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Subalpine Rhapsody in Dynamics

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SOME RESULTS ON PERIODIC POINTS AND CHAOTIC DYNAMICS ARISING FROM THE STUDY OF THE NONLINEAR HILL EQUATIONS

Abstract. We study fixed point theorems for maps which satisfy a property of stretching a suitably oriented topological space Z along the paths connecting two disjoint subsets Z_l^- and Z_r^- of Z. Our results reconsider and extend previous theorems in [56, 59, 60] where the case of two-dimensional cells (that is topological spaces homeomorphic to a rectangle of the plane) was analyzed. Applications are given to topological horseshoes and to the study of the periodic points and the symbolic dynamics associated to discrete (semi)dynamical systems.

1. Introduction

1.1. A motivation from the theory of ODEs

In the study of boundary value problems for nonlinear ODEs, the shooting method, in spite of being sometimes considered as an old fashioned technique, is still a quite powerful and effective tool in various different situations. For instance, as a sample model, let us consider the generalized Sturm−Liouville problem for a second order equation of the form

$$u'' + f(t,u,u') = 0, \qquad (u(t_0),u'(t_0)) \in \Gamma_0\,, \quad (u(t_1),u'(t_1)) \in \Gamma_1\,,$$

where $f = f(t,x,y) : [t_0,t_1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying a locally Lipschitz condition with respect to (x, y) and Γ_0 and Γ_1 are two unbounded closed connected subsets of the plane $\mathbb{R}^2$. Using the shooting method, one can start from the Cauchy problem (for which we have the uniqueness of the solutions and their continuous dependence upon the initial values)

$$u'' + f(t,u,u') = 0, \qquad (u(t_0),u'(t_0)) = (x_0,y_0) := z_0\,,$$

with $z_0 \in \Gamma_0$ and, having denoted by $\zeta(\cdot\,; t_0, z_0)$ *the* corresponding solution of the equivalent first order system in the phase-plane

$$x' = y, \quad y' = -f(t,x,y), \tag{1}$$

with $\zeta(t_0) = z_0$, look for the intersections between Γ_1 and the set $\Gamma_0' := \{\zeta(t_1; t_0, z_0) : z_0 \in \Gamma_0\}$. Clearly, any point $z_1 \in \Gamma_0' \cap \Gamma_1$ is the value $(u(t_1), u'(t_1))$ corresponding to a solution $u(\cdot)$ of the original Sturm−Liouville problem and different points in the intersection of Γ_1 with Γ_0' are associated to different solutions as well. In the simpler

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case in which Γ_0 is the image of a continuous curve $\theta_0 : [0, 1[\to \mathbb{R}^2$, the set Γ_0' can sometimes be described as the image of a continuous curve too, by means of the map $\theta_1 : [0, 1[\ni s \mapsto \zeta(t_1; t_0, \theta_0(s))$ (this, of course, is not always guaranteed, in fact we are not assuming in this example that all the solutions of the Cauchy problems can be defined on $[t_0, t_1]$ and therefore, without further assumptions on f, it could happen that θ_1 may be not defined on the whole interval $[0, 1[$). See also [9, 10, 13, 23] for different topological approaches where the uniqueness of the solutions for the Cauchy problem is not assumed.

In [55], dealing with some boundary value problems associated to the nonlinear scalar ODE

$$u'' + q(t)g(u) = 0 \tag{2}$$

and assuming that $q : [t_0, t_1] \to \mathbb{R}$ is a continuous and piece-wise monotone function and $g : \mathbb{R} \to \mathbb{R}$, with

$$g(s)s > 0, \quad \text{for } s \neq 0,$$

a locally Lipschitz continuous mapping satisfying a condition of superlinear growth at infinity, we considered the case in which there is $\tau \in]t_0, t_1[$ such that

$$q(t) > 0 \text{ for } t \in]t_0, \tau[\quad \text{and } q(t) < 0 \text{ for } t \in]\tau, t_1[$$

and then we found in the phase-plane two conical shells $W(+) := W(r, R)$ and $W(-) := -W(r, R)$, with

$$W(r, R) := \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ r^2 \leq x^2 + y^2 \leq R^2\}, \tag{3}$$

such that the following path-stretching property holds:

