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G.A. Maugin

BASICS OF THE MATERIAL MECHANICS OF MATERIALS (M3) FOR ARBITRARY CONTINUA

Abstract. It is shown that the canonical balance of momentum of continuum mechanics can be formulated in a general way, but not independently of the usual balance of linear momentum, even in the absence of specified constitutive equations. A parallel construct can be made for the accompanying time-like canonical energy equation. On specifying the energy, previous particular cases can be deduced including pure elasticity, inhomogeneous thermoelasticity of conductors, and the case of dissipative solid-like materials described by means of a diffusive internal variable (such as in damage or weakly nonlocal plasticity). A redefinition of the entropy flux is necessarily accompanied by a redefinition of the Eshelby stress tensor.

1. Introduction

There exist two opposite viewpoints concerning the status of the equation of material (or canonical) momentum in continuum mechanics. The viewpoint of the author [1]-[2] is that this equation is never independent of the classical (physical) equation of linear momentum, in Cauchy or Piola-Kirchhoff form, being essentially deduced from the latter by a complete pull-back to the reference configuration whenever constitutive equations are known for the material. It is, therefore, an identity at all regular material points - but it still is extremely useful on any singular manifold [3]. This is in agreement with the application of Noether's identity [4] when one considers a variational formulation for a nondissipative material, a point of view shared by J.D.Eshelby in his original works, e.g.,[5]. The second viewpoint is that of Gurtin [6] who claims that this equation is an a priori statement independent of the classical balance laws although in the end it is, for sure, always shown to be related to the physical balance of momentum so that Gurtin's statement is somewhat inappropriate.

Let us be more inclusive. Several phantasms and fallacies are at work in the field that is our concern. The present work has for main purpose to correct these by pondering the basics of the material formulation of continua. First, it was for long thought that canonical equations of motion or equilibrium such as obtained initially by Eshelby can be deduced only when a variational formulation is at hand to start with, i.e., in the absence of dissipation and when the kinetic and potential energies are prescribed since a Lagrangian density is needed to start with. This is the belief of, if we may say, those who know "too much". Indeed these authors know elements of field theory including the celebrated theorem of Noether [4] according to which a "conservation law" is associated with any of the parameters describing the field theory under study when a variational basis is considered to formulate balance laws. One must obviously distinguish between the **field equations** (one for each scalar component of the involved fields) - these are the Euler-Lagrange equations of motions, and the **conservation laws** that follow from Noether's identity. In pure continuum mechanics the fields are the actual components of the placement or of the displacement. The description parameters are a set of coordinates - usually the so-called material coordinates in order to avoid any confusion with any other system of coordinates-, and Newtonian time, a scalar. Infinitesimal variations of the latter generate the so-called equation of canonical - or material momentum - and the equation of energy conservation (Maugin and Trimarco [7]). If there are more fields, then there are more field equations, as many as fields, but the space-time parametrization remaining unchanged, the canonical conservation equations are still four in number (the three components of the canonical momentum equation, and the scalar energy equation). Accordingly, in general and working in Newtonian physics, there may be 3 + n field equations and 4 scalar conservation laws. In classical continuum mechanics where the medium involves no inner structure (such as in micropolar, Cosserat or micromorphic media), it happens that the three field equations left for the displacement and the material momentum equation can, at all regular material points, be placed in a one-to-one correspondence by the operations of material convection (pull-back and push-forward). Accordingly, one has the correct feeling that nothing is gained from having a conservation of canonical momentum -as a pure identity - in addition to that of momentum in physical space. The situation is altogether different when (i) there exist material points in the spatial domain of interest where the fields suffer a singularity of an appropriate order. The writing of the global canonical balance laws will then make additional terms emerge that correspond to the driving force (a **material** or **configurational** force acting on the singularity set - this may be a line or a surface) and an energy sink (so-called **energy-release rate**) such as at a crack tip line or at a surface of phase change) [8]-[9]. The situation is also more interesting even in the case (ii) where there are more fields than three but no singularity present, because both the canonical momentum conservation, then remaining essentially three dimensional, and the energy equation, as it should, but simultaneously with the canonical momentum conservation, involve all fields. This fact is exploited in perturbations of solutions of systems of partial differential equations (such as in soliton theory) [10] and also for checking the accuracy of numerical schemes of various nature [11]. Notice that when there are more fields than the classical placement, the canonical momentum equation is obtained by constructing a linear combination of field equations, each of these being first multiplied in the appropriate way by the material gradient of the corresponding field. In that sense the "canonical momentum" concept is additive and will include contributions from all fields including those of an electromagnetic nature [1] or even more surprisingly, rotational internal degrees of freedom although the canonical momentum itself reflects a translational invariance in material space (this is most nicely illustrated by the case of polar continua [12] and liquid crystals [13]). All these aspects have been duly examined in works by the author and co-workers. Still, one keeps on mind that constitutive equations have been suggested, perhaps only through a proposed dependence of the potential energy, in the relevant construct.

