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DIFFERENTIAL GEOMETRY, LEAST ACTION PRINCIPLES AND IRREVERSIBLE PROCESSES

"*Les phénomènes irréversibles et le théorême de Clausius ne sont pas explicables au moyen des équations de Lagrange"*. Poincaré, 1908.

Abstract. This contribution is intended as both a course on differential geometry and an illustration of the involvement of differential geometry in the calculus of variations, in articulation with the occurrence of irreversibility. Potential applications in terms of the continuous symmetries of the constitutive laws of dissipative materials shall be mentionned, leading potentially to a systematic and predictive approach of the construction of the so-called master curves.

1. Introduction

The contribution of differential forms to mathematics and physics is considerable, due to fact that they allow the unification, generalization and conception of notions encountered in a wide range of disciplines: mention amongst others elementary geometry, analysis, thermodynamics, continuum mechanics, electromagnetism, and analytical mechanics, (see [1, 2, 9, 11, 12, 16, 26, 28, 29, 30, 31]). The first part of the contribution gives the essentials of differential geometry in a synthetic manner. The proofs shall most of the time be omitted (the reader shall refer to one of the references related to differential geometry).
The following notations shall be used in the sequel: the partial derivative of a quantity a with respect to the variable x shall be noted a_x, or $a_{,x}$, or $\partial_x a$. The transpose of a vector or a tensor A is noted with a superscript A^t. The convention of summation of the repeated index in monomials is implicitly used (unless explicitly stated). The following abbreviations shall be used: w.r. for with respect to; s.t. for such that; r.h.s. for right-hand side; iff for iff and only if; notation := stands for the definition (expressed on the r.h.s.) of the quantity placed on the left hand-side.

2. Differential geometry: a reminder of the essential notions

2.1. Differentiable manifolds (submanifolds)

Consider M a set of points endowed with a topology and E_n a finite dimensional vector space (dimension n). A local chart on M is the pair (U_i, ϕ) consisting of an open set U_i of M and an homeomorphism ϕ: $U_i \longrightarrow \phi(U_i) \subset E_n$: one says that U_i is the domain of the chart (fig1).

Since a point in M can belong to 2 distinct open sets U_j, U_k, with the charts $\left(U_j, \phi_j\right)$ and (U_k, ϕ_k), a C^q-compatibility condition between the 2 charts is defined

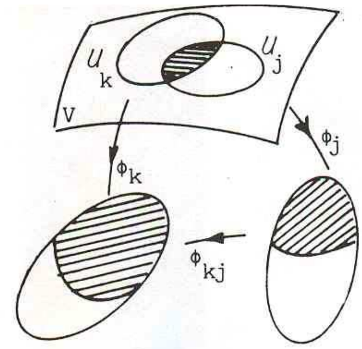

Figure 1: Chart of a manifold.

as $U_j \cap U_k \neq \emptyset \Rightarrow \phi_k \circ \phi_j^{-1}\Big|_{U_j \cap U_k}$ is a diffeomorphism of class C^q between the open sets $\phi_j\left(U_j \cap U_k\right)$ and $\phi_k\left(U_j \cap U_k\right)$. Points p on M are conveniently labeled by local coordinates x^i, which are the coordinates in $\mathbb{R}^n$ of the point $\phi(p)$. An atlas of class C^q on M is a set of charts $(U_i, \phi_i)_i$, s.t. the domains of the charts cover M; all charts of the atlas are C^q-compatible.

EXAMPLE 1. Consider the sphere of radius unity in $3D$ space, defined by

$$S^2 = \left((x_1, x_2, x_3) \in \mathbb{R}^3 / \sum_{i=1}^{3} x_i^2 = 1\right)$$

the stereographic projection of the North pole n onto the plane defined by $x_3 = 0$ is a bijection between $S^2 \setminus \{n\}$ and this plane. A similar projection of the South Pole can be defined.

In the following, the base of the topology of M is supposed countable, thus the manifold M is supposed separable. Submanifolds of $\mathbb{R}^{n+k}$ can be defined from the notions of submersion and immersion. For U open in $\mathbb{R}^n$, a C^∞ map $\psi : U \to \mathbb{R}^{n+k}$ is an immersion if its differential $d\psi(u) \in L\left(T_u\mathbb{R}^n \to T_{\psi(u)}\mathbb{R}^{n+k}\right)$ is a one-to-one map at every $u \in U$. The linear algebra characterization of an immersion is that the differential $d\psi(u)$ induces a one to one linear map from $\mathbb{R}^n$ to $\mathbb{R}^{n+k}$ (equivalently, the differential map $d\psi(u)$ has rank n). The dual notion of submersion is defined in the following manner: for V open in $\mathbb{R}^{n+k}$, and $f :\to \mathbb{R}^k$ a smooth map, f is a *submersion* if its differential $Df(x) \in L\left(T_x\mathbb{R}^{n+k} \to T_{f(x)}\mathbb{R}^k\right)$ in an onto map, thus when the matrix $Df(x)$ has rank k.

EXAMPLE 2. Consider the case $n = 2$ and $k = 1$; for $h \in C^\infty(\mathbb{R})$ a strictly positive function, the map which rotates the curve $x = h(z)$ around the z axis, namely

$\psi(u,\theta) = (h(u)\cos(\theta), h(u)\sin\theta, u)$ gives a parameterized surface of revolution. The differential is

$$D\psi\,(u,\,\theta) = \begin{pmatrix} h'(u)\cos\theta & -h(u)\sin\theta \\ h'(u)\sin\theta & h(u)\cos\theta \\ 1 & 0 \end{pmatrix},$$

the column of which being independent, thus ψ is an immersion. As an example of an submersion, let consider $V = \left\{(x_1, x_2, x_3)\, x_1^2 + x_2^2 + x_3^2 > 0\right\}$, and the function $f\,(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. Its differential is $Df\,(x_1, x_2, x_3) = 2\,(x_1, x_2, x_3)$, which is not zero on V, thus f is a submersion.

Manifolds can be parametrized as curves and surfaces traditionally; consider M as a subset of $\mathbb{R}^{n+k}$; an n-dimensional parametrization of M is given by a one-to-one immersion $\psi : W \to U \subset \mathbb{R}^{n+k}$, with U an open subset of $\mathbb{R}^{n+k}$ with $U \cap M \neq \emptyset$, and $\psi(W) = U \cap M$. The image of a $1D$ parametrization is a parametrized curve, and that of a $2D$ parametrization is a parametrized surface. For instance, the application

$$\begin{array}{rcl} \theta : (0, 2\pi) & \to & U = \mathbb{R}^2(1,0) \\ \theta & \mapsto & (\cos\theta, \sin\theta) \end{array}$$

gives a $1D$ parametrization of the unit circle in $\mathbb{R}^2$.

The implicit function theorem gives a convenient *implicit function parametrization*, i.e. one having the special form $\psi\,(x_1, \ldots, x_n, h_1(x), \ldots, h_n(x))$, with h an implicit function.

For example, a $2D$ implicit function parametrization at the point $(0, 0, 1)$ of the sphere S^2 in $\mathbb{R}^3$ is given by $\psi(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$, with domain

$$W = \left\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 < 1\right\}$$

and range $W \times (0, +\infty)$.

2.2. Transformations, Lie groups and Lie derivatives

Generally speaking, transformations map a set into itself, and a mathematical structure cam be characterized by those transformations that leaves it invariant (for instance, Euclidean geometry is invariant under orthogonal transformations, whereas special relativity has a structure compatible with invariance w.r. to the Lorentz group). Very often, transformations establish as a group, and the prime tool there is the infinitesimal transformation, which is described by a vector field (an infinitesimal generator of the group).

To each point of the manifold can be attached an n-dimensional vector space, called the tangent space (local notion). At a point $p_0 \in M$, let define the *germ* of a differentiable function g as the equivalence class of differentiable functions that coincide in an open neighborhood of p_0. Furthermore, a *tangent vector* is an equivalence class of curves having the same tangency at p_0; the curves $c : I \subset \mathbb{R} \to M$ are tangent at a point p_0, if, in a given chart $(U\phi)$, they give the same value $\frac{d}{dt}(\phi \circ c)(0)$, with $p_0 = c(0)$. Using further the composition of functions theorem gives the rate of variation of the function g along the curve c (i.e. the composite map $g \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$), as $\frac{d}{dt}(g \circ c)(0) = \sum_{I=1}^{n} \left(\frac{\partial g}{\partial xi}\right)_{x_0} \frac{dx_i}{dt}(0)$. Thereby, the notion of tangent vector receives a second definition: the tangent vector also is derivation acting on the set of germs of functions defined in an open neighborhood of p_0, i.e. a linear application $X_{p_0} : g \mapsto X_{p_0}(g) = \frac{d}{dt}(g \circ c)(0)$. Thus, the vector field X_{p_0} has the coordinates $\frac{dx_i}{dt}(0)$ (in the local basis $(x_i)_i$), and is given intrinsically by $X_{p_0} = \left(\frac{dx_i}{dt}\right)_{x_0} \frac{\partial}{\partial x_i}$. It is easy to see that the value of the action of X_{p_0} on any function is the same for any representant in the class of curves having he same tangent at p_0.

DEFINITION 1. *The tangent space to M at the point p_0 is the set of equivalence classes of tangent curves to M at p_0; it is also the set of tangent vectors to M at p_0. It is noted $T_{p_0}M$, and its dimension is n. The notion of tangent space to a manifold allows an intrinsic definition of the differential (independent from the local coordinates).*

For V_n, W_m differentiable manifolds, $f : V_n \to W_m$ differentiable, X_0 a tangent vector to V_n at point x_0, with $z_0 = f(x_0) \in W \in W_m$, the differential of f at x_0 is the linear application $df_{x_0} : T_{x_0}V_n \to T_{z_0}W_m$; $X_0 \mapsto df_{x_0}X_0$, s.t. $\forall h,\ df_{x_0}X_0(f^*h)$, see fig 2. The application f^* therein is the reciprocal image

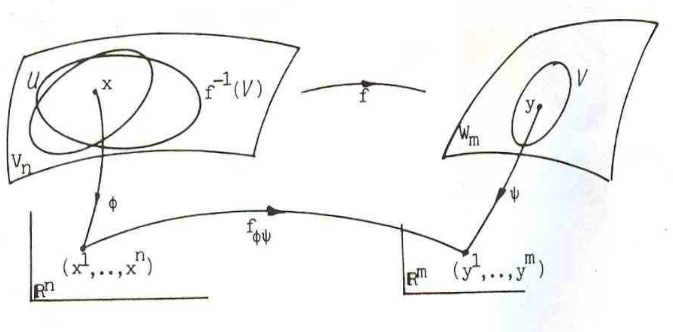

Figure 2: Differentiable functions and tangent mappings between manifolds.

The vector $Z_0 := df_{x_0}X_0$ is tangent to W_m at z_0. In local coordinates, one simply has

$$Z^j = \frac{\partial f^j}{\partial x_i}(x_0)\, X^i$$

and the matrix of elements $\left\{\dfrac{\partial f^i}{\partial x_i}(x_0)\right\}_{i,j}$ therein is the Jacobean matrix. This notion obviously reminds the transformation gradient in continuum mechanics.

DEFINITION 2 (Derivation). *A derivation is a first-order differential operator, which is sensitive only to linear terms, thus a derivation X shall operate on products of functions according to the rule $X(fg) = fX(g) + gX(f)$. This defines the Leibniz rule for derivatives that warrants X being insensitive to quadratic and higher-order terms (as shown earlier, vector fields act as derivations).*

For two vector fields X, Y acting on functions, the double operation XY also maps functions to functions, but is not a derivation. As an example, any derivation at the point $(0, 0)$ acting on the function $f(x, y) = x^2 + y^2$ must give zero (use Leibniz rule). But, composing ∂_x with itself gives $\partial_x \partial_x f(x, y) = 2$, thus $\partial_x \circ \partial_x$ is not a derivation (and the composition of derivations does not give a derivation in general). However, the operation of *Lie bracket* restores the property of being a derivation: it is defined by $[XY - YX]$, which in 3-vector notation would read $[X, Y] = (X \cdot \nabla) Y - (Y \cdot \nabla) X$. The Lie bracket receives an important geometric interpretation, in connection to **Frobenius Theorem**: if at every point, the Lie bracket of tangent vectors to two families of curves are a linear combination of the two vectors, the curves then fit together to define a 2-surface. This can easily be generalized to higher dimension.

