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PERIODIC SOLUTIONS OF FIRST ORDER NONLINEAR DIFFERENCE EQUATIONS

Abstract. This paper surveys some recent results on the existence and multiplicity of periodic solutions of nonlinear difference equations of the first order under Ambrosetti-Prodi or Landesman-Lazer type conditions.

1. Introduction

Periodic solutions of first and second order nonlinear difference equations have been widely studied, and the reader can consult [1, 9] for references. In some recent work with C. Bereanu, we have adapted the topological approach to the upper and lower solutions method to this class of problems and used it, together with Brouwer degree, to obtain new existence and multiplicity results of the Ambrosetti-Prodi and Landesman-Lazer type [2, 3]. In [4], we have used the same methodology to prove similar results for second order nonlinear difference equations with Dirichlet boundary conditions. The present paper surveys some of those results and is restricted, for the sake of simplicity, to the case of periodic solutions of first order difference equations. Some of the arguments of [2, 3] are simplified, and some of the conclusions are sharpened.

2. Periodic solutions

Let $n \geq 2$ be a fixed integer. For $(x_1, \ldots, x_n) \in \mathbb{R}^n$, define the first order difference operator $(Dx_1, \ldots, Dx_{n-1}) \in \mathbb{R}^{n-1}$ by

$$Dx_m := x_{m+1} - x_m \quad (1 \leq m \leq n-1).$$

Let $f_m : \mathbb{R}^n \to \mathbb{R} \quad (1 \leq m \leq n-1)$ be continuous functions. We study the existence of solutions for the periodic boundary value problem

$$(1) \qquad Dx_m + f_m(x_1, \ldots, x_n) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

Let

$$(2) \qquad U^{n-1} = \{x \in \mathbb{R}^n : x_1 = x_n\},$$

so that $U^{n-1} \simeq \mathbb{R}^{n-1}$ because an element of U^{n-1} can be characterized by the coordinates $x_1, \ldots, x_{n-1}$. The restriction $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ of D to $\mathbb{R}^{n-1}$ is given by

$$(3) \qquad (Lx)_m = x_{m+1} - x_m \quad (1 \leq m \leq n-2), \quad (Lx)_{n-1} = x_1 - x_{n-1},$$

or, in matrix form, by the circulant matrix [6]

$$(4)\qquad \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}.$$

If we define $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_m(x_1, \cdots, x_{n-1}) = f_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \le m \le n-1),$$

problem (1) is equivalent to study the zeros of the continuous mapping $H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$(5)\qquad H_m(x) = (Lx)_m + F_m(x) \quad (1 \le m \le n-1).$$

3. Bounded nonlinearities

Let us first consider the linear periodic problem

$$(6)\qquad Dx_m + \alpha x_m = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n,$$

where $\alpha \in \mathbb{R}$. The solutions of the corresponding difference system are given by

$$x_m = (1-\alpha)^{m-1} x_1 \quad (1 \le m \le n),$$

and hence (6) has a solution if and only if

$$x_1 = (1-\alpha)^{n-1} x_1.$$

This immediately implies the following

LEMMA 1. *Problem (6) has only the trivial solution if n is even and $\alpha \neq 0$ or if n is odd and $\alpha \notin \{0, 2\}$. When $\alpha = 0$, the solutions are of the form $x_m = c \quad (1 \le m \le n)$, and when n is odd and $\alpha = 2$, they have the form $x_m = (-1)^{m-1} c \quad (1 \le m \le n)$, with $c \in \mathbb{R}$ arbitrary.*

Let

$$(7)\qquad b_m : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto b_m(x_1, \ldots, x_n) \quad (1 \le m \le n-1)$$

be continuous and bounded, and consider the semilinear periodic problem

$$(8)\qquad Dx_m + \alpha x_m + b_m(x_1, \ldots, x_n) = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n.$$

If L is defined like above and $B : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$B_m(x_1, \ldots, x_{n-1}) = b_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \le m \le n-1),$$

then problem (8) is equivalent to the semilinear problem in $\mathbb{R}^{n-1}$

$$Lx + \alpha x + B(x) = 0. \tag{9}$$

We have the following existence result.

THEOREM 1. *Assume n odd and $\alpha \neq 0$ or n even and $\alpha \notin \{0, 2\}$ and assume that the functions b_m in (7) are continuous and bounded. Then problem (8) has at least one solution and, for all sufficiently large R,*

$$d_B[L + \alpha I + B, B(R), 0] = \pm 1.$$

Proof. Let $M > 0$ is such that $\|B(v)\| \leq M$ for all $v \in \mathbb{R}^{n-1}$. For each $\lambda \in [0, 1]$, each possible zero u of $L + \alpha I + \lambda B$ is such that, using Lemma 1,

$$\|u\| = \lambda\|(L + \alpha I)^{-1}B(u)\| \leq \|(L + \alpha I)^{-1}\|M.$$

Hence, if we take any $R > \|(L + \alpha I)^{-1}\|M$, and denote the Brouwer degree by d_B (see [7]), the homotopy invariance of the degree implies that