The view point of Gurtin [6] adopted by some authors is that there exists a priori a balance of configurational forces, in a sense, a new law of physics. We would say that this represents the view point of the philistines because they seem to ignore that the number of descriptive parameters, and therefore the number of balance laws of classical continuum mechanics, is limited so that there should not exist a balance law of the momentum type independently of the one already and generally first written in the spatial framework. These authors generally ignore these rules of invariance that are the tenets of modern physics. They are thus led to introducing the energy pressure-like term in the Eshelby material stress through a dubious argument. They claim, to their defense, that this is a way to arrive at the material momentum equation, or its jump in the case of a singular surface, without previous knowledge of the constitutive equations of the medium, and, therefore, even in the presence of dissipation.

Here we expand the view that the balance of canonical or material momentum, albeit following from the balance of physical momentum, can be formulated independently of any constitutive behavior. Moreover, accounting for the fact that this equation is the space-like equation associated with a particular form of the energy equation, it is shown that the former and the latter can be used in parallel to build a consistent thermomechanics of many behaviors, especially in the presence of dissipation. It in fact is dissipative terms that help us construct a true invariant thermomechanics of rather general continua. In other words we would like to show how far one can first proceed in the construction of canonical material conservations laws without previous specialization to a certain behavior, it being understood that dissipation is not an obstacle to the formulation of such equations.

2. Reminder of classical local balance laws of continuum mechanics

We shall use the standard notation of nonlinear continuum mechanics such as in Eringen [14] and Eringen and Maugin [15], and a fortiori in Maugin [1]-[2]. In particular, $\mathbf{x} = \bar{\mathbf{x}} = (\mathbf{X}, t)$ is the time-parametrized motion mapping of material space onto physical Euclidean space. ∇_R and div_R denote the referential (material) nabla and divergence, and $d/dt = \partial/\partial t|_X$ or a superimposed dot denote the material time derivative. We suppose that the following three local balance laws have been deduced from a global statement for sufficiently smooth fields (see any book on the thermomechanics of continua). Here we consider the Piola-Kirchhoff formulation of the balance of mass, physical (linear) momentum, and energy (no external supply of energy apart from that related to the body force) at any regular material point X in a continuous body in the presence of a body force $\mathbf{f}_0$ per unit reference volume

$$\left.\frac{\partial \rho_0}{\partial t}\right|_X = 0 \tag{1}$$

$$\left.\frac{\partial (\rho_0 \mathbf{v})}{\partial t}\right|_X - div_R \mathbf{T} = \mathbf{f}_0 \tag{2}$$

$$\left.\frac{\partial (K + E)}{\partial t}\right|_X - \nabla_R (\mathbf{T}.\mathbf{v} - \mathbf{Q}) = \mathbf{f}_0.\mathbf{v}, \tag{3}$$