DEFINITION 3. *A Lie group G is a set that both has the structure of a group and of a manifold. Thus, it is a differentiable manifold of class C^∞, and the group structure is characterized by the following operations (product and inversion)*

$$G \times G \to G;\, (x, y) \mapsto xy; \qquad G \to G;\, x \mapsto x^{-1}.$$

Every neighborhood of e is then sent by L_y to a neighborhood of y by the left translation; the differential application $dL_y : T_eG \to T_yG$ allows the definition of left invariant vector fields: $\forall y \in G,\ dL_y X(e) = X(y)$.

It can easily be shown that the left invariant vector fields on G have a vectorial structure (same dimension as G), and that this vectorial space is isomorphic to the tangent space T_eG. Furthermore, the bracket of two left invariant vector fields is itself a left invariant vector field, namely one has $dL_y[X, Y](e) = \left[dL_y X, dL_y Y\right](e) = [X, Y](y)$. The left invariant vector fields on the Lie group G thus have the structure of *Lie algebra*.

DEFINITION 4. *The Lie algebra of the Lie group G is the Lie algebra of the left invariant vector fields. A Lie group action on a manifold M is given by a map $\mu : G \times M \to M;\, (a, q) \mapsto \mu_a(q)$, satisfying $\mu_e(q) = q$ and the composition rule $\mu_a \circ \mu_b = \mu_{ab}$. Lie groups and Lie algebra are most of the case discussed in terms of their matrix representations.*

EXAMPLE 3. The Lorentz group action in ED space-time can be represented by the one-parameter family of matrices

$$\begin{bmatrix} ch\psi & sh\psi, \\ sh\psi & ch\psi, \end{bmatrix}$$

which defines a one-dimensional group (identity element is given by $\psi = 0$), with group manifold $\mathbb{R}$. The group action on $\mathbb{R}^2$ is defined by the application

$$(t, x) \mapsto (t, ch\psi + x\, sh\psi, t\, sh\psi + x\, ch\psi)$$

The Lie algebra is endowed with the bracket operation (of vector fields); it satisfies the properties of linearity, anticommutativity ($[X, Y] = -[Y, X]$), and the Jacobi identity ($[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$).
Once the action of a transformation on points is defined, the action on tangent vectors (or more generally on elements of the tangent bundle), on tensors and differential forms (on elements of the cotangent bundle) are determined. The change operated on those objects is called the *Lie derivative*. Recall that any differentiable vector field X on a manifold M generates a 1-parameter local group of diffeomorphisms ϕ_t relating neighborhoods of M, viz $\phi_t : M \to M\ x \mapsto \phi_t x$, that satisfies the differential equation

$$\frac{d\phi_t}{dt} = X\,(\phi_t x)$$

with the initial condition $\phi_0(x) = x$. The orbit of the group passing through point $x_0 = \phi(x_0)$ is the integral curve $\mathbb{R} \to M; \mapsto x(t) = \phi_t x_0$ tangent to the vectors $X\,(\phi_t x_0)$ of the field at each point $\phi_t x_0$. Since the manifolds (contrary to the Euclidean spaces) do not allow an easy comparison of vector fields attached at different points (thus leaving in different vectorial spaces), a novel derivative needs to be introduced. Consider g a differentiable function on M; the tangent vector to the group ϕ_t at the point x_0 is $X_0 = \left(\frac{d}{dt}x(t)\right)_{t=0} = \left(\frac{d}{dt}\phi_t x_0\right)_{t=0}$. The derivative of (the germ of) g in the direction of X at the point x_0 is the real $X_0\, g = \left(\frac{d}{dt}\,(f \circ \phi_t)\,(x_0)\right)_{t=0} = X^I \left(\frac{\partial}{\partial x_i} g\right)_{x_0^I} \left(\frac{dx^i}{dt}\right)_0$.

- The Lie derivative of the function g in the direction of X at point x_0, is defined as the directional derivative $L_{x_0} g = X_0 g = \lim_{t\to 0} \frac{g\,(\phi_t x_0) - g\,(x_0)}{t}$. The operation achieved therein means a pull-back along the orbit to the point x_0, comparing the value $g\,(\phi_t x_0)$ to the value $g\,(x_0)$ at the same point. In a set of local coordinates, one writes
$$L_X g = X^I \partial_i\, g$$
- The Lie derivative of the vector field Y in the direction of the vector field X at point x_0 is $L_X Y = \lim_{t\to 0} \frac{1}{t} \left(d\phi_t^{-1} Y_{\phi_t x_0} - Y_{x_0}\right) = \left(\frac{d}{dt} d\phi t^{-1} Y\right) t = 0$. It can be

proven that the Lie derivative coincides with the Lie bracket $L_X Y = [X, Y]$. In a set of local coordinates, one writes

$$L_X Y = \left(X^j \partial_j Y^I - Y^j \partial_j X^i \right) \partial i.$$

- The Lie derivative of a $\begin{pmatrix} q \\ 0 \end{pmatrix}$ tensor field is similarly defined as

$$L_X T := \lim_{t \to 0} \frac{1}{t} \left(d\phi_t^{-1} T_{\phi_t x_0} - Y_{x_0} \right)$$

- Noting $\phi_t^* T_{\phi_t x_0}$ the pull-back at point x_0 of the tensor $T_{\phi_t x_0}$, the Lie derivative of a $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor field is elaborated as

$$L_X T := \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* T_{\phi_t x_0} - Y_{x_0} \right) = \left(\frac{d}{dt} \phi_t^* T \right)_{t=0}$$

- Similarly, for completely antisymmetrical tensors of the previous type, i.e. for differential forms ω, the Lie derivative is defined as

$$(L_X \omega)_{x_0} = \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* \omega_{\phi_t x_0 - \omega_{x_0}} \right) = \left(\frac{d}{dt} \phi_t^* \omega \right)_{t=0}$$

  In a set of local coordinates, one writes for the Lie derivative of a 1-form $(L_X \omega)_I = X^j \partial_j \omega i + \omega_j \partial_i X^j$.

Properties

Only the essential properties of the differential operations so far introduced are listed in the sequel. A vector field ν s.t. $L_w \nu = 0$ is said to be **Lie-transported** or dragged along the vector field w. Since the Lie derivative is a local approximation, it shall satisfy Leibniz rule, thus

$$L_w \left(a \otimes b \right) = L_w a \otimes b + a \otimes L_w b.$$

Using the same rule gives the Lie derivative of a 1-form α: differentiating the function $f = \alpha.\nu$ renders $(L_w \alpha) .\nu = d \left(\alpha.\nu \right) w - \alpha \left[w, \nu \right]$, thus in terms of a set of coordinates $L_w \alpha = \left(\alpha\mu, \theta w \nu + \alpha \theta w^{\nu}_{,} \nu \right) dx^{\nu}$. Lie derivatives inform about symmetries of geometrical objects; so for instance, the infinitesimal symmetries of the 1-form field dx are given by those vector fields $w = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z}$, that satisfies $L_w dx = 0$, thus X has to be constant.

2.3. Calculus on differential forms

Differential forms find their origin in 1889 in the work of Elie Cartan $(1869-1951)$, and in the third volume of *Les Methodes Nouvelles de la Mecanique Celeste* by Henri Poincaré $(1854-1912)$. The program of writing the laws of physics in an invariant form (using differential forms), was started by g. Ricci-Curbastro $(1853-1925)$ and his student T. Levi-Civita $(1873-1941)$; it provided the useful framework for A. Einstein $(1879-1955)$ to develop the theory of relativity.
Consider M an n-dimensional manifold and $(x_1, x_2, \ldots, x_n)$ a coordinate system on this manifold.

DEFINITION 5. *A p-form or exterior form of degree p on M is alternated or completely antisymmetrical if it is the antisymmetrical part of a multilinear application from M to $\mathbb{R}$. Thus, for t_x a p-linear form (at point x of M), the operation of antisymmetrization renders*

$$A_p\, t_x\,(V_1, \ldots, V_n) = \frac{1}{p!} \sum_{\sigma} \epsilon_\sigma t_x \left(V_{\sigma(1)}, \ldots, V_{\sigma(p)} \right),$$

with σ a permutation having the signature ϵ_σ.

A p-form can be built from the tensorial product of n one-forms on M, according to the rule: the tensorial product of p 1-forms is the p-form, the components of which are identified with the components of the tensorial product if the p associated vectors.

EXAMPLE 4. Consider $\alpha(x_1\, x_2) = 3x_2 - x_1$ and $\beta(x_1\, x_2) = 2x_2 + x_1$; one then has

$$\alpha \otimes \beta = (x_1, x_2) \left[\begin{pmatrix} -1 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= (x_1\, x_2) \begin{pmatrix} -1 & -2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -x_1^2 + x_1 x_2 + 6x_2^2$$

DEFINITION 6. *The exterior product (notation $\wedge$) of p 1-forms A_{ik}, is the p-form obtained by the anti symmetrization of the tensorial product, viz*

$$A_{i1} \wedge A_{i2} \ldots \wedge A_{ip} = \delta^{i_1 \ldots i_p}_{I_1 \ldots I_p} A_{I_1} \otimes \ldots \otimes A_{I_p}$$

(summation of the indices I_k), with

$$\delta^{i_1 \ldots i_p}_{I_1 \ldots I_p} = \begin{cases} 0 \text{ if the } i_k \text{ are not a permutation of the } I_k \\ 1 \text{ if the } i_k \text{ are an even permutation of the } I_k \\ -1 \text{ if the } i_k \text{ are an odd permutation of the } I_k \end{cases}$$

EXAMPLE 5. One has in $\mathbb{R}^4$ the equality

$$x_3 \wedge x_1 = x_3 \otimes x_1 - x_1 \otimes x_3 = -x_1 \wedge x_3$$

The canonical basis of the set of p-forms of order $p \le n$ at the point $x \in M$, noted $\Omega_x^p(M)$, is given by the C_n^p exterior products $x_{i_1} \wedge \ldots \wedge x_{i_p}$, with the following ordering of the indices $i_1 < i_2 < \ldots < i_p$.. Thus, one can express any p-form ω in a basis $\theta^{i_1} \wedge \ldots \wedge \theta^{i_1}$ of $\Omega_x^p(M)$ using the *strict components* of the p-form $\omega = \omega_{i_1 \ldots i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}$, with the summation done only on the ordered indices $i_1 < i_2 < \ldots < i_p$.

EXAMPLE 6. Consider $\omega \in \Omega_x^2(M)$; the action of ω on a couple of tangent vectors $X, Y \in T_x M$ is given by $\omega(X, Y) = \omega\left(X^i e_i, Y^j e_j\right) = \omega_{ij} X^i Y^j$, noting $\omega_{ij} \equiv \omega(e_i, e_j)$ the action of ω on the basis vectors (e_i, e_j). Expanding the result and using the antisymmetry of the matrix $\omega_{ij} + \omega_{ji} = 0$, renders

$$\omega(X, Y) = \sum_{I<j} \left(X^i Y^j - X^j Y^i\right) = \sum_{I<j} \omega_{ij} \theta^i \wedge \theta^j (X, Y).$$

Thus, one has $\omega = \sum_{i<j} \omega_{ij} \theta^i \wedge \theta^j$, and the products $\theta^i \wedge \theta^j$, $i < j$, generate any 2-form.

The exterior product of forms has the following properties: the exterior product $\wedge$ is bilinear, associative, non commutative. For two forms ω of order p, and μ of order q, one has $\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega$. The space exterior product of the two spaces $\Omega_x^p(M)$ and $\Omega_x^q(M)$ is then the vectorial space $\Omega_x^{p+q}(M)$.

