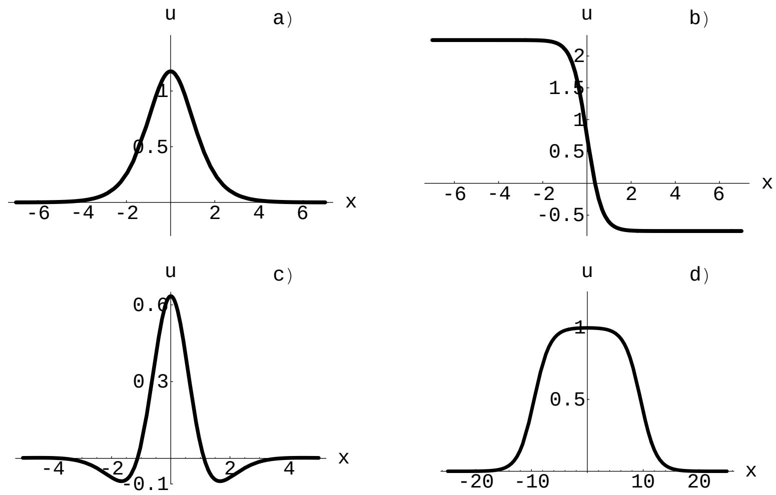

Figure 1: Various profiles of travelling wave solutions: a) bell-shaped solitary wave; b) kink-shaped wave; c) oscillatory vanishing solitary wave ; d) "fat" solitary wave.

of the equation obtained via analytical solutions describe the conditions when one or another stable profile of permanent shape is realized in numerics?

An attempt is done in this paper to answer these questions. Since the size of the paper does not allow to study the whole problem, the presentation is designed using some instructive examples that illustrate main ideas. First, the employment of exact solutions is considered. Usually exact solutions of nonlinear non-integrable partial differential equations are obtained using various direct methods. Here the attention is paid to one of them- the method of an ansatz. It looks the most efficient for non-integrable equations and rather simple in use. More information and useful references regarding direct methods may be found in [1, 2, 3, 4, 5]. Some exact solutions are presented to illustrate the power of the ansatz method. Then it is shown how the solutions obtained may help to understand some results of numerical simulations. Next section is devoted to the employment of asymptotic solutions in the same manner. The procedure used for finding asymptotic solutions is familiar [1, 5], hence it is not explained here in details. Instead, again some examples are considered where the asymptotic solutions describe important numerical results. The evolution of the bell-shaped solitary wave is studied on the basis of one seismic waves model. Then the attention is paid to the use of the kink-shaped solutions of the Burgers equation modified by weak dispersion and higher-order nonlinearity. It is important to mention that the moderate values of the small parameter responsible for the weak dissipation or dispersion may be formally used in numerics. It is found that even in this case the features of the asymptotic solutions remain valid.

2. Exact solutions

2.1. Direct methods for finding exact solutions

Sometimes, a solution may be obtained using a transformation of variables that allows us to reduce our equation to an equation whose solution is known. Another method, method of ansatz, allows us to reduce our differential equation to algebraic equations for the parameters of the solution.

Important steps in employment of these methods are:

- Reduce non-linear partial differential equation (PDE) to an ordinary differential equation (ODE) considering only travelling wave solutions (or self-similar);
- Reduce the obtained ODE to an ODE whose solution is already known, by means of a suitable transformation of variables or
- Reduce ODE to coupled algebraic equations for the parameters of the solution by means of a suitable ansatz.

As an example, consider the double-dispersive equation (DDE) [4, 5],

$$u_{tt} - \alpha_1 u_{xx} - \alpha_2 (u^2)_{xx} - \alpha_3 u_{xxtt} + \alpha_4 u_{xxxx} = 0. \tag{1}$$

that describes, in particular, longitudinal strain waves evolution in an elastic rod. Its travelling wave solution depending on the phase variable $\theta = x - V\,t$ is obtained from the ODE reduction of Eq.(1),

$$(V^2 - \alpha_1) u_{\theta\theta} - \alpha_2 (u^2)_{\theta\theta} + (\alpha_4 - \alpha_3 V^2) u_{\theta\theta\theta\theta} = 0 \tag{2}$$

Using substitution of variables,

$$u = \frac{6(\alpha_3 V^2 - \alpha_4)}{\alpha_2} v(\theta) + \frac{V^2 - \alpha_1}{2\alpha_2},$$

equation (2) is transformed to the Weierstrass equation,

$$\{v'(\zeta)\}^2 = 4v^3 - g_2 v - g_3,\ g_2 = \frac{(V^2 - \alpha_1)^2}{12(\alpha_4 - \alpha_3 V^2)} \tag{3}$$

whose known exact solution is expressed through the elliptic Weierstrass function, $v = \wp(\theta,\ g_2,\ g_3)$.

