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A DYNAMICAL APPROACH TO THE STUDY OF RADIAL SOLUTIONS FOR P-LAPLACE EQUATION

Abstract. In this paper we give a survey of the results concerning the existence of ground states and singular ground states for equations of the following form:

$$\Delta_p u + f(u, |\mathbf{x}|) = 0$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$ is the p-Laplace operator, $\mathbf{x} \in \mathbb{R}^n$ and f is continuous, and locally Lipschitz in the u variable. We focus our attention mainly on radial solutions.

The main purpose is to illustrate a dynamical approach, which involves the introduction of the so called Fowler transformation. This technique turns to be particularly useful to analyze the problem, when f is spatial dependent, critical or supercritical and to detect singular ground states.

1. Introduction

Let $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$ denote the p-Laplace operator. The aim of this paper is to discuss the existence and the asymptotic behavior of positive solutions of equation of the following family

$$\Delta_p u + f(u, |\mathbf{x}|) = 0 \tag{1}$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$, denotes the p-Laplace operator, $\mathbf{x} \in \mathbb{R}^n$ and $f(u, |\mathbf{x}|)$ is a continuous nonlinearity such that $f(0, |\mathbf{x}|) = 0$. The interest in equation of this type started from the classical Laplacian that is $p = 2$:

$$\Delta u + f(u, |\mathbf{x}|) = 0 \tag{2}$$

and is motivated by mathematical reasons, but also by the relevance of some equations of this type as model to describe phenomena coming from applied area of research. In particular Eq. (2) is important in quantum mechanic, astronomy and chemistry, while (1) is connected to problems arising in theory of elasticity, see e.g. [26]. Our purpose is to give a short, and not exhaustive, survey of the results which can be found in the wide literature concerning this argument, and in particular to discuss a method which is suitable to study radial solutions.

We think is worthwhile to stress that Eq. (2) can be regarded as the Euler equation of the following energy functional $E : \mathbb{R} \times W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$,

$$E(\mathbf{x}, u, \nabla u) = \int_\Omega \Big(\frac{|\nabla u|^2}{2} - F(u, |\mathbf{x}|)\Big) d\mathbf{x}$$

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where $F(u, |\mathbf{x}|) = \int_0^u f(s, |\mathbf{x}|)ds$. The p-Laplace operator arises naturally when we want to extend this functional to $W^{1,p}(\mathbb{R}^n)$ functions. In fact (1) is the Euler equation for the functional $E_p : \mathbb{R} \times W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$,

$$E_p(\mathbf{x}, u, \nabla u) = \int_\Omega \Big(\frac{|\nabla u|^p}{p} - F(u, |\mathbf{x}|)\Big) d\mathbf{x}.$$

We will focus our attention mainly on radial solutions, hence we will reduce (1) to the following singular O.D.E.

$$(u'|u'|^{p-2}r^{n-1})' + f(u, r)r^{n-1} = 0 \tag{3}$$

where $r = |\mathbf{x}|$ and we commit the following abuse of notation: we write $u(r)$ for $u(\mathbf{x})$ when $|\mathbf{x}| = r$ and u has radial symmetry; here and later $'$ denotes derivation with respect to r. Observe that (3) is singular when $r = 0$ and when $u' = 0$, unless $p = 2$.

We introduce now some notation that will be in force throughout all the paper. We will use the term "regular solution" to refer to a solution $u(r)$ of Eq. (3) satisfying $u(0) = u_0 > 0$ and $u'(0) = 0$. We will use the term "singular solution" to refer to a solution $\upsilon(r)$ of Eq. (3) such that $\lim_{r\to 0} \upsilon(r) = +\infty$.

A basic question in this kind of PDE is the existence and the asymptotic behaviour of ground states (G.S.), that are solutions $u(\mathbf{x})$ of (1) which are nonnegative for any $\mathbf{x} \in \mathbb{R}^n$ and such that $\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$. We are also interested in detecting singular ground states (S.G.S.), that is solutions $\upsilon(\mathbf{x})$ which are well defined and nonnegative for any $\mathbf{x} \in \mathbb{R}^n\backslash\{0\}$ and such that $\lim_{|\mathbf{x}|\to\infty} \upsilon(\mathbf{x}) = 0$ and $\lim_{|\mathbf{x}|\to 0} u(\mathbf{x}) = +\infty$. Other interesting family of solutions for the radial equation (3) is the one of crossing solutions, that is regular solutions $u(r)$ which are positive for r smaller than a certain value $R > 0$ and become null with nonzero slope at $r = R$. So they can also be regarded as solutions of the Dirichlet problem in the ball of radius R. Finally we individuate solutions $u(r)$ of the Dirichlet problem in the exterior of the ball of radius R, that is $u(R) = 0$, $u(r) > 0$ for $r > R$, and $u(r)$ has fast decay. We say that a positive solution $u(r)$ of (3) has fast decay if $\lim_{r\to\infty} u(r)r^{(n-p)/(p-1)} < +\infty$ and that it has slow decay if $\lim_{r\to\infty} u(r)r^{(n-p)/(p-1)} = +\infty$.

This article has the following structure: in section 1 we introduce the generalized Fowler transformation, and we apply it to a toy example, mainly for illustrative purpose. In sections 2 and 3 we introduce the Pohozaev function, that is one of the main tool for the analysis of equation of type (1), and we consider the case where respectively $f(u, r) = k(r)u|u|^{q-1}$ and $f(u, r) = k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1}$ where $q > p$, $q_2 > q_1 > p$, the functions $k(r)$, $k_1(r)$, $k_2(r)$ are positive and continuous for $r > 0$. In both the cases we assume that the corresponding Pohozaev functions have constant sign. In section 4 we discuss the case $f(u, r) = k(r)u|u|^{q-1}$ when the Pohozaev function changes sign, stressing in particular the case $q = p^*$. In section 5 we explain briefly few results concerning Eq. (2) when $f(u, r) = u|u|^{q_1-1} + u|u|^{q_2-1}$, when $p_* < q_1 < p^* < q_2$ and $p = 2$. We remark that in this case there are still many open problems. In section 6 we discuss the case $f(u, r) = -k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1}$, where $q_1 < q_2$, and the functions k_1 and k_2 are positive and continuous for $r > 0$.

Finally in the appendix we show how some more general equations can be reduced to (3), and we explain the concept of natural dimension, introduced in [20].

2. Preliminary results and autonomous case

The main purpose of this paper is to explain the method of investigation of positive solution of (3) which has been used in [2], [3], [4], [11], [1], [12], [13], [14], [15], [16], [17]. The advantage in the use of this method lies essentially on the fact that we can benefit of a phase portrait, and of the use of techniques typical of dynamical systems theory, such as invariant manifold theory and Mel'nikov functions. Moreover, restricting ourselves to the study of radial solutions, we overcome the difficulties deriving from the lack of compactness of the critical and supercritical case. With our method we can also naturally detect and classify singular solutions, which are not easily found by variational techniques or by standard shooting arguments. The main fault of the method is that it can just give information on radial solutions. However we wish to stress that, when the domain has radial symmetry (e.g. it is the whole $\mathbb{R}^n$), G.S. and solutions of the Dirichlet problem, if they exist, are radial in many different situations, which will be discussed in details in the following sections, see [6], [9], [42], [44].

Furthermore radial solutions play a key role also for many parabolic equations associated to (2). In fact in many cases the ω-limit set is made up of the union of radial solutions, see e. g. [39], [23].

The first step in this analysis consists in applying the following change of coordinates

$$\alpha_l = \frac{p}{l-p}, \quad \beta_l = \frac{p(l-1)}{l-p} - 1, \quad \gamma_l = \beta_l - (n-1), \quad l > p$$

$$x_l = u(r)r^{\alpha_l} \quad y_l = u'(r)|u'(r)|^{p-2}r^{\beta_l} \quad r = e^t \tag{4}$$

where $l > p$ is a parameter. This tool allows us to pass from (3) to the following dynamical system:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g(x_l, t) \end{pmatrix} \tag{5}$$

Here and later "$\cdot$" stands for $\frac{d}{dt}$, and

$$g_l(x_l, t) := f(x_l \exp(-\alpha_l t), \exp(t))e^{\alpha_l(l-1)t}. \tag{6}$$

This transformation was introduced by Fowler in the 30s for the case $p = 2$, and we generalized it to the case $p > 1$ just recently in [12], [13], [15] [14], [16], [17]. It will be useful to embed system (5), and in general all the dynamical systems that will be introduced in the paper, in a one-parameter family as follows:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g_l(x_l, t+\tau) \end{pmatrix} \tag{7}$$

We start from the special nonlinearity $f(u,r) = k(r)u|u|^{q-2}$. In such a case, setting $l = q$ and $\phi(t) = k(e^t)$, system (5) reduces to the following:

$$\begin{pmatrix} \dot{x}_q \\ \dot{y}_q \end{pmatrix} = \begin{pmatrix} \alpha_q & 0 \\ 0 & \gamma_q \end{pmatrix} \begin{pmatrix} x_q \\ y_q \end{pmatrix} + \begin{pmatrix} y_q |y_q|^{\frac{2-p}{p-1}} \\ -\phi(t) x_q |x_q|^{q-2} \end{pmatrix} \tag{8}$$

At the beginning of this section we will also assume that $k = \phi > 0$ is a constant, both for illustrative purpose and because the results will be useful later on in more difficult situations. We think it is worthwhile to recall that most of the results are well known in this rather trivial situation; however our method gives a new point of view on the problem and allows to to clarify and complete some aspects concerning singular solutions even in this easy setting. A first advantage in this change of coordinates consist in the fact that it allows us to pass from a singular non-autonomous ODE to an autonomous dynamical system from which the singularity has been removed (obviously this is not the case for every type of nonlinearity). Moreover now we can apply to the problem techniques typical of dynamical system theory, thus exploiting a different point of view.

We recall the value of two exponents that are critical for this equation. When $n > p$, we denote by $p^* = np/(n-p)$ the Sobolev critical exponent and by $p_* = p\frac{n-1}{n-p}$; when $p \geq n$ we set both p^* and p_* equal to $+\infty$. Let $\boldsymbol{\Omega}$ be an open bounded domain with non-empty smooth boundary $\partial\boldsymbol{\Omega}$, then p^* is the largest $q > p$ such that the embedding $W^{1,p}(\boldsymbol{\Omega}) \subset L^q(\boldsymbol{\Omega})$ holds, while p_* is the largest q such that the trace operator $\gamma : W^{1,p}(\boldsymbol{\Omega}) \to L^q(\partial\boldsymbol{\Omega})$ is continuous.

REMARK 1. We stress that, whenever $q > p$, $\alpha_q > 0$, $\gamma_q = -\frac{(p_*-q)(n-p)}{q-p}$ has the same sign as $p_* - q$, and $\alpha_q + \gamma_q = \frac{p^*-q}{(n-p)(q-p)}$, has the same sign as $p^* - q$. Also observe that (8) is C^1 if and only if $1 < p \leq 2$ and $q \geq 2$.

Notation.

In the whole paper we will use bold letters for vectorial objects. We denote by $u(d,r)$ a regular solution of (3) such that $u(d,0) = d$ and $u'(d,0) = 0$. Moreover if $\bar{u}(r)$ is solution of (3) we denote by $\bar{\mathbf{x}}_l(t) = (\bar{x}_l(t), \bar{y}_l(t))$ the corresponding trajectories of (5). For any $\mathbf{Q} \in \mathbb{R}^2$ we denote by $\mathbf{x}_l(\mathbf{Q},t) = (x_l(\mathbf{Q},t), y_l(\mathbf{Q},t))$ the trajectories of (5) passing through $\mathbf{Q}$ at $t = 0$, and by $\mathbf{x}_l^\tau(\mathbf{Q},t) = (x_l^\tau(\mathbf{Q},t), y_l^\tau(\mathbf{Q},t))$ the trajectory of (5) passing through $\mathbf{Q}$ at $t = \tau$ or equivalently the trajectory of (7) passing through $\mathbf{Q}$ at $t = 0$. Finally we denote by $\mathbb{R}^2_+$ the subset $\{(x,y), |x \geq 0\}$.

In this section we will always assume $q > p$. From a straightforward computation it is easy to observe that the system (8) admits three critical points whenever $q > p_*$: the origin $\mathbf{O}$, $\mathbf{P}= (P_x, P_y)$ and $-\mathbf{P}$, where $P_x = |\gamma_q \alpha_q^{p-1}/k|^{1/(q-p)}$, and $P_y = -|\gamma_q/k\alpha_q^{q-1}|^{(q-1)/(q-p)}$. Note that the critical point $\mathbf{P}$ is a center when $q = p^*$, it is asymptotically stable for $q > p^*$ and it is asymptotically unstable for $q < p^*$.