DEFINITION 7. *The exterior differentiation is the application d that associates to a p-form ω a $(p+1)$ -form $d\omega$, satisfying:*

- *for a function g from M to $\mathbb{R}$, the exterior derivative dg is simply the differential of g;*

it has the following properties

- *d is a linear operator;*
- *d is 2-nilpotent, viz $d \circ d = 0$ (iteration rule);*
- *d is an antiderivation, viz $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu$.*

A practical formula for the calculus of the exterior derivative is given in the following

THEOREM 1. *Set $\omega = \omega_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ a p-form; its exterior derivative is given by $d\omega = d\omega_{i_1 \ldots i_p} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} = \dfrac{\partial \omega_{i_1 \ldots i_p}}{\partial x_r} dx_r \wedge dx_{i_1} \ldots \wedge dx_{i_p}$.*

EXAMPLE 7. On $\mathbb{R}^3$, consider the 2-form $\omega = x_3 \, x_1 \, dx_2 \wedge dx_3$. One has

$$d\omega = x_3 \, dx_1 \wedge dx_2 \wedge dx_3.$$

It is interesting to relate exterior differentiation on $\mathbb{R}^3$ to classical vector calculus. Given a vector field $X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z}$, one defines the work form and the flux form of X respectively, as $\omega_X := \xi_1\, dx + \xi_2\, dy + \xi_3\, dz$, and $\phi_X := \xi_1\, dy\, dz + \xi_2\, dz\, dx + \xi\, dx\, dy$. These coinages come from the fact that the line integral of a one-form along a path measures the work done by X, while the surface integral of the flux form measures the flux of the field through the surface. Lastly, let define the density form of a smooth function f on the open set $U \subset \mathbb{R}^3$, as $\rho_f := f\, dx\, dy\, dz$. The table 2.1 then gives the correspondence between the differential forms ω_X, ϕ_X, ρ_f, and the vector components given as components of the exterior derivatives of ω_X, ϕ_X.

Differential form ω	Exterior derivative $d\omega$	Vectorial operator
$f = f(x, y, z)$	$df = \omega_{grad\ f} = f_{,x}dx + f_{,y}dy + f_{,z}dz$	$grad$
Work form $\omega_X := \xi_1\, dx + \xi_2\, dy + \xi_3\, dz$	$d\omega_X = \phi_{curl\ X} = \left(\frac{\partial \xi_3}{\partial y} - \frac{\partial \xi_2}{\partial z}\right) dy\, dz + \left(\frac{\partial \xi_1}{\partial z} - \frac{\partial \xi_3}{\partial x}\right) dzdx + \left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y}\right) dx\, dy$	$curl$
Flux form $\phi_X := \xi_1 dydz + \xi_2 dzdx + \xi_3 dxdy$	$d\phi_X = \rho_{divX} = \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}\right) dxdydz$	Div
$f\, dx\, dy\, dz$	0	

Table 2.1: exterior differentiation of forms and vector calculus

The set of differential forms of arbitrary order on a manifold M defines the *exterior algebra* on M, otherwise called the Grasmann algebra. The exterior algebra $\Omega_x(M)$ at M is the direct sum $\Omega_x(M) = \Omega_x^0(M) \oplus \Omega_x^1(M) \oplus \ldots \oplus \Omega_x^n(M)$. Elements in $\Omega_x^0(M)$ are functions, and the maximum order is $p = n$ (having only one element, the volume form). The dual to the tangent space to a manifold is called the *cotangent space*; it consists of the 1-forms (otherwise called *covectors* or covariant vectors - the tangent vectors being called contravariant) acting on the tangent space.
Another notion of differentiation of forms is given in the following

DEFINITION 8. *The vertical differential of a function $f = f(q_i, \dot{q}_i)$ (notation d_v) expresses locally as $d_v f = \dfrac{\partial f}{\partial \dot{q}_i} dq_i$, with $d_v(dq_i) = 0 = d_v(d\dot{q}_i)$.*

The inverse operation of decreasing the order of forms is given in the following

DEFINITION 9. *Consider a vector field X defined on M, and a p-form $\omega = \omega_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. The interior product of ω by X, noted $i_X\omega$, is the*

$(p-1)$-*form* $i_X\omega := \dfrac{1}{(p-1)!}\,\omega_{ki_2\ldots i_p} X_k\, dx_{i_2} \wedge \ldots \wedge dx_{i_p}$.

EXAMPLE 8. consider on $\mathbb{R}^2$ the 1-form $\omega = x_2\, dx_1$ and the vector field

$$X = x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2};$$

the direct application of previous definition gives $i_X\, \omega = \omega_k\, X_k = x_2^2$.

Properties

The following identity is often referred to as *Cartan identity*

$$L_\nu \omega = i_\nu d\omega + d\,(i_\nu \omega)\,.$$

One further list some basic properties (without proof):

$$i_\nu(\omega \wedge v) = (i_\nu\omega) \wedge v + (-1)^p \omega \wedge (I_\nu v)$$

$$L_\nu d\omega = dL_\nu\omega;\, L_\nu\,(i_u\omega) = i_{[\nu,\mu]}\omega + i_u L_\nu \omega;$$

$$L_{f\nu}\alpha = f\, L_\nu\alpha + df \wedge (i_\nu\alpha)$$

$$L_\nu(\omega \wedge v) = (L_\nu\omega) \wedge v + \omega \wedge (L_\nu v)\,.$$

The pullback of a p-form ω defined on the manifold W_m by the differentiable function $f\,:\,V_n \to W_m;\, x \mapsto z = f(x)$ is the *induced p-form,* $f^* : \Omega^p\,(W_m) \to \Omega^p\,(V_n)$, s.t.

$$\forall x \in V_n, \forall V_1, \ldots, V_p \in T_x V_n,\ \left(f^*\omega\right)_x \left(V_1, \ldots, V_p\right) = \omega_z \left(df_x V_1, \ldots, df_x V_p\right)$$

The representation of $f^*\omega$ in local coordinates is given by

$$f^*\omega = \omega_{j_1\ldots j_p}\,(z(x))\, \frac{D\left(z^{j_1}, \ldots, z^{j_p}\right)}{D\left(x^{j_1}, \ldots, x^{j_p}\right)}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

with $\frac{D\left(z^{j_1}, \ldots, z^{j_p}\right)}{D\left(x^{j_1}, \ldots, x^{j_p}\right)}$ the Jacobean of the transformation from the $\left(x^i\right)_i$ to the $\left(z^j\right)_j$. The operators d and f^* commute, viz $d \circ f^* = f^* \circ d$.

EXAMPLE 9. The pullback of the 3-form $dx\, dy\, dz$ (omitting here the symbol $\wedge$) under the change of coordinates $x = r\ \cos\theta$, $x = r\ \sin\theta$, $z = z$ (cylindrical coordinates $r,\ \theta,\ z$) is $f^*(dx\, dy\, dz) = d(r\ \cos\theta)\, d(r\ \sin\theta)\, dz \equiv r\, dr\, d\theta\, dz$ (the Jacobean is thus r).

The pullback of forms is used to evaluate integrals on manifolds (change of variables). Both the differential and the pullback operation find simple interpretations in terms of the Jacobean matrix: suppose $\omega = A_1\, dy^1 + \ldots + A_1\, dy^m$ is a 1-form on the open set $V \subset \mathbb{R}^n$, and $\eta = \phi^*\omega$ expresses as $\eta = B_1\, dx^1 + \ldots + B_n\, dx^n$ as an

1-form on the open set $U \subset \mathbb{R}^n$, with $\phi : U \rightarrow V$ a diffeomorphism. Representing ω and η as the row vectors $\vec{\omega} = [A_1, \ldots, A_m]$ and $\vec{\eta} = [B_1, \ldots, B_n]$ then gives

$$[B_1, \ldots, B_n] = [A_1, \ldots, A_m] \begin{bmatrix} D_1\phi^1 & \ldots & D_n\phi^1 \\ \ldots & \ldots & \ldots \\ D_1\phi^m & \ldots & D_n\phi^m \end{bmatrix}$$

Thus, the pullback of a 1-form corresponds to matrix post-multiplication. As an application, when $p = n = m$, one recovers the change of variable formula used in the theory of integration, viz $\omega = A\left(dy^1 \wedge \ldots \wedge dy^n\right) \Rightarrow \phi^*\omega = |D\phi|\, A\left(dx^1 \wedge \ldots \wedge dx^n\right)$. The exterior derivative of a 1-form corresponds to the pre-multiplication by the Jacobean matrix, since the operation corresponds to the tangent mapping associated to the differential. The pullback of 1-forms is related to the dual operation of the push-forward of vector fields: for X a vector field on U, the *push-forward* of X under ϕ is defined as the vector field $\phi \cdot X$ on V, s.t. $\phi \cdot X(y) = d\phi(x)\left(X\left(\phi^{-1}(y)\right)\right)$ The pullback of the 1-form ω then relates to the *push-forward* of X as: $(\phi^*\omega)\,.X(x) = \omega.\phi \cdot X(y)$.

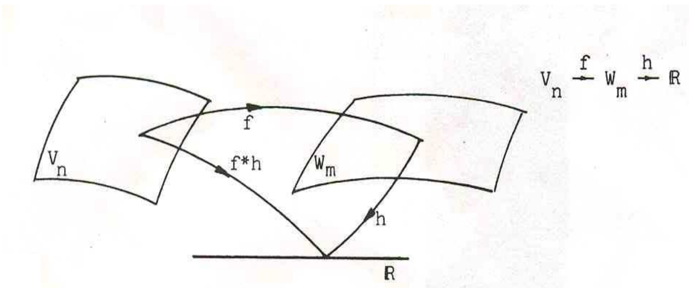

Figure 3: Diagrammatic representation of the pullback operation.

These two operations find useful applications in continuum mechanics, see [1].

The Hodge star operator

Let V_n be an n-dimensional vector space equipped with an inner product $\langle ., . \rangle$. Since $dim\left(\Omega^{n-p}(V_n)\right) = \dim\left(\Omega^p(V_n)\right)$, for $p \leq n$, one can define a natural isomorphism between both these spaces. For any $\lambda \in \Omega^p(V_n)$, and assuming a given choice of the orientation of space has been done, there exists a unique element - noted $^*\lambda \in \Omega^{n-p}(V_n)$ - s.t. $\forall \mu \in \Omega^{n-p}(V_n)\,,\ \lambda \wedge \mu = \langle ^*\lambda,\ \mu \rangle_{n-p}\,\sigma$, with σ the volume form on V_n. The Hodge star operator is the application that sends $\lambda \rightarrow^* \lambda$.
As an application, the correspondence between the exterior algebra and the 3D vector algebra is shown in the following Table.

Further applications of the Hodge star operator shall be given later on. Note lastly that differential forms receive a geometrical interpretation [31]. So, for instance, a 1-form can be represented by two parallel lines (planes in $3D$) in $2D$, representing the density of lines being cut. Just think of the gradient of a function as the 1-form giving the intensity of the slope between neighboring contours on a topographic map

Vector algebra expression	Exterior algebra expression
Cross product $(u \times v)$	$^* (u \wedge v)$
Triple product $u.(v \times w)$	$^* (u \wedge v \wedge w)$
$\lvert u \times v\rvert^2 = \lvert u\rvert^2 \lvert v\rvert^2 - (u.v)w$	$\langle u \wedge v,\ u \wedge v\rangle = \langle u,\ u\rangle\langle v,\ v\rangle - \langle u,\ v\rangle^2$
$u \times (v \times w) = (u.w)v - (u.v)w$	$u \wedge^* (v \wedge w) = \langle u,\ w\rangle\,(^*v) - \langle u,\ v\rangle\,(^*w)$

Table 2.2: relation between the exterior algebra and the 3D vector algebra

[31]. The gradient of the function f (here the height of the contour) is orthogonal to the 1-form representation. The condition $i_v df = 0$ for a vector v then means that v is orthogonal to the components $grad\ f$ of the 1-form df. In electricity, a 2-form in $3D$ space (represented by a box aligned by the current flow direction) gives the current density (section of the box).

