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ZERO SHEAR VISCOSITY LIMIT FOR THE NAVIER-STOKES EQUATIONS OF COMPRESSIBLE ISENTROPIC FLUIDS WITH CYLINDRIC SYMMETRY[*]

Abstract. We study the problem of the limit process as the shear viscosity goes to zero for global weak solutions to the Navier-Stokes equations of compressible isentropic fluids with cylindric symmetry between two circular cylinders. We prove that the limit of the global weak solutions is a weak solution of the corresponding system with zero shear viscosity.

1. Introduction

We shall study the convergence of solutions of the the Navier-Stokes equations for a compressible isentropic fluid with cylindric symmetry, as the shear viscosity goes to zero. In this paper we restrict ourselves to isentropic flows between two circular coaxial cylinders and assume that the motion of the flows depends only on the radial variable and the time variable. The corresponding symmetric form of the compressible isentropic Navier-Stokes equations, which express the conservation of mass and the balance of momentum, can be written as [35]

$$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0, \tag{1}$$

$$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\rho v^2}{x} + P_x = (\lambda + 2\epsilon)\left(u_x + \frac{u}{x}\right)_x, \tag{2}$$

$$(\rho v)_t + (\rho u v)_x + \frac{2\rho u v}{x} = \epsilon\left(v_x + \frac{v}{x}\right)_x, \tag{3}$$

$$(\rho w)_t + (\rho u w)_x + \frac{\rho u w}{x} = \epsilon\left(w_{xx} + \frac{w_x}{x}\right). \tag{4}$$

Here ρ is the density, u, v and w are the radial, angular and axial components of the velocity vector $\vec{v}$, respectively, x is the radial variable;

$$P \equiv P(\rho) = a\rho^\gamma, \quad \gamma > 1$$

denotes the pressure, and λ, ϵ, a and γ are positive constants which stand for the bulk (expansion) viscosity coefficient, shear viscosity coefficient, gas constant and specific heat ratio, respectively.

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We shall consider the following initial boundary value problem for (1)–(4) in the domain $Q_T := (0,T)\times\Omega$ with $\Omega := \{0 < r_1 < x < r_2 < \infty\}$:

$$u = 0,\quad v = v_i(t),\quad w = w_i(t)\quad \text{for } x = r_i,\quad i = 1,2, \tag{5}$$

$$(\rho, u, v, w)|_{t=0} = (\rho_0, u_0, v_0, w_0). \tag{6}$$

The boundary conditions (5) imply that the fluid sticks at the bounding cylinders which move in such a way that the axis of symmetry is fixed. For simplicity, we take here $v_i(t) = w_i(t) \equiv 0$, since otherwise we can use

$$v - \Big(\frac{r_2 - x}{r_2 - r_1} v_1 + \frac{x - r_1}{r_2 - r_1} v_2\Big) \quad\text{and}\quad w - \Big(\frac{r_2 - x}{r_2 - r_1} w_1 + \frac{x - r_1}{r_2 - r_1} w_2\Big)$$

to replace v and w, respectively, and thus the proof only needs minor modifications.

The asymptotic behavior of viscous flows, as the viscosity vanishes, is one of the important topics in the theory of compressible flows, and the problem of small viscosity finds many applications, for example, in the boundary layer theory [43].

Assuming that

$$\rho_0 \in H^1(\Omega),\quad \inf_{\Omega} \rho_0 > 0,\quad u_0,\ v_0,\ w_0 \in H_0^1(\Omega), \tag{7}$$

Shelukhin studies the zero shear viscosity limit for flows with heat-conducting between two parallel plates [47, 48], while in [49] he investigates the passage to the limit for a free-boundary problem of describing a joint motion of two compressible fluids with different viscosities, as the shear viscosity of one of the fluids vanishes.

In [17] Frid and Shelukhin investigate the cylinder symmetric isentropic problem (1)–(6). For the cylinder symmetric case, the velocity components influence each other through the momentum equations. This is one of the main differences between the systems considered in [47, 48, 17]. Under the conditions (7), Frid and Shelukhin prove that the problem (1)–(6) possesses a unique strong solution $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ satisfying

$$\rho_\epsilon \in L^\infty(0,T;H^1),\quad \partial_t\rho_\epsilon \in L^\infty(0,T;L^2),\quad \inf_{Q_T} \rho_\epsilon > 0,$$
$$(u_\epsilon, v_\epsilon, w_\epsilon) \in L^\infty(0,T;H_0^1)\cap L^2(0,T;H^2),\quad (\partial_t u_\epsilon, \partial_t v_\epsilon, \partial_t w_\epsilon) \in L^2(Q_T).$$

Furthermore, they use the method developed in [47] to obtain the following uniform in ϵ estimates:

$$\int_\Omega \Big[\rho_\epsilon(u_\epsilon^2 + v_\epsilon^2 + w_\epsilon^2) + \rho_\epsilon^\gamma\Big]dx + \int_0^T\int_\Omega \Big[(\lambda + 2\epsilon)(\partial_x u_\epsilon)^2 + \epsilon(\partial_x v_\epsilon)^2 + \epsilon(\partial_x w_\epsilon)^2\Big]dxdt \le C,$$

$$C^{-1} \le \rho_\epsilon \le C,\quad \|(v_\epsilon, w_\epsilon)\|_{L^\infty(Q_T)} \le C,$$

$$\|\partial_x\rho_\epsilon\|_{L^\infty(0,T;L^2)} \le C,\quad \|\partial_t\rho_\epsilon\|_{L^\infty(0,T;L^2)} \le C,$$

$$\|(\partial_t u_\epsilon, \partial_x^2 u_\epsilon)\|_{L^2(Q_T)} + \|\partial_x u_\epsilon\|_{L^\infty(0,T;L^2)} \le C,$$

where C is a positive constant independent of ϵ. Thus, using these uniform estimates, they can prove by the standard compactness imbedding arguments that as $\epsilon \to 0$,

$$(\rho_\epsilon, u_\epsilon) \to (\rho, u) \quad \text{strongly in} \quad C(\bar{Q}_T),$$
$$(v_\epsilon, w_\epsilon) \rightharpoonup (v, w) \quad \text{weak-}* \text{ in } L^\infty(Q_T). \tag{8}$$

Then, with the help of (8), they utilize a framework suitable for transport equations which allows one to improve the weak convergence to the strong one by analyzing and comparing the equations deduced for $\overline{\Phi(z)}$ and $\Phi(z)$, where z is any of the two velocity components v or w, $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function, and $\overline{\Phi(z)}$ denotes the weak limit of $\Phi(z_\epsilon)$ with $z_\epsilon = v_\epsilon$ or $z_\epsilon = w_\epsilon$. This idea of improvement of weak convergence goes back to the notion of the renormalized solutions introduced by Diperna and Lions [11], and applied and further developed in [25, 38, 39, 33, 34, 29, 14, 16, 48, 18, 30, 15, 50] and among others.

The problem of vanishing both shear and bulk (expansion) viscosity coefficients (λ, μ) is much more complex and the situation becomes delicate. It is expected that a general weak entropy solution to the Euler equations should be (strong) limit of solutions to the corresponding compressible Navier-Stokes equations with the same initial data as the viscosity and heat conductivity tend to zero. Indeed, the vanishing viscosity limit for the Cauchy problem for the compressible Navier-Stokes equations has been studied by several researchers.