Positive and decreasing solutions $u(r)$ of (3) correspond to trajectories such that $y_q(t) \leq 0 < x_q(t)$. Moreover, trajectories $\mathbf{x}_q(t)$ which are bounded and such that $x_q(t)$ is uniformly positive for $t > 0$ (resp. for $t < 0$) correspond to solutions $u(r)$ which have slow decay (resp. are singular for $r = 0$), that is $u(r)r^{\alpha_q}$ is uniformly positive and bounded as $r \to \infty$ (resp. as $r \to 0$). Now we want to give a rough picture of the phase portrait of (8), in the autonomous case $\phi \equiv k > 0$. For this purpose we need to introduce a function which plays a key role in all our analysis. Let us denote by

$$H_q(x, y, t) := \frac{n-p}{p} xy + \frac{p-1}{p} |y|^{\frac{p}{p-1}} + \phi(t) \frac{|x|^q}{q}. \tag{9}$$

This function is a translation in this dynamical context of the well known Pohozaev function

$$P(u, u', r) = r^n \left[\frac{n-p}{p} \frac{uu'|u'|^{p-2}}{r} + \frac{p-1}{p} |u'|^p + k(r) \frac{|u|^q}{q} \right]$$

which is one of the main tool in the analysis of equations of these type, see e.g. [38], [34], [35]. Observe in fact that, if $\mathbf{x}_{p^*}(t) = (x_{p^*}(t), y_{p^*}(t))$ is the trajectory of (8) corresponding to $u(r)$, then

$$P(u(r), u'(r), r) = H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = H_q(x_q(t), y_q(t), t) e^{-(\alpha_q + \gamma_q)t} .$$

When k is differentiable from a simple computation we get the following

$$\frac{d}{dt} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) := \frac{d}{dt} \left[e^{\alpha_{p^*}(q-p^*)t} \phi(t) \right] \frac{|x_{p^*}|^q(t)}{q} . \tag{10}$$

Note that the function H_{p^*} does not depend explicitly on t, when k is a constant and $q = p^*$; so in this case it is a first integral for the system. Therefore, using some elementary argument, it is possible to draw each trajectory of the system, see Lemma in [12], and to give a picture of the phase portrait see fig. 1.

Then we easily get a lot of information on the original equation (3).

We stress that in this easy situation we have an explicit formula for all the regular solutions, that is

$$u(d, r) = d \left[1 + \left(\frac{p-1}{n-p}\right)^{\frac{p}{p-1}} \left(\frac{1}{2n}\right)^{\frac{1}{p-1}} d^{\frac{p^2}{(p-1)(n-p)}} r^{\frac{p}{p-1}} \right]^{-\frac{n-p}{p}} k^{-\frac{n-p}{p^2}} . \tag{11}$$

It can be shown easily that system (3) with $q = p^*$ and $\phi(t) \equiv k > 0$ and with $q = s$ and $\phi(t) \equiv e^{\alpha_{p^*}(p^*-s)t} k$, are topologically equivalent. In fact we can push much further this kind of identification. This is done in the appendix where the concept of natural dimension is introduced, see [20], [33].

We will see that an unstable set for (8) exists for any $q > p$, while a stable set exists just when $p_* < q < p^*$. It can be shown that the former existence result is equivalent to the existence of regular solutions of (3), while the latter is equivalent to the existence of solutions $u(r)$ with fast decay.

From now on we will commit the following abuse of notation: we will call stable and unstable sets (or manifolds) the branches which depart from the origin and

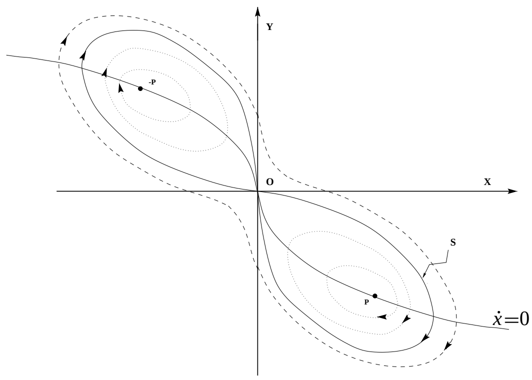

Figure 1: A sketch of the phase portrait for the autonomous system (8) when $\phi \equiv k > 0$, and $q = p^*$. The lines also represents the level curves for the function H_q for t fixed, when $g_q(x_q, t) = \phi(t) x_q |x_q|^{q-2}$.

gets into $\mathbb{R}^2_+$, which correspond to the positive solutions $u(r)$ of (3) we are interested in. The existence of trajectories converging to the origin either in the past or in the future can be inferred from invariant manifold theory, whenever $1 < p \le 2$ and $q \ge 2$. In such a case we directly prove the existence of a stable and an unstable manifold, denoted respectively by W^s and by W^u, see [12].

When these regularity hypotheses are not satisfied the proofs become more difficult, due to the lack of local uniqueness of the trajectories crossing the coordinate axes. But using Wazewski's principle and the fact that the trajectories we are interested in do not cross the coordinate axes, it is possible to obtain a similar result. However, with this different proof, a priori W^u and W^s are just compact and connected sets. But in the autonomous case $k \equiv \text{const} > 0$, we can exploit the invariance of the system with respect to t, to conclude that W^s and W^u are in fact graph of a trajectory having the origin respectively as ω-limit set and α-limit set. Therefore, even in this case, they are 1 dimensional manifolds, see [15], [17]. We think it is worth mentioning the fact that, when the system is not Lipschitz, a priori the trajectories could reach the origin at some $t = T$ finite, either in the past or in the future. However it is easy to show that this possibility cannot take place when $q \ge p$, see [17] for a detailed proof.
Note that, if $k > 0$ is a constant, we also have that $H_q(x_q(t), y_q(t), t)$ is increasing along the trajectories if and only if $p_* < q < p^*$, and it is decreasing if and only if $q > p^*$. Moreover for any trajectory converging to the origin as $t \to \pm\infty$, we have $\lim_{t\to\pm\infty} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = 0$. Putting together all these results, we can draw fig. 2, and classify positive solutions in one of the following structures.

A All the regular solutions are monotone decreasing G.S. with slow decay. There are uncountably many solutions of the Dirichlet problem in the exterior of the ball. More precisely, for any $R > 0$ there is a solution $v(r)$ such that $v(R) = 0$, $v(r)$ is positive for any $r > R$ and it has fast decay. There is at least one S.G.S. with slow decay.

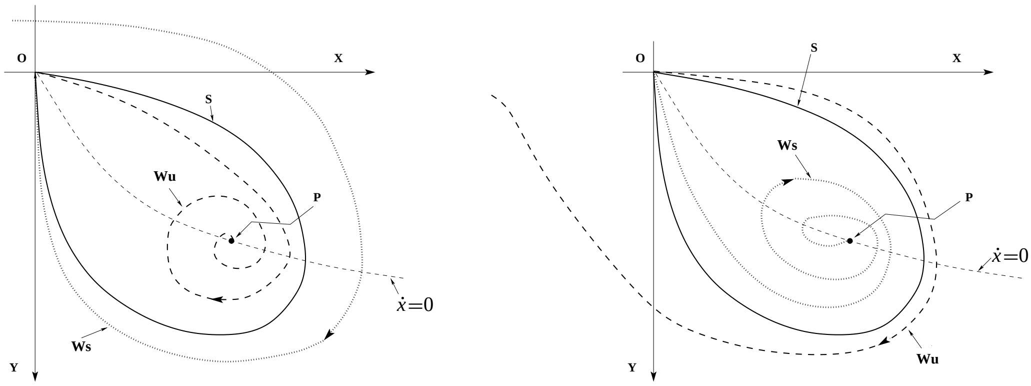

Figure 2: A sketch of the phase portrait for the autonomous system (8) when $\phi \equiv k > 0$, $1 < p \leq 2$ and $q \geq 2$. The figures show the stable manifold W^s (dotted line) and the unstable manifold W^u (dashed line). The solid curve S indicates the set $\{(x_q, y_q) \mid x_q \geq 0 \; H_q(x_q, y_q) = 0\}$. Figure 2A refers to the case $q \geq p_*$ while 2B to the case $p_* < q < p^*$.

B All the regular solutions $u(d, r)$ are crossing solutions, and there are uncountably many S.G.S. with fast decay $v(r)$. There is at least one S.G.S. with slow decay.

Namely, if $q > p^*$ positive solutions have structure **A**, while if $p_* < q < p^*$ they have structure **B**. In both the cases the S.G.S. with slow decay is unique and can be explicitly computed. If $q = p^*$ we are in the border situation, so all the regular solutions are G.S. with fast decay, see (11), there are uncountably many S.G.S. with slow decay, and uncountably many oscillatory solutions, see [12]. When $q \leq p_*$, it is easy to show that all the regular solutions $u(r)$ of (3) are crossing solutions.

We conclude this section with some basic results concerning the existence of regular solutions and positive fast decay solutions for (3) and a wide class of functions $f(u, r)$. First of all we recall that, if $f(u, r)$ is continuous and locally Lipschitz continuous in the u variable, the existence of regular solution is ensured, and if $f(d, 0) > 0$ we also have local uniqueness of $u(d, r)$. The proof of this standard result can be found in [19] for the spatially independent case, but the argument can be easily adapted to the general case, see [16], [17]. We give now a result concerning the asymptotic behaviour of positive solutions. The proof of this result can be found in [19], [13], [17].

PROPOSITION 1. *Consider a solution $u(r)$ of (3) such that $u'(r) \leq 0 \leq u(r)$ for any $r > R$ for a certain $R > 0$, and* $\lim_{r\to\infty} u(r) = 0$.

A *Assume that there are $U > 0$ and $g(u) \in \mathcal{L}^1_{loc}$ such that $|f(u, r)| < g(u)$ for $r \geq 0$ and $0 \leq u \leq U$, and denote by $G(u) = \int_0^u g(s)ds$. Moreover assume that $\int_0 |G(s)|^{-1/p} ds < \infty$. Then the support of $u(r)$ is bounded.*

B *Assume that there are $C > 0$, $U > 0$ and $q_1 \geq p$ such that $|f(u, r)| < Cu^{q_1 - 1}$ for*

$0 \le u \le U$ and $r \ge 0$. Then $u(r) > 0$ for $r > R$ and the limit $\lim_{r\to\infty} u(r) r^{\frac{n-p}{p-1}} = \lambda$ exists. Moreover, if $f(u,r) > 0$ for u small and r large, then $\lambda > 0$, while if $f(u,r) < 0$ for u small and r large, then $\lambda < \infty$.

When Hypothesis B is satisfied we can go a bit further. Now we distinguish between the case in which $f(u,r)$ is always positive and the case in which it is negative for u small.

COROLLARY 1. *Assume that Hypothesis **B** of the previous Proposition is satisfied. First assume that $f(u,r) > 0$ for u small and r large.*

1 *If $q_1 \le p_*$, and there are $U > 0$, $c > 0$ and $Q_1 \in (p, q_1]$ such that $f(u,r) > cu^{Q_1-1}$ for r large and $0 \le u < U$. Then $\lambda = \infty$.*

Assume now that $f(u,r) < 0$ for u small and r large.

2 *If $q_1 > p_*$, then $\lambda > 0$.*

3 *If $q_1 \le p_*$, and there are $U > 0$, $c > 0$ and $Q_1 \in (p, q_1]$ such that $-f(u,r) > cu^{Q_1-1}$ for r large and $0 \le u < U$. Then $\lambda = 0$ and $\limsup_{r\to\infty} u(r) r^{-\frac{p}{Q_1-p}} < \infty$. Furthermore if $Q_1 = p_*$ we also have $\limsup_{r\to 0} u(r) r^{\frac{n-p}{p-1}} |\ln(r)|^{-\frac{n-p}{p(p-1)}} < \infty$.*

4 *Assume that the following limit exists is bounded and negative:*

$$\lim_{r\to\infty} \frac{f(ur^{-\frac{p}{Q_1-p}}, r)}{\left|ur^{-\frac{p}{Q_1-p}}\right|^{Q_1-1}} = -k(\infty).$$

If $Q_1 < p_*$, then $\lim_{r\to\infty} u(r) r^{-\frac{p}{Q_1-p}} = P_x > 0$ where $\boldsymbol{P} = (P_x, P_y)$ is the critical point of system (5) where $l = q$ and $g \equiv k(\infty) x |x|^{q-2}$. If $Q_1 = p_*$ then $u(r) r^{\frac{n-p}{p-1}} |\ln(r)|^{-\frac{n-p}{p(p-1)}}$ is uniformly positive and bounded for r large.

Exploiting the knowledge of the autonomous case (8) with $\phi \equiv k > 0$, it is possible to prove the existence of a local stable and unstable manifold also for the non-autonomous system (5), under suitable hypotheses on $g_l(x_l, t)$, or equivalently on $f(u,r)$.

PROPOSITION 2. *Assume that $f(u,r)$ is continuous for $r = 0$ and consider system (5) where $l > p$; then there is a local unstable set*

$$\tilde{W}^u(\tau) := \{\boldsymbol{Q} \in \mathbb{R}^2_+ \mid \boldsymbol{x}^\tau_l(\boldsymbol{Q}, t) \in \mathbb{R}^2_+ \text{ for any } t \le 0 \text{ and } \lim_{t\to-\infty} \boldsymbol{x}^\tau_l(\boldsymbol{Q}, t) = \boldsymbol{O}\}.$$

*This sets contains a closed connected component to which **O** belongs and whose diameter is positive, uniformly in τ.*

Assume that there are $\nu > 0$ and $q_2 > p_*$ such that, for any $r \in [0, \nu]$, we have $\limsup_{u\to\infty} \frac{f(u,r)}{u^{q_2-1}} < a(r)$ where $0 < a(r) < \infty$. Moreover assume that one of the following hypotheses are satisfied

- *$f(u, r) > 0$ for r large and $u > 0$; moreover there is $q_1 > p_*$ such that $\frac{f(u,r)}{u^{q_1-1}}$ is bounded for u positive and small and r large.*
- *$f(u, r) < 0$ for r large and $u > 0$.*

Then there is a local stable set

$$\tilde{W}^s(\tau) := \{\boldsymbol{Q} \in \mathbb{R}^2_+ \mid \boldsymbol{x}^\tau_{q_2}(\boldsymbol{Q}, t) \in \mathbb{R}^2_+ \text{ for any } t \le 0 \text{ and } \lim_{t\to-\infty} \boldsymbol{x}^\tau_{q_2}(\boldsymbol{Q}, t) = \boldsymbol{O}\}\,.$$

*This sets contains a closed connected component to which **O** belongs and whose diameter is positive, uniformly in τ.*

Proof. Consider first the case $f(u, r) = k(r)u|u|^{q-2}$, and assume that $k(r)$ is uniformly continuous. Then the existence of these stable and unstable sets follows from invariant manifold theory for non-autonomous system, see [28], [30]. Moreover in such a case we also know that these sets are indeed smooth manifolds which depend smoothly on τ. In the general case the existence of the unstable set easily follows from the existence of regular solutions for (3). The existence of the stable set is more complicated and can be proved through Wazewski's principle, see [15] and [17]. In [17] the proof is given for the case $f(u, r) < 0$ for u small and r large, but the argument can be easily extended also to the case $f(u, r) < ku|u|^{q_1-2}$ with $q_1 > p_*$, for u small and r large. □

We give now a result proved in [13] and [17] which explains the relationship between stable and unstable sets of (5) and solutions of (3).