2.4. Contact structures and symplectic mechanics

The contact structure is a manifold suitable for the description of unparameterized curves. The line element contact bundle, called CM in the sequel, consists of a pair, namely a point in the manifold and a line element at that point. The line element itself gives the local approximation of the unparameterized curve, as a tangent vector of unspecified length (in fact a class of equivalence of tangent vectors, under the relation $v \sim kv$). Considering a submanifold - the pair $(N,\ \psi)$ - as being represented by a map $\psi : N \to M$ (s.t. both ψ and its differential are one-to-one), the first order contact between two submanifolds $(N,\ \psi)$ and $(N',\ \psi')$ at a common point $\psi(p) = \psi'(p')$ is traduced by the equality of tangent mappings $T_\psi \left[T_p(N)\right] = T\psi' \left[T_{p'}(N')\right]$ (this is not a point by point equality, but rather an equality between sets). The equivalence class of submanifolds in contact at a point $q \in M$ is called a *contact element* at q, and is noted $[N,\ \psi]$, for any submanifold N; it is in fact a linear subspace of the tangent space. Note that this notion of contact is weaker than the related notion of tangency. The contact structure is both a bundle (it has a projection onto the base space) and it has a contact structure: for each n-dimensional submanifold $(N,\ \psi)$ in M, one can define the natural lift $\sigma : M(M,\ n); q \mapsto (q,\ [N,\ \psi])$, with $[N,\ \psi]$ a contact element.

A simple chart for $C(M,\ N)$ is given by selecting n of the coordinates of M (labeled q^μ), and considering the remaining $(m - n)$ coordinates Y^a as functions of

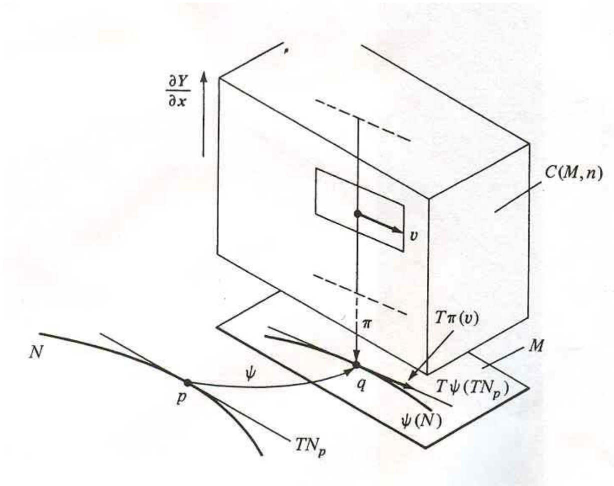

Figure 4: the geometric structure of the contact bundle (from [31]).

the q^{μ}. The contact elements are then represented by the partial derivatives $p^a_{\mu} = \frac{\partial Y^a}{\partial q^{\mu}}$ (note that the index position is here coherent with that chosen for the manifolds CM and C^*M).

DEFINITION 10. *Note that not all curves are lifts; a lifted submanifold is a submanifold for which the p^a_{μ} coincide with the partial derivatives, as written above. This condition can be expressed using the* 1*-forms $\theta^a = dy^a - p^a_{\mu} dq\mu$, that we pull-back onto a submanifold of $C(M, N)$, with $\psi : N \to C(M, N)$; $q \mapsto (q, Y(q), P(q))$. Using the pull-backs $\psi^* \cdot dq = dq = dq$; $\psi^* \cdot dy^a = \frac{\partial Y^a}{\partial q^{\mu}} dq^{\mu}$, one gets $\psi^* \cdot \theta^a = \left(\frac{\partial Y^a}{\partial q^{\mu}} - P^a_{\mu} \right)$. These pull-backs do vanish when N is a lifted submanifold; these lifted submanifolds shall then be called integral submanifolds of the contact ideal.*

EXAMPLE 10. Consider functions on the plane as the maps $F : \mathbb{R}^2 \to \mathbb{R}$. The graphs of these functions are the sections described by the map $\psi : \mathbb{R}^2 \to \mathbb{R}^3$; $(x, y) \mapsto (x, y, F(x, y))$, that define 2-sumanifolds on $\mathbb{R}^3$. A contact element at the point
$p\,(x_0, y_0, F\,(x_0, y_0))$ is an equivalence class of 2-submanifolds that have first order contact at p. It can be represented by the linear submanifold

$$(x, y) \mapsto \left(x, y, F + \frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) \right)$$

The partial derivatives (here evaluated at the point (x_0, y_0)) are natural coordinates for the contact elements; the coordinates for the jet bundle are here $(x, y, f, f_x\ f_y)$, in which f_x, f_y are coordinates of the contact element (and not partial derivatives). The pull-back of the contact 1-form $\theta := df - f_x dx - f_y dy$ onto a submanifold defined by the application $\psi : (x, y) \mapsto \left(x,\ y,\ F,\ F_x\ F_y\right)$ gives $\psi^* \cdot \theta = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy - F_x dx - F_y dy$, which vanishes when $F_x = \frac{\partial F}{\partial x}$; $F_y = \frac{\partial F}{\partial y}$. The geometric approach to the calculus of variations presents some interest because of its originality, compared to the standard approach. The standard formulation of the classical mechanics of unconstrained conservative systems states the existence of a Lagrangian function L, depending upon the state variables and possibly upon time, such that the action integral $\int L\, dt$ is extremized. On the contact bundle of the configuration space, having the natural coordinates $(t,\ q,\ \dot{q})$, the integrand $\int L\, dt$ is a one-form. The possible motions of the system are then described by the curves in the contact bundle on which the contact 1-forms $\alpha := dq - \dot{q}\, dt$ pull back to zero. The variation of the action integral around the extrema is further performed, using the vector field $v = Q(t) \dfrac{\partial}{\partial q} + \dot{Q}(t) \dfrac{\partial}{\partial \dot{q}}$, restricting to isochronal variations. Restricting further to variations with fixed end conditions, the variation is given by

$$\int_\Gamma L_v\,(L\, dt) = \int_\Gamma i_v\,(dL\, dt) = \int_\Gamma \left[Q \frac{\partial L}{\partial q} dt + \dot{Q} \frac{\partial L}{\partial q} dt \right]$$

In order now for the variations to satisfy the previous constraint condition, the vector field v has to move the initial path into a path that is parallel to the α; instead of pushing the path forward, one can equivalently pull back the 1-forms α, as $\alpha(\epsilon) = \alpha + \epsilon L_v \alpha$. The condition that the 1-form $\alpha(\epsilon)$ pulls backs to zero on the integral curve, viz $i_{\dot{\gamma}} \alpha(\epsilon) = 0$, renders $i_{\dot{\gamma}} L_v \alpha = 0$, thus $i_{\dot{\gamma}} \left(dQ - \dot{Q}\, dt \right) = 0$. Since integration over the optimal path is equivalent to contracting the integrand with $\dot{\gamma}$, using the previous equation allows to replace $\dot{Q}\, dt$ by dQ, thus

$$L_v \int_\Gamma L\, dt = \int_\Gamma \left[Q \frac{\partial L}{\partial q} dt + \frac{\partial L}{\partial \dot{q}} dQ \right]$$

Integrating the second term by part and omitting the perfect differential (since we consider fixed ends) renders

$$L_v \int_\Gamma L\, dt = \int_\Gamma Q \left[\frac{\partial L}{\partial q} dt + d\left(\frac{\partial L}{\partial \dot{q}} \right) \right]$$

The arbitrariness in the choice of the functions $Q(t)$ then leads to the condition

$$i_{\dot{\gamma}} \left\{ \frac{\partial L}{\partial q} dt + d\left(\frac{\partial L}{\partial \dot{q}} \right) \right\} = 0$$

for $\dot{\gamma}$ to be tangent to the path.
This condition together with the constraint

$$I_{\dot{\gamma}}\ (dq - \dot{q} dt) = 0$$

gives $2n$ relations for the $2n$ components of the line element.
Since $L\, dt$ is not a general 1-form (its exterior derivative $dL\, dt$ being in the differential

ideal generated by $L\,dt$) , and $d\alpha$ also is in the ideal generated by $L\,dt$, a more general viewpoint is needed. Let then enlarge the previous considerations, starting first from the extemum condition of unconstrained integrals of the form $I = \int_\gamma \omega$, with the smooth enough 1-form ω defined on a manifold M, and Γ a curve in M (between two points A and B) s.t. the integral I does not vary at the first order as Γ is deformed. Considering a curve parameterized by s, the tangent vector to the curve $\dot{\gamma}$ expresses as the push forward of the basis vector $\frac{\partial}{\partial s}$, viz $\dot{\gamma} = \Gamma_*.\left(\frac{\partial}{\partial s}\right)$. Let then continuously deform the curve Γ by a vector field v; the previous condition that the integral I does not change under this deformation expresses as

$$\int_\Gamma L_v \omega$$

Using the properties of the Lie derivatives further renders

$$\int_\Gamma i_v\, d\omega + \int_\Gamma i_v\, \omega = 0$$

Since v vanishes at the end points (A and B being fixed), the boundary term vanishes. From the definition of a line integral, we further have

$$\int_\Gamma i_v\, d\omega = \int_a^b \Gamma^*.\left(i_v\left(i_{\dot{\gamma}\, d\omega}\right)\right) ds = 0, \forall v,$$

thus the local condition $i_v\left(i_{\dot{\gamma}\, d\omega}\right) = 0$, $\forall v$, that further gives $i_{\dot{\gamma}\, d\omega} = 0$, which is an ordinary differential equation for Γ.

Using the local coordinates $x^\mu = x^\mu(s)$ along the curve, one can further elaborate previous condition: the integral $I = \int \omega_\mu\, dx^\mu ds$ renders the Euler-Lagrange equations $\omega_{\mu,v}\dot{x}^\mu - \frac{d}{ds}\omega_v = 0$ thus giving the Schwarz condition $\left(\omega_{\mu,v} - \omega_{v,\mu}\right)\dot{x}_\mu = 0$. A first insight into Noether's theorem can here be given: suppose an infinitesimal symmetry exists, having the vector field k, such that $L_k\omega = 0$. This clearly gives $\int_\Gamma L_k\omega = 0$; along any piece of a curve that satisfies the Euler-Lagrange equations, the condition $i_{\dot{\gamma}} d\omega = 0$ gives

$$\int_\Gamma L_k\omega = \int_\Gamma d\left(i_k\omega\right) + \int_\Gamma i_k d\omega = \int_\Gamma i_k\omega = 0$$

which means that the quantity $i_k\omega$ is constant along the solution curves. This is an illustration of Noether's theorem, articulating infinitesimal symmetries and conservation laws.

The case of constrained variations is next treated, whereby the constraint is expressed by the vanishing of some function $\phi : M \to \mathbb{R}$, in the case of holonomic variations: we require that the variation of the integral $I = \int_\Gamma \omega$ vanishes, viz $L_v \int_\Gamma \omega = 0$, for all deformations of the curve that satisfy the constraint $L_v\phi = i_v d\phi = 0$. Summarizing, the variation of the integral must vanish at any point, viz $i_v\left(i_{\dot{\gamma}} d\omega\right) = 0$, $\forall v$, for all v

satisfying the condition $i_v d\phi = 0$. This implies the existence of a multiplier function $\lambda = \lambda(s)$, s.t. the following condition holds along the curve: $i_{\dot\gamma} d\omega = \lambda d\phi$, where $\phi = 0$.
This condition forms a set of determining equations for the curve and the Lagrange multiplier λ. In the case of anholonomic variations, the constraints can be expressed in the form $i_{\dot\gamma}\alpha = 0$, for a set of prescribed 1-forms α. The extension of the vector field $\dot\gamma$ off the curve is done using the condition $L_v\dot\gamma = 0$, thus the constraint condition $i_{\dot\gamma} L_v\alpha = L_v\left(i_{\dot\gamma}\alpha\right) = 0$.
The optimal path Γ is again determined by the condition $i_{\dot\gamma} d\left(\omega + \lambda\alpha\right) = 0$, along with $i_{\dot\gamma}\alpha = 0$. To be complete, one can evidence a Lagrange multiplier λ s.t. an unconstrained minimum exists for the problem having the one-form $\omega + \lambda\alpha$, see e.g [31]. For that purpose, a vector field w is selected s.t. the deformation of the curve can be written in the form $v = v_c + \xi w$, where v_c satisfies the constraint, and ξ is a scalar function that restores the degree of freedom lost in the constraint: it is found $\xi = \dfrac{i_{\dot\gamma} L_v \epsilon}{i_{\dot\gamma}\left(i_w d\alpha\right)}$, under the condition $i_w d\alpha \neq 0$ (with $d\alpha \neq 0$, otherwise the constraint would be integrable). Using next Cartan identity and neglecting the exact differentials gives the multiplier $\lambda = -\dfrac{i_{\dot\gamma}\left(i_w d\omega\right)}{i_{\dot\gamma}\left(i_w d\alpha\right)}$.