For the one-dimensional isentropic compressible Navier-Stokes equations, DiPerna [9] uses the method of compensated compactness and establishes a.e. convergence as viscosity goes to zero of admissible solutions of the Navier-Stokes equations to an admissible solution of the corresponding Euler equations, provided that solutions of the Navier-Stokes equations are bounded and the density is bounded away from zero uniformly with respect to the viscosity coefficients. However, this uniform boundedness is difficult to verify in general, and the abstract analysis in [9] gets little information on the qualitative nature of the viscous solutions. In [27] Hoff and Liu investigate the inviscid limit problem in the case that the underlying inviscid flow is a single weak shock wave, and they show that solutions of the compressible Navier-Stokes equations with shock data exist and converge to the inviscid shocks, as viscosity vanishes, uniformly away from the shocks. Based on [19, 27], Xin in [52] shows that the solution to the Cauchy problem of the one-dimensional compressible isentropic Navier-Stokes equations with weak centered rarefaction wave data exists for all time and converges to the weak centered rarefaction wave solution of the corresponding Euler equations, as viscosity tends to zero, uniformly away from the initial discontinuity. Moreover, for a given centered rarefaction wave to the Euler equations with finite strength, he constructs a viscous solution to the compressible Navier-Stokes system with initial data depending on the viscosity, such that the viscous solution approaches the centered rarefaction wave at the rate $1/4$ of viscosity uniformly for all time away from $t = 0$. Recently, Jiang, Ni and Sun [32] extended this result to the non-isentropic case by using some ideas from the stability study of rarefaction waves [40, 41] and the time-decay property of initial discontinuities [24].

In the vanishing viscosity limit, the existence and stability of multidimensional shock fronts for the multidimensional compressible Navier-Stokes equations are proved in [23] and the Prandtl boundary layers (characteristic boundaries) are studied for the linearized case in [53, 54, 51] by using asymptotic analysis, while the boundary layer stability in the case of non-characteristic boundaries and one spatial dimension is discussed in [46, 42]. We also mention that there is an extensive literature on the vanishing artificial viscosity limit for hyperbolic systems of conservation laws, see, for example, [9, 10, 19, 37, 36, 55, 20, 45, 5, 21, 22, 3], also cf. the monographs [4, 8, 44] and the references therein.

Concerning the zero shear viscosity limit for (1)–(6), to our best knowledge, the known results are concerned with strong solutions under the conditions (7). The aim of this paper is to prove a similar vanishing shear viscosity limit result under weaker regularity assumptions on the initial data. Namely, we will study the limit as $\epsilon \to 0$ of (1)–(6) under the following conditions on the initial data:

$$\inf_{\Omega} \rho_0 > 0, \quad \rho_0 \in L^\infty(\Omega), \quad u_0 \in L^2(\Omega), \quad (v_0, w_0) \in L^\infty(\Omega). \tag{9}$$

Under (9), it is not difficult to prove that there exists at least one global weak solution $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ with $\rho_\epsilon > 0$ to the problem (1)–(6) by using arguments similar to those in, for example, [1, 2, 6, 39, 56, 26, 31, 28]. Moreover, $(\rho_\epsilon, u_\epsilon)$ is a renormalized solution of the equation (1), see [39]. On the other hand, for the vanishing shear viscosity limit for the weak solutions here, compared with the strong solutions dealt with in [17], the main difficulty lies in the derivation of the strong convergence of the density ρ_ϵ, due to lack of uniform a priori estimates on derivatives of ρ_ϵ. To overcome such difficulties, we use the techniques in the study of the global existence of weak solutions to the multidimensional compressible Navier-Stokes equations (see, e.g., [38, 14, 16, 29]), and exploit the feature of the equation (2).

Before stating our main result, we introduce the definition of weak solutions.

DEFINITION 1. *(i) We call $(\rho, u, v, w)(x, t)$ a global weak solution of (1)–(6), if for any $T > 0$, $\rho(x, t) \geq 0$ on $[0, T] \times \Omega$, and*

$$\rho, v, w \in L^\infty(Q_T), \quad u, v, w \in L^2(0, T; H_0^1), \quad u \in L^\infty(0, T; L^2),$$

and the following equations hold:

$$\int_0^T \int_\Omega \rho(\varphi_t + u\varphi_x) x\,dx\,dt + \int_\Omega \rho_0 \varphi(x, 0) x\,dx = 0, \tag{10}$$

$$\begin{aligned} \int_0^T \int_\Omega \Big\{ x\rho u \phi_t + x\rho u^2 \phi_x + \rho v^2 \phi + \Big[P(\rho) - (\lambda + 2\epsilon)\Big(u_x + \frac{u}{x}\Big)\Big](x\phi)_x \Big\} dx\,dt& \\ + \int_\Omega x\rho_0 u_0 \phi(x, 0)\,dx = 0,& \end{aligned} \tag{11}$$

$$\begin{aligned} \int_0^T \int_\Omega \Big\{ x\rho v \phi_t + x\rho u v \phi_x - \rho u v \phi - \epsilon\Big(v_x + \frac{v}{x}\Big)(x\phi)_x \Big\} dx\,dt& \\ + \int_\Omega x\rho_0 v_0 \phi(x, 0)\,dx = 0,& \end{aligned} \tag{12}$$

$$\text{(13)} \qquad \int_0^T \int_\Omega \Big\{ x\rho w \phi_t + x\rho u w \phi_x - \epsilon x w_x \phi_x \Big\} dx dt + \int_\Omega x\rho_0 w_0 \phi(x, 0) dx = 0,$$

for any $\varphi, \phi \in C^1(\bar{Q}_T)$, $\phi \in C([0, T], H_0^1)$ *and* $\varphi(\cdot, T) = \phi(\cdot, T) = 0$.

ii) We call $(\rho, u, v, w)(x, t)$ *a global weak solution of (1)–(6) with* $\epsilon = 0$, *if for any* $T > 0$, $\rho(x, t) \geq 0$ *on* $[0, T] \times \Omega$, *and*

$$\rho, v, w \in L^\infty(Q_T), \quad u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1),$$

and ρ, u, v, w *satisfy the equations (10)–(13) with* $\epsilon = 0$.

Thus, the main result of this paper reads:

THEOREM 1. *Assume that the initial data satisfy (9). Then there exists a global weak solution* $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ *of the problem (1)–(6). Moreover, there is a sequence* $\epsilon_n \downarrow 0$, *such that as* $\epsilon_n \to 0$,

$$(\rho_{\epsilon_n}, v_{\epsilon_n}, w_{\epsilon_n}) \to (\rho, v, w) \text{ strongly in } L^p(Q_T), \quad u_{\epsilon_n} \to u \text{ strongly in } L^s(Q_T),$$
$$\partial_x u_{\epsilon_n} \to u_x \text{ strongly in } L^2(Q_T)$$

for any $p \in [1, \infty)$ *and* $s \in [1, 6)$. *In addition, the limit* (ρ, u, v, w) *is a global weak solution of (1)–(6) with* $\epsilon = 0$.

REMARK 1. (i) If $\inf \rho_0 = 0$ and (ρ_0, u_0) satisfies a natural compatibility condition, then we can prove that the problem (1)–(6) has a unique global smooth solution. For the proof, see [7] when $\gamma \geq 2$ and [13] when $1 < \gamma \leq 2$.

(ii) A similar result has been obtained recently for the magnetohydrodynamic equations by Fan [12].

The next section gives the uniform estimates which will be used in the final section to complete the proof of Theorem 1.