$(H_\pm)$ *for every path $\sigma(\pm)$ contained in $W(\pm)$ and meeting the inner and the outer circumferences at the boundary of $W(\pm)$ there are sub-paths $\gamma_i(\pm)$ ($i = 1, 2$) contained in the domain D_φ of the map $\varphi : \mathbb{R}^2 \ni z_0 \mapsto \zeta(t_1; t_0, z_0) \in \mathbb{R}^2$ and such that $\varphi(\gamma_1(\pm))$ is contained in $W(\pm)$ and meets both the inner and the outer circumferences, as well as $\varphi(\gamma_2(\pm))$ is contained in $W(\mp)$ and meets both the inner and the outer circumferences.* *

The points $z_0 = (x_0, y_0)$ belonging to each of the sub-paths $\gamma_i(\pm)$ ($i = 2, 1$) are initial points of system

$$x' = y, \quad y' = -q(t)g(x), \tag{4}$$

for $t = t_0$ for which the corresponding solution $\zeta(\cdot\,; t_0, z_0) = (u(\cdot), u'(\cdot))$ is such that $u(t)$ has a sufficiently large (but fixed in advance) number of zeros in the interval $]t_0, \tau[$ and then, either exactly one zero or no zeros at all (and a zero for its derivative) in the interval $]\tau, t_1[$.

*In [55] we used a definition of path which is slightly different from the one considered here in Section 1.5; however, this does not effect the validity of property $(H_\pm)$. Actually, for the moment, as long as we are in the introductory part of this work and also for the sake of simplicity in the exposition, we prefer to be a little vague about the precise concept of path we are going to use.

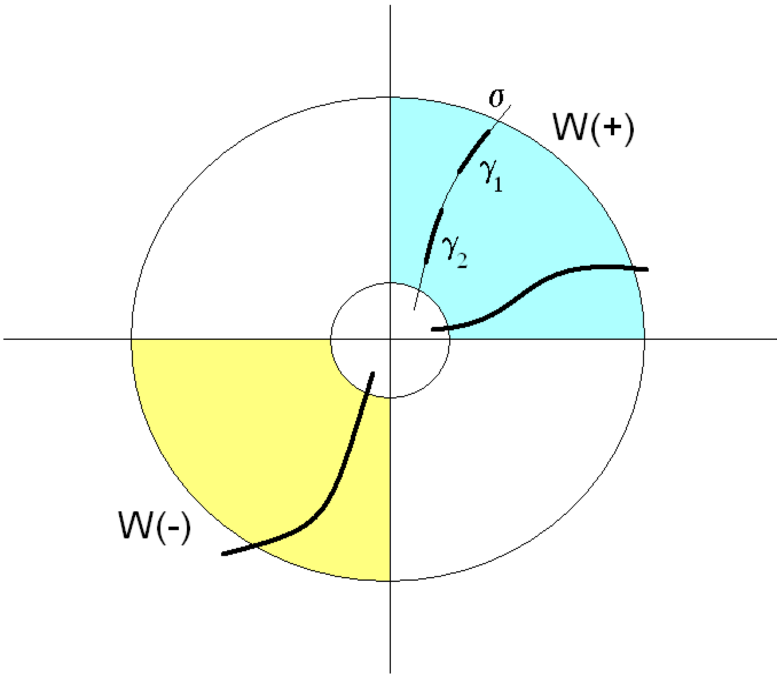

Figure 1: An illustration of the situation which occurs in [55]. Any path $\sigma = \sigma(+)$ contained in the first quadrant and meeting the inner and the outer circumferences of the conical shell $W(+)$ (like the one drawn with a thinner line) contains (at least) two sub-paths $\gamma_1 = \gamma_1(+)$ and $\gamma_2 = \gamma_2(+)$ (like those drawn with a thicker line) satisfying the following property: through the map φ, one of the two sub-paths (namely, γ_1) is transformed to a path (drawn with a thicker line) contained in the first quadrant and crossing the set $W(+)$, while the other sub-path (that is, γ_2) is transformed to a path (drawn with a thicker line) contained in the third quadrant and crossing the set $W(-)$. The same happens with respect to any path $\sigma(-)$ contained in $W(-)$ and intersecting the inner and the outer circumferences at the boundary of $W(-)$.