$$d_B[L + \alpha I + B, B(R), 0] = d_B[L + \alpha I, B(R), 0] = \pm 1,$$

and the existence follows from the existence property of Brouwer degree [7]. □

4. Upper and lower solutions

Let $f_m : \mathbb{R} \to \mathbb{R}$ $(1 \leq m \leq n - 1)$ be continuous functions, and let us consider the periodic problem

$$Dx_m + f_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n. \tag{10}$$

DEFINITION 1. *$\alpha = (\alpha_1, \dots, \alpha_n)$ (resp. $\beta = (\beta_1, \dots, \beta_n)$) is called a* lower solution *(resp.* upper solution*) for (10) if*

$$\alpha_1 \geq \alpha_n \quad (\textit{resp. } \beta_1 \leq \beta_n),$$

and the inequalities

$$D\alpha_m + f_m(\alpha_m) \geq 0 \quad (\textit{resp. } D\beta_m + f_m(\beta_m) \leq 0) \tag{11}$$

hold for all $1 \leq m \leq n - 1$. Such a lower or upper solution will be called strict *if the inequality (11) is strict for all $1 \leq m \leq n - 1$.*

The basic theorem for the method of upper and lower solutions goes as follows. The proof given here is a simplification of that given in [2], which is modeled on the corresponding one for differential equations in [11].

THEOREM 2. *If (10) has a lower solution* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and an upper solution* $\beta = (\beta_1, \ldots, \beta_n)$ *such that* $\alpha_m \le \beta_m$ $(1 \le m \le n)$, *then (10) has a solution* $x = (x_1, \ldots, x_n)$ *such that* $\alpha_m \le x_m \le \beta_m$ $(1 \le m \le n)$. *Moreover, if* α *and* β *are strict, then* $\alpha_m < x_m < \beta_m$ $(1 \le m \le n-1)$.

Proof. I. A modified problem.
Let $\gamma_m : \mathbb{R} \to \mathbb{R}$ $(1 \le m \le n-1)$ be the continuous functions defined by

$$\gamma_m(x) = \begin{cases} \beta_m & \text{if} \quad x > \beta_m \\ x & \text{if} \quad \alpha_m \le x \le \beta_m \\ \alpha_m & \text{if} \quad x < \alpha_m. \end{cases} \tag{12}$$

We consider the modified problem

$$Dx_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n, \tag{13}$$

and show that if $x = (x_1, \ldots, x_n)$ is a solution of (13) then $\alpha_m \le x_m \le \beta_m$ $(1 \le m \le n)$, and hence x is a solution of (10). Suppose by contradiction that there is some $1 \le i \le n$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max_{1 \le j \le n}(\alpha_j - x_j) > 0$. If $1 \le m \le n-1$, then

$$\alpha_{m+1} - x_{m+1} \le \alpha_m - x_m,$$

which gives

$$D\alpha_m \le Dx_m = x_m - \alpha_m - f_m(\alpha_m) \le x_m - \alpha_m + D\alpha_m < D\alpha_m,$$

a contradiction. Now the condition $\alpha_1 \ge \alpha_n$ shows that the maximum is reached at $m = n$ only if it is reached also at $m = 1$, a case already excluded. Analogously we can show that $x_m \le \beta_m$ $(1 \le m \le n)$. We remark that if α, β are strict, the same reasoning gives $\alpha_m < x_m < \beta_m$ $(1 \le m \le n-1)$.
II. Solution of the modified problem.
We use Brouwer degree to study the zeros of the continuous mapping $G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$G_m(x) = (Lx)_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) \quad (1 \le m \le n-1). \tag{14}$$

By Lemma 1, $L - I : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is invertible. On the other hand the mapping with components $f_m \circ \gamma_m + \gamma_m$ $(1 \le m \le n-1)$ is bounded on $\mathbb{R}^{n-1}$. Consequently, Theorem 1 implies the existence of $R > 0$ such that, for all $\rho > R$, one has

$$|d_B[G, B(\rho), 0]| = 1, \tag{15}$$

and, in particular, G has a zero $\widetilde{x} \in B(\rho)$. Hence, $x = (\widetilde{x}, x_1)$ is a solution of (13), which means that $\alpha_m \le x_m \le \beta_m$ $(1 \le m \le n)$ and x is a solution of (10). Moreover if α, β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \le m \le n-1)$. $\square$

Suppose now that α (resp. β) is a strict lower (resp. upper) solution of (10). Define the open set

$$\Omega_{\alpha\beta} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \alpha_m < x_m < \beta_m \quad (1 \leq m \leq n-1)\}, \tag{16}$$

and the continuous mapping $H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$H_m(x) = (Lx)_m + f_m(x_m) \quad (1 \leq m \leq n-1). \tag{17}$$

COROLLARY 1. *Assume that the conditions of Theorem 2 hold with strict lower and upper solutions. Then*

$$|d_B[H, \Omega_{\alpha\beta}, 0]| = 1, \tag{18}$$

with $\Omega_{\alpha\beta}$ defined in (16).