In order to transform the problem of finding exact solutions of ODE to the problem of finding solutions of algebraic equations, it would be nice to express an ansatz through the functions whose derivatives are expressed only through these functions. That is why various elliptic functions are widely used to obtain periodic solutions. One possibility is to use the set of the Weierstrass function $\wp$ and its first derivative $\wp_\theta$. Indeed, we have $\wp_{\theta\theta} = 6\wp^2 - 0.5 g_2$, $\wp_{\theta\theta\theta} = 12\wp_\theta$ etc. An important limit of the Weierstrass function corresponds to the choice $g_2 = 8k^4/3$, $g_3 = -8k^6/27$. In this case $\wp = k^2/3 - k^2 \mathrm{Sech}^2(k\theta)$ that accounts for a bell-shaped solitary wave solution. Also the solitary wave solutions are obtained directly using the hyperbolic functions for an ansatz. One can find a lot of papers where the hyperbolic tangent is employed. Indeed, we have $\text{Tanh}(k\theta)_\theta = k(1 - \text{Tanh}(k\theta)^2)$, $\text{Sech}^2(k\theta)_\theta = 2(\text{Tanh}^3(k\theta) - \text{Tanh}(k\theta))$ etc. A more complicated ansatz through the Ricatti functions was suggested in [6]. These functions satisfy the Riccati equations,

$$\sigma' = -\sigma\tau,\ \tau' = -\tau^2 - A\sigma + 1. \tag{4}$$

Certainly any derivatives of the Riccati functions σ and τ are expressed through themselves. Eqs. (4) possess the exact solution that allows us to express the Riccati functions through the hyperbolic functions,

$$\sigma = \frac{1}{A + C_1\text{Cosh}(k\theta) + C_2\text{Sinh}(k\theta)},\ \tau = \frac{C_2\text{Cosh}(k\theta) + C_1\text{Sinh}(k\theta)}{A + C_1\text{Cosh}(k\theta) + C_2\text{Sinh}(k\theta)} \tag{5}$$

A power series in Tanh or/and in Sech is often used to construct the ansatz. Use of the power series in the Riccati functions allows us to look for a solution as a rational function of the hyperbolic functions while power series approximation in terms of the hyperbolic functions appears as a special case. Indeed, when $C_1 = A$, $C_2 = 0$, we get from Eq. (5)

$$\sigma = \frac{1}{2A}\text{Sech}^2(\frac{k\theta}{2}),\ \tau = \text{Tanh}(\frac{k\theta}{2}).$$

However, substitution of an infinite power series into the equation yields a complicated algebra to find the coefficients of the series. Further simplification may be done using the pole analysis of the solution to define the functional form of the ansatz. One can see that the critical points of the elliptic and hyperbolic functions are poles (in the complex plane). Let us consider the DDE (1), for example, and assume that its solution possesses a pole of order n, $u \sim \theta^{-n}$. Then $u^2_{\theta\theta} \sim \theta^{-2n-2}$, $u_{\theta\theta\theta\theta} \sim \theta^{-n-4}$. Comparing higher order derivative (dispersion) and nonlinear terms one finds $n = 2$. This provides a balance between nonlinearity and dispersion required for existence of localized bell-shaped solitary wave solution. Then the ansatz may be suggested,

$$u = B\ \text{Sech}^2(k\theta). \tag{6}$$

Substituting (6) into (2) integrated two times one obtains

$$\text{Sech}^2(k\theta)B[\alpha_1 - V^2 - 4k^2(\alpha_4 - \alpha_3 V^2)]+$$
$$\text{Sech}^4(k\theta)B[\alpha_2\ B + 6k^2(\alpha_4 - \alpha_3 V^2)] = 0$$