EXAMPLE 11. The dynamic equations of motion of a conservative system described by a Lagrangian $L\left(q,\ \dot q, t\right)$ is given as the stationary conditions of the functional $\int_\Gamma L\left(q,\ \dot q,\ t\right)\ dt$, under the constraints $i_{\dot\gamma}\left(\dot q dt - dq\right) = 0$. Application of previous general methodology renders the multiplier $\lambda = -\dfrac{\partial L}{\partial \dot q}$, and the equivalent unconstrained problem is $\displaystyle\int_\Gamma \left[L dt - \frac{\partial L}{\partial \dot q}\left(\dot q dt - dq\right)\right]$, the Euler-Lagrange equations of which being

$$i_{\dot\gamma}\left\{dL dt - d\left(\tfrac{\partial L}{\partial \dot q}\right)\left(\dot q dt - dq\right) - \tfrac{\partial L}{\partial \dot q}\right\} = i_{\dot\gamma}\left\{\tfrac{\partial L}{\partial q} dq\, dt - d\left(\tfrac{\partial L}{\partial \dot q}\right)\left(\dot q dt - dq\right)\right\} = 0.$$

It holds true that

$$i_{\dot\gamma} dq\, dt = i_{\dot\gamma}\left(dq - \dot q dt\right) dt = -\left(dq - \dot q dt\right) dt = -\left(dq - \dot q dt\right)\left(i_{\dot\gamma} dt\right)$$

the previous equation then gives

$$i_{\dot\gamma}\left\{dL dt - d\left(\tfrac{\partial L}{\partial \dot q}\right)\left(\dot q dt - dq\right) - \tfrac{\partial L}{\partial \dot q} d\dot q dt\right\} = i_{\dot\gamma}\left\{\tfrac{\partial L}{\partial q} dq dt - d\left(\tfrac{\partial L}{\partial \dot q}\right)\left(\dot q dt - dq\right)\right\} = 0$$

$$i_{\dot\gamma}\left\{\frac{\partial L}{\partial q} dt - \left(\frac{\partial L}{\partial \dot q}\right)\right\}\left(\dot q dt - dq\right) = 0$$

Consequently, the extremals are the integral curves of the 1-forms $\dfrac{\partial L}{\partial q} dt - d\left(\dfrac{\partial L}{\partial \dot q}\right)$ and $dq - \dot q dt$.

The invariance of the constrained problem under infinitesimal symmetries again leads to the evidence of a conservation law: let indeed the vector k be an infinitesimal symmetry of both the variational principle and of the constraint, viz $L_v\omega = 0$ and $L_v\alpha = 0$. Thus, the Lie derivative of the one-form α has to be in the ideal generated by α (since $i_{\dot{\gamma}}\alpha = 0$). Incorporating the multiplier into the Lie differentiation then gives the condition $L_k(\omega + \lambda\epsilon) \subset I[\alpha]$, which traduces into a differential format as $\int_\Gamma \{d(\omega + \lambda\alpha) + i_k d(\omega + \lambda\alpha)\} = 0$. The second term is identified as the Euler-Lagrange equation (it vanishes), thus one obtains the conservation law $i_k d(\omega + \lambda\alpha) = constant$, along the extremals.

3. Lagrangian formalism and irreversibility

3.1. Differential structure of thermodynamics

A few words related to the geometrical setting of thermodynamics are first in order. Thermodynamic systems are described in the contact bundle sustained by the following coordinates:

- The total energy and entropy;
- the extensive variables (such as the volume, the number of particle, or the electric charge), that are the measurable degrees of freedom;
- the intensive associated variables, which are forces or potentials that describe the energy transfer between the various extensive variables.

EXAMPLE 12. An ideal gas is described in a 2D state space, with coordinates entropy and volume. An open gaseous system would require the additional coordinate of the amount of gas present in the system.

A thermodynamic system shall then be described by a linear structure, a contact structure and a convexity structure: the *linear structure* is a model for the physical idea of short-range interactions and existence of homogeneous systems with a scaling symmetry. The *contact structure* is associated with the energy conservation (first law of thermodynamics), while the *convexity structure* accounts for the Second Law and the entropy increase due to mixing. As a starting point, the *fundamental equation* consists of the expression of the stored internal energy of the system for all possible states, versus the set of state variables. For instance, the fundamental equation of an aggregate of N molecules of an ideal gas is

$$U(S\,V) = N^{5/3}V^{-2/3}\exp(2S/3NK),$$

with k the Boltzmann's constant. Linear structures rely on the assumption that the system size is much larger than the range of its interactions, thus the internal energy is proportional to the size of any subsystem (the shape of the subsystem does not matter):

by Euler's theorem (traducing here the homogeneity of degree one), this scaling symmetry gives $U = S\dfrac{\partial U}{\partial S} + V\dfrac{\partial U}{\partial V}$. Contrary to the status of the extensive variables, the intensive variables do not change versus size (homogeneity of degree zero). The graph of the fundamental equation is an n-surface in an $(n-1)$-space, with the potentials (partial derivatives of the internal energy w.r. to the extensive variables) the components of the contact elements to that surface. The contact bundle consists of the $(2n+1)$-space with coordinates the extensive variables, their associated intensive variables, and the internal energy. The contact ideal is generated by the 1-form

$$\alpha = dU + \sum (\text{forces})\, d\, (\text{extensive variables})\,.$$

EXAMPLE 13. The previously introduced ideal gas is modeled in the 5-dimensional contact bundle with coordinates $(U,\ T,\ S,\ P,\ V)$, and a contact ideal generated by the 1-form $\alpha = dU - TdS + Pdv$, with $P = -\partial - VU$; $T = \partial_S U$. The previous homogeneity condition becomes $U = TS - PV$, which is the *Gibbs-Duhem relation*. In addition to the fundamental equation, the equation of state expressing the intensive variables has to be specified. As an example, consider the ideal gas, which obeys the relation $PV = NkT$, together with the internal energy expression $U = 3/2NkT$. In the contact manifold, the system is described by the following map

$$\psi : (S,\ V) \mapsto (U,\ T,\ S,\ P,\ V) = (U\,(S,\ V)\,,\ T\,(S,\ V)\,,\ S,\ P\,(S,\ V)\,,\ V)$$

which expresses as

$$\psi : (S,\ V) \mapsto (U,\ T,\ S,\ P,\ V) = (3/2NkT,\ T\,(S,\ V)\,,\ S,\ NkT/V,\ V)\,.$$

The functions $U,\ Y,\ P$ shall satisfy the two previous conditions, as well as $\psi^*.\theta = 0$, with the 1-form $\theta = dU - U_V\, dV - u_p\, dP = dU - TdS + PdV$. Accounting for the pullbacks

$$\psi^*.dU = (3/2)\ Nk\left(\frac{\partial T}{\partial S}dS + \frac{\partial T}{\partial V}dV\right);\ \ \psi^*.dS = dS;\ \ \psi^*.dV = dV$$

one obtains

$$\psi^*.\theta = (3/2)\ Nk\left(\frac{\partial T}{\partial S}dS + \frac{\partial T}{\partial V}dV\right) - TdS + \left(\frac{NkT}{V}\right)dV \equiv 0$$

The independence of the differential elements $(dV,\ dT)$ then implies

$$\frac{1}{T}\frac{\partial T}{\partial S} = 2/3Nk;\ \ \frac{1}{T}\frac{\partial T}{\partial V} = -2/3V$$

The integration of these two equations gives $T = AV^{-2/3}exp(2S/3Nk)$, thus we recover the fundamental equation $U\,(U,\ V) = N^{5/3}V^{-2/3}exp\,(2S/3Nk)$. Note that the factor $N^{5/3}$ therein ensures the satisfaction of the homogeneity condition of U (a system of twice the volume, with twice the number of molecules has twice the energy).

Having so far developed what could be called a differential thermodynamics [31], the use of Frobenius theorem closes the characterization of the structure of thermodynamics: consider for instance the energy 2-surface as the map $\mathbb{R}^2 \rightarrow \mathbb{R}^3$; $(S, V) \mapsto (U, S, V)$, with the intensities given by $T = \partial_S U$ and $-P = \partial_V U$. The symmetry of the partial derivatives also leads to $\partial_V T + \partial_S P = 0$. The system can locally be described by a 2D surface element, spanned by the two vectors $A = a\frac{\partial}{\partial U} + \frac{\partial}{\partial T} + b\frac{\partial}{\partial S} + c\frac{\partial}{\partial P}$ and $A = d\frac{\partial}{\partial U} + \frac{\partial}{\partial T} + e\frac{\partial}{\partial S} + f\frac{\partial}{\partial V}$, that must lie in the zero surface of α, thus the conditions: $i_A\alpha = 0 = i_B\alpha$. This gives $a = bT$ and $d = eT - fP$. The fact that it is a differential ideal also implies that $d\alpha$ is one generator of the ideal, thus $i_B(i_A d\alpha) = 0$. Combining the previous relations gives the Maxwell relation $(c_V - C_P)/T + \left(\frac{dV}{dT}\right)_P \left(\frac{dP}{dT}\right)_V = 0$, with C_V, C_P the heat capacities at constant volume and pressure respectively.

3.2. Differential geometric setting for dynamical systems

Noether's theorem embodies the fact that to every symmetry is associated a conservation law. For an exterior differential system, a conservation law is a differential form whose restriction to the integral manifold is closed. Any closed generator of the ideal leads to a conservation law.

EXAMPLE 14. [31] The heat-equation $\kappa\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial t} = 0$ (with $\phi = \phi(x, t)$ the temperature, κ the specific thermal diffusivity), can be equivalently rephrased as the first order system $u = \kappa\frac{\partial\phi}{\partial x}$; $\frac{\partial u}{\partial x} - \frac{\partial\phi}{\partial t} = 0$. The last equation of this system rewrites as $d\alpha = 0$, having defined the 1-form $\alpha = \phi\, dx + u\, dt$, that represents the *heat flux*. T hus, the equation $d\alpha = 0$ describes the *conservation of energy*. The geometrical picture of the heat transport equation can be given, using the following sharp operator: $\sharp dx = \frac{\partial}{\partial x}$; $\sharp dt = 0$, leading to the Hodge star operator ${}^*1 = dx\, dt$; ${}^*dt = 0$; ${}^*dx = dt$; ${}^*dx\, dt = 0$. We then have ${}^*\alpha = \phi\, dt$; $d^*\alpha = \phi_{,x} dx\, dt$. The heat flux can further be written $u = i_{\frac{\partial}{\partial t}}\alpha$. Therefore, the geometric form of the heat equation is given by the following differential system

$$\left| \begin{array}{c} d\alpha = 0 \\ d\kappa^*\alpha = i_{*\lambda}\alpha \end{array} \right.$$

Note that the 1-form field α describes a field of conserved flux lines, but the 1-form ${}^*\alpha$ is not the gradient of any function, thus the flux lines are not cut by a regular family of orthogonal hypersurfaces. The problem can further be formulated as the integral submanifold of the ideal generated by the two 2-forms $\omega = d\phi\, dx + du\, dt$ and $\beta = u\, dx\, dt - \kappa\, d\phi\, dt$: for a $2D$ submanifold $\psi : (t, x) \mapsto (t, x, \phi, u)$, the condition of zero pull-back, viz $\phi^*\dot{\beta} = 0$, traduces the relationship $u = \kappa\, \phi_{,x}$. Note that the ideal generated by the previous 2-forms is a differential ideal, due to the relationship

$$d\beta = du\, dx\, dt = -dx \wedge \omega.$$

In the ideal defined by the forms $\omega,\ \beta$, the form ω is closed, and the 1-form $j := i_S\omega$ satisfies the differential identity $dj = L_S\omega - i_S d\omega$ which vanishes for every isovector S, since $d\omega$ pulls back to zero. This leads to conservation laws, for instance the conservation of heat, viz

$$i_{\phi\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial u}}\omega = \phi\, dx + u\, dt.$$