Proof. If ρ is large enough, then, using the additivity-excision property of Brouwer degree [7], we have

$$|d_B[G, \Omega_{\alpha\beta}, 0]| = |d_B[G, B(\rho), 0]| = 1.$$

On the other hand, H is equal to G on $\overline{\Omega_{\alpha\beta}}$, and then

$$|d_B[G, \Omega_{\alpha\beta}, 0]| = |d_B[H, \Omega_{\alpha\beta}, 0]|.$$

□

A simple but useful consequence of Theorem 2, goes as follows.

COROLLARY 2. *Assume that there exists numbers $\alpha \leq \beta$ such that*

$$f_m(\alpha) \geq 0 \geq f_m(\beta) \quad (1 \leq m \leq n-1).$$

Then problem (10) has at least one solution with $\alpha \leq x_m \leq \beta$ $(1 \leq m \leq n-1)$.

Proof. Just observe that $(\alpha, \ldots, \alpha)$ is a lower solution and $(\beta, \ldots, \beta)$ an upper solution for (10). □

COROLLARY 3. *For each $p > 0$, $a_m > 0$ and $b_m \in \mathbb{R}$ $(1 \leq m \leq n-1)$ the problem*

$$Dx_m - a_m|x_m|^{p-1}x_m = b_m \quad (1 \leq m \leq n-1), \quad x_1 = x_n$$

has at least one solution.

Proof. If $R \geq \left(\max_{1 \leq m \leq n-1} \frac{|b_m|}{a_m}\right)^{1/p}$, then $(-R, \ldots, -R)$ is a lower solution and $(R, \ldots, R)$ an upper solution. □

REMARK 1. *When* $\beta_m \leq \alpha_m$ $(1 \leq m \leq n-1)$, *one can try to repeat the argument of Theorem 2 by defining*

$$\delta_m(x) = \begin{cases} \alpha_m & \text{if} \quad x > \alpha_m \\ x & \text{if} \quad \beta_m \leq x \leq \alpha_m \\ \beta_m & \text{if} \quad x < \beta_m, \end{cases}$$

and considering the modified problem

$$(19) \qquad Dx_m + x_m + f_m \circ \delta_m(x_m) - \delta_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

As $L + I$ *is invertible, the degree argument still gives the existence of at least one solution for (19). If one tries to show that, say,* $\beta_m \leq x_m$ $(1 \leq m \leq n-1)$, *and assume by contradiction that* $\beta_i - x_i > 0$ *for some* $1 \leq i \leq n$, *one gets no contradiction with* $D\beta_m + f_m(\beta_m) \leq 0$. *This is in contrast with the ordinary differential equation case, for which the argument works independently of their order [13]. The reason of this difference comes from the fact that a local extremum is characterized by an equality (vanishing of the first derivative) in the differential case and by two inequalities (with only one usable in the argument) in the difference case. This raises the question of the validity of the method of upper and lower solutions with reversed upper and lower solutions in the difference case. This question is solved by the negative in the next two sections.*

5. Spectrum of the linear part

The construction of the counter-example proving the last assertion above is clarified by analyzing the spectral properties of the first order difference operator with periodic boundary conditions.

DEFINITION 2. *An* eigenvalue *of the first order difference operator with periodic boundary conditions is any* $\lambda \in \mathbb{C}$ *such that the problem*

$$(20) \qquad Dx_m = \lambda x_m \quad (1 \leq m \leq n-1), \quad x_1 = x_n$$

has a nontrivial solution.

Explicitly, system (20) can be written as

$$(21) \qquad \begin{aligned} x_1 - x_n &= 0 \\ x_2 - (1+\lambda)x_1 &= 0 \\ \cdots \ \cdots \ \cdots \ \cdots & \ \ \cdots \\ x_n - (1+\lambda)x_{n-1} &= 0 \end{aligned}$$

and is equivalent to the matrix eigenvalue problem

$$
(22) \qquad \begin{pmatrix} -1-\lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1-\lambda & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -1-\lambda & 1 \\ 1 & 0 & \cdots & \cdots & 0 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = 0.
$$

Hence the eigenvalues λ_k are $\lambda_k = -1 + \mu_k$ $(0 \le k \le n-2)$, where the μ_k are the eigenvalues of the (permutation, unitary, circulant) matrix

$$
\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},
$$

namely (see e.g. [6]),

$$
(23) \qquad \lambda_k = -1 + e^{\frac{2k\pi i}{n-1}} \quad (0 \le k \le n-2).
$$

The corresponding eigenvectors φ^k $(0 \le k \le n-2)$ have components

$$
\varphi_m^k = e^{\frac{2km\pi i}{n-1}} \quad (1 \le m \le n-1).
$$

In particular, $\lambda_0 = 0$ is always a real eigenvalue, and all the other eigenvalues have negative real part. If $n = 2$, 0 is the unique eigenvalue; if $n > 2$ is even, 0 is the unique real eigenvalue; if n is odd, $\lambda_{\frac{n-1}{2}} = -2$ is the unique nonzero real eigenvalue.