Equating to zero combinations at each power of Sech, one obtains *algebraic* equations for B, and k whose solutions are

$$B = \frac{3(V^2 - \alpha_1)}{2\alpha_2},\ k^2 = \frac{\alpha_1 - V^2}{4(\alpha_4 - \alpha_3 V^2)}. \tag{7}$$

There may be more than one higher-order derivative or non-linear terms in the equation. Consider the Korteweg-de Vries-Burgers (KdVB) equation,

$$u_t + u_x^2 + b\ u_{xx} + s\ u_{xxx} = 0, \tag{8}$$

In this case we get $n = 2$ comparing nonlinearity and dispersion u_{xxx} and $n = 1$ comparing nonlinearity and dissipation u_{xx}. We have to satisfy both possibilities in order to provide a common balance between nonlinearity, dispersion and dissipation. Hence the solution should contain both the second and the first order poles. The suitable ansatz is

$$u = B\,\mathrm{Sech}^2(k\theta) + F\,\mathrm{Tanh}(k\theta) + C. \tag{9}$$

It allows us to find the well-known kink-shaped solution of the KdVB equation with parameters defined by

$$B = 6k^2\,s,\; F = \frac{6b\,k}{5},\; C = \pm\frac{3b^2}{25s},\; k = \pm\frac{b}{10s},\; V = \pm\frac{6b^2}{25s}.$$

More examples regarding the method of ansatz may be found in Refs. [3, 4, 5].

2.2. Exact bell-shaped solitary wave solutions

Following the procedure described before one can find exact solutions to many non-integrable equations. However, most of them are single travelling wave solutions that require special initial conditions. In particular, the form of the initial condition for the exact solitary wave solution of the DDE is defined by Eq. (6), (7) with $t = 0$ in θ. What happens when an initial condition differs from the "Sech" shape? Certainly, it is unlikely to describe *whole* evolution of an arbitrary input analytically. But if we find such a solution numerically, what is the reason for finding special exact travelling wave solution?

Numerical simulation of the DDE has been performed in [5]. It was found that for $\alpha_2 > 0$ rather arbitrary initial pulse with positive amplitude splits into a train of solitary waves with different amplitudes while negative input is dispersed, and no travelling localized wave appears. The higher is the amplitude of the wave, the larger is its velocity. The distance between the localized waves increases in time, hence the waves interaction becomes weaker and weaker. Hence each wave may be considered as a single travelling wave and comparison with the exact solution may be done. It is found that each localized wave generated by positive input evolves according to the exact travelling wave solution (6), (7). Moreover, reality of the parameters in (7) gives rise to the conclusion that only positive amplitude solitary waves may exist for $\alpha_2 > 0$. Similarly, negative amplitude solitary waves arise from a negative amplitude input for $\alpha_2 < 0$ Hence exact solution allows us to choose suitable sign of the input amplitude to provide generation of localized stable solitary waves and to describe each solitary wave thus confirming numerical results.

Sometimes numerical simulations yield rather unusual results. Recently, the Gardner equation,

$$u_t + a\,u_x^2 + cu_x^3 + bu_{xxx} = 0, \tag{10}$$

was studied in [8]. It was found that a train of solitary waves appears from certain initial pulse. However, there is an input that produces rather wide solitary wave followed by a sequence of usual bell-shaped waves. These results may be explained using known solitary wave solution of the Gardner equation [7]:

$$u_s = \frac{3b\,k^2}{a(B_1 \mathrm{Cosh}(k\theta) + 1)}, \tag{11}$$

where

$$B_1 = \sqrt{1 + \frac{9bck^2}{2a^2}}, \theta = x - bk^2\,t.$$

The solution has an interesting feature for negative c: tendency to the extensive or "fat" shape at $k \to \sqrt{-2a^2/(9b\,c)}$. The amplitude of the wave tends to the limiting value equal to $-2a/3c$, while the width growths without limits, see Fig.1(d). Like in the case of the DDE, numerical simulations are successfully checked by the exact solution (11) both to account for the usual bell-shaped waves and the "fat" solitary wave [8].