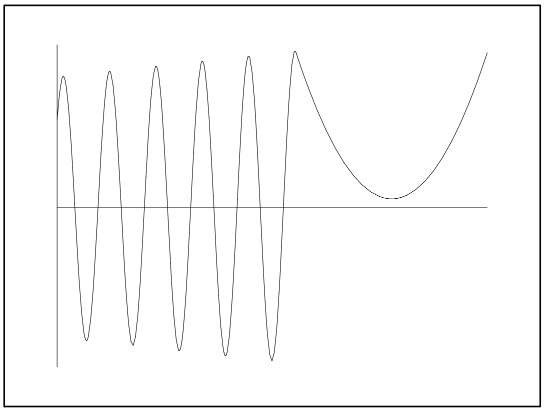

Figure 2: An illustration of the possible behavior of a solution $u(t)$ of (2) for $t \in [t_0, t_1]$, with $(u(t_0), u'(t_0))$ belonging to $\gamma_1(+)$. The solution oscillates a certain (large) number of times in $]t_0, \tau[$. At the time τ of switching from $q > 0$ to $q < 0$ the point $(u(\tau), u'(\tau))$ lies in the interior of the fourth quadrant of the phase-plane. Then, in the interval $[\tau, t_1]$ we have $u(t) > 0$ with $u'(t)$ vanishing exactly once.

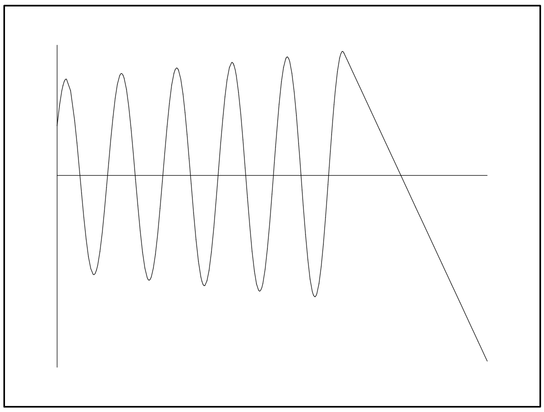

Figure 3: An illustration of the possible behavior of a solution $u(t)$ of (2) for $t \in [t_0, t_1]$, with $(u(t_0), u'(t_0))$ belonging to $\gamma_2(+)$. The solution oscillates a certain (large) number of times in $]t_0, \tau[$. At the time τ of switching from $q > 0$ to $q < 0$ the point $(u(\tau), u'(\tau))$ lies in the interior of the fourth quadrant of the phase-plane. Then, in the interval $[\tau, t_1]$ we have $u'(t) < 0$ with $u(t)$ vanishing exactly once.

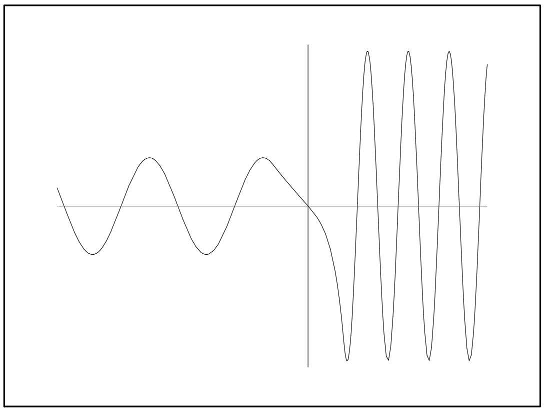

Figure 4: An illustration of the possible behavior of a solution $u(t)$ of (2) along an interval made by two subintervals where $q > 0$ which are separated by a subinterval where $q < 0$. In this latter subinterval $u' < 0$ and u vanishes exactly once.