6. Reversing the order of upper and lower solutions

For $n \ge 2$ odd and $\lambda = -2$, system (21) becomes

$$
(24) \qquad \begin{aligned} x_1 - x_n &= 0 \\ x_2 + x_1 &= 0 \\ \cdots \ \cdots & \ \cdots \\ x_n + x_{n-1} &= 0 \end{aligned}
$$

and has the solution φ associated to $\varphi^{(n-1)/2}$ with components

$$
\varphi_m = (-1)^{m-1} \quad (1 \le m \le n).
$$

The adjoint system

$$
(25) \qquad \begin{aligned} x_1 + x_2 &= 0 \\ \cdots \ \cdots & \ \cdots \\ x_{n-1} + x_n &= 0 \\ -x_1 + x_n &= 0 \end{aligned}
$$

has the same nontrivial solution φ. As $b_m = \delta_{nm}$ $(1 \leq m \leq n)$ (Kronecker symbol) is not orthogonal to the kernel of the adjoint system (25), the problem

$$\begin{aligned} x_1 - x_n &= 0 \\ x_2 + x_1 &= 0 \\ \ldots \quad \ldots & \quad \ldots \\ x_{n-1} + x_{n-2} &= 0 \\ x_n + x_{n-1} &= 1 \end{aligned}$$

has no solution, or, equivalently *the problem*

$$(26) \quad Dx_m + 2x_m = 0 \quad (1 \leq m \leq n-2), \quad Dx_{n-1} + 2x_{n-1} = 1, \quad x_1 = x_n$$

has no solution. However, $\alpha = (1, \ldots, 1)$ *is a lower solution and* $\beta = (0, \ldots, 0)$ *is an upper solution of (26) such that* $\beta_m \leq \alpha_m$ $(1 \leq m \leq n)$.

If now $n > 2$ is even, the problem

$$(27) \quad \begin{aligned} Dx_m + 2x_m &= 0 \quad (1 \leq m \leq n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \\ Dx_{n-1} &= 0, \quad x_1 = x_n \end{aligned}$$

is of course equivalent to the problem

$$Dx_m + 2x_m = 0 \quad (1 \leq m \leq n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \quad x_1 = x_{n-1}.$$

As $n-1$ is odd, it follows from the counter-example (26) that *problem (27) has no solution. However* $\alpha = (1, \ldots, 1)$ *is a lower solution and* $\beta = (0, \ldots, 0)$ *is an upper solution of (27) such that* $\beta_m \leq \alpha_m$ $(1 \leq m \leq n)$. Those counter-examples were first given in [3].

For $n = 2$, problem (10) is equivalent to the unique scalar equation

$$f_1(x_1) = 0$$

and, in this case, the validity of the method of upper and lower solutions, independently of their order, follows from its equivalence with Bolzano's theorem applied to the real function f_1.

REMARK 2. *Notice that, in contrast to the periodic problem for difference equations, whose eigenvalues are in the left half-plane, all the eigenvalues* $\lambda_k = \frac{2k\pi i}{T}$ $(k \in \mathbb{Z})$ *of the differential operator* $\frac{d}{dt}$ *with periodic boundary conditions on* $[0, T]$ *are on the imaginary axis. This explains that the method of upper and lower solutions works irrespectively to the order of the lower and the upper solution.*

7. Ambrosetti-Prodi type multiplicity result

Let $f_1, \ldots, f_{n-1} : \mathbb{R} \to \mathbb{R}$ be continuous functions, $s \in \mathbb{R}$. Consider the problem, with $n \geq 2$,

$$(28) \quad Dx_m + f_m(x_m) = s \quad (1 \leq m \leq n-1), \quad x_1 = x_n,$$

with the *coercivity condition*

$$f_m(u) \to \infty \quad \text{as} \quad |u| \to \infty \quad (1 \le m \le n-1). \tag{29}$$

When $n = 2$, problem (28) is equivalent to the scalar equation

$$f_1(x_1) = s \tag{30}$$

and, under condition (29) with $m = 1$, it is clear that there exists s_1 $(= \min_{\mathbb{R}} f_1)$ such that for $s < s_1$, equation (30) has no solution, for $s = s_1$, equation (30) has at least one solution, and for $s > s_1$, equation (30) has at least two solutions. We show that a similar result holds for any $n \ge 2$. Problems of this type were initiated by Ambrosetti-Prodi for second order semilinear Dirichlet problems and the approach given here slightly simplifies the one given in [2], modeled on the method introduced in [12, 13] for periodic solutions of first and second order ordinary differential equations.