PROPOSITION 3. *Consider system (5) and assume that $g_l(x_l, t)$ is bounded as $t \to -\infty$, for any $x_l > 0$. Then each regular solution $u(r)$ of (3) corresponds to a trajectory $\boldsymbol{x}^\tau_l(\boldsymbol{Q}^u, t)$ such that $\boldsymbol{Q}^u \in \tilde{W}^u(\tau)$, and viceversa. Moreover any trajectory $\boldsymbol{x}^\tau(\boldsymbol{Q}^s, t)$ where $\boldsymbol{Q}^s \in \tilde{W}^s(\tau)$, corresponds to a solution $u(r)$ of (3) with fast decay.*

REMARK 2. Take $f(u, r) = -k_1(r)u|u|^{q_1-2} + k_2(r)u|u|^{q_1-2}$, where $q_1 < q_2$ and the functions $k_i(r)$ are continuous, uniformly positive and bounded as $r \to \infty$. Then if $q_1 < p$ we are in the Hypotheses of claim A of Proposition 1, while if $q_1 \ge p$ Hyp. B is satisfied. Moreover if $q_1 > p_*$, then we are in Hyp. 2 of Corollary 1, while if $p < q_1 \le p_*$ Hyp. 3 of Corollary 1 holds. To satisfy Hyp. 4 we need to assume that $p < q_1 \le p_*$ and $\lim_{r\to\infty} k_1(r) = k(\infty) > 0$.

We recall that, roughly speaking, positive solutions of (3) can have two asymptotic behaviours, both as $r \to 0$ and as $r \to \infty$. Obviously the asymptotic behavior as $r \to 0$ is influenced by the behaviour of f for u large and r small, while their behavior as $r \to \infty$ depends on the behaviour of f for u small and r large. Generally speaking, when f is positive for u small, we have seen that solutions with fast and slow decay may coexist, while when it is negative we can have either solutions with fast decay or oscillatory solutions. Analogously when $f(u,r)$ is positive and supercritical with respect to p_*, for u large and r small, we can have regular solutions $u(d,r)$ such that $u(d,0) = d$ and $u'(d,0) = 0$, and singular solutions $\upsilon(r)$ that are such that $\lim_{r\to 0}\upsilon(r) = +\infty$. More precisely

PROPOSITION 4. *Assume that there are $s > p_*$, $\rho > 0$ and positive functions $b(r) \geq a(r)$ such that, for any $0 \leq r \leq \rho$ we have*

$$0 < a(r) \leq \liminf_{u\to+\infty} \frac{f(u,r)}{u^{s-1}} \leq \limsup_{u\to+\infty} \frac{f(u,r)}{u^{s-1}} \leq b(r) < \infty\,.$$

If $\boldsymbol{Q} \in W^u(\tau)$ then the solution $u(r)$ corresponding to $\boldsymbol{x}_s^\tau(\boldsymbol{Q},t)$ is a regular solution. Moreover any singular solution, if it exists, is such that $u(r)r^{p/(s-p)}$ is bounded for r small and, if $s \neq p^*$, $u(r)r^{p/(s-p)}$ is uniformly positive, too.

Assume further that $s \neq p^*$ and that the limit $\lim_{u\to+\infty} \frac{f(u,0)}{u^{s-1}} = k(\infty) > 0$ exists and is finite. Then $\lim_{r\to\infty} u(r)r^{p/(s-p)} = P_x > 0$ where $\boldsymbol{P}= (P_x, P_y)$ is the critical point of system (5) where $l = q$ and $g \equiv k(\infty)x|x|^{s-2}$.

Assume that there are $q > p_*$, $R > 0$ and positive functions $B(r) \geq A(r)$ such that, for any $r > R$ we have

$$0 < A(r) \leq \liminf_{u\to 0} \frac{f(u,r)}{u^{q-1}} \leq \limsup_{u\to 0} \frac{f(u,r)}{u^{q-1}} \leq B(r) < \infty\,.$$

Then, if $\boldsymbol{Q} \in \tilde{W}^s(\tau)$, the solution $u(r)$ corresponding to $\boldsymbol{x}_q^\tau(\boldsymbol{Q},t)$ has fast decay, that is the limit $\lim_{r\to\infty} u(r)r^{(n-p)/(p-1)} > 0$ exists and is finite. A slow decay solution (if it exists), is such that $u(r)r^{p/(q-p)}$ is bounded for r large; moreover if $q \neq p^*$, $u(r)r^{p/(q-p)}$ is uniformly positive, too.

Assume further that the limit $\lim_{u\to 0} \frac{\lim_{r\to\infty} f(u,r)}{u^{q-1}} = k(\infty) > 0$ exists and is finite. Then $\lim_{r\to\infty} u(r)r^{p/(q-p)} = P_x > 0$ where $\boldsymbol{P}= (P_x, P_y)$ is the critical point of system (5) where $l = q$ and $g \equiv k(\infty)x|x|^{q-2}$.

These results are proved in [12], [13], [17] using dynamical arguments.

3. When the Pohozaev function does not change sign

3.1. The case $f(u,r) = k(r)u|u|^{q-2}$

In this subsection we discuss positive solutions of equation (3) in the case $f(u,r) = k(r)u|u|^{q-2}$ and $q > p_*$. This problem has been subject to rather deep investigations in the '90s also for the relevance it has in different applied areas. First of all, when $p = 2$ eq. (1) can be regarded as a nonlinear Schroedinger equation. Moreover, when $q = p^*$ and again $p = 2$, this equation is known with the name of scalar curvature equation. In fact the existence of a G.S. $u(\mathbf{x})$ amounts to the existence of a metric g conformal to a standard metric g_0 on $\mathbb{R}^n$ ($g = u^{\frac{4}{n-2}} g_0$), whose scalar curvature is $k(|\mathbf{x}|)$. Furthermore, if the G.S. has fast decay, the metric g gives rise, via the stereographic projection, to a metric on the sphere deprived of a point $S^n \setminus \{\text{a point}\}$ which is equivalent to the standard metric.

Moreover when $q > 1$ and $k(r)$ takes the form $k(r) = \frac{r^\alpha}{1+r^\beta}$ eq. (2) is also known as Matukuma equation and it was proposed as a model in astrophysics. This problem will be investigated in details also in section 4, where we will assume that the Pohozaev functions change their sign, so that positive solutions have a richer structure.

We begin by some preliminary results concerning forward and backward continuability and long time behaviour for positive solutions, in relation with the Pohozaev function. These results can be proved using directly the Pohozaev identity, or through a dynamical argument exploiting our knowledge of the level sets of the function $H(\mathbf{x}, t)$, see [34] and [13].

LEMMA 1. *Let $u(r)$ be a solution of (3), and $\boldsymbol{x}_{p^*}(t)$ the corresponding trajectory. Assume that* $\liminf_{t \to \pm\infty} H(\boldsymbol{x}_{p^*}(t), t) > 0$*, then $\boldsymbol{x}_{p^*}(t)$ has to cross the coordinate axes indefinitely as $t \to \pm\infty$, respectively.*

Assume that $\limsup_{t \to \pm\infty} H(\boldsymbol{x}_{p^*}(t), t) < 0$*, then $\boldsymbol{x}_{p^*}(t)$ cannot converge to the origin or cross the coordinate axes.*

When $p = 2$, the standard tool to understand the behaviour of solutions with fast decay is the Kelvin transformation. Let us set

$$s = r^{-1} \quad \tilde{u}(s) = r^{n-2} u(r) \quad \tilde{K}(s) = r^{2\lambda} K(r^{-1}) \quad \lambda = \frac{(n+2)(q-2^*)}{2}\,; \tag{12}$$

Then (3) is transformed into

$$[\tilde{u}_s(s) s^{n-1}]_s + \tilde{K}(s) \tilde{u} |\tilde{u}|^{q-2}(s) s^{n-1} = 0\,. \tag{13}$$

Note that a regular solution $u(d, r)$ of (3) is transformed into a fast decay solutions $\tilde{u}(s)$ of (13) such that $\lim_{r \to \infty} \tilde{u}(r) r^{\frac{n-p}{p-1}} = d$, and viceversa. So we can reduce the problem of discussing fast decay solutions to an analysis of regular solutions for the transformed problem.

However we do not have an analogous result for the case $p \neq 2$, so we need the following Lemma, that, when $p = 2$, is a trivial consequence of the existence of the Kelvin inversion.

LEMMA 2. *Assume that $f(u, r) > 0$ for any $u > 0$ and consider a solution $u(r)$ which is positive and decreasing for any $r > R$. Then $u(r) r^{\frac{n-p}{p-1}}$ is increasing for any $r > R$.*

Proof. Consider system (8) where $l = p_*$ and the trajectory $\mathbf{x}_{p_*}(t)$ corresponding to $u(r)$. Note that $x_{p_*}(t) = u(r) r^{\frac{n-p}{p-1}}$ and that $\gamma_l = 0$; hence $\dot{y}_{p_*}(t) < 0$ whenever $x_{p_*}(t) > 0$. Assume for contradiction that there is $t_1 > T = \ln(R)$ such that $\dot{x}_{p_*}(t_1) <$

0, then, from an elementary analysis on the phase portrait either there is $t_2 > t_1$ such that $x_{p_*}(t_2) < 0$, or $\lim_{t\to\infty} x_{p_*}(t) = 0$. Assume the latter, then $\lim_{r\to\infty} u(r) r^{\frac{n-p}{p-1}} = \lim_{t\to\infty} x_{p_*}(t) = 0$; but from (3) it follows that $u'(r)^{\frac{n-1}{p-1}}$ is decreasing and admits limit $\lambda \leq 0$. Using de l'Hospital rule we find that $u'(r)^{\frac{n-1}{p-1}} \to \frac{p-1}{n-p}\lambda$; so we get $\lambda = 0$ and $u'(r) \equiv u(r) \equiv 0$ for $r > R$, so the claim is proved. □

Recall that, when $k(r)$ is differentiable, the Pohozaev identity can be reformulated in this dynamical context as (10). Therefore we can think of H_{p^*} as an energy function, which is increasing along the trajectories when $\phi(s)e^{\alpha_{p^*}(p^*-q)t}$ is increasing and decreasing when $\phi(s)e^{\alpha_{p^*}(p^*-q)t}$ is decreasing. This observation can be refined combining it with the fact that all the regular solutions $u(r)$ are decreasing, and fast decay solutions are such that $x_{p_*}(t)$ is increasing, whenever they are positive. For this purpose we define two auxiliary functions, which are closely related to the Pohozaev identity, and which were first introduced in [35]. In this subsection we will always assume (without mentioning) that $e^{nt}\phi(t) \in \mathcal{L}^1\big((-\infty, 0]\big)$ and $e^{(n-q\frac{n-p}{p-1})s}\phi(t) \in \mathcal{L}^1\big([0, +\infty)\big)$, so that we can define the following functions:

$$
\begin{aligned}
J^+(t) &:= \frac{\phi(t)e^{nt}}{q} - \frac{n-p}{p}\int_{-\infty}^{t} \phi(s)e^{ns}ds \\
J^-(t) &:= \frac{\phi(t)e^{(n-q\frac{n-p}{p-1})t}}{q} - \frac{n-p}{p(p-1)}\int_{-\infty}^{t} \phi(s)e^{(n-q\frac{n-p}{p-1})s}ds
\end{aligned} \tag{14}
$$

We will see, that the sign of these functions play a key role in determining the structure of positive solutions for (3). When ϕ is differentiable we can rewrite J^+ and J^- in this form, from which we can more easily guess the sign:

$$
\begin{aligned}
J^+(t) &:= \frac{1}{q}\int_{-\infty}^{t} \frac{d}{ds}[\phi(s)e^{\alpha_{p^*}(p^*-q)s}]e^{\alpha_{p^*}qs}ds \\
J^-(t) &:= \frac{1}{q}\int_{t}^{+\infty} \frac{d}{ds}[\phi(s)e^{\alpha_{p^*}(p^*-q)s}]e^{-\frac{(n-p)q}{p(p-1)}s}ds
\end{aligned} \tag{15}
$$

Let $u(r)$ be a solution of (3) and let $\mathbf{x}(t)$ be the corresponding trajectory of (8). Using (10) and integrating by parts we easily find the following

$$
\begin{aligned}
&H_{p^*}(\mathbf{x}_{p^*}(t), t) + \lim_{t\to-\infty} H(\mathbf{x}_{p^*}(t), t) \\
&\qquad = J^+(t)\frac{|u|^q(e^t)}{q} - \int_{-\infty}^{t} J^+(s)u'(e^s)u|u|^{q-2}(e^s)ds \\
&H_{p^*}(\mathbf{x}_{p^*}(t), t) - \lim_{t\to\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) \\
&\qquad == J^-(t)\frac{|x_{p_*}|^q(t)}{q} + \int_{t}^{+\infty} J^-(s)\dot{x}_{p_*}(s)x_{p_*}|x_{p_*}|^{q-2}(s)ds
\end{aligned} \tag{16}
$$

From (16) we easily deduce the following useful result.