If there are several adjacent intervals like $[t_0, t_1]$ in which the weight function $q(t)$ changes its sign, we can repeat the same argument for each of such intervals and obtain solutions which have a large number of zeros in each interval when $q(t) > 0$ and either exactly one zero, or no zeros at all (according to any finite sequence in $\{0, 1\}$ which is fixed in advance). Figure 4 and Figure 5, below, describe the situation of two positive intervals for the weight which are separated by an interval where $q < 0$. According to [55] there exist solutions with a large number of oscillations in the intervals when $q > 0$ and vanishing either once or never in the interval when the weight is negative. Both kind of solutions coexist for the same equation, the different outcome depending by small differences in the initial conditions. The graphs drawn in Figures 2-5 (using Maple software) represent an idealized situation that we use as a description of the main result in [55] and do not concern a specific equation like (2).

In [55], using a key lemma involving the path-stretching property $(H_{\pm})$, and some results adapted from Hartman [28] and Struwe [69] (which allow to find infinitely many solutions for generalized Sturm−Liouville superlinear problems when the weight function is positive), we obtained the existence of (infinitely many) solutions for various boundary value problems associated to equation (2) in the case of a nonlinear function g satisfying a condition of superlinear growth at infinity. A possible choice of g is given by $g(s) = |s|^{\alpha-1}s$ with $\alpha > 1$, which makes equation (2) a nonlinear analogue of Hill's equation (cf. [5]). Our results in [55] allow to obtain solutions having an arbitrarily large number of zeros in the intervals when $q > 0$ and either no zeros or exactly one zero (following any prescribed rule) in the intervals when $q < 0$. In some related works (see, e.g., [14, 58, 59]) it was then proved that similar results hold also with respect to other kinds of nonlinear ODEs or under different growth assumptions for $g(s)$ in (2). The interested reader can find in [4, 6, 70] as well as in [8, 14, 24, 45, 52, 53, 54, 57, 59] and the references therein further information and

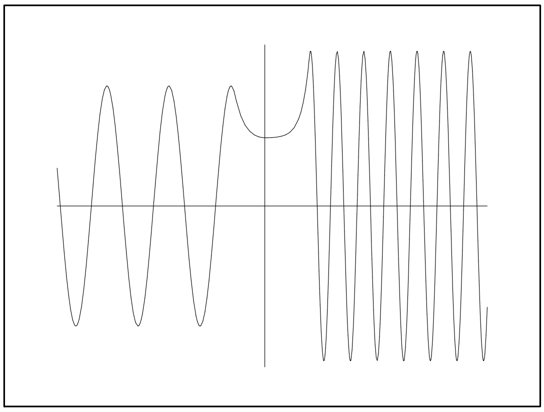

Figure 5: An illustration of the possible behavior of another solution $u(t)$ of (2) along an interval made by two subintervals where $q > 0$ which are separated by a subinterval where $q < 0$. In this latter subinterval $u > 0$ and u' vanishes exactly once.

results about the rich structure and the complex dynamics of the solutions of the nonlinear equation (2) with a sign changing weight $q(t)$.

In the case when $q(t)$ is a periodic function such that its interval of periodicity can be decomposed into a finite number of adjacent subintervals where $q(t)$ alternates its sign, a natural problem turns out to be that of the search of periodic (harmonic and subharmonic) solutions to (2). Results about the existence of infinitely many periodic solutions for the superlinear case were obtained by Butler in his pioneering work [6]. For a nonlinearity having superlinear growth at infinity, Terracini and Verzini in [70] proved the existence of periodic solutions which have an arbitrarily large (but possibly fixed in advance) number of zeros in the intervals when $q > 0$ and precisely one zero in the intervals when $q < 0$. At the best of our knowledge, this is the first result giving evidence of a very complicated behavior for the solutions of the nonlinear Hill's equations with a sign changing weight.

In view of the path-stretching property $(H_{\pm})$ and the above quoted results for the periodic problem, as a next step, one can raise the question whether it is possible to obtain fixed points (as well as periodic points) for a map like the φ considered in $(H_{\pm})$. This goal was achieved in [56] where we obtained a fixed point theorem for planar maps (subsequently reconsidered and generalized in [59, 60]) that, when applied to the case of a periodic weight function, shows that the path-stretching condition $(H_{\pm})$ implies the existence of infinitely many periodic solutions (harmonic and subharmonic) as well as the presence of a chaotic-like dynamics for the solutions of (2).