Even more interesting unusual profiles appear studying numerically the equation

$$u_t + 2b\,u\,u_x + 3c\,u^2 u_x + r\,u\,u_{xxx} + s\,u_x\,u_{xx} + d\,u_{3x} + f\;u_{5x} = 0, \tag{12}$$

which is often called the extended KdV equation [5, 9]. A review of its exact bell-shaped solitary wave solutions may be found in [3, 5]. An appearance of the solitary waves described by the exact solutions from rather arbitrary input was studied numerically in [5, 9]. It was found that sometimes there is a good agreement with the exact solutions, namely, in the shape of generated solitary waves and in dependence of their parameters upon the equation coefficients. Also the conditions required for existence of exact solutions were realized in numerics. At the same time, the localized waves were obtained that differ from those described by the analytical solutions. In particular, an oscillatory vanishing at infinity, see Fig.1(c), and a multi-humps localized waves have been discovered in [5, 9] as well as the "fat" solitary wave [9]. One has to note that all known exact solutions of Eq.(12) either do not contain free parameters or exist under special relationships between the equation coefficients. So the absence of free parameters or additional restrictions do not allow us to use exact solutions so efficiently as in the case of the DDE whose solution (6) contains the free parameter V and in the case of the Gardner equation having free parameter k.

2.3. Kink-shaped solutions

The Burgers equation

$$u_t + (u^2)_x + bu_{xx} = 0, \tag{13}$$

is widely used in many physical problems [10, 11]. In particular, it possesses the well-known shock-wave solution (or a kink),

$$u_0 = b\;p\;\mathrm{Tanh}(p(x - Vt)) + V/2. \tag{14}$$

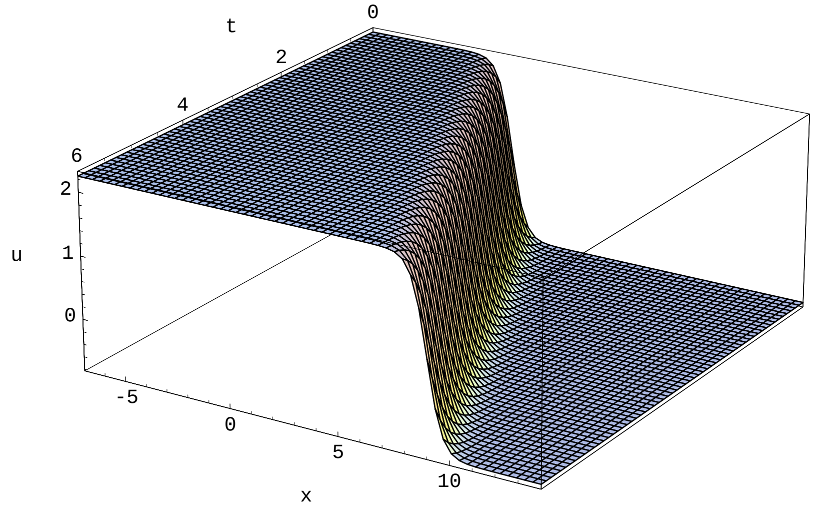

Figure 2: Evolution of the exact solution of the Burgers equation for $b < 0$.

where p and V are free parameters to be defined by the boundary conditions for x. The Burgers equation is integrable, and a more general solution of the Cauchy problem may be obtained. However, integrability fails for most of its generalizations caused by an inclusion of the additional terms like dispersion, higher-order non-linearity etc.

Figure 2 shows stable movement of the kink wave of permanent shape (14). This is because the shock-wave solution of the Burgers equation arises as a result of a balance between nonlinearity and dissipation.

The same simulations for the KDVB equation (8) yield a profile different from that of the exact solution since it contains oscillations on the upper or on the lower parts of the step depending upon the sign of s, see Fig.3. A possible reason of it lies in the fact that the exact solution of the KDVB equation (9) does not contain free parameters in contrast to the two-parameter solution of the Burgers equation (14).

In order to check this idea let us add an additional non-linear term to the KdVB equation,