As the end of this section, we introduce the notation used throughout this paper. $L^p(I, B)$ respectively $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p = \infty$) functions from I to B respectively its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. $C(I, B - w)$ is the space of all functions which are in $L^\infty(I, B)$ and continuous in t with values in B endowed with the weak topology. We will use the abbreviation:

$$L^q(0, T; W^{m,p}) \equiv L^q(0, T; W^{m,p}(\Omega)),$$
$$\|\cdot\|_{L^q(0,T;W^{m,p})} \equiv \|\cdot\|_{L^q(0,T;W^{m,p}(\Omega))}, \quad \|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\Omega)}.$$

The same letter C will denote various positive constants which do not depend on ϵ.

2. Uniform a priori estimates

We denote the weak solution of (1)–(6) by $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ throughout the rest of this paper. This section is devoted to the derivation of a priori estimates of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$

which are independent of ϵ .

We start with the following conservation identity which is obtained by multiplying (1) by x, integrating the resulting equation over $(0,t)\times\Omega$ and using the boundary conditions (5):

$$\int_\Omega x\rho_\epsilon(x,t)dx = \int_\Omega x\rho_0(x)dx. \tag{14}$$

The following lemma gives an elementary energy estimate which is proved in [17] by multiplying the system (2)–(4) by $(u_\epsilon, v_\epsilon, w_\epsilon)$ in $L^2((0,t)\times\Omega)$ and using (1).

LEMMA 1. *The following energy estimate holds.*

$$\begin{aligned}&\sup_{t\in[0,T]}\int_\Omega\Big[\frac{x}{2}\rho_\epsilon(u_\epsilon^2+v_\epsilon^2+w_\epsilon^2)+\frac{ax}{\gamma-1}\rho_\epsilon^\gamma\Big](x,t)dx\\&\quad+\int_0^T\int_\Omega x\Big[(\lambda+2\epsilon)(\partial_x u_\epsilon)^2+\epsilon(\partial_x v_\epsilon)^2+\epsilon(\partial_x w_\epsilon)^2\\&\quad+(\lambda+2\epsilon)\frac{u_\epsilon^2}{x^2}+\epsilon\frac{v_\epsilon^2}{x^2}\Big]dxdt\le C.\end{aligned}\tag{15}$$

As in [17], we rewrite the equation (2) in the form

$$(\rho_\epsilon u_\epsilon)_t+\Big[\rho_\epsilon u_\epsilon^2+P_\epsilon-(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)+\sigma_\epsilon\Big]_x=0,$$

where

$$\sigma_\epsilon:=\int_{r_1}^x\frac{\rho_\epsilon(u_\epsilon^2-v_\epsilon^2)}{z}dz,\qquad P_\epsilon:=a\rho_\epsilon^\gamma.$$

It is easy to see that by (15),

$$\|\sigma_\epsilon\|_{L^\infty(Q_T)}\le C. \tag{16}$$

Introducing the function

$$\varphi_\epsilon(t,x):=\int_0^t\Big\{(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)-\rho u_\epsilon^2-P_\epsilon-\sigma_\epsilon\Big\}(x,\tau)d\tau+\int_{r_1}^x\rho_0u_0d\xi,$$

one has

$$\partial_x\varphi_\epsilon=\rho_\epsilon u_\epsilon,\quad \partial_t\varphi_\epsilon=(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)-\rho_\epsilon u_\epsilon^2-P_\epsilon-\sigma_\epsilon. \tag{17}$$

Observe that by virtue of (15), (17), the Cauchy-Schwarz inequality and (14),

$$\begin{aligned}\|\partial_x\varphi_\epsilon\|_{L^\infty(0,T;L^1)}&\le\|\rho_\epsilon u_\epsilon\|_{L^\infty(0,T;L^1)}\\&\le C\|\sqrt{x\rho_\epsilon}u_\epsilon\|_{L^\infty(0,T;L^2)}\|\sqrt{x\rho_\epsilon}\|_{L^\infty(0,T;L^2)}\\&\le C\end{aligned}$$

and

$$\sup_{t\in[0,T]}\left|\int_\Omega \varphi(x,t)dx\right| \le C.$$

Hence, the generalized Poincaré inequality implies

$$\|\varphi_\epsilon\|_{L^\infty(Q_T)} \le C. \tag{18}$$

Utilizing the equations (17), and the estimates (16) and (18), following the same arguments as in [17, Lemma 2.2], we obtain the following lemma, the proof of which is therefore omitted.

LEMMA 2. *There are positive constants $\underline{\rho}$, $\overline{\rho}$ independent of ϵ, such that*

$$\underline{\rho} \le \rho_\epsilon(x,t) \le \overline{\rho} \qquad \forall\, x \in \bar{\Omega},\ t \ge 0. \tag{19}$$

As a consequence of Lemmas 1 and 2, one has by the Cauchy-Schwarz inequality that

$$\begin{aligned}\int_0^T \|u_\epsilon\|_{L^6}^6 dt &\le \|u_\epsilon\|^2_{L^\infty(0,T;L^2)}\int_0^T \|u_\epsilon\|^4_{L^\infty}dt\\ &\le C\int_0^T\left(\int_{r_1}^{r_2}|u_\epsilon\partial_x u_\epsilon|d\xi\right)^2 dt\\ &\le C\int_0^T \|u_\epsilon\|^2_{L^2}\|\partial_x u_\epsilon\|^2_{L^2}dt\\ &\le C.\end{aligned} \tag{20}$$

Now, one can apply Lemma 1 and (19) to the parabolic equations (2)–(4) to obtain bounds on the time derivative of $(\rho_\epsilon, \rho_\epsilon\vec{v}_\epsilon)$:

LEMMA 3.

$$\|\partial_t\rho_\epsilon\|_{L^\infty(0,T;H^{-1})} + \|\partial_t(\rho_\epsilon\vec{v}_\epsilon)\|_{L^2(0,T;H^{-1})} \le C, \tag{21}$$

where $\vec{v}_\epsilon = (u_\epsilon, v_\epsilon, w_\epsilon)$.

The following lemma gives us uniform bounds of (w_ϵ, v_ϵ) in L^∞-norm, the proof of which is based on using the properties of transport equations and Lemmas 1 and 2, and can be found in [17].

LEMMA 4.

$$\begin{aligned}\|v_\epsilon\|_{L^\infty(Q_T)} &\le \|v_0\|_{L^\infty(\Omega)}\exp\left(C\int_0^T\|u_\epsilon\|_{L^\infty(\Omega)}dt\right)\\ &\le C\|v_0\|_{L^\infty(\Omega)},\\ \|w_\epsilon\|_{L^\infty(Q_T)} &\le \|w_0\|_{L^\infty(\Omega)}.\end{aligned}$$

3. Proof of Theorem 1

In this section we pass to the limit for $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ as $\epsilon \to 0$ in (1)–(6). First, it is easy to see by the uniform a priori estimates established in the last section and Lemma C.1 in [38] that one can extract a subsequence of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$, still denoted by $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ for simplicity, such that as $\epsilon \to 0$,

(22) $$\rho_\epsilon \rightharpoonup \rho \quad \text{weak-}* \text{ in } L^\infty(Q_T), \quad \underline{\rho} \le \rho(x,t) \le \overline{\rho}, \quad \text{a.e.},$$

(23) $$\rho_\epsilon \to \rho \quad \text{in } C([0,T], L^\gamma(\Omega) - w) \quad \text{for any } \gamma > 1,$$

$$u_\epsilon \rightharpoonup u \quad \text{weak-}* \text{ in } L^\infty(0,T; L^2(\Omega))$$