One of the approaches suitable for the generalization of the Lagrange formalism to dissipation is the differential geometry of manifolds: the interest of this generalized Lagrangian formulation lies in the fact that it follows from the structure of the chosen manifold, and naturally introduces the notion of a Rayleigh potential. In order to illustrate this method, let consider a discrete system of n punctual masses m_I, having the d.o.f. $q = \{q_I(t),\ I = 1 \ldots 3n\}$ in $3D$ Euclidean space. Such a mechanical system is characterized by (Godbillon):

- a differentiable manifold generated by the d.o.f. $q = \{q_i(t),\ i = 1 \ldots 3n\}$, called configuration manifold (the integer $m = 3n$ is the number of d.o.f.);
- a differentiable function K on the tangent space to M (here noted $T(M)$), called kinetic energy;
- a pfaffian π (differential form of degree one) defined on $T(M)$, that takes the form of the work $\pi = F_i\,(q,\ \dot{q})\, dq_i$ of the force F_i. The fundamental form of the mechanical system is defined as the exterior differential of the vertical differential of K, viz

$$\omega = \frac{\partial^2 K}{\partial q_k \partial \dot{q}_i} dq_k \wedge dq_i + \frac{\partial^2 K}{\partial \dot{q}_k \partial \dot{q}_i} d\dot{q}_k \wedge dq_i$$

Assuming this 2-form is closed and regular, and introducing the Liouville vector field $v = \dot{q}_i \dfrac{\partial}{\partial \dot{q}_i}$, the manifold structure implies the following

THEOREM 2. *There is a unique vector field* $X = a_i \dfrac{\partial}{\partial q_i} + b_i \dfrac{\partial}{\partial \dot{q}_i}$ *defined on* $T(M)$ *s.t.*

$$\begin{aligned} i_X\omega &= \frac{\partial^2 K}{\partial q_k \partial \dot{q}_i} a_k\, dq_i - \frac{\partial^2 K}{\partial q_k \partial \dot{q}_i} a_i\, dq_k + \frac{\partial^2 K}{\partial \dot{q}_k \partial \dot{q}_i} b_k\, dq_i \\ &- \frac{\partial^2 K}{\partial \dot{q}_k \partial \dot{q}_i} a_i\, d\dot{q}_k = d\,(K - vK) + \pi \end{aligned}$$

The integral curve of X (as a dynamical system) are solutions of the Lagrange equations

$$\frac{d}{dt}\frac{\partial K}{\partial q_i} - \frac{\partial K}{\partial q_i} = F_i\,(q,\,\dot{q})$$

The force field π is further decomposed into a contribution due to conservative forces F_I^C, deriving from a potential energy V, according to $-dV = F_i^c\,dq_i \to F_i^c = -\dfrac{\partial V}{dq_i}$, and a non conservative contribution $F_i^{nc}dq_i$, viz

$$\pi = -dV + F_i^{nc}dq_i.$$

Introducing the Lagrangian of the system given by $L := K - V$, previous equation rewrites as

(1) $$\frac{d}{dt}\frac{\partial L}{\dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i^{nc}\,(q,\,\dot{q})$$

EXAMPLE 15. **Differential geometry of the oscillatory mass**
In the case of single mass m evolving on a straight line, with position $q = q(t)$, submitted to an elastic force $F^c = -kq = -\dfrac{\partial V}{\partial q}$ (with clearly $V = \dfrac{1}{2}kq^2$) and a viscous force $F^{nc} = -\lambda\dot{q}$ (with λ a constant), the kinetic energy is $K = \dfrac{1}{2}m\dot{q}^2$, and the force field associated to F^c and F^{nc} is $\pi = -\lambda\dot{q}\,dq - k\,q\,dq$. Thus, the fundamental form ω and the Liouville vector field are respectively given by
$\omega = m\,d\dot{q} \wedge dq$; $v = \dot{q}\dfrac{\partial}{\partial\dot{q}}$.
Application of previous Theorem then leads to the search of a vector field X under the form

$$X = a\,(q,\,\dot{q})\,\frac{\partial}{\partial q} + b\,(q,\,\dot{q})\,\frac{\partial}{\partial\dot{q}}$$

satisfying the differential form identity

$$\begin{aligned} i_X\omega &= -m\,a\,(q,\,\dot{q})\,d\dot{q} + -m\,b\,(q,\,\dot{q})\,d\dot{q} \\ &= d\,(K - vK) + \pi = -m\,\dot{q}d\dot{q} + (-\lambda\dot{q} - kq)\,dq. \end{aligned}$$

The identification of the coefficients of the one-forms dq and $d\dot{q}$ then leads to

$$X = \dot{q}\frac{\partial}{\partial q} - \left(\frac{\lambda}{m}\dot{q} + \frac{k}{m}q\right)\frac{\partial}{\partial\dot{q}}$$

The integral curves of X are the solutions of the differential system
$\dfrac{dq}{dt} = d\dot{q}$; $\dfrac{d\dot{q}}{dt} = -\dfrac{\lambda}{m}\dot{q} - \dfrac{k}{m}q$ that condenses into the dynamical equation of motion $m\ddot{q} + \lambda\dot{q} + kq = 0$.

Defining the Lagrangian as the difference $L = K - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$, the equivalence between the Lagrange equation and the integral curves of X easily appears:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F^{nc} \Leftrightarrow m\ddot{q} + kq = -\lambda\dot{q}$$

Previous theorem can be considered as a generalization of the Lagrangian formalism, since the previous equation (1) results from the Lagrange-d'Alembert principle

$$(2)\quad \delta\int_{t_0}^{t_1} L\,(q,\,\dot{q})\;dt + \int_{t_0}^{t_1} F_i^{nc}\,(q,\,\dot{q})\,\delta q_i\;dt = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i^{nc}\,(q,\,\dot{q})$$

In the conservative case ($\pi = -dV$), previous equation resumes to the stationary condition of the action integral. Note furthermore that the non conservative forces are usually supposed to derive from a pseudo-potential dissipation (also called Rayleigh potential) R, as $R = -\frac{1}{2}\lambda\dot{q}^2 \Rightarrow F_i^{nc} = \frac{\partial R}{\partial \dot{q}_i}\,(= -\lambda\dot{q})$

The application of the Lagrange-d'Alembert principle also shows that the variation of the action integral $\int_{t_0}^{t_1} L\,(q,\,\dot{q})\;dt$ does not vanish, evidencing thereby a closure defect of the pfaffian $L\,(q,\,\dot{q})\;dt$ along its extremal (topological torsion of the configuration manifold, according to [21]).

The notion of Rayleigh potential introduced in the dynamics evokes the nearby concept of dissipation potential, that plays a role essentially in the thermomechanics of continuous media. Various attempts towards the formulation of the state laws and evolution equations of a viscoelastic and / or viscoplastic solid under a Lagrangian form have been addressed in the literature. Those approaches rely on the setting up of the Helmholtz free energy - here noted ψ that essentially depends upon two types of variables:

- observable variables (one can measure them), being in general the temperature T and a deformation like variable ϵ;
- hidden variables that describe the internal state of the material. These variables are otherwise called internal variables, here noted α (of a scalar or tensorial nature).

Accordingly, the potential takes a priori the general expression $\psi = \psi\,(\epsilon,\,\alpha,\,T)$, from which the state laws follow from the use of Clausius-Duhem inequality, as

$$\sigma_r = \rho_0\psi_{,\epsilon};\;\; A = -\rho_0\psi_{,\alpha};\;\; s = -\rho_0\psi_{,T}$$

with ρ_0 the density in the reference configuration, σ_r the reversible part of the stress, A the thermodynamical affinity (conjugated to the internal variables), and s the entropy density. These state laws shall be completed in the case of dissipative media by the information related to the irreversible behavior, via a pseudo potential of dissipation

$\Omega(\dot{\epsilon})$, s.t. the irreversible part of the stress is given by $\sigma_{ir} = \Omega_{,\dot{\epsilon}}$, considering the additive decomposition $\sigma = \sigma_r + \sigma_{ir}$. Adopting a viscoelastic behavior, the affinity A derives from a second pseudo-potential $\Phi(\dot{\alpha})$, as $A = \Phi_{,\dot{\alpha}}$.

The Lagrangian formalism established in [33] relies on the definition of a pseudo-potential D, being the sum $D = \Omega(\dot{\epsilon}) + \Phi(\dot{\alpha})$. The author next defines a functional $S = S[u, \dot{u}, \alpha, T]$ with

$$\begin{aligned} S \;&:=\; \int_{t_0}^{t_1} \left(\int_V \left[\frac{1}{2}\rho_0 \left(\frac{du}{dt} \right)^2 - \rho_0 \psi\left(\epsilon(u), \alpha, T\right) \right] dV \right) dt \\ &+\; \int_{t_0}^{t_1} \left(\lambda \int_{S_f} T^d . u \, dS \right) dt \end{aligned}$$

with T^d the given imposed traction on the portion of boundary S_f, and λ a loading parameter that explicitly depends upon time. The variational principle associated to the extremality conditions of S can be viewed as a generalization of the Lagrange d'Alembert principle to continuous dissipative media; its formulation w.r. to the sole displacement is

$$\delta S + \int_{t_0}^{t_1} \left[\int_V \left(\frac{\partial D}{\partial \dot{\epsilon}} \epsilon \, \delta u \right) dV \right] dt = 0$$

leading to a relation analogous to (2):

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} = \frac{\partial D}{\partial \dot{\epsilon}} \epsilon$$

This equation in turn leads to the dynamical equations of equilibrium

$$div\left(\sigma_r + \sigma_{ir}\right) = \rho_0 \frac{d^2 u}{dt^2}; \;\; \left(\sigma_r + \sigma_{ir}\right) . n = \lambda T^d$$

with n the outward normal to S_f. The complementary information relative to the thermodynamic forces (that traduces the internal evolution of the body) is given by the Lagrange equations relative to the internal variables, viz

$$\frac{\partial L}{\partial \alpha} = \frac{\partial L}{\partial \dot{\alpha}} \Leftrightarrow A = -\rho_0 \frac{\partial \psi}{\partial \alpha} = \frac{\partial \Phi}{\partial \dot{\alpha}}$$

One shall note the strong analogy between the description of the dynamics of a discrete system of dissipative punctual masses and the writing of the Lagrange equations in presence of non conservative forces F^{nc}: the first involves a Rayleigh potential R, while the second approach requires the functional S to be supplemented by the pseudo-dissipation potential D.

Going further in that direction, an attempt to extend the Lagrange formalism for dissipative media is further elaborated, in connection with the associated variational symmetries.

4. Lagrange formalism and TIP

Following the axioms of classical thermodynamics as stated in [4], let assume the existence of a functional E, called the internal energy, being extensive w.r. to its arguments, viz $E = (E(V\epsilon, S, N)$, whereby the introduced arguments reflect the different forms of energy:

- mechanical energy (we here focus on small deformations ϵ, with a nearby constant volume V);
- calorific energy, represented by the total entropy S;
- chemical energy, represented by the number of moles $N = \{N_k, k = 1 \ldots n\}$ of the various species.