1.2. Fixed points and periodic points for planar mappings

In order to present the results in [56, 59, 60], first of all we put in a more abstract form the situation described in $(H_{\pm})$. For sake of simplicity, we give here only some of the main features of our approach, with a few comments. The interested reader is referred to [59, 60] for all the details, as well as for some remarks and comments [61] relating our results to some developments of the Conley−Ważewski theory [17, 25, 26, 47, 49, 65, 66, 67, 68, 79, 80, 81, 82]. We also notice that the terminology here is a little different than in [56, 59, 60]. The different choice in the presentation is made in order to employ some terms that can be easily adapted to the higher dimensional case, that is the scope of the second part of this article.

Let X be a Hausdorff topological space. A subset $\mathcal{R} \subseteq X$ is called a *generalized rectangle* or a *two-dimensional cell* if there is a homeomorphism η of $[0,1]^2 \subseteq \mathbb{R}^2$ onto $\mathcal{R} \subseteq X$. Given η, we put in evidence the sets

$$\mathcal{R}^-_l := \eta(\{0\} \times [0,1]), \quad \mathcal{R}^-_r := \eta(\{1\} \times [0,1]), \quad \text{and } \mathcal{R}^- := \mathcal{R}^-_l \cup \mathcal{R}^-_r .$$

We define also the pair

$$\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$$

as a (generalized) *oriented rectangle* or *oriented cell*. The indexes " l " and " r " stand for " left " and " right ", respectively. Clearly, any other way of labelling two objects (like " 0 " and " 1 ") fits well. By setting

$$\mathcal{R}^+_b := \eta([0,1] \times \{0\}) \quad \text{and } \mathcal{R}^+_t := \eta([0,1] \times \{1\}),$$

one can also define in a dual manner the " base " and the " top " of the generalized rectangle $\mathcal{R}$. By convention, we take the orientation through the $[\,\cdot\,]^-$-set.

As an example, the conical shell $\mathcal{R} := W(+) = W(r,R)$ defined in (3) is a generalized rectangle and we can take the homeomorphism η in order to have

$$\mathcal{R}^-_l := W(+) \cap \{x^2 + y^2 = r^2\}, \quad \mathcal{R}^-_r := W(+) \cap \{x^2 + y^2 = R^2\}.$$

Next, we present a fixed point theorem for continuous maps which stretch an oriented rectangle. The main feature of our result is that we look for fixed points which belong to a given subset $\mathcal{D}$ of the domain of the map ψ under consideration. By reason of this requirement, we consider pairs $(\mathcal{D}, \psi)$.
Let $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ be two oriented rectangles (in the same Hausdorff topological space X), let $\psi : X \supseteq D_\psi \to X$ be a continuous map and let $\mathcal{D} \subseteq D_\psi \cap \mathcal{A}$.

We say that $(\mathcal{D}, \psi)$ *stretches* $\widehat{\mathcal{A}}$ *to* $\widehat{\mathcal{B}}$ *along the paths* and write

$$(\mathcal{D}, \psi) : \widehat{\mathcal{A}} \Join\!\!\rightarrow \widehat{\mathcal{B}},$$

if there is a compact set $\mathcal{K} \subseteq \mathcal{D}$ such that the following conditions are satisfied:

$\psi(\mathcal{K}) \subseteq \mathcal{B}$,

for every path $\sigma \subseteq \mathcal{A}$ with $\sigma \cap \mathcal{A}^-_l \neq \emptyset$ and $\sigma \cap \mathcal{A}^-_r \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{K}$ with $\psi(\gamma) \cap \mathcal{B}^-_l \neq \emptyset$ and $\psi(\gamma) \cap \mathcal{B}^-_r \neq \emptyset$.