where ρ_0 is the mass density, $\mathbf{v} = \partial\bar{\mathbf{x}}\partial t|_X$ is the physical velocity, $\mathbf{T}$ is the first Piola-Kirchhoff stress, $K = \rho_0 v^2/2$ is the kinetic energy, E is the internal energy per unit reference volume, and $\mathbf{Q}$ is the material heat flux. This is complemented by the second law of thermodynamics written as

$$(4) \qquad \left.\frac{\partial S}{\partial t}\right|_X + \nabla_R.\mathbf{S} \geq 0, \;\; \mathbf{S} = (\mathbf{Q}/\theta) + \mathbf{K},$$

where S is the entropy density, θ is the absolute temperature ($\theta > 0, \;\inf\theta = 0$), and $\mathbf{S}$ is the entropy flux. The "extra entropy flux" $\mathbf{K}$ vanishes in most cases. We note $\mathbf{F} = \partial\bar{\mathbf{x}}/\partial X|_t = \nabla_R\bar{\mathbf{x}}$ the deformation gradient.

3. Canonical balance laws of momentum and energy

3.1. A canonical form of the energy conservation

First we shall formulate an interesting form of the energy conservation equation. A part of the reasoning is standard. In effect, taking the scalar product of both sides of eqn. (2) by $\mathbf{v}$ and performing some elementary manipulations we obtain the so-called *theorem of the kinetic energy* as

$$(5) \qquad \frac{dK}{dt} - \nabla_R.(\mathbf{T}.\mathbf{v}) + \mathbf{T}:\dot{\mathbf{F}} - \mathbf{f}_0.\mathbf{v} = 0.$$

Combining this with the first law of thermodynamics (3) we obtain the so-called *theorem of internal energy*:

$$(6) \qquad \frac{dE}{dt} - \mathbf{T}:\dot{\mathbf{F}} + \nabla_R.\mathbf{Q} = 0.$$

Further, in the case where $\mathbf{K} = \mathbf{0}$, introducing the Helmholtz free energy function by $W = E - S\theta$, we transform the inequality $(4)_1$ into the celebrated *Clausius-Duhem inequality*

$$(7) \qquad -\left(\frac{dW}{dt} + S\frac{d\theta}{dt}\right) + \mathbf{T}:\dot{\mathbf{F}} - \mathbf{S}.\nabla_R\theta \geq 0.$$

As we know, this is exploited as a constraint in the formulation of thermodynamically admissible constitutive equations, while the "conservation equation " (6) is the equation governing heat propagation in a disguise. This can be given several transformed forms. A most interesting form is obtained straightforwardly by noting that $E = W + S\theta$, yielding

$$(8) \qquad \frac{d(S\theta)}{dt} + \nabla_R.\mathbf{Q} = h^{int}, \quad h^{int} := \mathbf{T}:\dot{\mathbf{F}} - \left.\frac{\partial W}{\partial t}\right|_X$$

This is of special interest because of the expression in the right-hand side which a priori appears as an *internal heat source*. Indeed, for a typically thermodynamically reversible behavior such as pure nonlinear elasticity (hyperelasticity), where $W = \bar{W}(\mathbf{F})$

depends only on **F**, we have from the exploitation of (7),

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}} \Rightarrow h^{int} \equiv 0 \tag{9}$$

Note that in the situation where (8) holds good, the inequality (7) can also be written in the following enlightening form

$$S\dot{\theta} + \mathbf{S}.\nabla_R \theta \leq h^{int} \tag{10}$$

We claim that $(8)_1$ in fact is the most interesting form of the energy conservation equation for our purpose (i.e., establishing canonical equations). This we discover by constructing the canonical equation of momentum as follows.