The extensity of E (homogeneity of degree one) expresses as (Euler's theorem):

$$E(\lambda V\epsilon, \lambda S, \lambda N) = \lambda E(V\epsilon, S, N), \quad \forall \lambda \in \mathbb{R}$$

Deriving previous equation w.r. to λ at $\lambda = 1$ leads to the Euler identity

$$\frac{\partial E}{\partial (V\epsilon)} : (V\epsilon) + \frac{\partial E}{\partial S} S + \frac{\partial E}{\partial N} N = E(V\epsilon, S, N) \Rightarrow$$

$$E(V\epsilon, S, N) = \sigma(V\epsilon, S, N) : (V\epsilon) + T(V\epsilon, S, N) S + \mu(V\epsilon, S, N) .N$$

wherein the intensive quantities conjugated to the independent intensities have been introduced: the stress $\sigma(V\epsilon, S, N)$, the temperature $T(V\epsilon, S, N)$, and the chemical potentials $\mu(V\epsilon, S, N)$. Accounting for these relationships then leads to the fundamental Gibbs relation:

$$dE = \sigma : d(V\epsilon) + TdS + \mu.dN$$

The differentiation of Euler's identity leads to the Gibbs-Duhem relation

$$(V\epsilon) : d\sigma + SdT + N.d\mu = 0$$

Both the Gibbs and Gibbs-Duhem relations are at the roots of thermodynamics; Gibbs-Duhem relation expresses the adjustment of the intensive variables during the variation of the extensities. When sufficient mechanical energy is brought to the system as input, may lead to a change of the internal configuration of the body, due to the fact that the system escapes from equilibrium. Assuming that the internal energy still has the status of a potential function, and replacing the variables N_k by extensive internal variables Ω_i, one has $E = E(V\epsilon, S, \Omega)$. The thermodynamic driving force A (or affinity in the language of De Donder) associated to the internal variable α expresses as $A_i(V\epsilon, S, \Omega) = -\dfrac{\partial E(V\epsilon, S, \Omega)}{\partial \Omega_i}$. In the sequel, we shall rather work with densities, thus writing the generalized fundamental Euler's relation as

$$e(\epsilon, s, \alpha) = \sigma(\epsilon, s, \alpha) : \epsilon + T(\epsilon, s, \alpha) s - A(\epsilon, s, \alpha) .\alpha$$

here introducing the energy and entropy density e and s respectively, and the density of the internal variables extensities Ω, noted α. The Gibbs-Duhem relation then rewrites as

$$\epsilon : d\sigma + sdT - \alpha.dA = 0$$

The state laws that give the constitutive behavior of the body then express in rate form as

$$\text{(3)} \qquad \begin{pmatrix} \dot{\sigma} \\ \dot{T} \\ -\dot{A}_i \end{pmatrix} = \begin{pmatrix} e_{,\epsilon\epsilon} & e_{,\epsilon s} & e_{,\epsilon\alpha_k} \\ e_{,s\epsilon} & e_{,ss} & e_{,s\alpha_k} \\ e_{,\alpha_k\epsilon} & e_{,\alpha_k s} & e_{,\alpha_k\alpha_k} \end{pmatrix} \cdot \begin{pmatrix} \dot{\epsilon} \\ \dot{s} \\ -\dot{\alpha}_k \end{pmatrix},$$

In the vicinity of equilibrium, the matrix of second order partial derivatives can be considered as made of constant entries. In order to be synthetic, let introduce the vector $y = (\epsilon,\ s)^t$ of the controlled extensities (their densities), being conjugated to the dual observable, noted $Y = (\sigma,\ T)^t$. Previous system then rewrites $P = 0$, with

$$\text{(4)} \qquad P := \begin{cases} P_Y(y,\ \alpha) = Y - e_{,yy}.\dot{y} - e_{,y\alpha}.\dot{\alpha} = 0 \\ P_A(y,\ \alpha) = -\dot{A} - e_{,\alpha y}.\dot{y} - e_{,\alpha\alpha}.\dot{\alpha} = 0 \end{cases}$$

Elementary calculations show that the previous system satisfies the self-adjunction condition of the Frechet derivative of P, viz $D_p = D^* P$, being equivalent to the Maxwell's relations for the internal energy e [19, 32]. Recall that the *Frechet derivative* of a vector of functions $P_i\left(x,\ u^{(n)}\right)$, depending upon the independent variable x and the dependent variable u, up to its derivatives to the order n, is the differential operator D_p given by $\left(D_p\right)_{ij} = \sum_J \frac{\partial P_i}{\partial u_{j,J}} D_J,\ i = 1 \ldots r,\ \ j = 1 \ldots q$. The multiindex J of dimension k consist of a set of k indices each less than 4, viz $J = (j_1, \ldots,\ j_k),\ 1 \leq j_k \leq 4$. Accordingly, one expresses the partial derivative $u_{i,j} = \frac{\partial^k u_i}{\partial x_{j1} \ldots . \partial x_{jk}}$.

EXAMPLE 16. For $P = u + u_x^2$, one has $D_p = \frac{\partial P}{\partial u} + \frac{\partial P}{\partial u_x} D_x = 1 + 2u_x D_x$, with D_x the total derivative operator w.r.t. x.

THEOREM 3. *The adjunct of the Frechet derivative is the matrix of differential*

$$\left(D^* P\right)_{ij} = \sum_J (-D)_J \frac{\partial P_j}{\partial u_{i,J}},\ \ i = 1 \ldots q;\ \ j = 1 \ldots r$$

Given the scalar products of two elements $P = \left\{P_i\left(x,\ u^{(n)}\right)\right\},\ \ Q = \left\{Q_i\left(x,\ u^{(n)}\right)\right\}$ *as* $\langle P,\ Q\rangle := \int_\Omega \sum_{I=1}^q P_i Q_i dx$, *the adjunct satisfies the following condition*

$$\langle P,\ DQ\rangle = \langle Q,\ D^* P\rangle,\ \ \forall P = \left\{P_i\left(x,\ u^{(n)}\right)\right\},\ \ \forall Q = \left\{Q_i\left(x,\ u^{(n)}\right)\right\}$$

EXAMPLE 17. For $D = \dfrac{d}{dt}$ the operator acting on functions with compact support in $\Omega =]0,\ 1[$, one writes

$$\langle u,\ Dv\rangle = \int_0^1 v\frac{dv}{dt}dt = -\int_0^1 u\frac{dv}{dt}dt + [uv]_0^1 = \langle v,\ D^* u\rangle,$$

thus the adjunct $D^* \equiv -\frac{d}{dt}$.

The existence and construction of a Lagrangian for a system described by a set of PDE's is expressed in the following

THEOREM 4. [27] *A system of PDE on the dependent variables u of the form* $P(u) = \left\{P_i\left(x,\ u^{(n)}\right),\ \ I = 1\ldots q\right\} = 0$ *realizes the extremum of a functional integral* $S = \int_\Omega L\ d\Omega$, *i.e.* $P_i = E_i(L)$, *with* $E_i(.)$ *the Euler-Lagrange operator, iff its Frechet derivative is self-adjunct. In this case, a possible Lagrangian is given by the line integral* $\displaystyle\int_0^1 \sum_{i=1}^{q} u_i.P_i(\lambda u)d\lambda$. *Equivalent Lagrangian are obtained up to the generalized divergence of a vector* $P = \{P_t,\ P_X,\ P_Y,\ P_z\}$, *defined as* $Div\ P = \displaystyle\sum_{i=1}^{4}\frac{\partial P_i}{\partial x_i}$.

EXAMPLE 18. (The vibrating string) The transverse vibrations of a string of length l_0 are described by the PDE $\lambda\, u_{tt} - T\, u_{xx} = 0$, with λ the lineic mass, and T the tension applied to the string. It is immediate to see that this EDP is self-adjunct, and a possible Lagrangian is set up as $L = \dfrac{1}{2}u\left(\lambda\, u_{,tt} - T\, u_{,xx}\right)$, however lacking a physical significance. It can further be worked out as

$$L = -\frac{1}{2}\lambda\, u_{,t}^2 + \frac{1}{2}T\, u_{,x}^2 + \frac{d}{dt}\left(\frac{1}{2}\lambda\, u\, u_{,t}\right) - \frac{d}{dx}\left(\frac{1}{2}T\, u\, u_{,x}\right).$$

An equivalent Lagrangian is $L = \dfrac{1}{2}\lambda\, u_{,t}^2 - \dfrac{1}{2}T\, u_{,x}^2$, thus the action integral

$$S = \int_0^\tau dt \int_0^{l_0}\left(\frac{1}{2}\lambda\, u_{,t}^2 - \frac{1}{2}T\, u_{,x}^2\right) dx = K - V,$$

difference of the kinetic energy $K = \displaystyle\int_0^\tau dt\int_0^{l_0}\left(\frac{1}{2}\lambda\, u_{,t}^2\right) dx$ and of the potential energy V, which itself results from the linearization of the expression

$$V = T\,(l - l_0) \equiv T\left(\int_0^{l_0}\sqrt{1 + u_x^2}dx - l_0\right).$$

Application of the previous theorem shows that the self-adjunction condition of the state laws is satisfied, thus the Lagrangian

$$L = \int_0^1 [y.P_Y\,(\lambda y,\ \lambda\alpha) + \alpha.P_A(\lambda y,\ \lambda\alpha)]\ d\lambda.$$

Accounting for the homogeneity of degree -1 of the second order partial derivatives of $e(y, \alpha)$, and the homogeneity of degree zero of the intensities $Y(y, \alpha)$ and $A(y, \alpha)$ then leads to

$$L = y.\dot{Y} - \alpha.\dot{A} + e_{,y}.\dot{y} + e_{,\alpha}.\dot{\alpha} - \frac{d}{dt}\left(e_{,y}.y + e_{,\alpha}.\alpha\right)$$

The last contribution can be removed (it is a total derivative), and the first contribution vanishes, according to Gibbs-Duhem relation, thus an equivalent Lagrangian is given by $L = e_{,y}.\dot{y} + e_{,\alpha}.\dot{\alpha}$, as independently obtained in [23].The stationarity of the action integral

$$S = \int e_{,y}.\dot{y} + e_{,\alpha}.\dot{\alpha} = \int \frac{de}{dt} \equiv e[y, \alpha]$$

(it is indeed a functional, due to the history dependence of the potential $e = e(y, \alpha)$ upon the internal variables α) simply means that the internal energy keeps its status of potential function during the evolution of the system. The postulate of existence of a thermodynamic potential E outside equilibrium thereby generates a stationarity principle, equivalent to the state laws. Note that adapted potentials can be built using the Legendre transformation, when a given set of control variables have been chosen. The Lagrangian so far established incorporates the thermodynamical information related to the state laws, but it does not consider the evolution laws of the internal variables. These can be written for GSM (generalized standard material) as the following subdifferential identities: $(-\dot{\alpha}) = \partial_A \phi^* (\sigma, T, A)$, with $\phi^* (\sigma, T, A)$ the pseudo-potential of dissipation [14]. Thus, using this last equation as a constraint via a set of Lagrange multipliers yields the unconstrained problem:

$$\delta \int_{t_0}^{t} \left[\dot{e} + \sum_{k=1}^{n} \lambda_k \left(\dot{\alpha}_k - \partial_{A_k} \phi^* (\sigma, T, A)\right)\right] dt = 0$$

where the subdifferential is taken w.r. to the affinity A_k, for the augmented Lagrangian

$$\dot{e} + \sum_{k=1}^{n} \lambda_k \left(\dot{\alpha}_k - \partial_{A_k} \phi^* (\sigma, T, A)\right)$$

sum of a thermodynamic Lagrangian $L_{thermo} := \dot{e}$ and a kinetic Lagrangian

$$L_{kin} := \sum_{k=1}^{n} \lambda_k \left(\dot{\alpha}_k - \partial_{A_k} \phi^* (\sigma, T, A)\right).$$

Note that the subdifferential reduces to the partial derivative in a 'smooth' case.

4.1. Continuous symmetries of dissipative constitutive laws and master curves

A reminder of variational symmetries is first in order: when a differential problem admits a variational formulation in terms of the stationarity of a functional, Noether's theorem associates to each variational symmetry a conservation law. Recall that the one-parameter (μ is the parameter) Lie group of transformations $G : \bar{x_i} = \bar{x_i}(x,\ u,\ \mu)$; $\bar{u_i} = \bar{u_i}(x,\ u,\ \mu)$ is a symmetry group for the functional integral

$$S = \int_{\Omega} L\left(x,\ u^{(n)}\right)\, d\Omega$$

iff S keeps the same value in the set of transformed variables, viz

$$\bar{S} = \int_{\bar{\Omega}} \bar{L}\left(\bar{x},\ \bar{u}^{(n)}\right)\, d\Omega = S = \int_{\Omega} L\left(x,\ u^{(n)}\right)\, d\Omega$$

The vector field (symmetry generator)

$$v = \sum_{k=1}^{4} \xi_k(x,\ u)\frac{\partial}{\partial x_k} + \sum_{k=1}^{q} \phi_k(x,\ u)\frac{\partial}{\partial u_k} \equiv \sum_{i=1}^{4} \frac{\partial \bar{x_i}}{\partial \mu_{|\mu=0}}\frac{\partial}{\partial u_i}$$

defines a variational symmetry group iff the following condition is satisfied:

$$pr^{(n)} + L div\, \xi = 0.$$

The prolongation of the vector field v, alias $pr^{(n)}$, is defined as the extended vector field

$$pr^{(n)}v = v + \sum_{k=1}^{q}\sum_{J} \phi_k^J\left(x,\ u^{(n)}\right)\frac{\partial}{\partial u_{k,J}}$$

$$\phi_k^J\left(x,\ u^{(n)}\right) = D_J\left(\phi_k - \sum_{I=1}^{4} \xi_i u_{k,i}\right) + \sum_{I=1}^{4} \xi \frac{\partial}{\partial x_i}\left(D_J u^k\right)$$

J being an arbitrary multiindex or order less than 4.