REMARK 3. Assume that there is T such that $J^+(t) \geq 0$ (resp. $J^+(t) \leq 0$), but $J^+(t) \not\equiv 0$ for any $t \leq T$, and consider a regular solution $u(r)$ which is positive and decreasing for any $0 < r < R = \ln(T)$. Then $H_{p^*}(\mathbf{x}_{p^*}(t), t) \geq 0$ (resp. $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$) for any $t \leq T$.

Analogously assume that $J^-(t) \geq 0$ (resp. $J^-(t) \leq 0$) but $J^-(t) \not\equiv 0$ for any $t \geq T$, and consider a solution $u(r)$ which is positive and decreasing for any $r > R = \ln(T)$ and has fast decay. Then $H_{p^*}(\mathbf{x}_{p^*}(t), t) \geq 0$ (resp. $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$) for any $t \geq T$.

Using Remark 3 and Lemma 1 we obtain the following result.

THEOREM 1. *Assume that either $J^+(r) \geq 0$ and $J^+(r) \not\equiv 0$, or $J^-(r) \geq 0$ and $J^-(r) \not\equiv 0$ for any $r > 0$. Then all the regular solutions are crossing solutions and there exists uncountably many S.G.S. with fast decay.*

Assume that either $J^+(r) \leq 0$ and $J^+(r) \not\equiv 0$ or $J^-(r) \leq 0$ and $J^-(r) \not\equiv 0$ for any $r > 0$. Then all the regular solutions are G.S. with slow decay. Moreover there are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of a ball.

The proof of the result concerning regular solutions can be find in [34], and involves just a shooting argument and the use of J^+ and J^- in relation with the Pohozaev identity. Translating this argument in this dynamical context we easily get a classification also of singular solutions, see also [12].

Proof. Assume that $J^+(r) \leq 0$ for any $r > 0$, but $J^+ \not\equiv 0$; consider a regular solution $u(r)$ which is positive and decreasing in the interval $[0, R)$ and the corresponding trajectory $\mathbf{x}_{p^*}(t)$. Using (16) we easily deduce that $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$ for any $t < T = \ln(R)$. From our assumption we easily get that there is $l < p^*$ such that $g(x_l, t)$ is uniformly positive for t large and $\lim_{t\to\infty} H_l(\mathbf{x}_l(t), t) < 0$. It follows that $\mathbf{x}_l(t)$ is forced to stay in a compact subset of the open 4^{th} quadrant for t large, so $u(r)$ is a G.S. with slow decay.

Analogously consider a trajectory $\bar{\mathbf{x}}_l(t)$ converging to $\mathbf{O}$ as $t \to +\infty$. Then the corresponding solution $\bar{u}(r)$ has fast decay, is positive and decreasing for any $r > R$ where $R > 0$ is a constant. Assume for contradiction that $R = 0$; then from (16) we find that $\liminf_{t\to-\infty} H_{p^*}(\bar{\mathbf{x}}_{p^*}(t), t) > \lim_{t\to\infty} H_{p^*}(\bar{\mathbf{x}}_{p^*}(t), t) = 0$. Hence, from Lemma 1, we deduce that $\bar{\mathbf{x}}_{p^*}(t)$ has to cross the coordinate axes indefinitely as $t \to -\infty$. Thus $R > 0$ and $\bar{u}(r)$ is a solution of the Dirichlet problem in the exterior of a ball.

The other claims can be proved reasoning in the same way, see again [12]. □

Reasoning similarly we can complete the previous result proving the existence of S.G.S. with slow decay, to obtain the following Corollary.

COROLLARY 2. *Assume that $J^-(r) \geq 0$ for any r but $J^-(r) \not\equiv 0$, and that there is $p_* < m \leq p^*$ such that the limit $\lim_{t\to\infty} \phi(t) e^{\alpha_m(m-q)t} = k(\infty)$ exists is positive and finite. Then positive solutions have a structure of type **A**.*

*Analogously assume that $J^+(r) \leq 0$ for any r but $J^+(r) \not\equiv 0$, and that there is $s \geq p^*$ such that the limit $\lim_{t\to-\infty}\phi(t)e^{\alpha_s(s-q)} = k(0)$ exists is positive and finite. Then positive solutions have a structure of type **B**.*

Note that if the limits $\lim_{r\to\infty}k(r) = k(\infty)$ and $\lim_{r\to 0}k(r) = k(0)$ exist are positive and finite we can simply set $m = q = s$. In [12] there is a condition sufficient to obtain the uniqueness of the S.G.S. with slow decay. Roughly speaking this result is achieved respectively when $s \neq p^*$ and $m \neq p^*$.

We give some examples of application of Theorem 1 and Corollary 2.

REMARK 4. Assume that $k(r)$ is uniformly positive and bounded and that the limit $\lim_{r\to\infty}k(r)$ exists. Then, if $q < p^*$ and $k(r)$ is nondecreasing, positive solutions have a structure of type **A**, while if $q > p^*$ and $k(r)$ is nonincreasing, positive solutions have a structure of type **B**.

Consider the generalized Matukuma equation, that is (3) where $f(u,r) = \frac{1}{1+r^\tau}u|u|^{q-2}$. Then, if $\tau \leq p$ positive solutions have a structure of type **A** when $p < q < p(n-\tau)/(n-p)$, and of type **B** when $q > p^*$, see also [35]. The remaining cases will be analyzed in section 4.

3.2. The generic case: $f(u,r) = k_1(r)f_1(u) + k_1(r)f_2(u)$

Now we try to extend the results of the previous subsection to a wider class of functions $f(u,r)$:

$$f(u,r) = \sum_{i=1}^{N} k_i(r)u|u|^{q_i-2}, \qquad p < q_1 < q_2 < \cdots < q_N \tag{17}$$

where $N \geq 1$ and the functions $k_i(r)$, are continuous and positive. We will see that, under natural conditions on the functions $k_i(r)$, when $p_* < q_1 < q_N \leq p^*$ positive solutions have a structure of type **A**, while when $q_1 \geq p^*$ they have a structure of type **B**. The behavior of regular solutions have been classified directly using Pohozaev identity in [35]. In [13] we have completed the results by classifying the behaviour of singular solutions, using dynamical methods. In fact we have followed the path paved by Johnson and Pan in [29], for the analogous problem in the case $p = 2$.

In this paper we generalize slightly the techniques used in [34] and [13], combining them with some ideas of [17], to obtain more general results. As usual we set $\phi_i(t) = k_i(e^t)$, and we always assume (without mentioning) that $e^{nt}\phi_i(t) \in \mathcal{L}^1\big((-\infty,0]\big)$ and $e^{(n-q_i\frac{n-p}{p-1})s}\phi_i(t) \in \mathcal{L}^1\big([0,+\infty)\big)$ for any $i = 1,\ldots,N$, so that we can define functions similar to $J^\pm$ of the previous subsections:

$$J_i^+(t) := \frac{\phi_i(t)e^{nt}}{q_i} - \frac{n-p}{p}\int_{-\infty}^{t}\phi_i(s)e^{ns}ds$$
$$J_i^-(t) := \frac{\phi_i(t)e^{(n-q_i\frac{n-p}{p-1})t}}{q} - \frac{n-p}{p(p-1)}\int_{t}^{+\infty}\phi_i(s)e^{(n-q_i\frac{n-p}{p-1})s}ds$$

Observe that, if $f_1(u) = u|u|^{q-1}$ and $k = 1$, the functions $J_1^{\pm}(t)$ defined in (3.2) coincide with the functions $J^{\pm}(t)$ defined in section 3.1. As we did in section 3.1, if $\phi_i \in C^1$ we can rewrite the functions $J_i^{\pm}$ in a form similar to (15) from which we can more easily guess the sign. So we find the analogous of (16):

$$H_{p^*}(\mathbf{x}_{p^*}(t), t) + \lim_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) = \sum_{i=1}^{N} \Big[J_i^+(t) \frac{u^{q_i}(e^t)}{q_i} - \int_{-\infty}^{t} J_i^+(s) u'(e^s) u^{q_i-1}(e^s) ds \Big]$$

$$H_{p^*}(\mathbf{x}_{p^*}(t), t) - \lim_{t\to\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) = \sum_{i=1}^{N} \Big[J_i^-(t) \frac{x_{p_*}^{q_i}(t)}{q_i} + \int_{t}^{\infty} J_i^-(s) \dot{x}_{p_*}(t) x_{p_*}^{q_i-1}(s) \Big] ds$$

Therefore we have a result analogous to Remark 3 and repeating the argument of the proof of Theorem 1, we obtain the following generalization.

THEOREM 2. *Assume that either $J_i^+(t) \geq 0$ for any i and $\sum_{i=1}^N J_i^+(t) \not\equiv 0$ or $J_i^-(t) \geq 0$ for any i and $\sum_{i=1}^N J_i^-(t) \not\equiv 0$. Then all the regular solutions are crossing solutions; moreover if $q_1 > p_*$, and $k_i(r)$ is uniformly positive for r large, there are uncountably many S.G.S. with fast decay.*

Assume that either $J_i^+(t) \leq 0$ for any i and $\sum_{i=1}^N J_i^+(t) \not\equiv 0$ or $J_i^-(t) \leq 0$ for any i and $\sum_{i=1}^N J_i^-(t) \not\equiv 0$. Then all the regular solutions are G.S. with slow decay. Moreover there are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of a ball.

This result is proved in [13] for the case $1 < p \leq 2$. However it can be easily extended to the case $p > 2$ putting together the construction of a stable set $\tilde{W}^s(\tau)$ developed in [17] (and quoted in Theorem 2), and the argument of [13] concerning the function H_{p^*} (that we have sketched in this section). In fact the minimal requirement for the fast decay solution to exist, is that there are $c > 0$ and $m > p_*$ such that $g(x_m(t), t) > c x_m(t)|x_m(t)|^{m-2}$ for t large.

Repeating the argument in Corollary 2 we easily obtain also this result:

COROLLARY 3. *Assume that all the functions $J_i^-(t) \geq 0$ for any r but $\sum_{i=1}^N J_i^-(t) \not\equiv 0$, and that there is $p_* < m \leq p^*$ such that the limit $\lim_{t\to\infty} g(x_m(t), t)$ $/|x_m(t)|^{m-1} = k_m(\infty)$ exists, is positive and finite. Then positive solutions have a structure of type **A**.*

*Analogously assume that all the functions $J_i^+(t) \leq 0$ for any r but $\sum_{i=1}^N J_i^+(t) \not\equiv 0$, and that there is $s \geq p^*$ such that the limit $g(x_s(t), t)/|x_s(t)|^{s-1} = k_s(0)$ exists, is is positive and finite. Then positive solutions have a structure of type **B**.*

Once again if $m \neq p^*$ and $s \neq p^*$ respectively, and a further technical condition is satisfied the S.G.S. with slow decay is unique, see [13]. From Theorem 2 and

Corollary 3 we easily get the following.

REMARK 5. Consider (3) where f is as in (17) and assume that the functions $k_i(r)$ are uniformly positive and bounded. Then, if $p_* < q_1 < q_N \leq p^*$ and the functions $k_i(r)$ are nondecreasing, positive solutions have a structure of type **A**, while if $q_1 \geq p^*$ and the functions $k_i(r)$ are nonincreasing, positive solutions have a structure of type **B**.

4. When the Pohozaev function changes sign

In this section we discuss equation (3) when $f(u,r) = k(r)u|u|^{q-2}$, and we assume that $e^{nt}\phi(t) \in \mathcal{L}^1\big((-\infty, 0]\big)$ and $e^{(n-q\frac{n-p}{p-1})s}\phi(t) \in \mathcal{L}^1\big([0, +\infty)\big)$, so that the functions $J^{\pm}(r)$ are well defined. We discuss now the case when $J^{+}(r)$ and $J^{-}(r)$ change sign. In such a case positive solutions may exhibit the following rich structure:

C There are uncountably many G.S. with slow decay and crossing solutions, and at least one G.S. with fast decay. There are uncountably many S.G.S. with fast decay, S.G.S. with slow decay, and solutions of the Dirichlet problem in the exterior of the ball.

Let us recall that, for $n > 2$ we denote by $2^* = 2n/(n-2)$ and by $2_* = 2(n-1)/(n-2)$. We start from this interesting result proved by Bianchi in [6].