3.2. Canonical (material) momentum conservation

Guided by what is valid for pure finite-strain elasticity (Noether's identity; see Maugin [1]), we apply **F** to the right of eqn. (2) and note that (T = transpose)

$$\left(\frac{\partial(\rho_0 \mathbf{v})}{\partial t}\right).\mathbf{F} = -\left.\frac{\partial \mathbf{P}}{\partial t}\right|_X - \nabla_R\left(\frac{1}{2}\rho_0 \mathbf{v}^2\right) + \left(\frac{1}{2}\mathbf{v}^2\right)(\nabla_R \rho_0), \tag{11}$$

and

$$(div_R \mathbf{T}).\mathbf{F} = div_R(\mathbf{T}.\mathbf{F}) - \mathbf{T} : (\nabla_R \mathbf{F})^T, \tag{12}$$

where we have set

$$\mathbf{P} := -\rho_0 \mathbf{v}.\mathbf{F} \tag{13}$$

the material momentum. Introducing plus and minus the material gradient of an (unspecified) free energy density $W = \bar{W}(., ., ..., \mathbf{X})$, we then check that eqn. (2) yields the following material balance of momentum

$$\frac{d\mathbf{P}}{dt} - div_R \mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh}, \tag{14}$$

in which we have defined the material *Eshelby stress* **b**, the material *inhomogeneity force* $\mathbf{f}^{inh}$ (cf. [1]-[2] for this notion), the material *external* (or body) force $\mathbf{f}^{ext}$, and the material *internal* force $\mathbf{f}^{int}$ by

$$\mathbf{b} = -(L_W \mathbf{1}_R + \mathbf{T}.\mathbf{F}), \quad L_W := K - W \tag{15}$$

$$\mathbf{f}^{inh} := \left.\frac{\partial L_W}{\partial \mathbf{X}}\right|_{expl} \equiv \left.\frac{\partial L_W}{\partial \mathbf{X}}\right|_{fixed\ fields} = (\mathbf{v}^2/2)\nabla_R \rho_0 - \left.\frac{\partial \bar{W}}{\partial \mathbf{X}}\right|_{expl}, \tag{16}$$

$$\mathbf{f}^{ext} := -\mathbf{f}_0.\mathbf{F}, \quad \mathbf{f}^{int} := \mathbf{T} : (\nabla_R \mathbf{F})^T - \nabla_R W|_{impl}, \tag{17}$$

where the subscript notations *expl* and *impl* mean, respectively, the material gradient keeping the fields fixed (and thus extracting the explicit dependence on **X**), and taking the material gradient only through the fields present in the function.

Equation (14) is the *canonical* balance of momentum of continuum mechanics *in the absence of specification of constitutive equations*. It is a mathematically strict conservation equation only when all source terms in its right-hand side vanish. Here the new notion is that of *material internal force* which appears in parallel and total analogy with the internal heat source $(8)_2$, the action of the material gradient replacing that of the material time derivative. We note that there is no "time-like" scalar equivalent to $\mathbf{f}^{inh}$ in equation $(8)_1$ because this inhomogeneity force which is automatically captured by that equation, has no dissipative nature. An explicit dependence of W on time (in a nonholonomous system) would yield a nonzero term h^{inh}, but this is hardly considered in continuum mechanics except perhaps in growing and ageing materials such as soft tissues (inhomogeneity of the material in time!). Similarly, there is no equivalent to the external material force $\mathbf{f}^{ext}$ in $(8)_1$ because this equation governs essentially the internal energy. It would be easy to rewrite eqns. $(8)_1$ and (14) as a single four-dimensional space-time equation (see [8]) but this serves no special purpose, except for an aesthetic satisfaction, in engineering applications. Still the *consistency* between the space-like co-vectorial equation (14) and the time-like equation $(8)_1$ is a fundamental requirement in the thermodynamical study of the progress of singularity sets (e.g., defects).

Still, in the present approach, in order to proceed further we need to specify the full functional dependence of W. The general expressions $(8)_1$ and (14) are the most general canonical equations for momentum and energy we can write down without a postulate of the full dependency of W. However, just like for other equations if continuum mechanics, we could also write the jump relations associated with $(8)_1$ and (14) at a singular surface by using elements of the theory of hyperbolic systems or a more naive method such as the pill-box method. But since the "conservation laws" $(8)_1$ and (14) already exhibit source terms in the bulk (i.e., they are not conservation laws in a strict mathematical sense), the associated jump relations will also contain surface source terms. The latter, a priori unknown but responsible for the dissipation at the singularity, have to be computed with the help of the standard jump relations associated with eqns. (1)-(3).