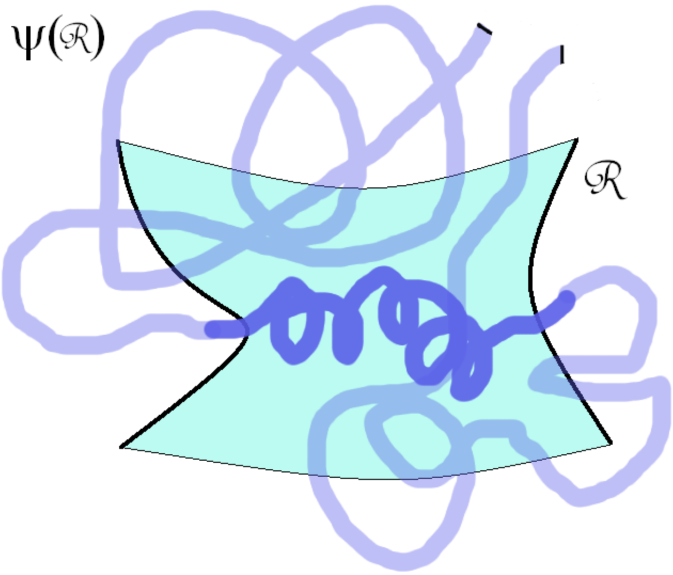

Figure 6: Suppose that a continuous mapping ψ transforms the generalized rectangle $\mathcal{R}$ (the "fat" cheese-like object) to the worm-like set $\psi(\mathcal{R})$. The two components of the $[\,\cdot\,]^-$-set of $\mathcal{R}$ as well as their images under ψ are represented by segments (arcs) with a darker color at the contour of the corresponding figures. The stretching property is visualized by the fact that there is a "crossing" of the "worm" through the "cheese".

To put emphasis on the role of the compact set $\mathcal{K}$ in the stretching definition, sometimes we also write

$$(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{B}}.$$

Using a result about plane continua previously applied in a different context also by Conley [11] and Butler [6] (for a proof, see [63] as well as [59, 60]), the following fixed point theorem was obtained.

THEOREM 1. *Let* $\widehat{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented rectangle in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{R}} \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{R}}$, *then there is* $w \in \mathcal{D}$ *(actually* $w \in \mathcal{K}$*) such that* $\psi(w) = w$.

An illustration of Theorem 1 is given in Figure 6.

Of course, not all the crossings fit to our purposes. For instance, Figure 7 shows an example of nonexistence of fixed points.

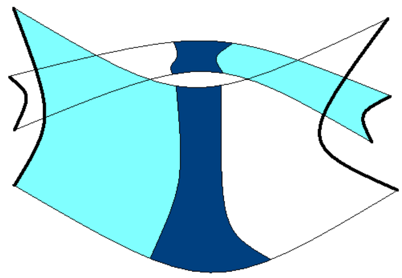

Figure 7: An example showing a case in which the stretching property is not satisfied. The white region is moved to the white one and the darker to the darker one by a homeomorphism which does not have fixed points.

In order to better understand which are the good crossings between $\mathcal{R}$ and $\psi(\mathcal{R})$ that permit to apply our fixed point theorem, we considered the following definitions of slabs of an oriented rectangle (called " slices " in [60]).

Let $\widehat{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ and $\widehat{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ be two oriented cells in X. We say that $\widehat{\mathcal{M}}$ is a *horizontal slab* of $\widehat{\mathcal{N}}$ and write

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}},$$

if $\mathcal{M} \subseteq \mathcal{N}$ and, either

$$\mathcal{M}_l^- \subseteq \mathcal{N}_l^- \quad \text{and} \quad \mathcal{M}_r^- \subseteq \mathcal{N}_r^- \,,$$

or

$$\mathcal{M}_l^- \subseteq \mathcal{N}_r^- \quad \text{and} \quad \mathcal{M}_r^- \subseteq \mathcal{N}_l^- \,.$$

Similarly, we say that $\widehat{\mathcal{M}}$ is a *vertical slab* of $\widehat{\mathcal{N}}$ and write

$$\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}},$$

if $\mathcal{M} \subseteq \mathcal{N}$ and, either

$$\mathcal{M}_b^+ \subseteq \mathcal{N}_b^+ \quad \text{and} \quad \mathcal{M}_t^+ \subseteq \mathcal{N}_t^+ \,,$$

or

$$\mathcal{M}_b^+ \subseteq \mathcal{N}_t^+ \quad \text{and} \quad \mathcal{M}_t^+ \subseteq \mathcal{N}_b^+ \,.$$