THEOREM 3. *Consider (2) and define $g(r) = k(r)|r^2 - c^2|^{\frac{2^*-q}{2(n-2)}}$; assume that there is $c > 0$ such that $g(r)$ is non-increasing for $0 < r < c$ and non-decreasing for $r > c$. Then all the G.S. and the S.G.S. are radial.*

This fact gives more relevance to the study of radial solutions. Note that, if $q = 2^*$ then the Theorem simply requires that there is c such that $k(r)$ is non-increasing for $r < c$ and non-decreasing for $r > c$. In fact we think that these results may be extended also to the case $p \neq 2$; but it cannot be extended to any kind of potential $k(r)$ in fact, modifying the nonnegative potential $K_0 = \big(1 - (r/\delta)^{\rho_1}\big)_+ + \big(1 - (\delta r)^{\rho_2}\big)_+$, where $\delta, \rho_1, \rho_2 > 0$, Bianchi in [6] constructed a positive potential $k = \bar{k}(r)$ so that (2) admits no radial G.S. with fast decay, but it admits non-radial G.S. with fast decay. The potential $\bar{k}(r)$ is obtained from a potential satisfying the hypotheses of Theorem 3 and subtracting an arbitrarily small bump at $r = 0$ and at $r = \infty$.

For completeness we also quote the following result, borrowed from [5], concerning potential $k(\mathbf{x})$ which are not necessarily radial.

THEOREM 4. *Consider (2) where $q = 2^*$ and assume $n \geq 4$. Choose two points $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and two numbers $C_{\boldsymbol{y}}, C_{\boldsymbol{z}} > 0$. Then there is a positive potential $k = \tilde{k}(\boldsymbol{x})$ of the form*

$$\tilde{k}(\boldsymbol{x}) = C_{\boldsymbol{y}} - \epsilon|\boldsymbol{x} - \boldsymbol{y}|^{\rho} \quad \textit{for } \boldsymbol{x} \textit{ in a neighborhood of } \boldsymbol{y}$$
$$\tilde{k}(\boldsymbol{x}) = C_{\boldsymbol{z}} - \epsilon|\boldsymbol{x} - \boldsymbol{z}|^{\rho} \quad \textit{for } \boldsymbol{x} \textit{ in a neighborhood of } \boldsymbol{z} \tag{18}$$

where $\epsilon > 0$ is small enough and $\rho = n - 2$, such that (2) admits no G.S. with fast decay.

In the same article Bianchi has also proved that a potential satisfying condition (18), with $\rho > n-2$ and ϵ arbitrary chosen, and some further sufficient conditions necessarily admits a G.S. with fast decay. This result shows how sensitive to small changes in the potential k the behaviour of positive solutions is.

Now we turn again to the case $p \neq 2$, and we focus our attention on radial solutions. In order to find a G.S. with fast decay we need to find a balance between the gain of energy, due to the values for which $k(r)r^{p^*-q}$ is increasing, and the loss of energy, due to the values for which it is decreasing. A first important result concerning the structure of positive solutions is the following:

THEOREM 5. *Consider (3) and assume that there is $R > 0$ such that one of the following conditions is satisfied*

$J^+(r) \geq 0$ *for any* $0 \leq r \leq R$ *and it is decreasing for* $r \geq R$

$J^-(r) \leq 0$ *for any* $r \geq R$ *and it is increasing for* $0 \leq r \leq R$.

Then regular solutions have one of the following structure.

1. *They are all crossing solutions*
2. *They are all G.S. with slow decay*
3. *There is $D > 0$ such that $u(d, r)$ is a crossing solution for $d > D$, it is a G.S. with slow decay for $d < D$, and a G.S. with fast decay for $d = D$.*

The result concerning $J^+(r)$ has been proved in [35], evaluating the Pohozaev function on regular solutions. The part concerning $J^-(r)$ is not explicitly stated in [35], however it can be easily obtained as follows, see also [18]. We can construct a stable set $\tilde{W}^s(\tau)$ through Proposition 2, and then deduce the existence of solutions with fast decay. Then, applying the argument of [35] to these solutions, we conclude. We think that one could easily reach a classification result also for S.G.S. in this situation, combining the argument in [35] with a dynamical argument. In fact, if we restrict to regular solutions, structure **A** and **B** give back structure 1 and 2 respectively, and structure 3 is a special case of **C**.

We have already seen that, when either $J^+(r)$ or $J^-(r)$ are positive for any $r > 0$ we have structure **A** (so we are in the first case), while when they are negative we have structure **B** (so we are in the second case). In order to derive a sufficient condition for structure **C** to exist, we start from the case $p = 2$ and following [43] we introduce the function:

$$Z(t) := e^{-\frac{(n-2)p}{2}t} J^+(t) - e^{\frac{(n-2)p}{2}t} J^-(t) \tag{19}$$

Then we define $\rho_+ = \inf\{r \in (0,\infty) \mid J^+(r) < 0\}$, and $\rho_- = \sup\{r \in (0,\infty) \mid J^-(r) < 0\}$, setting $\rho_+ = \infty$ if $J^+(r) \geq 0$ for any $r > 0$ and $\rho_- = 0$ if $J^-(r) \geq 0$ for any

$r > 0$. Now we can state the following result proved in [43], using the Pohozaev identity and the Kelvin transformation.

THEOREM 6. *Consider (3) where* $p = 2$ *and assume* $\rho_+ > 0$ *and* $\rho_- < \infty$. *If* $Z(r_1) > 0$ *for some* $r_1 \in (0, \rho_+]$ *and* $Z(r_2) > 0$ *for some* $r_2 \in [\rho_-, \infty)$*, there is* $D > 0$ *such that* $u(D, r)$ *is a G.S. with fast decay.*

Let us set $\lambda := \frac{(n-p)(q-p^*)}{p}$; following [43], we get the following more explicit result.

COROLLARY 4. *Consider (3) where* $p = 2$ *and suppose* $q \neq p^*$ *and that* $k(r)$ *is nonnegative and satisfies:*

$$k(r) = Ar^\sigma + o(r^\sigma) \ \ \text{at } r = 0 \qquad k(r) = Br^l + o(r^l) \ \ \text{at } r = \infty,$$

where $A, B > 0$ *and* $l < \lambda < \sigma$*, then there is a G.S. with fast decay.*

Now assume $q = p^*$ *and that* $k(r)$ *satisfies*

$$k(r) = A_0 + A_1 r^\sigma + o(r^\sigma) \ \ \text{at } r = 0 \qquad k(r) = B_0 + B_1 r^l + o(r^l) \ \ \text{at } r = \infty,$$

where $A_1, B_1 > 0$, $A_0, B_0 > 0$, $-n < l < 0 < \sigma < n$. *Then there is a G.S. with fast decay.*

Note that the case $q = p^*$ is more delicate; we stress that the restriction $|l|, |\sigma|$ smaller than n is needed even if it was not required in [43]. However when $A_0 = 0$ we do not need the restriction on $|l|$ and when $B_0 = 0$ we do not need the restriction on $|\sigma|$.

Using a similar argument Kabeya, Yanagida and Yotsutani, in [32] found an analogous result for the case $p \neq 2$.

THEOREM 7. *Consider (3) where* $k(r) \geq 0$ *for any* r. *Assume that either* $\liminf_{r\to 0} \frac{rk'(r)}{k(r)} > \lambda$ *or* $k(r) = Ar^\sigma + o(r^\sigma)$ *at* $r = 0$ *for some* $A > 0$ *and* $\sigma > \lambda$. *Moreover assume that either* $\limsup_{r\to\infty} \frac{rk'(r)}{k(r)} < \lambda$ *or* $k(r) = Br^l + o(r^l)$ *at* $r = 0$ *for some* $B > 0, l < \lambda$. *Then there is a strictly increasing sequence* $d_j > 0$, $j = 0, \ldots, \infty$, *such that* $u(d_j, r)$ *has exactly* j *zeroes and has fast decay. So in particular* $u(d_0, r)$ *is a G.S. with fast decay.*

Note that the conditions of the previous Theorem at $r = 0$ (and at $r = \infty$) are similar, but they do not imply each other. In fact the former is useful when $k(r)$ has a logarithmic term, e. g. $k(r) = |\ln(r)|r^\sigma$, and the latter when $k(r)$ behaves like a power at $r = 0$ (and at $r = \infty$). However, in both the cases, when $q = p^*$ we have $k(0) = k(\infty) = 0$.

We wish to mention that Bianchi and Egnell in [5], [7] have some other sufficient conditions for the existence of G.S. with fast decay. Each condition, as the ones of Corollary 4 and of Theorem 7, in some sense, requires a change in the sign of the function $J^+(r)$ which is “sufficiently large to be detected”. We stress that the situation becomes more delicate when $q = p^*$ and $k(r)$ is uniformly positive and bounded, see [5], [7], for a careful analysis. In fact, as suggested from Theorem 8 stated below and borrowed from [7], we are convinced that the condition on the smallness of $|l|$, $|\sigma|$ of Corollary 4 is not technical.

THEOREM 8. *Consider (2) where $q = 2^*$ and take two numbers $\rho_1, \rho_2 > n(n-2)/(n+2)$, such that $1/\rho_1 + 1/\rho_2 \geq 2/(n-2)$. Then there is a function $k(r)$ such that $k(r) = 1 - M_1 r^{\rho_1}$ near the origin and $k(r) = 1 - M_2 r^{-\rho_2}$ near ∞, where M_1, M_2 are positive large constants so that (2) admits no radial G.S. with fast decay.*

We also stress that the previous result shows that the situation is much more clear when $J^+(r)$ is positive for r small and negative for r large, than when we are in the opposite situation.

We introduce now some perturbative results, proved with dynamical techniques, that help us to understand better also what happens to singular solutions, and also which is the difference between the case in which we have a subcritical behaviour for r small and supercritical for r large (easier situation), and the opposite case (difficult situation). We focus on the case $q = p^*$: in the autonomous case this is a border situation between a structure of type **A** and **B**. So it is the best setting in order to have new phenomena as the existence of G.S. with fast decay.

Let us assume that $k(r)$ has one of the following two form:

$k(r) = 1 + \epsilon K(r)$, where K is a bounded smooth function,

$k(r) = K(r^\epsilon)$, where K is a bounded smooth function, positive in some interval,

where $\epsilon > 0$ is a small parameter and we assume $K \in C^2$. In the former case we say that $k(r)$ is a regular perturbation of a constant (k changes little), in the latter we say that it is a singular perturbation of a constant (k changes slowly). It is worthwhile to note that, in the latter case k may change sign.

This problem was studied in the case $p = 2$ by Johnson, Pan and Yi in [30] using the Fowler transformation, invariant manifold theory for non-autonomous system and Mel'nikov theory. In both the cases they found a non-degeneracy condition of Mel'nikov type, related to some kind of expansion in ϵ of the Pohozaev function, which is sufficient for the existence of G.S. with fast decay. They also proved that when $K(e^t)$ is periodic and the Mel'nikov condition is satisfied, there is a Smale horseshoe for the associated dynamical system. Then they inferred the existence of a Cantor set of S.G.S. with slow decay.

In the singular perturbation case the condition is easy to compute: there is a G.S. with fast decay for each non-degenerate positive critical point of $K(r)$. These results have been completed by Battelli and Johnson in [2], [3], [4], and eventually they proved the existence of a Smale horseshoe also in this case. Thus they inferred again the existence of a Cantor set of S.G.S. with slow decay, assuming that $K(e^t)$ is periodic.

These results have been extended to the case $2n/(n+2) \leq p \leq 2$ in [14], and completed to obtain a structure result for positive solutions. First we have introduced a dynamical system of the form (8) through (4) with $l = q = p^*$. In this section we will always set $l = q = p^*$ in (4) so we will leave the subscript unsaid, to simplify the notation. Since (8) is C^1 and uniformly continuous in the t variable, $\mathbf{O}$ admits local unstable and stable manifolds, denoted respectively by $W^u_{\epsilon,loc}(\tau)$ and $W^s_{\epsilon,loc}(\tau)$, see [30], [14]. From Proposition 3 we know that, if $\mathbf{Q}^u \in W^u_{\epsilon,loc}(\tau)$, then $\lim_{t\to-\infty}\mathbf{x}^\tau(\mathbf{Q}^u, t) = \mathbf{O}$ and the corresponding solution $u(r)$ of (3) is a regular solution, while if $\mathbf{Q}^s \in W^s_{\epsilon,loc}(\tau)$, then $\lim_{t\to\infty}\mathbf{x}^\tau(\mathbf{Q}^s, t) = \mathbf{O}$ and the corresponding solution $v(r)$ of (3) is a solution with fast decay. Using the flow it is possible to extend the local manifolds to global manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. As usual we commit the following abuse of notation: we denote by $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ just the branches of the manifolds that depart from the origin and get into $\mathbb{R}^2_+$. From [30] we also know that the leaves are C^1 and vary continuously in the C^1 topology with respect to τ and ϵ. Observe that for $\epsilon = 0$, both in the regular and in the singular perturbation case, the manifold $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ coincide and are the image of the homoclinic trajectory. We fix a segment L which is transversal to $W^u_0(\tau) \equiv W^s_0(\tau)$ and which intersects it in a point, say $\mathbf{U}$. Using a continuity argument, we deduce that, for $\epsilon > 0$ small enough, $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ continue to cross L transversally in points $\xi^s(\tau,\epsilon)$ and $\xi^u(\tau,\epsilon)$ close to $\mathbf{U}$. We want to find intersections $\mathbf{Q}$ between $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$; then the trajectory $\mathbf{x}^\tau(\mathbf{Q}, t)$ corresponds to a regular solution $u(r)$ having fast decay. Then it is easily proved that $\mathbf{x}^\tau(\mathbf{Q}, t) \in \mathbb{R}^2_+$ for any t so it is a monotone decreasing G.S. with fast decay.