(24) $$\text{and weakly in } L^2(0,T; H_0^1(\Omega)) \cap L^6(Q_T),$$

(25) $$(v_\epsilon, w_\epsilon) \rightharpoonup (v, w) \text{ weak-}* \text{ in } L^\infty(Q_T),$$

(26) $$(\epsilon\partial_x u_\epsilon, \epsilon\partial_x v_\epsilon, \epsilon\partial_x w_\epsilon) \to (0,0,0) \text{ strongly in } L^2(Q_T),$$

and from (23) and the Sobolev compact imbedding theorem, one gets

(27) $$\rho_\epsilon \to \rho \quad \text{in } C([0,T], H^{-1}(\Omega)).$$

Using (22)–(24), (21) and Lemma 5.1 in [39], we find that

(28) $$\begin{aligned} &\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ weak-}* \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ and} \\ &\qquad \text{weakly in } L^2(0,T; L^p(\Omega)) \text{ for all } p > 1, \end{aligned}$$

and by Lemma C.1 in [38],

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ in } C([0,T], L^2(\Omega) - w),$$

from which and the Sobolev compact imbedding theorem, it follows that

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ in } C([0,T], H^{-1}(\Omega)).$$

Hence, the above weak convergence together with (24) results in

(29) $$\rho_\epsilon u_\epsilon^2 \rightharpoonup \rho u^2 \text{ weakly in } L^2(Q_T).$$

On the other hand, noticing that by virtue of Lemma 1 and the Sobolev imbedding theorem, $\int_0^T \|u_\epsilon(t)\|_{L^\infty}^2 dt \le C$. Therefore,

$$\int_0^T \|u_\epsilon^2(t)\|_{H^1} dt \le C \quad \text{uniformly in } \epsilon,$$

which together with (27) implies

(30) $$\left| \int_0^T \int_\Omega (\rho_\epsilon - \rho) u_\epsilon^2 \phi \, dx dt \right| \to 0 \quad \text{as } \epsilon \to 0, \quad \phi \in C_0^\infty(Q_T).$$

So, recalling the pointwise boundedness of ρ, one immediately gets from (29) and (30) that $u_\epsilon^2 \rightharpoonup u^2$ weakly in $L^2(Q_T)$, which combined with (24) shows that

$$u_\epsilon \to u \text{ strongly in } L^2(Q_T).$$

Therefore, by interpolation and (20), we infer that

$$u_\epsilon \to u \text{ strongly in } L^s(Q_T), \quad \forall\, s < 6. \tag{31}$$

Now, we use and adapt the techniques in [39, 14, 16] (also cf. [29]) to prove the following strong convergence of ρ_ϵ.

LEMMA 5.

$$\rho_\epsilon \to \rho \textit{ strongly in } L^1(Q_T), \quad \textit{as } \epsilon \to 0.$$

Proof. First, multiplying (2) by $\phi \in C_0^\infty(\boldsymbol{\Omega})$ and integrating over (r_1, x), then multiplying the resulting equation by $\psi(t)\rho_\epsilon$, $\psi(t) \in C_0^\infty(0,T)$, and integrating over $(0,T)\times\boldsymbol{\Omega}$, we obtain after a straightforward calculation that

$$\begin{aligned}
&\int_0^T \psi(t)\int_\Omega \rho_\epsilon\left[a\rho_\epsilon^\gamma - (\lambda+2\epsilon)\left(\partial_x u_\epsilon + \frac{u_\epsilon}{x}\right)\right]\phi\, dx dt\\
&= \int_0^T \psi'(t)\int_\Omega \rho_\epsilon \int_{r_1}^x \rho_\epsilon u_\epsilon \phi\, d\xi dx dt - \int_0^T \psi \int_\Omega \frac{1}{x}\rho_\epsilon u_\epsilon \int_{r_1}^x \rho_\epsilon u_\epsilon \phi\, d\xi dx dt\\
&\quad + \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x (a\rho_\epsilon^\gamma + \rho_\epsilon u_\epsilon^2)\phi_\xi\, d\xi dx dt - \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \frac{\rho_\epsilon u_\epsilon^2}{\xi}\phi\, d\xi dx dt\\
&\quad + \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \frac{\rho_\epsilon v_\epsilon^2}{\xi}\phi\, d\xi dx dt\\
&\quad -(\lambda+2\epsilon)\int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \left(\partial_\xi u_\epsilon + \frac{u_\epsilon}{\xi}\right)\phi_\xi\, d\xi dx dt,
\end{aligned} \tag{32}$$

where $H := a\rho_\epsilon^\gamma - (\lambda+2\epsilon)(\partial_x u_\epsilon + u_\epsilon/x)$ is so-called the effective viscous pressure which possesses some smoothing property and plays an important role in the existence proof of global weak solutions to the multidimensional compressible Navier-Stokes equations, cf. [39, 29, 14, 15].

Now, passing to the limit in (32) as $\epsilon \to 0$, and making use of (22)–(29) and (31), we see that

$$\begin{aligned}
&\int_0^T \psi(t)\int_\Omega \left[a\overline{\rho^{\gamma+1}} - \lambda\left(\overline{\rho u_x} + \frac{\rho u}{x}\right)\right]\phi\, dx dt\\
&= \int_0^T \psi'(t)\int_\Omega \rho \int_{r_1}^x \rho u \phi\, d\xi dx dt - \int_0^T \psi \int_\Omega \frac{1}{x}\rho u \int_{r_1}^x \rho u \phi\, d\xi dx dt\\
&\quad + \int_0^T \psi \int_\Omega \rho \int_{r_1}^x (a\overline{\rho^\gamma} + \rho u^2)\phi_\xi\, d\xi dx dt - \int_0^T \psi \int_\Omega \rho \int_{r_1}^x \frac{\rho u^2}{\xi}\phi\, d\xi dx dt\\
&\quad + \int_0^T \psi \int_\Omega \rho \int_{r_1}^x \frac{\overline{\rho v^2}}{\xi}\phi\, d\xi dx dt - \lambda\int_0^T \psi \int_\Omega \rho \int_{r_1}^x \left(u_\xi + \frac{u}{\xi}\right)\phi_\xi\, d\xi dx dt.
\end{aligned} \tag{33}$$

Here and in what follows, $\overline{f(\eta)}$ denotes again the weak limit of $f(\eta_\epsilon)$ as $\epsilon \to 0$.

On the other hand, with the help of (22)–(29) and (31), one has by taking $\epsilon \to 0$ in (1) and (2) that

$$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0, \tag{34}$$

$$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\overline{\rho v^2}}{x} + \left(a\overline{\rho^\gamma}\right)_x - \lambda\left(u_x + \frac{u}{x}\right)_x = 0 \tag{35}$$

in $\mathcal{D}'((0,T)\times\mathbf{\Omega})$, where $\overline{\rho v^2}$ and $\overline{\rho^\gamma}$ denote the weak limits of $\rho_\epsilon v_\epsilon^2$ and ρ_ϵ^γ, respectively.