3.3. Case $\mathbf{K} \neq \mathbf{0}$

Without reporting the whole algebra, starting with $(4)_2$, we let the reader check that the thermodynamical inequality (7) is replaced by

$$-\left(\frac{dW}{dt} + S\frac{d\theta}{dt}\right) + \mathbf{T} : \dot{\mathbf{F}} + \nabla_R.(\theta \mathbf{K}) - \mathbf{S}.\nabla_R \theta \geq 0, \tag{18}$$

where **S** is still give by the general expression $(4)_2$. Equations (8) and (14) are left unchanged:

$$\frac{d(S\theta)}{dt} - \nabla_R.\mathbf{Q} = h^{int}, \quad h^{int} := \mathbf{T} : \dot{\mathbf{F}} - \left.\frac{\partial W}{\partial t}\right|_X, \tag{19}$$

(20) $$\frac{d\mathbf{P}}{dt} - div_R\mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh};$$

On account of (18), eqn. (10) is replaced by

(21) $$S\dot{\theta} + \mathbf{S}.\nabla_R\theta \leq h^{int} + \nabla_R.(\theta\mathbf{K}).$$

Now let us illustrate these general equations by specific cases; some trivial, and some nontrivial ones.

4. Examples without body force

4.1. Pure homogeneous elasticity

In that case $\rho_0 = const.$, and $W = \bar{W}(\mathbf{F})$ only. We have $h^{int} \equiv 0$, $\mathbf{f}^{int} \equiv 0$ since (9) holds good, and also $\mathbf{f}^{inh} = \mathbf{0}$, $\mathbf{Q} \equiv \mathbf{0}$ since the body is homogeneous and non conducting. Equations (8) and (14) reduce to the following [in fact Hamiltonian for a (3+1)-dimensional canonical momentum $(\mathbf{P}, \theta_0 S)$] system ($\theta_0 = const.$):

(22) $$\frac{d\mathbf{P}}{dt} - div_R\mathbf{b} = \mathbf{0}, \quad \theta_0\frac{dS}{dt} = 0.$$

In four-dimensional form this is the formulation of Kijowski and Magli [16].

4.2. Inhomogeneous thermoelasticity of conductors

In that case $\rho_0 = \bar{\rho}_0(\mathbf{X})$, and $W = \bar{W}(\mathbf{F}, \theta; \mathbf{X})$. We have the constitutive equations

(23) $$\mathbf{T} = \frac{\partial\bar{W}}{\partial\mathbf{F}}, \quad S = -\frac{\partial\bar{W}}{\partial\theta}$$

that follow from a standard exploitation of the Clausius-Duhem inequality. Accordingly, we obtain that

(24) $$\mathbf{f}^{int} \equiv \mathbf{f}^{th}, \quad h^{int} \equiv h^{th} := S\dot{\theta}$$

where

(25) $$\mathbf{f}^{th} := S\nabla_R\theta$$

is the material thermal force first introduced by Bui in small strains [17] and independently by Epstein and Maugin in their geometrical considerations [18], so that (8) and (14) are replaced by the following canonical (*non*-Hamiltonian) system of balance of momentum and energy:

(26) $$\frac{d\mathbf{P}}{dt} - div_R\mathbf{b} = \mathbf{f}^{inh} + \mathbf{f}^{th}, \quad \frac{d(S\theta)}{dt} + \nabla_R.\mathbf{Q} = h^{th},$$

as first found in by Maugin [8].