Let us rewrite (8) as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \tau + t, \epsilon)$. From now on we restrict our attention to the singularly perturbed system since the other can be treated similarly, see [30] and [14]. We define a Melnikov function which measures the distance with sign between $\xi^s(\tau,\epsilon)$ and $\xi^u(\tau,\epsilon)$ along L.

$$M(\tau) = \frac{d}{d\epsilon}\left[\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)\right]\lfloor_{\epsilon=0} \wedge \mathbf{f}(\mathbf{U}, \tau)$$

where "$\wedge$" denotes the standard wedge product in $\mathbb{R}^2$. Then define

$$h(\tau,\epsilon) = \begin{cases} M(\tau) & \text{for} \quad \epsilon = 0 \\ \frac{\xi^-(\tau,\epsilon)-\xi^+(\tau,\epsilon)}{\epsilon} \wedge \mathbf{f}(\mathbf{U},\tau) & \text{for} \quad \epsilon \neq 0. \end{cases}$$

We point out that the vector $\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)$ belongs to the transversal segment L, so we have $h(\tau,\epsilon) = 0 \iff \xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon) = 0$ for $\epsilon \neq 0$.

Suppose $M(\tau_0) = 0$ and $M'(\tau_0) \neq 0$, then, using the implicit function theorem, we construct a C^1 function $\epsilon \to \tau(\epsilon)$ defined on a neighborhood of $\epsilon = 0$, such that $\tau(0) = \tau_0$, for which we have $\xi^-(\tau(\epsilon),\epsilon) = \xi^+(\tau(\epsilon),\epsilon)$. Therefore we have a homoclinic solution of the system (8).

Following [30] and [14] we find that

$$(20) \qquad M(\tau) = -\phi'(\tau)\phi(\tau)^{-\frac{n}{p}} \int_{-\infty}^{+\infty} \frac{|x_1(t)|^\sigma}{\sigma} dt = -C\phi'(\tau)\phi(\tau)^{-\frac{n}{p}}$$

where $\mathbf{x}_1(t) = (x_1(t), y_1(t))$ is a homoclinic trajectory of (8) where $\phi \equiv 1$, so $C > 0$ is a computable positive constant. Note that $M(\tau)$ is closely related to the first term in the expansion in ϵ of the function $Z(t)$ defined in (19). It follows that for any positive non degenerate critical point of $k(r)$ there is a crossing between $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$, so we have a G.S. with fast decay.

Introducing a further Mel'nikov function depending on two parameters, it can be proved that such a crossing is transversal, see [30], [2], [14]. In order to use the Smale construction of the horseshoe, we need to prove that the functions $\xi^{\pm}(\epsilon, \tau)$ are C^2 even if the system is just C^1. This has been done in [4], using some fixed point theorems in weighted spaces, and observing that the first branch of $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ cannot cross the coordinate axes, where part of the regularity is lost.

Now we assume that ϕ is periodic and admits a non-degenerate positive critical point. Using the previous Lemma we find a point $\mathbf{Q}(\epsilon) \in W^u_\epsilon(\tau(\epsilon)) \cap W^s_\epsilon(\tau(\epsilon))$. Then, using the Smale construction, we find a Cantor set Λ close to the transversal crossing $\mathbf{Q}(\epsilon)$, such that the trajectories $\mathbf{x}^\tau(\mathbf{P}, t)$, where $\mathbf{P} \in \Lambda$ are bounded, and do not converge to the origin. With some elementary analysis on the phase portrait we can also show that $\mathbf{x}^\tau(\mathbf{P}, t) \in \mathbb{R}^2_+$ for any $t \in \mathbb{R}$. So we find the following, see [30], [2], [3], [4] [14] for the proof.

THEOREM 9. *Consider (3) where $q = p^*$, $2n/(n+2) \le p \le 2$, and $k \in C^2$ is a singular perturbation of a constant. Then there is a monotone decreasing G.S. with fast decay for each positive non-degenerate critical point of $k(r)$.*

Moreover assume that $k(e^t)$ is a periodic function and it admits a non degenerate positive extremum. Then there is a Cantor-like set of monotone decreasing S.G.S. with slow decay $\upsilon(r)$. Moreover if $k(r)$ is strictly positive, the S.G.S. are monotone decreasing.

When k is a regular perturbation of a constant, we proceed in the same way but we find a different Mel'nikov function:

$$\bar{M}(\tau) = \int_{-\infty}^{+\infty} \phi'(t+\tau)\frac{|x_1|^{p^*}}{p^*}dt\,, \qquad \bar{M}'(\tau) = \int_{-\infty}^{+\infty} \phi''(t+\tau)\frac{|x_1|^{p^*}}{p^*}dt$$

Then, arguing as above we find the following.

THEOREM 10. *Assume that $k(r) = 1 + \epsilon K(r)$ is a C^2 function and $\epsilon > 0$ is a sufficiently small parameter. Then equation (3) admits a G.S. with fast decay for each non degenerate zero of $M(\tau)$. Assume in addition that $K(e^t)$ is a periodic function. Then equation (3) admits a Cantor-like set of monotone decreasing S.G.S. with slow decay.*

Following [14], we point out that now it is possible to get further information on the structure of positive solutions, both regular and singular, with a careful analysis of the phase portrait. The idea is to construct a barrier set made up of branches of the manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. We illustrate it with an example, remanding to [14] for a detailed discussion. Let us assume that $k(r)$ admits 9 positive non degenerate critical points for $r > 0$, 5 maxima and 4 minima , see figure 3.

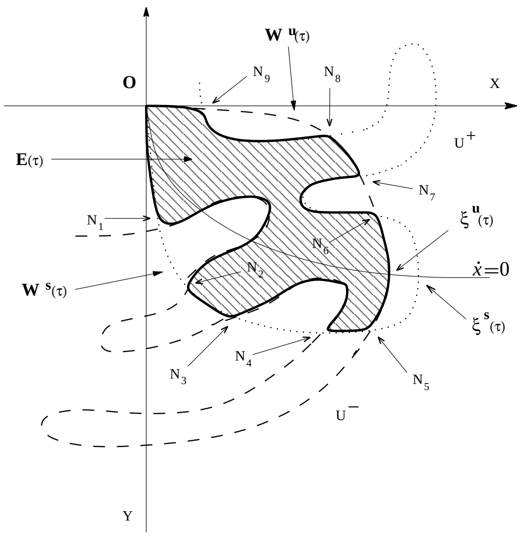

Figure 3: A sketch of the set $E(\tau)$, when $k(r) = K(r^{\epsilon})$ has 5 maxima and 4 minima. The solid line represents $B(\tau)$, and it is obtained joining segments of $W^u(\tau)$ (dotted line), and of $W^s(\tau)$ (dashed line).

First observe that if $\mathbf{Q}^\tau \in W^u_\epsilon(\tau)\cap W^s_\epsilon(\tau)$ then $\mathbf{x}^\tau(\mathbf{Q}^\tau, t) \in W^u_\epsilon(\tau+t)\cap W^s_\epsilon(\tau+t)$ for any t. So the number of intersection between $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ does not depend on τ. Therefore there are 9 functions $\tau_i(\epsilon)$ such that $\xi^u(\tau(\epsilon), \epsilon) = \xi^s(\tau(\epsilon), \epsilon)$ for $i = 1, \cdots, 9$ and 9 points $\mathbf{N}^i(\tau)$ of intersection between stable and unstable manifolds. We denote by $\mathbf{N}^1(\tau)$, the first point met following $W^s_\epsilon(\tau)$ from the origin towards $\mathbb{R}^2_+$, by $\mathbf{N}^2(\tau)$ the second, and so on. Let us denote by $B^0(\tau)$ the branch of $W^s_\epsilon(\tau)$ between the origin and $\mathbf{N}^1(\tau)$, by $B^1(\tau)$ the branch of $W^u_\epsilon(\tau)$ between $\mathbf{N}^1(\tau)$ and $\mathbf{N}^2(\tau)$, by $B^2(\tau)$ the branch of $W^s_\epsilon(\tau)$ between $\mathbf{N}^2(\tau)$ and $\mathbf{N}^3(\tau)$, and so on till the branch of $W^u_\epsilon(\tau)$ between $\mathbf{N}^9(\tau)$ and the origin which is denoted by $B^9(\tau)$. Finally we denote by $B(\tau) = \cup_{i=0}^{9} B^i(\tau)$, and by $E(\tau)$ the bounded open subset enclosed by $B(\tau)$. The key observation is that $B(\tau)$ is contained in $\mathbb{R}^2_+$ for any τ, and in $\{\mathbf{x} \,|\, y < 0 < x\}$ when ϕ is uniformly positive, see [14] for a detailed proof.

Observe that $E(\tau)\setminus (W^u_\epsilon(\tau) \cup W^s_\epsilon(\tau))$ contains uncountably many points and take $\mathbf{Q}$ in it. The trajectory $\mathbf{x}^\tau(\mathbf{Q}, t)$ is forced to stay in the interior of $E(\tau + t)$ for any t, therefore it corresponds to a S.G.S. with slow decay. With a careful analysis on the phase portrait it is possible to find points $\mathbf{Q} \in B(\tau)\setminus W^s_\epsilon(\tau)$ such that $\mathbf{x}^\tau(\mathbf{Q}, t)$ is forced to stay in the interior of $E(\tau + t)$ for any $t > 0$, and $\mathbf{P} \in B(\tau)\setminus W^s_\epsilon(\tau)$ such that $\mathbf{x}^\tau(\mathbf{P}, t)$ has to cross the y axis for some $t > 0$. Therefore they correspond respectively to G.S. with slow decay and to crossing solutions. Analogously we find $\mathbf{Q}, \mathbf{P} \in B(\tau)\setminus W^u_\epsilon(\tau)$ such that $\mathbf{x}^\tau(\mathbf{Q}, t) \in E(\tau + t)$ for any $t < 0$ and $\mathbf{x}^\tau(\mathbf{P}, t)$ has to cross the y axis for some $t < 0$, which correspond respectively to S.G.S. with fast decay and to solutions of the Dirichlet problem in the exterior of a ball, see [14] for more details. The results can be summed up as follows. Let us introduce the following hypotheses:

M$_1$ there exists $\rho > 0$ such that $k(\rho) > 0$ is a non degenerate maximum and $k(r)$ is uniformly positive and monotone increasing for $0 \leq r \leq \rho$.

M$_2$ there exists $R > 0$ such that $k(R) > 0$ is a non degenerate maximum and $k(r) > 0$ is uniformly positive and monotone decreasing for $r \geq R$.

O$_1$ $k(r)$ is oscillatory as $r \to 0$ and admits infinitely many positive non degenerate critical points.

O$_2$ $k(r)$ is oscillatory as $r \to \infty$ and admits infinitely many positive non degenerate critical points.

Then we have the following result:

THEOREM 11. *Consider equation (3) and assume that $k(r) = K(r^\epsilon)$ is bounded. Then, for $\epsilon > 0$ small enough, we have at least as many G.S. with fast decay as the non degenerate critical points of $k(r)$. Moreover*

1. *Assume that either* $\mathbf{M_2}$ *or* $\mathbf{O_2}$ *is satisfied. Then the there are uncountably many G.S. with slow decay and uncountably many crossing solutions.*
2. *Assume that either* $\mathbf{M_1}$ *or* $\mathbf{O_1}$ *is satisfied. Then there are uncountably many S.G.S. with fast decay and uncountably many solutions $\upsilon(r)$ of Dirichlet problem in the exterior of a ball.*
3. *Assume that both Hypotheses 1 and 2 are satisfied. Then the positive solutions of equation (3) have a structure of type* **C**.

Furthermore, if $k(r)$ is uniformly positive, then G.S. and S.G.S. are decreasing.

REMARK 6. Note that when $k(r)$ is decreasing for r small and increasing for r large, we are not able to state the existence of S.G.S. and of G.S. with slow decay. This is due to the fact that, in such a case it is not possible to construct a set $B(\tau)$ which is contained in $\mathbb{R}^2_+$ for any τ, so our argument fails. However also in this case we are able to prove the existence of G.S. with fast decay.

Following [14] we can easily obtain an analogous result for the regularly perturbed problem. The difference lies in the fact that the Melnikov condition is a bit more complicated, so we have to replace the assumption that $k(r)$ has a positive critical point by the condition that $\bar{M}(\tau) = 0$ and $\bar{M}'(\tau) \neq 0$.

Now we want to extend some of these results to the "in the large" case, so we want to see what happens when $\epsilon \to 1$. This in fact will shed some light on the reason for which positive solutions exhibit the same structure, under two completely different types of perturbation. The idea is to use our knowledge of the autonomous case to understand the non-autonomous one, replacing the Melnikov function by the energy function H. We will discuss the following Hypotheses

$\bar{\mathbf{M}}_{\mathbf{1}}^{+}$ $k(r)$ is increasing for r small and $k'(r)r^{-n/(p-1)} \notin L^1(0,1]$.

$\bar{\mathbf{M}}_{\mathbf{1}}^{-}$ $k(r)$ is decreasing for r small and $k'(r)r^{-n/(p-1)} \notin L^1(0,1]$.

$\bar{\mathbf{M}}_{\mathbf{2}}^{+}$ $k(r)$ is increasing for r large and $k'(r)r^{n} \notin L^1[1,\infty)$.

$\bar{\mathbf{M}}_{\mathbf{2}}^{-}$ $k(r)$ is decreasing for r large and $k'(r)r^{n} \notin L^1[1,\infty)$.

Now we can state the following theorem, see [18], [15].