LEMMA 2. *If condition (29) holds, then*

$$\sum_{m=1}^{n-1} f_m(x_m) \to +\infty \quad \textit{if} \quad \|x\| \to \infty.$$

Proof. From (29),

$$(\exists\, c \in \mathbb{R})(\forall\, u \in \mathbb{R})(\forall\, m \in \{1, \ldots, n-1\}) : f_m(u) \ge c, \tag{31}$$

and

$$\begin{aligned} &(\forall\, R > 0)(\exists\, r' > 0)(\forall u \in \mathbb{R} : |u| \ge r')(\forall\, m \in \{1, \ldots, n-1\}) : \\ &\qquad\qquad f_m(u) \ge R - (n-2)c. \end{aligned} \tag{32}$$

If $r = \sqrt{n-1}r'$ and if $x \in \mathbb{R}^{m-1}$ is such that $\|x\| \ge r$, then, for at least one $j \in \{1, \ldots, n-1\}$, one has $|x_j| \ge r'$, so that, using (31) and (32),

$$\begin{aligned} \sum_{m=1}^{n-1} f_m(x_m) &= \sum_{m=1}^{n-1} [\, f_m(x_m) - c] + (n-1)c \ge f_j(x_j) - c + (n-1)c \\ &\ge R - (n-2)c - c + (n-1)c = R. \end{aligned}$$

Consequently, $\sum_{m=1}^{n-1} f_m(x_m) \to +\infty$ if $\|x\| \to \infty$. □

LEMMA 3. *Let $b \in \mathbb{R}$. If condition (29) holds, there is $\rho = \rho(b) > 0$ such that each possible solution x of (28) with $s \le b$ is such that $\|x\| < \rho$.*

Proof. Let $s \le b$ and $(x_1, \ldots, x_n)$ be a solution of (28). We see that

$$\sum_{m=1}^{n-1} f_m(x_m) = (n-1)s \le (n-1)b. \tag{33}$$

and the result follows from Lemma 2. □

THEOREM 3. *If the functions* f_m $(1 \le m \le n-1)$ *satisfy (29), there exists* $s_1 \in \mathbb{R}$ *such that* (28) *has zero, at least one or at least two solutions according to* $s < s_1, s = s_1, s > s_1$.

Proof. Let

$$S_j = \{s \in \mathbb{R} : \text{(28) has at least j solutions } \} \quad (j \ge 1).$$

(a) $S_1 \neq \emptyset$.
Take $s^* > \max_{1 \le m \le n-1} f_m(0)$ and use (29) to find $R_-^* < 0$ such that

$$\min_{1 \le m \le n-1} f_m(R_-^*) > s^*.$$

Then α with $\alpha_j = R_-^* < 0$ $(1 \le j \le n)$ is a strict lower solution and β with $\beta_j = 0$ $(1 \le j \le n)$ is a strict upper solution for (28) with $s = s^*$. Hence, using Theorem 2, $s^* \in S_1$.
(b) If $\widetilde{s} \in S_1$ *and* $s > \widetilde{s}$ *then* $s \in S_1$.
Let $\widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_n)$ be a solution of (28) with $s = \widetilde{s}$, and let $s > \widetilde{s}$. Then $\widetilde{x}$ is a strict upper solution for (28). Take now $R_- < \min_{1 \le m \le n} \widetilde{x}_m$ such that $\min_{1 \le m \le n-1} f_m(R_-)$ $> s$ It follows that α with $\alpha_j = R_-$ $(1 \le j \le n)$ is a strict lower solution for (28), and hence, using Theorem 2, $s \in S_1$.
(c) $s_1 = \inf S_1$ *is finite and* $S_1 \supset]s_1, \infty[$.
Let $s \in \mathbb{R}$ and suppose that (28) has a solution $(x_1, \dots, x_n)$. Then (33) holds, from where we deduce that $s \ge c$, with $c \in \mathbb{R}$ given in (31). To obtain the second part of claim (c) $S_1 \supset]s_1, \infty[$ we apply (b).
(d) $S_2 \supset]s_1, \infty[$.
We reformulate (28) to apply Brouwer degree theory. Consider the continuous mapping $\mathcal{G} : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$\mathcal{G}_m(s, x) = (Lx)_m + f_m(x_m) - s \quad (1 \le m \le n-1).$$

Then $(x_1, \dots, x_{n-1}, x_1)$ is a solution of (28) if and only if $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ is a zero of $\mathcal{G}(s, \cdot)$. Let $s_3 < s_1 < s_2$. Using Lemma 3 we find $\rho > 0$ such that each possible zero of $\mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $\max_{1 \le m \le n-1} |x_m| < \rho$. Consequently, $d_B[\mathcal{G}(s, \cdot), B(\rho), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $\mathcal{G}(s_3, x) \neq 0$ for all $x \in \mathbb{R}^{n-1}$. This implies that $d_B[\mathcal{G}(s_3, \cdot), B(\rho), 0] = 0$, so that $d_B[\mathcal{G}(s_2, \cdot), B(\rho), 0] = 0$ and, by excision property, $d_B[\mathcal{G}(s_2, \cdot), B(\rho'), 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\widehat{x} = (\widehat{x}_1, \dots, \widehat{x}_n)$ be a solution of (28) (using (c)). Then $\widehat{x}$ is a strict upper solution of (28) with $s = s_2$. Let $R < \min_{1 \le j \le n} \widehat{x}_j$ be such that $\min_{1 \le m \le n-1} f_m(R) > s_2$. Then $(R, \dots, R) \in \mathbb{R}^n$ is a strict lower solution of (28) with $s = s_2$. Consequently, using Corollary 1, (28) with $s = s_2$ has a solution in $\Omega_{R\widehat{x}}$ and