$$u_t + u_x^2 + b\,u_{xx} + s\,u_{xxx} + q\,u_{xx}^2 \;=\; 0. \tag{15}$$

Its exact kink-shaped solution

$$u_0 = \frac{s\;p}{q}\mathrm{Tanh}(p(x - Wt)) + W/2, \tag{16}$$

contains is a free parameter p and fixed velocity

$$W \;=\; \frac{s - b\,q}{q^2}.$$

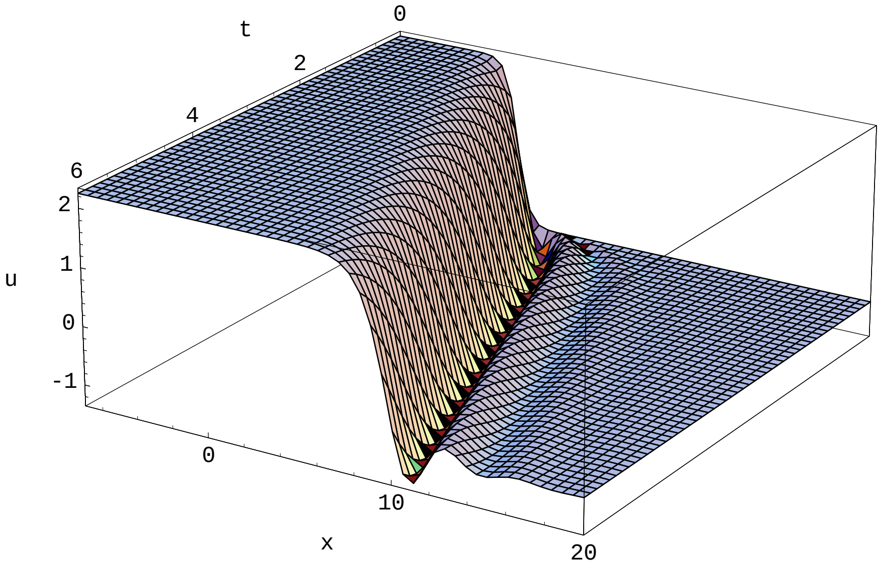

Figure 3: Evolution of the initial Burgers shock wave for $b < 0, s > 0$.

The solution (16) never coincides with the Burgers kink solution (14) since the equality in the amplitudes, $b = s/q$ yields $W = 0$ for the velocity in the solution (16). Now numerical simulation of Eq.(16) demonstrate almost identical to the Burgers kink evolution for certain values of s and q. However, the amplitude in the solution (16) does not depend on the value of b. It means that the wave shown in Fig.2 might propagate for positive values of b, and this prediction of the exact solution is realized also. There exist domains of the values of s and q where initial Burgers kink evolves like in the case of the KdVB equation with oscillations on the profile, see Fig.3. Another scenario is the smoothness of the initial profile shown in Fig.4. It is very likely that oscillations are caused by dispersion while higher-order nonlinearity is responsible for the smoothness, and observed deviations in the kink shape are caused by breach of the balance between dispersion and higher-order nonlinearity. So, addition of the higher-order nonlinearity allows us to provide two balances *separately*, between nonlinearity and dissipation and between higher-order nonlinearity and dispersion. In contrast to it, the solution (9) of the KdVB equation should satisfy two balances *simultaneously* that fixes its parameters and prevents the appearance of the solution from an arbitrary input.

3. Asymptotic solutions

The balance between nonlinearity and dispersion may be destroyed due to the influence of dissipation and/or accumulation. When this influence is weak, asymptotic solutions may be obtained to account for the bell-shaped solitary wave evolution. Introduction of the fast (usually phase) and slow (time or space) variables allows us to study more

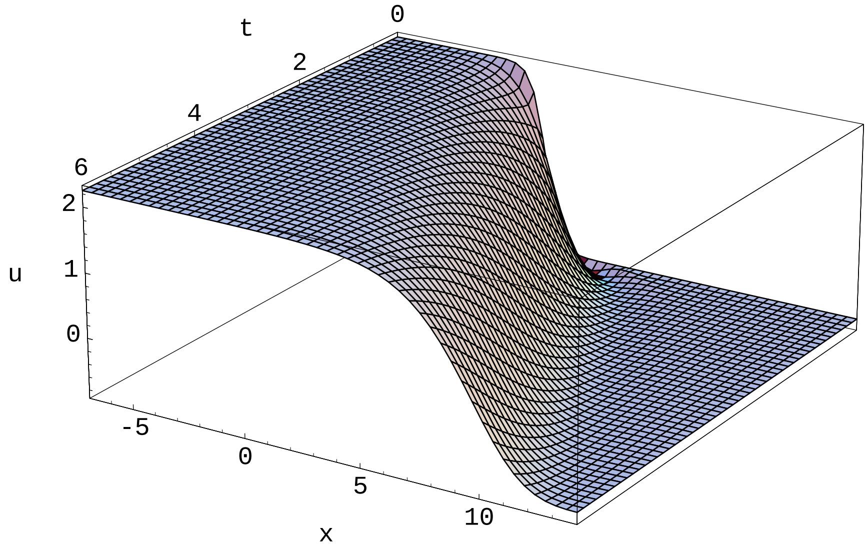

Figure 4: Smoothness of the initial Burgers shock wave.