THEOREM 12. *Consider (3) where $q = p^*$ and $k(r) \in [a,b]$ for any $r \geq 0$, for some $b > a > 0$. Assume that either hypotheses $\bar{\mathbf{M}}_{\mathbf{1}}^{+}$ and $\bar{\mathbf{M}}_{\mathbf{2}}^{-}$, or $\bar{\mathbf{M}}_{\mathbf{1}}^{-}$ and $\bar{\mathbf{M}}_{\mathbf{2}}^{+}$ are satisfied. Then there is a G.S. with fast decay. Moreover*

1. *If $\bar{M}_{-}^{2}$ is satisfied there are uncountably many G.S. with slow decay and uncountably many crossing solutions.*
2. *If $\bar{\mathbf{M}}_{\mathbf{1}}^{+}$ is satisfied, there are uncountably many S.G.S. with fast decay and uncountably many solutions of Dirichlet problem in the exterior of a ball.*
3. *If $\bar{\mathbf{M}}_{\mathbf{1}}^{+}$ and $\bar{M}_{-}^{2}$ are satisfied positive solutions have structure **C**.*

Proof. Consider the autonomous system (8) where $q = p^*$ and $\phi \equiv a$, or $\phi \equiv b$ respectively. Denote by $\mathbf{x}_a(t)$ and $\mathbf{x}_b(t)$ the trajectories of the former and the latter system such that $\dot{x}_a(0) = 0 = \dot{x}_b(0)$. Denote by $A^+ = \{\mathbf{x}_a(t)\,|\,t \leq 0\}$, $A^- = \{\mathbf{x}_a(t)\,|\,t \geq 0\}$, $B^+ = \{\mathbf{x}_b(t)\,|\,t \leq 0\}$, $B^- = \{\mathbf{x}_b(t)\,|\,t \geq 0\}$, by $\mathbf{A} = (A_x, A_y) = \mathbf{x}_a(0)$ and by $\mathbf{B} = (B_x, B_y) = \mathbf{x}_b(0)$. Let us denote by E^+ (respectively E^-) the bounded subsets enclosed by A^+, B^+ (resp. A^-, B^-) and the isocline $\dot{x} = 0$.

Note that the flow of the non autonomous system (8) on $A^+ \cup B^+$ points towards the interior of E^+ while on $A^- \cup B^-$ points towards the exterior of E^-. So, using Wazewski's principle, we can construct compact connected sets as follows, see [15].

$$W^u(\tau) := \{\mathbf{Q} \in E^+ \mid \lim_{t\to-\infty} \mathbf{x}^\tau(\mathbf{Q},t) = \mathbf{O} \text{ and } \mathbf{x}^\tau(\mathbf{Q},t) \in E^+ \text{ for } t \leq 0\},$$
$$W^s(\tau) := \{\mathbf{Q} \in E^- \mid \lim_{t\to+\infty} \mathbf{x}^\tau(\mathbf{Q},t) = \mathbf{O} \text{ and } \mathbf{x}^\tau(\mathbf{Q},t) \in E^- \text{ for } t \geq 0\}.$$

We denote by $\xi^u(\tau)$ and $\xi^s(\tau)$ the intersection of the isocline $\dot{x} = 0$ respectively with $W^u(\tau)$ and $W^s(\tau)$. In analogy to what we have done in the perturbative case we want to measure the distance with sign of the compact non-empty sets $\xi^u(\tau)$ and $\xi^s(\tau)$ evaluating the energy function H on these sets.

We wish to stress that we have committed a mistake in [15] in such evaluation, but we can correct it as follows, see [18]. Let us denote by L the line $x = B_x$, and by $\mathbf{C}^+$ the intersection of L with A^+; finally let L^+ be the segment of L between $\mathbf{C}^+$ and $\mathbf{B}$. Denote by $\mathbf{x}_a^\tau(t)$, the trajectory of the autonomous system where $\phi \equiv a$ such that $\mathbf{x}_a^\tau(0) = \mathbf{C}^+$, and by $\mathbf{x}_b^\tau(t)$, the trajectory of the autonomous system where $\phi \equiv b$ such that $\mathbf{x}_b^\tau(0) = \mathbf{B}$. Recall that we have explicit formulas for $x_a^\tau(t)$ and $x_b^\tau(t)$ and that we can find $C > c$ such that $\sqrt[p^*]{c}\,e^{\frac{n-p}{p}t} < x_b^\tau(t) < x_a^\tau(t) < \sqrt[p^*]{C}\,e^{\frac{n-p}{p}t}$ for

$t \le 0$. Consider a trajectory $\mathbf{x}^\tau(\mathbf{Q}^u(\tau), t)$ of the non-autonomous system (8) such that $\mathbf{x}^\tau(\mathbf{Q}^u(\tau), 0) = \mathbf{Q}^u(\tau) \in L^+$. It can be proved that

$$ce^{nt} < |x_b^\tau(t)|^{p^*} \le |x^\tau(\mathbf{Q}^u(\tau), t)|^{p^*} \le |x_a^\tau(t)|^{p^*} < Ce^{nt}$$

for any $t \le 0$, see [18]. Denote by $\bar{W}^u(\tau)$ and $\bar{W}^s(\tau)$ respectively the subset of $W^u(\tau)$ and $W^s(\tau)$ contained in $\{\mathbf{x} \mid 0 < x < L_x\}$. It can be shown easily that for any point $\mathbf{Q} \in \bar{W}^u(\tau)$ we have $c(\mathbf{Q})e^{nt} \le |x^\tau(\mathbf{Q}^u(\tau), t)|^{p^*} \le C(\mathbf{Q})e^{nt}$, where $C(\mathbf{Q})/C = K(\mathbf{Q}) = c(\mathbf{Q})/c > 0$.

Now assume that hypothesis $\bar{M}_2^+$ is satisfied; then there is $T_0 > 0$ such that $\dot{\phi}(t) > 0$ for any $t > T_0$. Hence for any $\mathbf{Q} \in \bar{W}^u(\tau)$ we have

$$\begin{aligned} H_{p^*}(\mathbf{Q}, \tau) &= \int_{-\infty}^{0} \dot{\phi}(\tau + t)\frac{|x^\tau(\mathbf{Q}; t)|^{p^*}}{p^*} dt \ge \\ &\ge \frac{e^{-n\tau} K(\mathbf{Q})}{\sigma}\Big[C(\phi(T_0) - b)e^{nT_0} + c\int_{T_0}^{\tau} \dot{\phi}(\zeta)e^{n\zeta} d\zeta\Big] \end{aligned} \tag{21}$$

Since $\dot{\phi}(\zeta)e^{n\zeta} \notin \mathcal{L}^1\big[[0, \infty)\big]$, we can find $N^+ > T_0$ such that $H_{p^*}(\mathbf{Q}, \tau) > 0$ for any $\mathbf{Q} \in \bar{W}^u(\tau)$ and $\tau > N^+$.

We denote by $\Phi_{\tau,t}(\mathbf{Q})$ the diffeomorphism defined by the flow of (8), precisely $\Phi_{\tau,t}(\mathbf{Q}) = x^\tau(\mathbf{Q}; t)$. Note that for any $\mathbf{Q} \in \bar{W}^u(\tau)$, where $\tau > N^+$, and any $t \ge 0$, we have $H(\Phi_{\tau,t}(Q), t + \tau) > H(Q, \tau) > 0$ since $\dot{\phi}(s) > 0$ for $s > \tau > N^+$.

Observe that there is a unique $t = T^u(\mathbf{Q}) > 0$ such that $\mathbf{x}^\tau(\mathbf{Q}; t) \in E^+$ for any $t < T^u(\mathbf{Q})$ and $\mathbf{x}^\tau(\mathbf{Q}; T^u(\mathbf{Q})) \in \xi^u(T^u(\mathbf{Q}) + \tau)$. We choose $T_\omega^+ = \min\{T^u(\mathbf{Q}) + N^+ \mid \mathbf{Q} \in \bar{W}^u(N^+)\}$; it follows that $\Phi_{N^+,t}[\bar{W}^u(N^+)] \supset \tilde{W}^u(N^+ + t)$, for any $t \ge T_\omega^+ - N^+$. Hence $H(Q, \tau) > 0$ for any $Q \in \tilde{W}^u(\tau)$ for any $\tau > T_\omega^+$.

Moreover, for any $\mathbf{P} \in \tilde{W}^s(\tau)$ we have

$$H_{p^*}(\mathbf{P}, \tau) = -\int_{\tau}^{+\infty} \dot{\phi}(t + \tau)\frac{|x^\tau(\mathbf{P}, t)|^{p^*}}{p^*} dt < 0\,,$$

since $\dot{\phi}(t) > 0$ for $t + \tau > T_0$. Therefore $H_{p^*}(\mathbf{P}, \tau) < 0 < H_{p^*}(\mathbf{Q}, \tau)$ for any $\mathbf{P} \in \tilde{W}^s(\tau)$ and any $\mathbf{Q} \in \tilde{W}^u(\tau)$. Analogously if $\bar{M}_1^-$ is satisfied, we can find $T_\alpha^- < 0$ such that $H_{p^*}(\mathbf{Q}, \tau) < 0 < H_{p^*}(\mathbf{P}, \tau)$ for any point $\mathbf{P} \in \tilde{W}^s(\tau)$ and $\mathbf{Q} \in \tilde{W}^u(\tau)$, for any $\tau < T_\alpha^-$. It follows that there is $\tau_0 \in (T_\alpha^-, T_\omega^+)$ such that $\xi^s(\tau_0) \cap \xi^u(\tau_0) \ne \emptyset$. So if $\mathbf{Q}^0 \in \xi^s(\tau_0) \cap \xi^u(\tau_0)$ we have that the solution $u(r)$ of (3) corresponding to $\mathbf{x}^{\tau_0}(\mathbf{Q}^0, t)$ is a G.S. with fast decay.

Then repeating the argument of the perturbative case we conclude the proof of the Theorem. □

This way we have proved structure results for positive solutions also in the case $p > 2$ and corrected the corresponding results in [15]. However we cannot correct the proof of the results concerning the existence of multiple G.S. with fast decay, published in [15].

Note that with this approach it is possible to prove the existence of G.S. with fast decay also when $\bar{M}_1^-$ and $\bar{M}_2^+$ are satisfied, while the approach of [43], [32] fails in that case. However the latter article is able to deal also with the case $q \neq p^*$. We wish to stress that the condition on the integrability of $k'(r)r^n$ and $k'(r)r^{-n/(p-1)}$ is in some sense optimal, in view of Theorem 8. Moreover observe that we can combine the existence results for G.S. with fast decay given in Theorem 6, 11, 12 with the structure result of Theorem 5 to obtain uniqueness. Furthermore we have the following, see [18], [15].

REMARK 7. Assume that hypotheses $\bar{\mathbf{M}}_\mathbf{1}^+$ and $\bar{\mathbf{M}}_\mathbf{2}^-$ are satisfied. Then there are $B \geq A > 0$ such that $u(d, r)$ is a crossing solution for any $d > B$ and it is a G.S. with slow decay for $0 < d < A$.

Assume that hypotheses $\bar{\mathbf{M}}_\mathbf{1}^-$ and $\bar{\mathbf{M}}_\mathbf{2}^+$ are satisfied. Then there are $B \geq A > 0$ such that $u(d, r)$ is a crossing solution for any $d > B$ and any $0 < d < A$. Moreover there are $R \geq \rho > 0$ such that the Dirichlet problem in the ball of radius r admits 2 solutions for $r > R$ and 0 solutions for $0 < r < \rho$.

Roughly speaking, if $k(r) \in C^1$ is uniformly positive and bounded, admits just one critical point which is a maximum and it is not too flat for r small and r large, regular solutions have structure 3 of Theorem 5 (and positive solutions have structure **C**). But if the critical point is a minimum, the situation is more complicated. We know from Theorem 12 a sufficient condition to have a G.S. with fast decay. However we conjecture, that, in such a case, we may have multiple G.S. with fast decay, perhaps even infinitely many.

Theorem 12 also helps to understand what happens in the perturbative case. When we have a regular perturbation, the stripes E^+ and E^- are very narrow. So, when we approximate the trajectory of the perturbed system with a trajectory of the unperturbed one, we commit a small mistake. In the singular perturbation case we have that ϕ varies slowly, so $\dot{\phi}$ has constant sign in long intervals. Since the trajectory of the stable and unstable sets have an exponential decay, the sign of the energy function H mainly depends on the sign of $\phi(\epsilon t + \tau)x^{p*}(t)$ evaluated when $\mathbf{x}(t)$ is far from the origin. Choose $\mathbf{Q}$ either in $\xi^s(\tau)$ or in $\xi^u(\tau)$. The idea hidden in Theorem 11 is that, playing with the values of the parameters τ and ϵ, we can make the sign of $H_{p*}(\mathbf{Q}, \tau)$ depend just on the sign of $\dot{\phi}$ evaluated at $t = \tau$.

5. f subcritical for u small and supercritical for u large

In this section we collect few results about an equation for which even some basic questions are still unsolved. We consider Eq. (1) where $f(u) = u|u|^{q_1-2} + u|u|^{q_2-2}$, and $p_* < q_1 < p^* < q_2$. In fact as far as we are aware there are only two articles, [11] and [1], concerning the argument and they deal with the case $p = 2$. Recall that $2^* = 2n/(n-2)$ and $2_* = 2(n-1)/(n-2)$.

Zhou in [44] established that G.S. for (2), in this case have to be radial. So we can in fact consider directly an equation of the form (3) (with $p = 2$).