4.3. Homogeneous dissipative solid material described by means of a diffusive internal variable

Let α the internal variable of state whose tensorial nature is not specified. This may relate to damage, or anelasticity of some sort with a possible diffusion of the said variable so that its material gradient must be taken into account (e.g., in strain-gradient plasticity). This is in the spirit of the thermodynamics developed at length in a book [19]. Then W is specified as the general sufficiently regular function

$$W = \bar{W}(\mathbf{F}, \theta, \alpha, \nabla_R \alpha). \tag{27}$$

First we assume that **K** vanishes. The *equations of state* (in a sense, mere definitions of the partial derivatives of the free energy) are given by *Gibbs'equation* as

$$\begin{gathered} \mathbf{T} = \frac{\partial \bar{W}}{\partial \mathbf{F}} \qquad S = -\frac{\partial \bar{W}}{\partial \theta} \\ A := -\frac{\partial \bar{W}}{\partial \alpha} \quad \mathbf{B} := -\frac{\partial \bar{W}}{\partial (\nabla_R \alpha)} \end{gathered} \tag{28}$$

Accordingly, we find that

$$\mathbf{f}^{int} = \mathbf{f}^{th} + \mathbf{f}^{intr}, \quad h^{int} = h^{th} + h^{intr}, \tag{29}$$

where the thermal sources have already been defined and the "intrinsic" sources are given by

$$\mathbf{f}^{intr} := A(\nabla_R \alpha)^T + B\left(\nabla_R\, (\nabla_R \alpha)^T\right)^T, \quad h^{intr} := A\dot{\alpha} + \mathbf{B}.(\nabla_R \dot{\alpha})^T, \tag{30}$$

so that we have the following consistent (non-Hamiltonian) system of canonical balance laws:

$$\frac{d\mathbf{P}}{dt} - div_R \mathbf{b} = \mathbf{f}^{th} + \mathbf{f}^{intr}, \quad \frac{d(S\theta)}{dt} + \nabla_R.\mathbf{Q} = h^{th} + h^{intr}, \tag{31}$$

while the dissipation reads

$$\Phi = h^{intr} - \mathbf{S}.\nabla_R \theta \geq 0, \quad \mathbf{K} \equiv \mathbf{0}. \tag{32}$$

Here the thermodynamical forces A and **B** are purely dissipative by virtue of the "internal" character of the state variable α.

This approach with **K**=**0** favors the *continuum mechanics* (Coleman-Noll) *standard viewpoint* by accepting the classical relationship between heat and entropy flux, and assuming that α and its material gradient are essentially independent. A more *field-theoretic* viewpoint is to envisage the set of eqns.(18) through (21) as holding true and selecting the non-zero **K** such that the divergence term in (18) be identically zero, after computation of dW/dt on account of (27), i.e.,

$$\mathbf{K} = -\theta^{-1}\mathbf{B}\dot{\alpha}. \tag{33}$$

This follows the scheme originally developed in [20] for materials with *diffusive* dissipative processes described by means of internal variables of state.

We let the reader check that eqns. (31) and (32) are then replaced by the following equations:

$$\frac{d\mathbf{P}}{dt} - div_R\tilde{\mathbf{b}} = \mathbf{f}^{\,th} + \tilde{\mathbf{f}}^{\,intr}, \quad \frac{d(S\theta)}{dt} + \nabla_R.\tilde{\mathbf{Q}} = h^{th} + \tilde{h}^{intr}, \tag{34}$$

and

$$\Phi = \tilde{h}^{intr} - \tilde{\mathbf{S}}.\nabla_R\theta \geq 0, \quad \tilde{h}^{intr} := \tilde{A}\dot{\alpha} \tag{35}$$

where we have introduced the new definitions

$$\tilde{A} \equiv := -\frac{\delta\bar{W}}{\delta\alpha} := -\left(\frac{\partial\bar{W}}{\partial\alpha} - \nabla_R.\frac{\partial\bar{W}}{\partial(\nabla_R\alpha)}\right) = A - \nabla_R.\mathbf{B}, \tag{36}$$

$$\tilde{\mathbf{S}} := \theta^{-1}\tilde{\mathbf{Q}}, \quad \tilde{\mathbf{Q}} = \mathbf{Q} - \mathbf{B}\dot{\alpha}$$

and

$$\tilde{\mathbf{b}} = -(L\mathbf{1}_R + \mathbf{T}.\mathbf{F} - \mathbf{B}.(\nabla_R\alpha)^T), \quad \tilde{\mathbf{f}}^{intr} := \tilde{A}\,(\nabla_R\alpha)^T\,. \tag{37}$$