$$|d_B[\mathcal{G}(s_2, \cdot), \Omega_{R\widehat{x}}, 0]| = 1.$$

Taking ρ' sufficiently large, we deduce from the additivity property of Brouwer degree that

$$\begin{aligned} |d_B[\mathcal{G}(s_2,\cdot), B(\rho')\setminus\overline{\Omega}_{R\widehat{x}},0]| &= |d_B[\mathcal{G}(s_2,\cdot), B(\rho'),0] - d_B[\mathcal{G}(s_2,\cdot),\Omega_{R\widehat{x}},0]| \\ &= |d_B[\mathcal{G}(s_2,\cdot),\Omega_{R\widehat{x}},0]| = 1, \end{aligned}$$

and (28) with $s = s_2$ has a second solution in $B(\rho')\setminus\overline{\Omega}_{R\widehat{x}}$.
(e) $s_1 \in S_1$.
Taking a decreasing sequence $(\sigma_k)_{k\in\mathbb{N}}$ in $]s_1,\infty[$ converging to s_1, a corresponding sequence $(x_1^k,\dots,x_n^k)$ of solutions of (28) with $s=\sigma_k$ and using Lemma 3, we obtain a subsequence $(x_1^{j_k},\dots,x_n^{j_k})$ which converges to a solution $(x_1,\dots,x_n)$ of (28) with $s = s_1$. □

COROLLARY 4. *If $p > 0$, $a_m > 0$ and $b_m \in \mathbb{R}$ $(1 \le m \le n-1)$, there exists $s_1 \in \mathbb{R}$ such that the periodic problem*

$$Dx_m + a_m|x_m|^p = s + b_m \quad (1\le m\le n-1), \quad x_1 = x_n$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

Similar arguments allow to prove the following result.

THEOREM 4. *If the functions f_m satisfy condition*

(34) $$f_m(x) \to -\infty \quad as \quad |x|\to\infty \quad (1\le m\le n-1).$$

then there is $s_1 \in \mathbb{R}$ such that (28) has zero, at least one or at least two solutions according to $s > s_1$, $s = s_1$ or $s < s_1$.

COROLLARY 5. *If $p > 0$, $a_m > 0$ and $b_m \in \mathbb{R}$ $(1 \le m \le n-1)$, there exists $s_1 \in \mathbb{R}$ such that the periodic problem*

$$Dx_m - a_m|x_m|^p = s + b_m \quad (1\le m\le n-1), \quad x_1 = x_n$$

has no solution if $s > s_1$, at least one solution if $s = s_1$ and at least two solutions if $s < s_1$.

8. One-side bounded nonlinearities

The nonlinearity in Ambrosetti-Prodi type problems is bounded from below and coercive or bounded from above and anticoercive. In this section, we consider nonlinearities which are bounded from below or above but have different limits at $+\infty$ and $-\infty$.

Let $n \ge 2$ be an integer and $f_m : \mathbb{R}\to\mathbb{R}$ continuous functions $(1\le m\le n-1)$. Consider the problem

(35) $$Dx_m + f_m(x_m) = 0 \quad (1\le m\le n-1), \quad x_1 = x_n.$$

It is easy to check that the linear mapping L defined in (3) is such that

$$\begin{aligned} N(L) &= \{(c,\dots,c)\in\mathbb{R}^{n-1} : c\in\mathbb{R}\},\\ R(L) &= \{(y_1,\dots,y_{n-1})\in\mathbb{R}^{n-1} : \sum_{m=1}^{n-1} y_m = 0\}. \end{aligned}$$

The projector $P:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$

$$\begin{aligned} P(x_1,\dots,x_{n-1}) &= \left(\frac{1}{n-1}\sum_{m=1}^{n-1} x_m,\dots,\frac{1}{n-1}\sum_{m=1}^{n-1} x_m\right)\\ &= \left(\frac{1}{n-1}\sum_{m=1}^{n-1} x_m\right)(1,\dots,1) \end{aligned}$$

is such that $N(P)=R(L)$, $R(P)=N(L)$. Let us finally define $F:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ by

$$F(x_1,\dots,x_{n-1}) = (f_1(x_1),\dots,f_{n-1}(x_{n-1})),$$

and let $H=L+F$, so that the solutions of (35) correspond to the zeros of H. To study them using Brouwer degree, we introduce, like in Theorem IV.13 of [10] the family of equations

(36) $$Lx + (1-\lambda)PF(x) + \lambda F(x) = 0, \quad \lambda\in[0,1].$$

LEMMA 4. *For each* $\lambda\in\,]0,1]$, *equation (36) is equivalent to equation*