general processes than those described by exact travelling wave solutions. In particular, asymptotic solutions may account for an amplification or attenuation of a wave. The ODE describing the solitary wave amplitude variation may predict the case when an increase or a decrease in the amplitude happens to some finite value defined by the values of the coefficients of the original PDE. We call this process the solitary wave selection. Selection from *below* is accompanied by the growth of the initial amplitude while selection from *above* is provided by the decrease of the initial solitary wave amplitude.

Similarly the case may be studied when the balance between nonlinearity and dissipation is destroyed by the presence of dispersion and higher-order nonlinearities. In this case, the asymptotic solution allows us to account for a propagation of a kink with stationary deviations on its front and to establish a connection between the shape of these deviations and the structure of the perturbation terms in the equation. Both for the bell-shaped and kink-shaped waves the features of the asymptotic solutions are realized in numerics even when the small parameter characterizing weak perturbations achieves moderate values.

3.1. Evolution of the bell-shaped waves

The asymptotic solution allows us to describe seismic waves selection on the basis of the model equation obtained in [12]:

$$u_t + u\, u_x + d\; u_{xxx} = -\varepsilon \left(a_1 u - a_2 u^2 + a_3 u^3 \right), \tag{17}$$

a_1, a_2, a_3 are positive constants and ε is a small parameter. One can see that Eq.(17) is nothing but the disturbed KdV equation that possesses exact bell-shaped solitary wave solution in the absence of disturbances. In the general case, Eq. (17) may describe an appearance of microseisms. Following [5, 13] assume the function u depends upon a fast variable ξ and a slow time T , such as

$$\xi_x = 1,\ \xi_t = -V(T),\ T = \varepsilon\, t.$$

The asymptotic solution is sought of the form

$$u(\xi, T) = u_0(\xi, T) + \varepsilon\, u_1(\xi, T) + ... \tag{18}$$

The bell-shaped solitary wave solution of the KdV equation arises in the leading order,

$$u_0 = 12\, d\ k(T)^2 \mathrm{Sech}^2\, (k(T)\, \xi) \tag{19}$$

However, now its parameters depend upon the slow time T. Next order solution yields the equation for the wave amplitude $Q = 12\, d\ k(T)^2$ of the form [5, 13]:

$$Q_T = -\frac{4}{105}\, Q(24a_3 Q^2 - 28a_2 Q + 35a_1). \tag{20}$$

The behavior of the solitary wave amplitude, Q, depends on the value of $Q_0 \equiv Q(T = 0)$. Indeed, Q will diverge at $Q_0 < Q_1$, when $Q_1 < Q_0 < Q_2$, Q will grow up to Q_2, while if $Q_0 > Q_2$, it will decrease by Q_2. Here $Q_1 < Q_2$ are the roots of equation $24a_3 Q^2 - 28a_2 Q + 35a_1 = 0$. Hence parameters of the solitary wave tends to the finite values prescribed by the equation coefficients a_i, and the *selection* of the solitary wave takes place.

Despite this solution requires special initial condition, numerical simulations [5, 13] confirm the behavior of the wave predicted by the theory even when an initial condition is arbitrary or in the presence of solitary waves interaction. In the former case, the situation is close to that of the DDE when the input is transformed into the train of solitary waves each separately being described by the asymptotic solution. In the latter case, it is found that the interaction does not prevent solitary waves amplification, vanishing or selection that are realized for the coefficients of Eq.(17) prescribed by the asymptotic solution. It is important that the amplitudes of the resulting localized waves in the numerical solution are equal to those of the selected waves obtained from the asymptotic solution. Moreover, the value of the amplitude of the selected solitary wave remain valid when ε is not small.