Now, multiplying (35) by $\phi \in C_0^\infty(\mathbf{\Omega})$, integrating then over (r_1, x), and multiplying the resulting equation by $\psi(t)\rho_\epsilon$, $\psi(t) \in C_0^\infty(0,T)$, and integrating over $(0,T)\times\mathbf{\Omega}$, we deduce by the same arguments as in the derivation of (32) that

$$\begin{aligned}
&\int_0^T \psi(t)\int_\Omega \rho\left[a\overline{\rho^\gamma} - \lambda\left(u_x + \frac{u}{x}\right)\right]\phi dx dt \\
&= \int_0^T \psi'(t)\int_\Omega \rho\int_{r_1}^x \rho u\phi d\xi dx dt - \int_0^T \psi\int_\Omega \frac{1}{x}\rho u\int_{r_1}^x \rho u\phi d\xi dx dt \\
&\quad + \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \left(a\overline{\rho^\gamma} + \rho u^2\right)\phi_\xi d\xi dx dt - \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \frac{\rho u^2}{\xi}\phi d\xi dx dt \\
&\quad + \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \frac{\overline{\rho v^2}}{\xi}\phi d\xi dx dt - \lambda\int_0^T \psi\int_\Omega \rho\int_{r_1}^x \left(u_\xi + \frac{u}{\xi}\right)\phi_\xi d\xi dx dt.
\end{aligned} \tag{36}$$

Comparing the right hand side of (33) with that of (36), we infer that

$$\begin{aligned}
&\int_0^T \psi\int_\Omega \left[a\overline{\rho^{\gamma+1}} - \lambda\left(\overline{\rho u_x} + \frac{\rho u}{x}\right)\right]\phi dx dt \\
&\qquad = \int_0^T \psi\int_\Omega \rho\left[a\overline{\rho^\gamma} - \lambda\left(u_x + \frac{u}{x}\right)\right]\phi dx dt,
\end{aligned} \tag{37}$$

whence

$$a\overline{\rho^{1+\gamma}} - \lambda\overline{\rho u_x} = a\rho\overline{\rho^\gamma} - \lambda\rho u_x. \tag{38}$$

In the sequel, we apply the idea of the renormalized solutions to the equation (1) introduced by DiPerna and Lions [39] to show that ρ satisfies

$$\left(\overline{\rho\log\rho}\right)_t + \left(u\,\overline{\rho\log\rho}\right)_x + \overline{\rho u_x} + \frac{\rho u}{x} + \frac{\overline{\rho\log\rho}}{x}u = 0, \tag{39}$$

$$(\rho\log\rho)_t + (u\,\rho\log\rho)_x + \rho u_x + \frac{\rho u}{x} + \frac{\rho\log\rho}{x}u = 0 \tag{40}$$

in the sense of distributions. In fact, since ρ_ϵ is uniformly bounded in $L^\infty(Q_T)$ and $u \in L^2(0,T;H_0^1(\mathbf{\Omega}))$, we get from Proposition 4.2 in [15] that ρ_ϵ is a renormalized solution of (1), i.e., ρ_ϵ satisfies

$$\partial_t b(\rho_\epsilon) + [b(\rho_\epsilon)u_\epsilon]_x + [b'(\rho_\epsilon)\rho_\epsilon - b(\rho_\epsilon)]\partial_x u_\epsilon + b'(\rho_\epsilon)\frac{\rho_\epsilon u_\epsilon}{x} = 0 \tag{41}$$

for any $b \in C^1(\mathbb{R})$, $b'(z) = 0$ for z large enough. It is not difficult to verify that one can take $b(z) = z \log z$ in (41) by an approximate argument and the uniform a priori estimates established for ρ_ϵ and u_ϵ. Thus, we have

$$(\rho_\epsilon \log \rho_\epsilon)_t + (u_\epsilon \rho_\epsilon \log \rho_\epsilon)_x + \rho_\epsilon \partial_x u_\epsilon + \frac{\rho_\epsilon u_\epsilon}{x} + \frac{u_\epsilon}{x}\rho_\epsilon \log \rho_\epsilon = 0.$$

Letting $\epsilon \to 0$ in the above equation and making use of (22), (28) and (31), we obtain (39) immediately.

Similarly, the limit functions ρ, u are still a renormalized solution to (34). That is, the equation (41) with $(\rho_\epsilon, u_\epsilon)$ replaced by (ρ, u) is still valid, and by an approximation one can take $b(z) = z \log z$, and hence, the equation (40) holds.

Subtraction of (40) from (39) leads to

$$[\overline{\rho \log \rho} - \rho \log \rho]_t + [u(\overline{\rho \log \rho} - \rho \log \rho)]_x + \frac{\overline{\rho \log \rho} - \rho \log \rho}{x} u = \rho u_x - \overline{\rho u_x}. \tag{42}$$

On the other hand, from (38) and the weak lower semicontinuity of convex functions, we find that

$$\overline{\rho u_x} - \rho u_x = \frac{a}{\lambda}\left(\overline{\rho^{1+\gamma}} - \rho\overline{\rho^{\gamma}}\right) \geq 0, \quad \text{a.e.} \tag{43}$$

and

$$\overline{\rho \log \rho} \geq \rho \log \rho, \quad \text{a.e.} \tag{44}$$

As in [14, 16], consider a sequence of functions $\phi_m \in C_0^\infty(\Omega)$, such that

$$\begin{aligned}
&0 \leq \phi_m \leq 1,\ \phi_m(x) = 1 \text{ for all } x \text{ such that } \operatorname{dist}(x, \partial\Omega) \geq m^{-1},\\
&|\partial_x \phi_m(x)| \leq 2m \ \text{ and } \ \operatorname{dist}(x, \partial\Omega)|\partial_x \phi_m(x)| \leq 2 \ \text{ for all } x \in \Omega,\\
&\phi_m(x) \to 1 \text{ as } m \to \infty \text{ for all } x \in \Omega.
\end{aligned}$$

Notice that by virtue of $u \in L^2(0, T; H_0^1)$, $|u|[\operatorname{dist}(x, \partial\Omega)]^{-1} \in L^2(0, T; L^2)$. Therefore, multiplying (42) by $x\phi_m(x)$ and integrating over $(0, t) \times \Omega$, then taking $m \to \infty$, and using (43) and (23), we infer

$$\int_\Omega x(\overline{\rho \log \rho} - \rho \log \rho)(x, t)dx \leq \int_\Omega x(\overline{\rho \log \rho} - \rho \log \rho)(x, 0) = 0, \quad \text{a.e. } t \in [0, T],$$

which combined with (44) gives $\overline{\rho \log \rho} = \rho \log \rho$ a.e. on Q_T. This identity, together with, for example, Theorem 2.11 in [15], yields $\rho_\epsilon \to \rho$ a.e. on Q_T, which combined with the Egorov theorem proves the lemma. □

By Lemma 5 and interpolation, it is easy to see that

(45) $$\rho_\epsilon \to \rho \quad \text{strongly in } L^p(Q_T) \quad \text{for any } 1 \le p < \infty.$$

Now, passing to the limit in (1)–(4) as $\epsilon \to 0$, and utilizing (22)–(29), (31) and (45), we see that the limit functions ρ, u, v, w satisfy the following equations in the sense of distributions:

(46) $$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0,$$

(47) $$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\overline{\rho v^2}}{x} + a(\rho^\gamma)_x = \lambda \left(u_x + \frac{u}{x}\right)_x,$$