Flores et al. in [11] and [1] use the classical Fowler transformation and change equation (3) into a dynamical system of the form (5). Then they face the problem using dynamical techniques such as invariant manifold theory. They set $l = q_2$ in (4) and obtain a system of the form (5) such that $g_{q_2}(x_{q_2}, t)$ is bounded as $t \to -\infty$, for any fixed x_{q_2}. In fact they consider the 3-dimensional autonomous system obtained from (5) adding the extra variable $z = e^{\xi t}$, where $\xi > 0$. As usual this system admits 3 critical points: the origin **O**, $\mathbf{P}(-\infty) = (P_x(-\infty), P_y(-\infty)$ and $-\mathbf{P}(-\infty)$, where $P_y(-\infty) < 0 < P_x(-\infty)$. In such a case regular solutions of the original problem correspond to trajectories of the $2-$dimensional unstable manifold of the origin, while the singular solutions corresponds to the trajectory whose graph is the $1-$dimensional unstable manifold of $\mathbf{P}(-\infty)$. Then they consider the system obtained from (5) with $l = q_1$, adding the extra variable $z = e^{\xi t}$, where $\xi < 0$, which again have three critical points: **O**, $\mathbf{P}(+\infty) = (P_x(+\infty), P_y(+\infty)$ and $-\mathbf{P}(+\infty)$, where $P_y(+\infty) < 0 < P_x(+\infty)$. In this case **O** admits a 2 dimensional stable manifold whose trajectories correspond to solutions with fast decay of (3), and $\mathbf{P}(+\infty)$ admits a $1-$dimensional stable manifold made up of a trajectory corresponding to a solution with slow decay. Then they use dynamical arguments in order to find intersections between these objects, and this way in [11] they prove the following very interesting results.

THEOREM 13. **a)** *Let $q_2 > 2^*$ be fixed. Then, given an integer $k \geq 1$, there is a number $s_k < 2^*$ such that if $s_k < q_1 < 2^*$, then (2) has at least k radial G.S. with fast decay.*

b) *Let $2_* < q_1 < 2^*$ be fixed. Then, given an integer $k \geq 1$, there is a number $S_k < 2^*$ such that if $2^* < q_2 < S_k$, then (2) has at least k radial G.S. with fast decay.*

They have also found a non-existence counterpart, which shows how sensitive to the variations of the exponents these existence results are.

THEOREM 14. *Let $q_2 > 2^*$ be fixed. Then there is a number $Q > 2_*$ such that if $1 < q_1 < Q$, then (2) admits no G.S neither S.G.S.*

This non-existence result is in some sense optimal. In fact Lin and Ni in [36] have constructed explicitly a G.S. with slow decay of the form $u(r) = A(B+r^2)^{-1/(p-1)}$, where A and B are suitable positive constants, in the special case $q_2 = 2(q_1 - 1) > 2^*$ (note that $2^* = 2(2_* - 1)$). However the existence of G.S. with slow decay probably is not a generic phenomenon. In fact it corresponds to the existence of 1 dimensional intersection of a $2-$dimensional object with a $1-$dimensional object in 3 dimensions.

Finally we have this result concerning S.G.S. and G.S. with slow decay.

THEOREM 15. **a)** *Given $q_2 > 2^*$, there is an increasing sequence of numbers $Q_k \to 2^*$ such that if $q_1 = Q_k$ then there is a radial S.G.S. of (2) with either slow or fast decay.*

b) *Given $2_* < q_1 < 2^*$, there is a decreasing sequence of numbers $S_k \to 2^*$ such that if $q_2 = S_k$ then (2) admits either a radial S.G.S. with slow decay or a radial G.S.*

with slow decay.

Moreover, exploiting the existence of the G.S. with slow decay in the case $q_2 = 2(q_1 - 1) > 2^*$, Flores was able to prove the following result in [1].

THEOREM 16. *Assume that* $2_* < q_1 < 2^* < q_2$ *and* $q_1 > 2\frac{N+2\sqrt{N-1}-2}{N+2\sqrt{N-1}-4}$*. Then, given any integer* $k \geq 1$*, there is a number* $\epsilon_k > 0$ *such that, if* $|q_2 - 2(q_1 - 1)| < \epsilon_k$*, then there are at least* k *radial G.S. with fast decay for (2). In particular if* $q_2 = 2(q_1 - 1)$ *there are infinitely many G.S. with fast decay.*

The condition $q_1 > 2\frac{N+2\sqrt{N-1}-2}{N+2\sqrt{N-1}-4}$ guarantees that $\mathbf{P}(+\infty)$ is a focus and this point is crucial for the proof.

We think that all the Theorems of this section could be generalized to the case $p \neq 2$ using the new change of coordinates (4). Moreover we think that these techniques could be adapted to generalize Theorems 13, 14, 15 also to the spatial dependent case, that is when $f(u,r) = k_1(r)u|u|^{q_1-2} + k_k(r)u|u|^{q_2-2}$, where k_1 and k_2 are actually functions. The last Theorem 16 crucially depends on the existence of the G.S. with slow decay, that seems to be structurally unstable. However we have been able to compute this solution also for the corresponding equation (1). So perhaps also Theorem 16 can be extended to the case $p \neq 2$.

REMARK 8. Consider (1) where $f(u,r) = u|u|^{q_1-1} + u|u|^{q_2-1}$, where $q_2 = \frac{(q_1-1)p}{p-1}$ and $p_* < q_1 < p^* < q_2$. Then there is a radial G.S. with slow decay

$$u(r) = A\left(\frac{1}{B + r^{\frac{p}{p-1}}}\right)^{\frac{p-1}{q_1-p}}$$

where $A = \left[\left|\frac{p}{q_1-p}\right|^{p-1}\left(n - \frac{p(q_1-1)}{q_1-p}\right)\right]^{\frac{1}{q_1-p}}$ and $B = \left(n - \frac{p(q_1-1)}{q_1-p}\right)A^{\frac{q_1-1}{p-1}}$.

6. f negative for u small and positive for u large

In this section we will consider (3), assuming that $f(u,r)$ is negative for u small and positive for u large and r small. The prototypical non-linearity we are interested in is the following

$$f(u,r) = -k_1(r)u|u|^{q_1-2} + k_2(r)u|u|^{q_2-2} \tag{22}$$

where the functions $k_i(r)$ are nonnegative and continuous. When $p = q_1 = 2$ (2) describes a Bose-Einstein condensate, and the G.S., if it exists, is the least energy solution. In order to have G.S. we need to have a balance between the gain of energy due to the negative terms and the loss of energy due to the positive terms. The strength of the contribution is proportional to the corresponding value of $|J_i^+(r)|$, so it depends strongly on the exponent q_i. When f is as in (22) and $q_1 < p^* \leq q_2$ the contribution given by the positive term is not strong enough, while when $q_1 \geq p^*$ the contribution of the positive term is too strong. When $q_1 < q_2 < p^*$ we expect to find a richer scenario, similar to the one depicted in Theorem 5 (3), where G.S. with slow decay are replaced by oscillatory solutions.

Also in this situation, roughly speaking, solutions $u(r)$ which are positive for r large can have two different behaviour: either they converge to 0, usually with fast decay (see Proposition 1 and Corollary 1), or they are uniformly positive, and typically they oscillate indefinitely between two values c_2, c_1 where $0 < c_1 < c_2 < \infty$.

Also in this case radial solutions are particularly important, since in many cases G.S. in the whole $\mathbb{R}^n$, S.G.S. and solutions of the Dirichlet problem in the ball for (1) have to be radial. This fact was proved when $p = 2$ and f is as in (22) and the functions $-k_1(r)$ and $k_2(r)$ are decreasing by Gidas, Ni, Nirenberg in [24], [25] using the moving plane method and the maximum principle. Afterwards this results have been extended to the case $1 < p \leq 2$ in [9], [10], and finally in [42] to the case $p > 1$, and to more general spatial independent nonlinearities $f(u)$: they simply assume that $f(0) = 0$, f is negative in a right neighborhood of $u = 0$ and it is positive for u large.

Once again the Pohozaev identity proves to be an important tool to face the problem of looking for positive solutions. In fact it was used by Ni and Serrin in [38] to construct obstructions for the existence of G.S. in the spatial independent case.

THEOREM 17. *Consider (3) where f has the following form*

$$f(u) = -\sum_{i=1}^{N} k_i u|u|^{q_i-2} + \sum_{i=N+1}^{M} k_i u|u|^{q_i-2} \qquad q_i < q_{i+1} \tag{23}$$

where $k_i > 0$ are constants for any $i = 1, \dots, M$, $q_{N+1} \geq p^*$, $M > N \geq 1$. Then there are no crossing solutions neither G.S.

Note that when $N = 1$ and $M = 2$ (23) reduces to (22). Recall that in such a case all the G.S. of (1) and also all the solutions of the Dirichlet problem in a ball have to be radial. Therefore the non existence result holds globally for the PDE (1).

When $k_i(r)$ behave like powers at $r = 0$ or at $r = \infty$, using the concept of natural dimension explained in the appendix (see [20] and [17]), it is possible to reduce the problem to an equivalent one in which the functions are uniformly positive and bounded either for r small, or for r large, or for both.

In [13] we have discussed a problem similar to the one of Theorem 17, but in the spatial dependent framework, using again dynamical techniques combined with the Pohozaev identity. In [13] we have analyzed functions f of the form (22), but the proofs work also when f is as in (23) and satisfies:

F0 $q_N \leq p^* \leq q_{N+1}$; $k_i(r)$ is a positive, continuous function for $r > 0$, for any $i \leq M$. $J_j^+(t) \leq 0 \leq J_i^+(t)$ for any t and $j \leq N < i$ and $\sum_{i=1}^{M} |J_i^+(t)| \not\equiv 0$.

REMARK 9. Assume that **F0** holds and that there is $s \geq p^*$ such that the limits $\lim_{t\to-\infty} \phi_i(t) e^{\alpha_s(p^*-q_i)t} = A_i \geq 0$ exists and are finite for $i \leq M$, and that

$\sum_{i=N+1}^{M} A_i > 0$. Then there is at least one singular solution $v(r)$ of (3).

THEOREM 18. *Consider (3) where f satisfies **F0**. Moreover assume that there are positive constants C_i and c_j such that $k_i(r) > C_i$ and $k_j(r) < C_j$ for r large and $i \leq N$ and $j > N$. Then all the regular and singular solutions are defined and positive for any $r \geq 0$ and $\limsup_{r\to\infty} u(d,r) > 0$ for any $d > 0$.*

Assume further that $-k_i(r)$ and $k_j(r)$ are decreasing and bounded for r large and $i \leq N$ and $j > N$, then there is is a computable constant b^*, such that all the regular solutions $u(r)$ (and the singular, if they exist) are such that

$$0 < \liminf_{r\to\infty} u(r) \leq \liminf_{r\to\infty} u(r) < b^*$$

REMARK 10. The Hypotheses of Theorem 18 are satisfied for example if we take f as in (22), $q_1 \leq p^* \leq q_2$, $q_1 < q_2$, and the functions $k_i(r)$ uniformly positive, bounded and $-k_1(r)$ and $k_2(r)$ are decreasing.

It is possible to give some ad hoc condition for the existence of G.S. even in the case $q_1 \leq p^* \leq q_2$ and $q_2 > q_1$. In fact we have to lower the contribution given by the negative term $-k_1(r)u|u|^{q_1-2}$, taking a strongly decreasing function $k_1(r)$, see [13]. More precisely

THEOREM 19. *Assume **F0**, and that the limits $\lim_{t\to\infty}\phi_i(t)e^{\alpha_{p^*}(p^*-q_i)t} = B_i \geq 0$ exist and are finite for $i \leq M$, and that $\sum_{i=N+1}^{M} B_i > 0$.*
Then all the regular solutions $u(r)$ are G.S. with slow decay. Finally, if there is a singular solution it is a S.G.S. with slow decay.

REMARK 11. The Hypotheses of Theorem 19 and Remark 9 are satisfied for example if we take f as in (22), $q_1 = p^* < q_2$, $k_1(r)$ uniformly positive, bounded and increasing, $k_2(r) = a + br^{\alpha_{p^*}(q_2-p^*)}$ where $a, b > 0$; or if we take $q_1 < p^* = q_2$, $k_2(r)$ uniformly positive, bounded and increasing, and $k_1(r) = a/(1 + br^{\alpha_{p^*}(p^*-q_1)})$, where $a, b > 0$.

As we said at the beginning of the section, the situation becomes more interesting when f is subcritical both as $u \to 0$ and as $u \to \infty$. A first important step to understand equation (2) in this setting was made in [25], where the authors proved the existence of a G.S. in the case $p = 2$ and assuming that $f(u,r)$ is as in (22), $q_1 = 2 < q_2 < 2^*$ and $-k_1(r)$ and $k_2(r)$ non-increasing. These results have been extended to more general operators, including the p-Laplacian for $p > 1$, in [19] and to a wider class of nonlinearities f. They just require that there is $A > 0$ such that $F(u) < 0$ for $0 < u < A$, $F(A) = 0$ and $f(A) > 0$, where $F(u) := \int_0^u f(s)ds$.

In [19] the non-linearity f is assumed to be spatially independent and sub-halflinear, namely either there is $b > A$ such that $f(b) = 0$, or $\liminf_{u\to\infty} \frac{F(u)}{u^p} < \infty$. If we consider the prototypical case (22) the assumptions of [19] reduce to $1 < q_1 < q_2 < p$, $k_1 \equiv 1 \equiv k_2$. They also proved the uniqueness of the G.S., and they have given good estimates of the asymptotic behaviour. The question of uniqueness has been discussed in many papers, see e.g. [19], [20], [8], but it is beyond the purpose