Similarly the amplification, attenuation and selection of the bell-shaped nonlinear waves may be studied using an equation

$$v_{tt} - v_{xx} - \varepsilon\alpha_1 (v^2)_{xx} - \gamma\alpha_2\, v_{xxt} + \delta(\alpha_3\, v_{xxxx} - \alpha_4\, v_{xxtt}) +$$

$$\gamma\delta(\alpha_5\, v_{xxxxt} + \alpha_6\, v_{xxttt}) + \gamma^2\alpha_7 v_{xxtt} = 0, \tag{21}$$

that appears to account for the strain waves $v(x, t)$ in a microstructured medium [5, 14]. When $\delta = O(\varepsilon)$, $\gamma << 1$, this equation is nothing but the DDE disturbed by

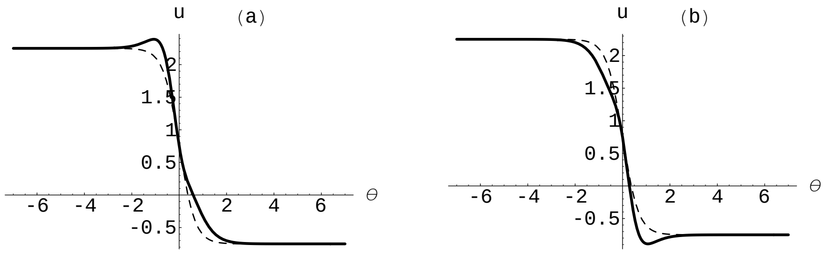

Figure 5: Influence of the weak dispersion on the Burgers kink for $b < 0$: a) $s > 0$; b) $s < 0$.

dissipative/active terms. Its asymptotic solution is obtained in [5, 14] similar to the solution of Eq. (17).

The same procedure may be applied to study the bell-shaped localized wave evolution in the two-dimensional case. An example of the two-dimensional selection of the lump of the Kadomtsev-Petviashvili equation may be found in [15].

3.2. Evolution of the kink-shaped waves

Let us consider the KdVB equation

$$u_t + (u^2)_x + bu_{xx} = -\delta s u_{xxx}, \tag{22}$$

when δ is a small parameter. The asymptotic solution accounting for the perturbation of the kink-shaped wave is sought in the form

$$u(\theta) = u_0(\theta) + \delta u_1(\theta) + ...$$

where $\theta = x - Vt$, and $u_1 \to 0$ for $\theta \to \pm\infty$. Substituting this series into Eq.(22) we obtain in the leading order an ordinary differential equation (ODE)

$$(-Vu_0 + u_0^2 + b\, u_{0,\theta})_\theta \;=\; 0,$$

which is satisfied by the travelling wave solution of the Burgers equation (14). In the next order an inhomogeneous linear ODE appears for the function u_1,

$$(-Vu_1 + 2u_0u_1 + b\, u_{1,\theta})_\theta \;=\; -su_{0,\theta\theta\theta},$$

whose solution is

$$u_1 = 2p^2 s\ \mathrm{Sech}^2(p\theta)\mathrm{Log}(\mathrm{Cosh}(p\,\theta)).$$

Figure 5 demonstrates affect of the weak dispersion on the shape of the Burgers shock wave, $u = u_0 + \delta u_1$. Here and in the following the unperturbed solution is shown by dashed line. One can note non-symmetric influence on the upper and lower parts of the wave. For positive s, a ”hat” appears at the upper part while the lower one is subjected

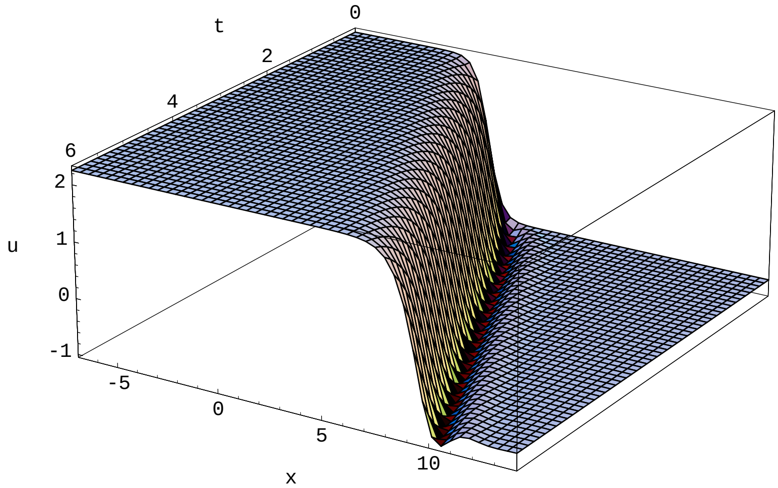

Figure 6: Evolution of the initial Burgers shock wave for $s > 0$ and small δ.

to a smoothness of the wave front. The mirror profile appears for negative values of s. We see that all deviations are concentrated around the wave front.