(48) $$(\rho v)_t + (\rho u v)_x + \frac{2\rho u v}{x} = 0,$$

(49) $$(\rho w)_t + (\rho u w)_x + \frac{\rho u w}{x} = 0.$$

In the sequel, we prove the strong convergence of $(v_\epsilon, w_\epsilon, \partial_x u_\epsilon)$ to (v, w, u_x) in $L^2(Q_T)$. To this end, we introduce the Lagrangian coordinates (y, t) or (z, t) which are connected to the Eulerian coordinates (x, t) by

$$y \equiv y(x,t) := r_1 + \int_{r_1}^{x} s\rho_\epsilon(s,t)ds$$

and

$$z \equiv z(x,t) := r_1 + \int_{r_1}^{x} s\rho(s,t)ds.$$

Without loss of generality, we may assume that

(50) $$\int_\Omega x\rho_0(x)dx = r_2 - r_1.$$

From (50) and the mass conservation, we find that $y, z \in \Omega$. Since ρ_ϵ and ρ are bounded and strictly away from 0, the mappings (the inverse mappings of $y(x,t)$ and $z(x,t)$) $x(y,t)$ and $x(z,t)$: $\Omega \to \Omega$ are surjective. Moreover,

$$y_t = -x\rho_\epsilon u_\epsilon, \quad y_x = x\rho_\epsilon, \quad dydt = x\rho_\epsilon dxdt.$$

Thus, the equations (1)–(4) in the new variables (y, t) read:

(51) $$(x\rho_\epsilon)_t + x^2\rho_\epsilon^2 \partial_y u_\epsilon = 0,$$

(52) $$\partial_t u_\epsilon - \frac{v_\epsilon^2}{x} = (\lambda + 2\epsilon)x\left(x\rho_\epsilon\partial_y u_\epsilon + \frac{u_\epsilon}{x}\right)_y - x\left[P(\rho_\epsilon)\right]_y,$$

(53) $$\partial_t v_\epsilon + \frac{u_\epsilon v_\epsilon}{x} - \epsilon x\left(x\rho_\epsilon\partial_y v_\epsilon + \frac{v_\epsilon}{x}\right)_y = 0,$$

(54) $$\partial_t w_\epsilon - \epsilon(x^2\rho_\epsilon\partial_y w_\epsilon)_y = 0,$$

(55) $$x_t = u_\epsilon, \quad \rho_\epsilon x x_y = 1,$$

(56) $$(u_\epsilon, v_\epsilon, w_\epsilon)|_{\partial\Omega} = (0,0,0), \quad (\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)|_{t=0} = (\rho_0, u_0, v_0, w_0).$$

First, we show the strong convergence of w_ϵ. Multiplying (54) by $2w_\epsilon\phi$ with $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ and $\phi(\cdot, T) = 0$, then integrating over $(0, T) \times \Omega$, integrating by parts and using the boundary conditions (56), we obtain

$$\int_0^T \int_\Omega w_\epsilon^2 \phi_t dy dt + \int_\Omega w_0^2 \phi(y, 0) dy = 2\epsilon \int_0^T \int_\Omega \Big[x^2 \rho_\epsilon (\partial_y w_\epsilon)^2 \phi + x^2 \rho_\epsilon \partial_y w_\epsilon \phi_y w_\epsilon \Big] dy dt.$$

Transforming this identity into the Eulerian coordinates, we see that

$$\int_0^T \int_\Omega x\rho_\epsilon w_\epsilon^2 (\phi_t + u_\epsilon \phi_x) dx dt + \int_\Omega x\rho_0 w_0^2 \phi(x, 0) dx = 2\epsilon \int_0^T \int_\Omega \Big[x(\partial_x w_\epsilon)^2 \phi + x \partial_x w_\epsilon \phi_x w_\epsilon \Big] dx dt.$$

Letting $\epsilon \to 0$ in the above equation, and using (26), (31) and (45), we find that

$$\int_0^T \int_\Omega x\rho \overline{w^2} (\phi_t + u\phi_x) dx dt + \int_\Omega x\rho_0 w_0^2 \phi(x, 0) dx = 2\langle \overline{\epsilon x w_x^2}, \phi \rangle \geq 0, \tag{57}$$

where $\overline{\epsilon x w_x^2}$, the weak limit of $\epsilon x(\partial_x w_\epsilon)^2$ in the space of signed Radon measures on Q_T, is a nonnegative Radon measure on Q_T. By transforming (57) into the Lagrangian coordinates (z, t), the inequality (57) in the variables (z, t) reads

$$\int_0^T \int_\Omega \overline{w^2} \phi_t dz dt + \int_\Omega w_0^2 \phi(z, 0) dz \geq 0. \tag{58}$$

On the other hand, transforming (49) into the Lagrangian coordinates (calculated in the weak form), testing then the resulting equation with $2w\phi$, we get

$$\int_0^T \int_\Omega w^2 \phi_t dz dt + \int_\Omega w_0^2 \phi(z, 0) dz = 0. \tag{59}$$

Subtracting (59) from (58) and noticing that ϕ_t can be nonpositive and arbitrary, we find that $\overline{w^2}(z, t) \leq w^2(z, t)$, a.e. in Q_T. Hence, $\overline{w^2}(z, t) = w^2(z, t)$ a.e. in Q_T, which implies

$$w_\epsilon \to w \quad \text{strongly in } L^2(Q_T). \tag{60}$$

As a consequence of (60) and (59), we see that the left hand side of (57) in the Lagrangian coordinates (z, t) is equal to zero, therefore,

$$\langle \overline{\epsilon x w_x^2}, \phi \rangle = 0, \qquad \forall\, \phi \in C^\infty(Q_T),\ \phi \geq 0 \text{ in } Q_T,\ \phi(\cdot, T) = 0.$$

Next, we show the strong convergence of v_ϵ in $L^2(Q_T)$ by similar arguments. Multiplying (53) by $2v_\epsilon\phi$ in $L^2(Q_T)$ with $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ in Q_T and $\phi(\cdot, T) = 0$, integrating by parts, and using (55) and (56), we infer

$$\int_0^T \int_\Omega v_\epsilon^2 \phi_t dydt + \int_\Omega v_0^2 \phi(y,0)dy = 2\epsilon \int_0^T \int_\Omega \Big\{ x^2\rho_\epsilon(\partial_y v_\epsilon)^2\phi + 2v_\epsilon \partial_y v_\epsilon \phi$$
$$+ x^2\rho_\epsilon v_\epsilon \partial_y v_\epsilon \phi_y + v_\epsilon^2 \phi_y + \frac{v_\epsilon^2\phi}{x^2\rho_\epsilon}\Big\} dydt + 2\int_0^T \int_\Omega \frac{u_\epsilon v_\epsilon^2}{x}\phi dydt,$$

which, in the Eulerian coordinates, turns out

$$\int_0^T \int_\Omega x\rho_\epsilon v_\epsilon^2(\phi_t + u_\epsilon\phi_x)dxdt + \int_\Omega x\rho_0 v_0^2\phi(x,0)dx = 2\epsilon \int_0^T \int_\Omega \Big\{ x(\partial_x v_\epsilon)^2\phi$$
$$+ 2v_\epsilon \partial_x v_\epsilon \phi + xv_\epsilon \partial_x v_\epsilon \phi_x + v_\epsilon^2\phi_x + \frac{v_\epsilon^2}{x}\phi\Big\} dxdt + 2\int_0^T \int_\Omega \rho_\epsilon u_\epsilon v_\epsilon^2 \phi dxdt.$$

Passing to the limit as $\epsilon \to 0$ in the above equation, similarly to (57), we deduce that

$$\int_0^T \int_\Omega x\rho\overline{v^2}(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 v_0^2 \phi(x,0)dx$$
$$= 2\langle \overline{\epsilon x v_x^2}, \phi\rangle + 2\int_0^T \int_\Omega \rho u\overline{v^2}\phi dxdt$$
$$\geq 2\int_0^T \int_\Omega \rho u \overline{v^2}\phi dxdt, \tag{61}$$

where, $\overline{\epsilon x v_x^2}$, the weak limit of $\epsilon x(\partial_x v_\epsilon)^2$ in the space of signed Radon measures on Q_T, is a nonnegative Radon measure. Transformation of (61) into the Lagrangian coordinates (z,t) results in

$$\int_0^T \int_\Omega \overline{v^2}\phi_t dzdt + \int_\Omega v_0^2\phi(z,0)dz \geq 2\int_0^T \int_\Omega \frac{u\overline{v^2}}{x}\phi dzdt. \tag{62}$$