(37) $$Lx + \lambda F(x) = 0.$$

For $\lambda = 0$, *equation (36) is equivalent to equation*

(38) $$PF(x) = 0, \quad x\in N(L).$$

Proof. We first notice that, applying P to both members of equation (36), we get

$$PF(x) = 0$$

and hence, for $\lambda\in\,]0,1]$, equation (36) implies equation (37), and, for $\lambda=0$, implies equation (38). Conversely, if equation (37) holds and $\lambda\in\,]0,1]$, then, applying P to both members, we get $PF(x)=0$ and we may add $(1-\lambda)PF(x)$ to the left-hand member to obtain (36). If equation (38) holds, then

$$PF(x) = 0, \quad Lx = 0,$$

and hence (36) with $\lambda = 0$ follows by addition. □

The following Lemma, taken from [2], adapts to difference equations an argument of Ward [14] for ordinary differential equations.

LEMMA 5. *If the functions* f_m $(1 \le m \le n-1)$, *are all bounded from below or all bounded from above, say by* c, *and if for some* $R > 0$

$$\sum_{m=1}^{n-1} f_m(x_m) \neq 0 \quad \textit{whenever} \quad \min_{1\le j\le n-1} x_j \ge R \quad \textit{or} \quad \max_{1\le j\le n-1} x_j \le -R, \tag{39}$$

then, for each $\lambda \in]0,1]$ *each possible zero* x *of* $L + \lambda F$ *is such that*

$$\max_{1\le j\le n-1} |x_j| < R + 2(n-1)|c|. \tag{40}$$

Proof. Let $(\lambda, x) \in]0,1] \times \mathbb{R}^{n-1}$ be a possible zero of $L + \lambda N$. It is a solution of the equivalent system

$$\sum_{m=1}^{n-1} f_m(x_m) = 0, \quad Dx_m + \lambda f_m(x_m) = 0, \quad x_1 = x_n, \quad (1 \le m \le n-1). \tag{41}$$

On the other hand, if we assume, say, that each f_m $(1 \le m \le n-1)$ is bounded from below, say by c, we have, for all $1 \le m \le n-1$, and all $u \in \mathbb{R}$,

$$|f_m(u)| - |c| \le |f_m(u) - c| = f_m(u) - c,$$

and hence

$$|f_m(u)| \le f_m(u) + 2|c|. \tag{42}$$

Consequently, using (41) and (42), we obtain

$$\begin{aligned} \sum_{m=1}^{n-1} |Dx_m| &= \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \le \sum_{m=1}^{n-1} |f_m(x_m)| \\ &\le \sum_{m=1}^{n-1} f_m(x_m) + 2(n-1)|c| = 2(n-1)|c|. \end{aligned} \tag{43}$$

We deduce

$$\begin{aligned} \max_{1\le m\le n-1} x_m &\le \min_{1\le m\le n-1} x_m + \sum_{m=1}^{n-1} |Dx_m| \\ &\le \min_{1\le m\le n-1} x_m + 2(n-1)|c|. \end{aligned} \tag{44}$$

Using (41) and assumption (39), we obtain $\min_{1\le m\le n-1} x_m < R$ and $-R < \max_{1\le m\le n-1} x_m$. Combined with (44), this gives

$$-[R + 2(n-1)|c|] < \min_{1\le m\le n-1} x_m \le \max_{1\le m\le n-1} x_m < R + 2(n-1)|c|.$$

If the f_m are bounded from above, it suffices to consider the equivalent problem $-Lx - F(x) = 0$ with all function $-f_m$ bounded from below, as $-L$ has the same null-space and range as L. □

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{n-1}\left(\sum_{m=1}^{n-1} f_m(u)\right),$$

so that, for $u(1,\dots,1) \in N(L)$,

$$PF(u(1,\dots,1)) = \varphi(u)(1,\dots,1).$$

The following theorem slightly sharpens a result of [2].

THEOREM 5. *Suppose that the functions* f_m $(1 \le m \le n-1)$ *are all bounded from below or all bounded from above, and that for some* $R > 0$ *and* $\epsilon \in \{-1, 1\}$,

$$\begin{aligned} \epsilon \sum_{m=1}^{n-1} f_m(x_m) &\geq 0 \quad \textit{whenever} \quad \min_{1\le j\le n-1} x_j \ge R \\ \epsilon \sum_{m=1}^{n-1} f_m(x_m) &\leq 0 \quad \textit{whenever} \quad \max_{1\le j\le n-1} x_j \le -R. \end{aligned} \tag{45}$$

Then, problem (35) has at least one solution.