Our travelling wave asymptotic solution requires special initial condition in the form of already perturbed kink, $u(t = 0)$. However, one can check numerically that the Burgers unperturbed kink transforms into another one whose shapes agrees well with that described by our solution, see Fig. 6. One can see that perturbations of the shock wave profile are stable, they are located near the wave front and do not evolve far from it. Moreover, even for $\delta = O(1)$, the initial Burgers kink wave still evolves into the profile predicted by the asymptotic solution which is not valid in this case in a strict mathematical sense (δ is not small). Typical evolution is shown in Fig.3, where this new wave continues to propagate with one and same velocity and the shape. Hence asymptotic solution explains what was not covered by the exact solution (9).

Next equation to be considered is similar to the extension of the KdVB equation (15),

$$u_t + (u^2)_x + bu_{xx} = -\delta(su_{xxx} + qu_{xx}^2). \tag{23}$$

Its asymptotic solution is

$$u = u_0 + \delta p^2 \mathrm{Sech}^2(p\theta)\left[2(s - b\,q)\mathrm{Log}(\mathrm{Cosh}(p\,\theta)) - V\,q\,\theta\right]$$

An influence of the higher-order quadratic nonlinearity is seen in Fig. 7 for $s = 0$. A smoothness is achieved for $q < 0$ the same happens for the numerical solution shown in Fig. 4. Again the asymptotic solution reveals the features was not discovered by the exact solution (16).

Similarly one can study an influence of the higher-order dissipation u_{xxxx} and other linear and non-linear perturbations on the kink solution of the Burgers equation.

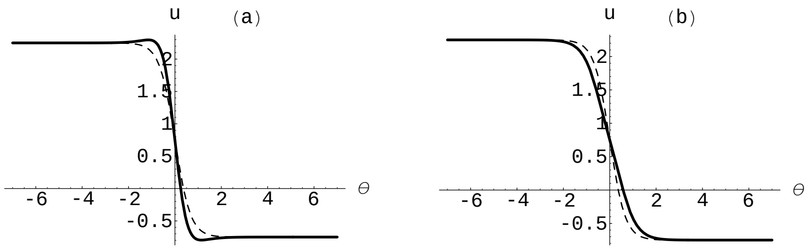

Figure 7: Influence of the weak higher-order nonlinearity on the Burgers kink for $b < 0$: a) $q > 0$; b) $q < 0$.

4. Conclusions

One can see that exact travelling solitary wave solutions with free parameters arise in a more general numerical solution, they may predict important features of an arbitrary input evolution. However, exact solutions with fixed parameters are not necessary realized in computations. In the case when exact solution with free parameters is unlikely to find, an asymptotic solution depending upon the fast and the slow variables may help to understand the behavior of the wave, in particular, amplification and selection of the bell-shaped solitary wave.

Even more special, *travelling wave* asymptotic solutions predict deviations in the profile of the Burgers shock wave that are realized in numerical simulations of unsteady processes. Like for the bell-shaped waves, these predictions remain valid even outside the formal applicability of the asymptotic solution at moderate values of the small parameter.

Both the exact and asymptotic solutions provide us with the relationships between the coefficients of the equation required to achieve one or another kind of the wave evolution. One can use this information in advance for a design of numerical study, it does not allow us to miss one or another scenario of the waves localization. On the other hand, analytical solutions may be used as a testing point for a design of numerical scheme. To sum up, they deserve time required for their finding.

5. Acknowledgment

The author thanks Professor F. Pastrone for his kind invitation to attend the Intensive Seminar and for his warmest hospitality during the work of the Seminar.

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AMS Subject Classification: 35Q51, 35Q53, 37K40.

Alexey PORUBOV, A.F.Ioffe Physical Technical Institute of the Russian Academy of Sciences
Polytekhnicheskaya st., 26, 194021, Saint-Petersburg, RUSSIA
e-mail: porubov@math.ioffe.ru