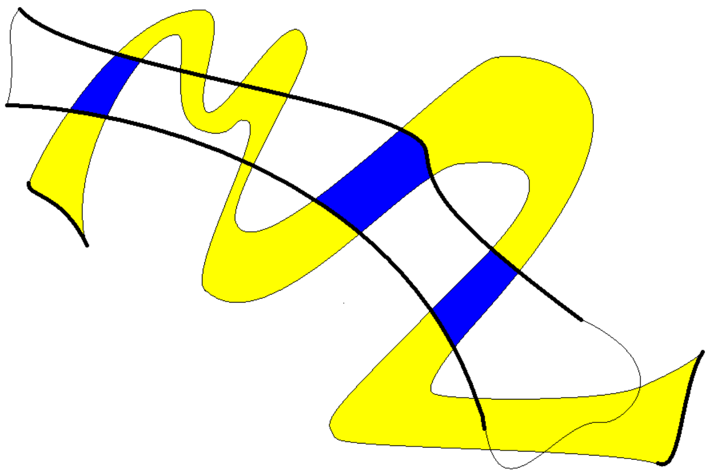

Figure 8: (taken from [60]). Example of oriented cells $\widetilde{\mathcal{R}}$ (white) and $\psi(\widetilde{\mathcal{R}})$ (light color) with crossings into three slabs (darker color). The $[\,\cdot\,]^-$-sets are indicated with a bold line. Among the five cells which are the connected components of the intersection $\psi(\mathcal{R}) \cap \mathcal{R}$, only the three painted with darker color are suitable to play the role of the $\mathcal{M}$'s for the application of Theorem 2.

Now, given three oriented rectangles (cells) in X, which are denoted by $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$, $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ and $\widehat{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$, we say that $\widehat{\mathcal{B}}$ *crosses* $\widehat{\mathcal{A}}$ *in* $\widehat{\mathcal{M}}$ and write

$$\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\},$$

if

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{A}} \quad \text{and} \quad \widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{B}}.$$

The symbol $\pitchfork$ is borrowed from the case of transversal intersections, however we point out that in our situation (although confined to sets which are two-dimensional in nature) we don't need any smoothness assumption. In fact, our setting is that of topological spaces. From the above definitions and by Theorem 1 the following result easily follows.

THEOREM 2. *Let* $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *be oriented cells in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Join\!\!\!\rightsquigarrow \widehat{\mathcal{B}}$ *and there is an oriented cell* $\widehat{\mathcal{M}}$ *such that* $\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}$, *then there exists* $w \in \mathcal{K} \cap \mathcal{M}$ *such that* $\psi(w) = w$.

A situation like that depicted in Figure 8 in which we have more than one good intersection between the domain and the image of a homeomorphism is typical of the horseshoe maps and thus, as a next step, we can look for the existence of a complete dynamics on m symbols, where $m \geq 2$ is the number of the crossings. With this respect, we have to recall that a very general topological theory has been developed in the recent years by Kennedy, Yorke and their collaborators in a series of fundamental papers in this area (see [33, 35, 36, 37, 38]). Our goal instead is to take advantage of our simplified framework in which we consider only sets which are homeomorphic to a square and prove the existence of periodic points of any order. To this aim, we have to apply Theorem 1 to the iterates of the map ψ and select carefully some subset of the domain $\mathcal{D}$ in order to find "true" periodic points (for instance those with a long minimal period). First, however, we need a further definition, taken from [39].

We say that $\psi : X \supseteq D_\psi \to X$ has a *chaotic dynamics of coin-tossing type on k symbols* if $k \geq 2$ and there is a metrizable space $Z \subseteq X$ and k pairwise disjoint compact sets $W_1, \ldots, W_k \subseteq Z \cap D_\psi$ such that, for each two-sided sequence $(s_n)_{n\in\mathbb{Z}}$ with

$$s_n \in \{1, \ldots, k\}, \quad \forall\, n \in \mathbb{Z},$$

there is a sequence of points $(z_n)_{n\in\mathbb{Z}}$ with

$$z_n \in W_{s_n} \quad \text{and} \quad z_{n+1} = \psi(z_n), \quad \forall\, n \in \mathbb{Z}.$$

In other words, any possible itinerary on the sets $W_1 \ldots, W_k$ is followed by some point.