On the other hand, transforming (48) into the Lagrangian coordinates (z,t), multiplying then the resulting equation by $2v\phi$ in $L^2(Q_T)$, $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ and $\phi(\cdot, T) = 0$, one obtains

$$\int_0^T \int_\Omega v^2\phi_t dzdt + \int_\Omega v_0^2\phi(z,0)dz = 2\int_0^T \int_\Omega \frac{uv^2}{x}\phi dzdt. \tag{63}$$

Subtracting (63) from (62), we have

$$\int_0^T \int_\Omega \left(\overline{v^2} - v^2\right)\left(\phi_t - 2\frac{u}{x}\phi\right) dzdt \geq 0, \quad \forall \phi \in C^\infty(Q_T),\ \phi \geq 0,\ \phi(\cdot, T) = 0,$$

which implies $\overline{v^2} \le v^2$ a.e. in Q_T, since ϕ_t can be nonpositive and arbitrary. Consequently,

$$v_\epsilon \to v \quad \text{strongly in } L^2(Q_T). \tag{64}$$

Hence, it follows from (63) and the form of (61) in Lagrangian coordinates that

$$\langle \overline{\epsilon x v_x^2}, \phi \rangle = 0.$$

Again multiplying (2) by $2xu_\epsilon \phi$ with $\phi \in C^\infty(Q_T)$, $\phi \ge 0$ and $\phi(\cdot, T) = 0$, then integrating over $(0, T) \times \Omega$, integrating by parts and employing the boundary condition for u, one gets after a straightforward calculation that

$$\begin{aligned} &\int_0^T \int_\Omega x\rho_\epsilon u_\epsilon^2(\phi_t + u_\epsilon \phi_x)dxdt + \int_\Omega x\rho_0 u_0^2 \phi(x,0)dx \\ &= 2(\lambda + 2\epsilon) \int_0^T \int_\Omega \Big\{ x(\partial_x u_\epsilon)^2 \phi + 2u_\epsilon \partial_x u_\epsilon \phi + xu_\epsilon \partial_x u_\epsilon \phi_x \\ &\quad + \frac{u_\epsilon^2}{x}\phi + u_\epsilon^2 \phi_x \Big\} dxdt - 2\int_0^T \int_\Omega \Big\{ \rho_\epsilon u_\epsilon v_\epsilon^2 \phi + P(\rho_\epsilon)(xu_\epsilon \phi)_x \Big\} dxdt. \end{aligned} \tag{65}$$

Letting $\epsilon \to 0$ in (65) and making use of (24), (31), (45) and (64), we arrive at

$$\begin{aligned} &\int_0^T \int_\Omega x\rho u^2(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 u_0^2 \phi(x,0)dx \\ &= 2\langle \overline{(\lambda + 2\epsilon)xu_x^2}, \phi \rangle + 2\lambda \int_0^T \int_\Omega \Big\{ 2uu_x \phi + xuu_x \phi_x \\ &\quad + \frac{u^2}{x}\phi + u^2 \phi_x \Big\} dxdt - 2\int_0^T \int_\Omega \Big\{ \rho u v^2 \phi + P(\rho)(xu\phi)_x \Big\} dxdt, \end{aligned} \tag{66}$$

where $\overline{(\lambda + 2\epsilon)xu_x^2}$, the weak limit of $(\lambda + 2\epsilon)x(\partial_x u_\epsilon)^2$, is a nonnegative Radon measure.

Now, multiplying (47) by $2xu\phi$ in $L^2(Q_T)$ with the same ϕ as in (65), and recalling $\overline{v^2} = v^2$, we find, in the same manner as in (65), that

$$\begin{aligned} &\int_0^T \int_\Omega x\rho u^2(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 u_0^2 \phi(x,0)dx \\ &= 2\langle \lambda xu_x^2, \phi \rangle + 2\lambda \int_0^T \int_\Omega \Big\{ 2uu_x \phi + xuu_x \phi_x \\ &\quad + \frac{u^2}{x}\phi + u^2 \phi_x \Big\} dxdt - 2\int_0^T \int_\Omega \Big\{ \rho u v^2 \phi + P(\rho)(xu\phi)_x \Big\} dxdt. \end{aligned} \tag{67}$$

Combining (66) with (67), we conclude

$$\langle \lambda xu_x^2, \phi \rangle = \langle \overline{(\lambda + \epsilon)xu_x^2}, \phi \rangle \ge \langle \overline{\lambda xu_x^2}, \phi \rangle, \quad \forall \phi \in C^\infty(Q_T),\ \phi \ge 0,\ \phi(\cdot, T) = 0,$$

which implies $u_x^2 = \overline{u_x^2}$. Therefore, $\partial_x u_\epsilon \to u_x$ strongly in $L^2(Q_T)$.

Finally, by interpolation, (60), (64) and Lemma 4, we easily conclude $(v_\epsilon, w_\epsilon) \to (v, w)$ in $L^p(Q_T)$ for all $1 \le p < \infty$.

Having had the strong convergence of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon, \partial_x u_\epsilon)$, we easily see, by testing (46)–(49) with C_0^∞-functions and employing a density argument, that the limit functions ρ, u, v, w are indeed a weak solution of the initial boundary value problem (1)–(6) with $\epsilon = 0$ in the sense of Definition 1. This completes the proof of Theorem 1.

References

[1] ANTONTSEV S.N., KAZHIKHOV A.V., MONAKHOV V.N., *Boundary value problem in mechanics of nonhomogeneous fluids*, Elsevier Science Publishers B.V., New York 1990.

[2] AMOSOV A.A. AND ZLOTNIK A.A., *Solvability "in the large" of a system of equations of the one-dimensional motion of an inhomogeneous viscous heat-conducting gas*, Math. Notes **52** (1992), 753–763.

[3] BIANCHINI S. AND BRESSAN A., *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. **161** (2005), 223–342.

[4] BRESSAN A., *BV-solutions to hyperbolic systems by vanishing viscosity*, S.I.S.S.A., Trieste 2000.

[5] BRESSAN A. AND YANG T., *On the convergence rate of vanishing viscosity approximations*, Comm. Pure Appl. Math. **57** (2004), 1075–1109.

[6] CHEN G.Q., HOFF D. AND TRIVISA K., *Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data*, Comm. Partial Diff. Equations **25** (2000), 2233–2257.

[7] CHOE H.J. AND KIM H., *Global existence of the radially symmetric solutions of the Navier-Stokes equations for the isentropic compressible fluids*, Math. Meth. Appl. Sci. **28** (2005), 1–28.

[8] DAFERMOS C.M., *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, Berlin, Heidelberg 2000.

[9] DIPERNA R.J., *Convergence of approximate solutions to conservation laws*, Arch. Rat. Mech. Anal. **83** (1983), 27–70.

[10] DIPERNA R.J., *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys. **91** (1983), 1–30.

[11] DIPERNA R.J. AND LIONS P.-L., *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511–547.

[12] FAN J., *Vanishing shear viscosity in the magnetohydrodynamic equations*, preprint 2005.