Proof. For definiteness, assume that each f_m is bounded from below by c. For each $k \ge 1$, let us define

$$f_m^{(k)}(x_m) = f_m(x_m) + \frac{\epsilon x_m}{k(1+|x_m|)} \quad (1 \le m \le n-1),$$

so that each $f_m^{(k)}$ is bounded from below by $c-1$ and, using assumption (45),

$$\begin{aligned} \epsilon \sum_{m=1}^{n-1} f_m^{(k)}(x_m) &> 0 \quad \text{whenever} \quad \min_{1\le j\le n-1} x_j \ge R \\ \epsilon \sum_{m=1}^{n-1} f_m^{(k)}(x_m) &< 0 \quad \text{whenever} \quad \max_{1\le j\le n-1} x_j \le -R. \end{aligned} \tag{46}$$

Define $F^{(k)} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_m^{(k)}(x_1,\dots,x_{n-1}) = f_m^{(k)}(x_m) \quad (1 \le m \le n-1).$$

Lemma 5 implies that each possible solution $x^{(k)}$ of each equation

$$Lx + \lambda F^{(k)}(x) = 0 \quad (k = 1,2,\dots), \quad (\lambda \in]0,1]) \tag{47}$$

is such that

$$\max_{1\le m\le n-1} |x_m^{(k)}| < R + 2(n-1)(|c|+1) := \rho \quad (k = 1,2,\dots).$$

Furthermore, condition (46) with $x_1 = \ldots = x_{n-1} = \pm\rho$ implies that

$$PF^{(k)}(\pm\rho(1,\ldots,1)) = \frac{1}{n-1}\sum_{m=1}^{n-1} f_m^{(k)}(\pm\rho) \neq 0.$$

If $C(\rho) = (]-\rho,\rho[)^{n-1}$, it follows then from Lemma 4 and the homotopy invariance of Brouwer degree, that, for each $k = 1, 2, \ldots,$

$$d_B[L + F^{(k)}, C(\rho)), 0] = d_B[L + PF^{(k)}, C(\rho), 0].$$

Now, $L + P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is an isomorphism and, for $z \in R(P)$, we have $(L+P)^{-1}z = z$, so that, using the multiplication property of the Brouwer degree and denoting the Brouwer index by i_B (see e.g. [7]), we obtain

$$\begin{aligned} d_B[L + PF^{(k)}, C(\rho), 0] &= d_B[(L+P)[I + (L+P)^{-1}(PF^{(k)} - P)], C(\rho), 0] \\ &= i_B(L+P, 0)\cdot d_B[I - P + PF^{(k)}, C(\rho), 0] \\ &= \pm d_B[I - P + PF^{(k)}, C(\rho), 0]. \end{aligned}$$

Now, the Leray-Schauder reduction formula (see e.g. [7]) implies that

$$\begin{aligned} d_B[I - P + PF^{(k)}, C(\rho), 0] &= d_B[PF^{(k)}|_{N(L)}, C(\rho)\cap N(L), 0] \\ &= d_B[\varphi^{(k)},]-\rho,\rho[, 0], \end{aligned}$$

where

$$\varphi^{(k)}(u) = \varphi(u) + \frac{\epsilon u}{k(1+|u|)}.$$

Now assumption (45) with $x_1 = \ldots = x_{n-1} = \pm\rho$ implies that $\varphi^{(k)}(-\rho)\varphi^{(k)}(\rho) < 0$ for all $k = 1, 2, \ldots,$ so that

$$d_B[\varphi^{(k)},]-\rho,\rho[, 0] = \pm 1.$$

Thus it follows from the existence property of Brouwer degree that equation (47) has at least one solution $x^{(k)}$ such that $x^{(k)} \in C(\rho)$ for all $k = 1, 2, \ldots$. Going if necessary to a subsequence, we can assume that $x^{(k)} \to x \in \overline{C(\rho)}$ which is a zero of $L + F$ and hence a solution of (35) □

Let $u^+ = \max\{u, 0\}$.

COROLLARY 6. *For* $p > 0$, $a_m > 0$, $b_m \in \mathbb{R}$ $(1 \le m \le n-1)$, *the periodic problem*

(48) $$Dx_m + a_m(x_m^+)^p - b_m = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n,$$

has at least one solution if and only if

$$\sum_{m=1}^{n-1} b_m \ge 0.$$

When $a_m < 0$ *and* $b_m \in \mathbb{R}$ $(1 \leq m \leq n-1)$, *problem (48) has at least one solution if and only if*

$$\sum_{m=1}^{n-1} b_m \leq 0.$$

Proof. For the necessity, if problem (48) has a solution x, then

$$\sum_{m=1}^{n-1} b_m = \sum_{m=1}^{n-1} a_m (x_m^+)^p \geq 0.$$

For the sufficiency, each function $f_m(x_m) = a_m(x_m^+)^p - b_m$ is bounded from below by $-b_m$. Furthermore, if

$$R \geq \left(\frac{\sum_{m=1}^{n-1} b_m}{\sum_{m=1}^{n-1} a_m} \right)^{1/p},$$

then $\sum_{m=1}^{n-1} f_m(x_m) \geq 0$ when $\min_{1 \leq m \leq n-1} x_m \geq R$. On the other hand, $\sum_{m=1}^{n-1} f_m(x_m) = -\sum_{m=1}^{n-1} b_m \leq 0$ when $\max_{1 \leq m \leq n-1} x_m \leq 0$. Hence the result follows from Theorem 5. The proof of the other case is similar. □

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