The two thermodynamical approaches just illustrated are to be compared to the recent constructive comments of Ireman and Nguyen Quoc-Son [21]. Here we additionally show that alteration in the entropy flux definition goes along with a parallel alteration in the expression of the Eshelby stress tensor, thus reinforcing the space-like complementarity of eqn. (34). More on this with the possible interpretation of α as an additional degree of freedom when it is equipped with its own inertia in a recent work [22].

5. Conclusion

The above-reported formal developments had for main purpose to show that, guided by an admissible form of the energy conservation, we are naturally led to the construction of the corresponding canonical equation of conservation for the material momentum, with no specific information on the functional dependence of the free energy. This obviously accommodates a large spectrum of dissipative behaviors, in particular when we adopt the thermodynamics of internal variables to formulate complex irreversible behaviors. This generality is encapsulated in the general expression (27). For instance, in finite-strain elastoplasticity, we would select only the elastic deformation "gradient" $\mathbf{F}^e$ instead of the full $\mathbf{F} = \mathbf{F}^e.\mathbf{F}^p$ and the set of internal variables α can be built up of the plastic deformation gradient $\mathbf{F}^p$ itself and a set β of hardening variables, yielding a sufficiently general framework.

The resulting canonical equations of conservation of material momentum and energy are those to be exploited to determine the driving forces on defects in materially homogeneous or inhomogeneous materials, including appropriate generalizations of the *J* integral of fracture and the driving force on shock waves of different types (true shock waves and phase-transition fronts). To do this one can follow the line taken in a paper by Dascalu and Maugin [23] for cracks and the author [3], [9] for singularity surfaces.

What we finally learn from the above analysis is to make a clear distinction between various concepts of field theory applied to continuum mechanics. These concepts are those of (i) field equations, (ii) balance laws, (iii) conservation laws, and (iv) strict conservation laws. The first type are those equations which govern the fields, the latter being understood in the sense of field theory, i.e., selected form the start. This procedure is particularly well defined in the Lagrangian-Hamiltonian variational approach. The second type relates to a scrupulous examination of what makes a basic physical quantity (which is not necessarily a basic field) vary in time and space on account of prescribed external actions in the bulk and at the surface of a body. The result of this generally is a partial differential equation exhibiting a time derivative, a divergence term, and source terms. A conservation law in the present setting is generated by a variation in the describing parameters of the fields. This is related to an invariance requirement. A strict conservation law involves no source terms and is typically written as a four-dimensional space-time divergence. The four types of equations have been illustrated in this paper on the case of continuum mechanics. In some problems such as in the theory of exactly integrable systems in soliton theory, the number of balance equations is usually small, the (scalar) components of the field equations may usually be few, and the conservation laws may be infinite in number! (see, e.g., [24]). The relationship between some of the members of this infinite series and the conservation equations addressed in the present paper has been examined by the author in the context of the wave mechanics of solids (see, e.g., [25]).

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References

[1] MAUGIN G.A., *Material inhomogeneities in elasticity*, Chapman, London 1993.

[2] MAUGIN G.A., *Material forces: concepts and applications*, A.S.M.E. Appl. Mech. Rev. **48** (1995), 213–245.

[3] MAUGIN G.A., *Thermomechanics of inhomogeneous-heterogeneous systems: application to the irreversible progress of two- and three-dimensional defects*, ARI **50** (1997), 41–56.

[4] NOETHER W., *Invariante variationsproblem*, Klg-Ges. Wiss. Nach. Göttingen. Math. Phys. **2** (1918).