THEOREM 5 (E. Noether, [20]). *When v generates a symmetry group for the functional $S[u] = \int_{\Omega} L\left(x,\ u^(1)\right)\, d\Omega$, the conservation law*

$$Div\, P = D_1 P_1 + \cdots + D_4 P_4 = 0$$

is satisfied, with the quadruplet $\{P_i,\ i = 1 \ldots 4\}$ given by

$$P_i = \sum_{k=1}^{q}\sum_{j=1}^{4} \xi_j u_{k,j}\frac{\partial L}{\partial u_{k,j}} - \sum_{j=1}^{q} \phi_j \frac{\partial L}{\partial u_{j,i}} - \xi_i L,\ \ \forall x \in \Omega.$$

Going back to the finding of the variational symmetries associated to the Lagrangian $L = L_{thermo} + L_{kin}$, the group generator

$$v = \xi\partial_t + \phi_\epsilon\partial_\epsilon + \phi_T\partial_T + \phi_{\alpha_k}\partial_{\alpha_k} + \phi_\sigma\partial_\sigma + \phi_S\partial_S + \phi_{A_k}\partial_{A_k}$$

maybe elaborated in such a way that the variational symmetry for L_{thermo} is automatically satisfied: just compute the components of the intensive variables s.t. they satisfy the state laws. Previous symmetry condition then simplifies to [19, 32]

$$pr(n)v + L_{kin} Div\xi = 0 \text{ with } \mathrm{div}\xi \equiv D_t\xi$$

Using TIP and the elegant formalism of differential geometry, balance laws for intrinsically dissipative continuous media can then be formulated, in articulation with symmetries. These can be obtained in the following manner: the variation of the functional S under an arbitrary group of transformations expresses as

$$\delta S = \mu \int_{\Omega} \left(\frac{\partial L}{\partial u_k} - D_i \frac{\partial L}{\partial u_{k,i}} \right) \left(\phi_k - \xi_j u_{k,j} \right) d\Omega + \mu \int_{\partial\Omega} \left(L\xi_I + \left(\phi_k - u_{k,j}\xi_j \right) \right) n_i \, d(\partial\Omega)$$

This form can be transcribed into the compact differential form identity (Cartan formula):

$$L_X\omega = i_X \, d\omega + d\,(i_X\omega)$$

which allows a condensed writing of Noether's theorem: under the conditions $L_X\omega = 0$ (invariance of $\int_\Omega \omega \equiv \int_\Omega L\, dx\, dt$ by the group generated by X) and $i_X d\omega = 0$ (validity of the Euler-Lagrange equations), the following conservation law is obtained:

$$Div \left(L\xi_i + \left(\phi_k - u_{k,j}\xi_j \right) \frac{\partial L}{\partial u_{k,i}} \right) = 0$$

This identity appears as a balance law for dissipative media, wherein the Lagrangian describes the kinetics of evolution of the internal variables (according to previous developments). This approach seems more natural compared to the work in [5], since the authors do not truly consider dissipative media per se.

EXAMPLE 19. (Conservation of Deborah number in linear viscoelasticity) As a simple illustrative example, let consider the linear viscoelasticity law relating the Cauchy stress rate $\dot{\sigma}$ to the strain and strain rates, written as the following first order PDE with initial condition:

$$Delta := \left| \begin{array}{c} \frac{\partial\sigma}{\partial t} - E^0\dot{\epsilon} + \frac{\sigma - E^{\infty}\epsilon(t)}{\tau(T)} = 0 \\ \sigma(0) = 0 \end{array} \right.$$

wherein $\tau(T)$ is a temperature dependent relaxation time, and E^0, E^∞ denote the instantaneous and relaxed moduli respectively. The parameters τ, ϵ are here considered as dependent variables, whereas the time t is the independent variable. An equivalence principle is defined as the prescription of two groups of transformations G_1, G_2, s.t.

$$G_1\,(t,\ \sigma,\ \mu_1) = G_2\,(t,\ \sigma,\ \mu_2)$$

when $\mu_1 = \mu_2$, with σ solution of Δ. In terms of the generators, previous condition is rephrased as $pr^{(1)}v_1(\Delta) = pr^{(2)}v_2(\Delta)$, when $\Delta = 0$. As a specific generator that satisfies the previous condition together with the initial condition $\sigma(0) = 0$, one obtains the time dilatation group (expressing the equivalence principle and integrating the resulting system of ODE satisfied by the coefficients of the two selected generators

$$v_1 = \xi\,(t,\ \tau,\ \dot{\epsilon})\,\frac{\partial}{\partial t} + \alpha\,(t,\ \tau,\ \dot{\epsilon})\,\frac{\partial}{\partial \dot{\epsilon}} \qquad \text{and} \qquad v_2 = \beta\,(t,\ \tau,\ \dot{\epsilon})\,\frac{\partial}{\partial t},$$

having the generators

$$v_1 = t\,\frac{\partial}{\partial t} - \dot{\epsilon}\,\frac{\partial}{\partial \dot{\epsilon}}; \qquad v_2 = -\tau\,\frac{\partial}{\partial \tau}.$$

They correspond to the two symmetry groups

$$G_1\,(t,\ \sigma,\ \mu) := \left| \begin{array}{c} \bar{t}_1 = e^{\mu_t} \\ \bar{\tau}_1 = \tau \\ \bar{\dot{\epsilon}}_1 = e^{-\mu_{\dot{\epsilon}}} \\ \bar{\sigma}_1 = \sigma \end{array} \right. \text{ and } G_2\,(t,\ \sigma,\ \mu) := \left| \begin{array}{c} \bar{t}_2 = t \\ \bar{\tau}_2 = e^{-\mu_\tau} \\ \bar{\dot{\epsilon}}_2 = \dot{\epsilon} \\ \bar{\sigma}_2 = \sigma \end{array} \right.$$

denoting the transformed variables with an over bar. Traducing the equivalence condition as $\bar{\sigma}_1\left(\bar{t}_1,\ \bar{\sigma}_1,\ \bar{\dot{\epsilon}}_1\right) = \bar{\sigma}_2\left(\bar{t}_2,\ \bar{\sigma}_2,\ \bar{\dot{\epsilon}}_2\right)$ gives the relation $\sigma\left(\frac{t}{\alpha},\ \alpha\tau,\ \dot{\epsilon}\right)$, with $\alpha = e^{\mu}$. Thereby, it appears that an identical response of the material is obtained, when a time contraction and a strain rate dilatation are operated, with the factors $1/\alpha$ and α respectively. This equivalence between time and strain rate leads to the conservation of Deborah number, defined as the ratio of the internal relaxation time (microscopic time) to the observer (macroscopic) time scale, viz

$$n_D := \frac{\tau}{t_{obs}} = \frac{\tau}{\epsilon/\,(\alpha\dot{\epsilon})} \equiv \frac{\alpha\tau}{t}$$

Applications of this methodology to the time-temperature equivalence principles have been further done [19], within a thermodynamic framework of relaxation [6]. Thereby, a systematic and predictive methodology for the setting up of master curves of dissipative media has been elaborated. Note that the symmetry groups act in the space of both independent variables (space and time) and dependent variables (that itself depend upon the selected thermodynamic framework); these symmetries shall further be exploited.

References

[1] ABRAHAM R, MARSDEN J.E AND RATIU T., *Manifolds, tensor analysis, and applications*, Springer, New York 1988.

[2] ANTHONY K.H., *Hamilton's action principle and thermodynamics of irreversible processes. A unifying procedure for reversible and irreversible processes*, J. Non-Newtonian Fluid. Mech. **96** (2001), 291–339.

[3] BIOT M.A., *Variational and lagrangian methods in viscoelasticity*, Springer-Verlag, New York 1956.

[4] CALLEN H., *Thermodynamics: an introduction to the physical theories of equilibrium thermostatics and irreversible thermodynamics*, Wiley, New York 1960.

[5] CHEN N., HONEIN T. AND HERMANN G., *Dissipative systems, conservation laws and symmetries*, Int. J. Solids Struct. **33** (20-22) (1993), 2959–2968.

[6] CUNAT C., *The DNLR approach and relaxation phenomena. Part I: historical account and DNLR formalism* , J. Mech. Time Dependent Mat. **5** (2001), 1013–1032.

[7] FEYNMANN R., *The strange history of light and matter*, Penguin, London 1990.

[8] GAO D.Y., *Duality principles in nonconvex systems*, Kluwer Academic Publishers, Dordrecht-Hardbound 1999.

[9] BERGER M. AND GOSTIAUX B., *Differential geometry: manifolds, curves, and surfaces*, Springer, New York 1988.

[10] CARTAN H., *Differential forms*, Hermann, Paris 1970.

[11] CURTIS W. D. AND MILLER F.R., *Differential manifolds and theoretical physics*, Academic Press, New York 1985.

[12] EDELEN D.G.B., *Applied exterior calculus*, Wiley, New York 1985.

[13] COLEMAN B.D. AND GURTIN M.E., *Thermodynamics with internal state variables*, J. Chem. Phys. **47** (1967), 597–613.

[14] GERMAIN P., NGUYEN Q.S. AND SUQUET P., *Continuum thermodynamics*, J. Appl. Mech. **50** (1983), 1010–1020.

[15] TRUESDELL C. AND NOLL W., *The non-linear field theories of mechanics*, Handbuch der Physik **III/3**, Springer-Verlag, Berlin 1965.

[16] FLANDERS H., *Differential forms with applications to the physical sciences*, Dover, New York 1989.

[17] LAGRANGE J.L., *Mecanique analytique*, Editions Jacques Gabay, Paris 1989.

[18] MAUGIN G.A., *Internal variables and dissipative structures*, J. Non-equ. Thermod. **15** (1990), 173–192.

[19] MAGNENET V., GANGHOFFER J.F., RAHOUADJ R. AND CUNAT C., *Master curves for viscous media predicted from symmetry analysis*, Proc. STAMM, Darmstadt 2004.

[20] OLVER P., *Applicaton of Lie group to differential equations*, Springer Verlag 1989.

[21] KIEHN R.M., *An extension of Hamilton's principle to include dissipative sys*, J. Math. Phys., **15** (1974), 9–13.

[22] KRUPKOVA O., *Hamiltonian field theory*, J. of Geometry and Physics, **778** (2001), 1–40.

[23] RAHOUADJ R., GANGHOFFER J.F. AND CUNAT C., *A thermodynamic approach with internal variables using Lagrange formalism. Part I and II*, Mech. Res. Comm. **30** (2003), 109–123.

[24] REIWE F., *Nonconservative Lagrangian and Hamiltonian mechanics*, Phys. Rev. E **53** (2) (1996), 1890–1899.

[25] POINCARE H., *Methodes nouvelles de la mecanique celeste*, Dover publications, 1957.

[26] LANG S., *Differentiable manifolds*, Addison-Wesley, Reading, MA. 1972.

[27] SANTILLI R. M., *Variational approach to self adjointness for tensorial field equations*, Annals of Phys. **103** (1977), 354–408.

[28] SPIVAK M., *A comprehensive introduction to differential geometry*, Vol. I-IV. Berkeley 1979.

[29] STRUIK D., *Lectures on classical differential geometry*, Dover, New York 1988.

[30] TALPAERT Y., *Lecons et applications de geometrie differentielle et de mecanique analytique*, Cepadues Editions 1993.

[31] BURKE W.L., *Applied differential geometry*, Cambridge University Press 1985.

[32] MAGNENET V., *Formulation thermodynamique de lois de comportement hors-equilibre: groupes de symmetries continues issus dune approce lagrangienne irreversible*, Ph.D. Thesis, INPL, Nancy 2005.

[33] STOLZ C., *Sur les equations generales de la dynamique des milieux continues anelastiques*, C.R. Acad. Sci. Serie II **307** (1988), 1997–2000.

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