[13] FAN J. AND NI G., *Uniform boundedness and global existence of radially symmetric strong solutions to the Navier-Stokes equations for compressible isentropic fluids*, preprint 2005.

[14] FEIREISL E., *On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable*, Comment. Math. Univ. Carlolinae **42** (2001), 83–98.

[15] FEIREISL E., *Dynamics of viscous compressible fluids*, Oxford Univ. Press, Oxford 2004.

[16] FEIREISL E., NOVOTNÝ A. AND PETZELTOVÁ H., *On the existence of globally defined weak solutions for the Navier-Stokes equations of isentropic compressible fluids*, J. Math. Fluid Mech. **3** (2001), 358–392.

[17] FRID H. AND SHELUKHIN V., *Boundary layers for the Navier-Stokes equations of compressible fluids*, Comm. Math. Phys. **208** (1999), 309–330.

[18] FRID H. AND SHELUKHIN V.V., *Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry*, SIAM J. Math. Anal. **31** (2000), 1144–1156.

[19] GOODMAN J. AND XIN Z., *Viscous limits for piecewise smooth solutions to systems of conservation laws*, Arch. Rat. Mech. Anal. **121** (1992), 235–265.

[20] GRENIER E. AND ROUSSET F., *Stability of one-dimensional boundary layers by using Green's functions*, Comm. Pure Appl. Math. **54** (2001), 1343–1385.

[21] GÙES O., MÉTIVIER G., WILLIAMS M. AND ZUMBRUN K., *Multidimensional viscous shocks II: The small viscosity limit*, Comm. Pure Appl. Math. **57** (2004), 141–218.

[22] GÙES O., MÉTIVIER G., WILLIAMS M. AND ZUMBRUN K., *A new approach to the stability of multidimensional viscous shocks*, reprint 2004.

[23] GÙES O., MÉTIVIER G., WILLIAMS M. AND ZUMBRUN K., *Navier-Stokes regularization of multidimensional Euler shocks*, Ann. Sci. Ecole Norm. Sup. **39** (2006), 75–175.

[24] HOFF D., *Global well-posedness of Cauchy problem for Navier-Stokes equations of nonisentropic flow with discontinuous initial data*, J. Diff. Eqs. **95** (1992), 33–74.

[25] HOFF D., *Global solutions of the Navier-Stokes equations for multi-dimensional compressible flow with discontinuous initial data*, J. Diff. Eqs. **120** (1995), 215–254.

[26] HOFF D., *Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states*, Z. Angew. Math. Phys. **49** (1998), 774–785.

[27] HOFF D. AND LIU T.P., *The inviscid limit for Navier-Stokes equations of compressible isentropic flow with shock data*, Indiana Univ. Math. J. **38** (1989), 861–915.

[28] HSIAO L. AND JIANG S., *Nonlinear hyperbolic-parabolic coupled systems*, in: "Handbook of Differential Equations", (Eds. Dafermos C. and Feireisl E.), Elsevier, Amsterdam 2004, 287–384.

[29] JIANG S. AND ZHANG P., *Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations*, Comm. Math. Phys. **215** (2001), 559–581.

[30] JIANG S. AND ZHANG P., *Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids*, J. Math. Pures Appl. **82** (2003), 949–973.

[31] JIANG S. AND ZHANG P., *Global weak solutions to the Navier-Stokes equations for a 1D viscous polytropic ideal gas*, Quart. Appl. Math. **61** (2003), 435–449.

[32] JIANG S., NI G. AND SUN W., *Vanishing viscosity limit to rarefaction waves for the Navier-Stokes equations of one-dimensional compressible heat-conducting fluids*, SIAM J. Math. Anal. **38** (2006), 368–384.

[33] KAZHIKHOV A.V. AND SHELUKHIN V.V., *The verification compactness method*, Novosibirsk: Actual Problems in Modern Math. **2** (1996), 51–60.

[34] KAZHIKHOV A.V. AND WEIGANT V.A., *On global existence of the two-dimensional Navier-Stokes equations of viscous compressible fluid*, Sib. Mat. Zhurn. **36** (1995), 1283–1316 (in Russian).

[35] LANDAU L.D. AND LIFSHITZ E.M., *Fluid mechanics*, Pergamon Press Ltd., 2nd Edition, Oxford 1987.

[36] LIONS P.L., PERTHAME B. AND SOUGANIDIS P.E., *Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Appl. Math. **49** (1996), 599–638.

[37] LIONS P.L., PERTHAME B. AND TADMOR E., *Kinetic formulation of the isentropic gas dynamics and p-system*, Comm. Math. Phys. **163** (1994), 415–431.

[38] LIONS P.-L., *Mathematical topics in fluid dynamics, Vol. 1, Incomrpessible models*, Oxford Science Publication, Oxford 1996.

[39] LIONS P.-L., *Mathematical topics in fluid dynamics, Vol. 2, Comrpessible models*, Oxford Science Publication, Oxford 1998.

[40] LIU T.P. AND XIN Z., *Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations*, Comm. Math. Phys. **118** (1988), 451–465.

[41] NISHIHARA K., YANG T. AND ZHAO H.J., *Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations*, SIAM J. Math. Anal. **35** (2004), 1561–1597.

[42] ROUSSET F., *Stability of small amplitude boundary layers for mixed hyperbolic-parabolic systems*, Trans. Amer. Math. Soc. **355** (2003), 2991–3008.

[43] SCHLICHTING H., *Boundary layer theory*, 7th ed., McGraw-Hill, New York 1979.

[44] SERRE D., *Systems of conservation laws I,II*, Cambridge Univ. Press, Cambridge 2000.

[45] SERRE D., *Sur la stabilité des couches limites de viscosité*, Ann. Inst. Fourier **51** (2001), 109–129.

[46] SERRE D. AND ZUMBRUN K., *Boundary layer stability in real vanishing viscosity limit*, Comm. Math. Phys. **221** (2001), 267–292.

[47] SHELUKHIN V.V., *A shear flow problem for the compressible Navier-Stokes equations*, Int. J. Nonlinear Mechanics **33** (1998), 247–257.

[48] SHELUKHIN V.V., *The limit of zero shear viscosity for compressible fluids*, Arch. Rat. Mech. Anal. **143** (1998), 357–374.

[49] SHELUKHIN V.V., *Vanishing shear viscosity in a free-boundary problem for the equations of compressible fluids*, J. Diff. Eqs. **167** (2000), 73–86.

[50] SUN W., JIANG S. AND GUO Z., *Helically symmetric solutions to the 3-D Navier-Stokes equations for compressible isentropic fluids*, J. Diff. Eqs. **222** (2006), 263–296.

[51] WANG Y. AND XIN Z., *Zero-viscosity limit of the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane*, SIAM J. Math. Anal. **37** (2005), 1256–1298.

[52] XIN Z., *Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations of compressible isentropic gases*, Comm. Pure Appl. Math. **46** (1993), 621–665.

[53] XIN Z., *Viscous boundary layers and their stability. I*, J. Partial Diff. Eqs. **11** (1998), 97–124.

[54] XIN Z. AND YANAGISAWA T., *Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane*, Comm. Pure Appl. Math. **52** (1999), 479–541.

[55] YU S., *Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws*, Arch. Rat. Mech. Anal. **146** (1999), 275–370.

[56] ZLOTNIK A.A. AND AMOSOV A.A., *On stability of generalized solutions to the equations of one-dimensional motion of a viscous heat-conducting gas*, Siberian Math. J. **38** (1997), 663–684.

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