[5] ESHELBY J.D., *Elastic energy-momentum tensor*, J. of Elasticity **5** (1975), 321–335.

[6] GURTIN M.E., *Configurational forces as basic concepts of continuum physics*, Springer, New York 1999.

[7] MAUGIN G.A. AND TRIMARCO C., *Pseudo-momentum and material forces in nonlinear elasticity: variational formulations and application to brittle fracture*, Acta Mechanica **94** (1992), 1–28.

[8] MAUGIN G.A., *On the universality of the thermomechanics of forces driving singular sets*, Arch. Appl. Mech. **70** (2000), 31–45.

[9] MAUGIN G.A., *On Shock waves and phase-transition fronts in continua*, ARI **50** (1998), 141–150.

[10] MAUGIN G.A., *Eshelbian continuum mechanics and nonlinear waves*, in: "Computational fluid dynamics (Anniversary Volume of K. Roesner)" (Eds. Leutloff D. and Srivastava R.C.), Springer-Verlag, Berlin 1995, 269–287.

[11] MAUGIN G.A., *Geometry of material space: its consequences in modern numerical means*, Technische Mechanik (Magdeburg) **20** (2000), 95–104.

[12] MAUGIN G.A., *On the structure of the theory of polar elasticity*, Phil. Trans. Roy. Soc. Lond., **A356** (1998), 1367–1395.

[13] MAUGIN G.A. AND TRIMARCO C., *On material and physical forces in liquid crystals*, Int. J. Engng. Sci. **33** (1995), 1663–1678.

[14] ERINGEN A.C., *Mechanics of continua*, Krieger, Melbourne, Florida 1980.

[15] ERINGEN A.C. AND MAUGIN G.A., *Electrodynamics of continua*, Vol. I, Springer-Verlag, New York 1990.

[16] KIJOWSKI J. AND MAGLI G., *Unconstrained hamiltonian formulation of general relativity with thermo-elastic sources*, Classical Quantum Grav. **15** (1998), 3891–3916.

[17] BUI H.D., *Mécanique de la rupture fragile*, Masson, Paris 1978.

[18] EPSTEIN M. AND MAUGIN G.A., *Thermoelastic material forces: definition and geometric aspects*, C. R. Acad. Sci. Paris, **II-320** (1995), 63–68.

[19] MAUGIN G.A., *Thermdynamics of nonlinear irreversible behaviors*, World Scientific, Singapore 1999.

[20] MAUGIN G.A., *Internal variables and dissipative structures*, J. Non-Equilib. Thermod. **15** (1990), 173–192.

[21] IREMAN P. AND NGUYEN QUOC-SON, *Using the gradients of the temperature and internal parameters in continuum thermodynamics*, C. R. Mécanique **332** (2004), 249–255.

[22] MAUGIN G.A., *On the thermodynamics of continuous media with diffusion and/or weak locality*, Arch. Appl. Mech. (Anniversary issue) **75** (2006), 723–738.

[23] DASCALU C. AND MAUGIN G.A., *Forces matérielles et taux de restitution de l'énergie dans les corps élastiques homogènes avec défauts*, C. R. Acad. Sci. Paris **II-317** (1993), 1135–1140.

[24] ABLOWITZ M.J. AND SEGUR H., *Solitons and the inverse scattering transform*, S.I.A.M., Philadelphia 1981.

[25] MAUGIN G.A. AND CHRISTOV C.I., *Nonlinear waves and conservation laws (Nonlinear Duality Between Elastic waves and Quasi-particles)*, in: "Topics in nonlinear wave mechanics", (Eds. Christov C.I. and Guran A.), Birkhauser, Boston 2002, 117–160.

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Gérard A. MAUGIN, Laboratoire de Modélisation en Mécanique, UMR 7607 CNRS, Université Pierre et Marie Curie, Tours 55-65, case 162, place Jussieu 4, 75252 Paris cedex O5, FRANCE
e-mail : gam@ccr.jussieu.fr