As an auxiliary tool in order to obtain at the same time the existence of such kind of chaotic trajectories and also the fact that all the periodic itineraries can be followed by some periodic point, we have the following theorem (see [59]), which is also reminiscent of some results in [34] and [73].

THEOREM 3. *Assume that there is a (double) sequence of oriented rectangles* $(\widehat{\mathcal{A}}_k)_{k\in\mathbb{Z}}$ *and maps* $((\mathcal{D}_k, \psi_k))_{k\in\mathbb{Z}}$, *with* $\mathcal{D}_k \subseteq \mathcal{A}_k$, *such that* $(\mathcal{D}_k, \psi_k) : \widehat{\mathcal{A}}_k \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{A}}_{k+1}$ *for each* $k \in \mathbb{Z}$. *Then the following conclusions hold:*

(a_1) *There is a sequence* $(w_k)_{k\in\mathbb{Z}}$ *with* $w_k \in \mathcal{D}_k$ *and* $\psi_k(w_k) = w_{k+1}$ *for all* $k \in \mathbb{Z}$;

(a_2) *For each* $j \in \mathbb{Z}$ *there is a compact and connected set* $\mathcal{C}_j \subseteq \mathcal{D}_j$ *satisfying*

$$\mathcal{C}_j \cap (\mathcal{A}_j)^+_b \neq \emptyset, \quad \mathcal{C}_j \cap (\mathcal{A}_j)^+_t \neq \emptyset$$

and such that for each $w \in \mathcal{C}_j$ *there is a sequence* $(y_\ell)_{\ell\geq j}$, *with* $y_\ell \in \mathcal{D}_\ell$ *and* $y_j = w$, $y_{\ell+1} = \psi_\ell(y_\ell)$ *for each* $\ell \geq j$;

(a_3) *If there are integers* h, k *with* $h < k$ *such that* $\widehat{\mathcal{A}}_h = \widehat{\mathcal{A}}_k$, *then there is a finite sequence* $(z_i)_{h\leq i\leq k}$, *with* $z_i \in \mathcal{D}_i$ *and* $\psi_i(z_i) = z_{i+1}$ *for each* $i = h, \ldots, k-1$, *such that* $z_h = z_k$, *that is,* z_h *is a fixed point of* $\psi_{k-1} \circ \cdots \circ \psi_h$.

The proof of (a_1) and partially also that of (a_2) could be given by adapting to our setting the argument in [33, Lemma 3 and Proposition 5]. As to (a_3), we apply Theorem 1. The existence of the continuum (compact connected set) $\mathcal{C}_j$ in (a_2) is a byproduct of the topological lemma that we employ also in the proof of Theorem 1. We give all the main details along the proof of Theorem 11 in Section 3.2 and refer to [59] for more information.

From Theorem 3 several corollaries can be obtained. Now we just recall a few of them which are taken from [59] and [60]. Due to space limitation, we don't give here other applications to ODEs. We just mention the recent thesis by Covolan [12] which contains a detailed description of the results in [33] and those in [59, 60] and where it is shown that our theorem, when applied to the search of fixed points for the iterates of a two-dimensional map, may add some useful information (about the existence of periodic points) to the conclusions obtained in some recent articles (like, e.g., [30, 75, 76, 77]), where the theory of topological horseshoes was applied to prove the existence of a chaotic dynamics in various different models.

THEOREM 4. *Suppose that* $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *are oriented cells in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Leftarrow\!\!\!\!\rightsquigarrow \widehat{\mathcal{B}}$ *and there are* $k \geq 2$ *oriented cells* $\widehat{\mathcal{M}}_1 \dots, \widehat{\mathcal{M}}_k$ *such that*

$$\widehat{\mathcal{M}}_i \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}, \quad \textit{for } i = 1, \dots, k,$$

with

$$\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{K} = \emptyset, \ \textit{for all } \ i \neq j, \ \textit{with } \ i, j \in \{1, \dots, k\},$$

then the following conclusion holds:

(b_1) ψ *has a chaotic dynamics of coin-tossing type on* k *symbols (with respect to the sets* $W_i = \mathcal{K}_i = \mathcal{K} \cap \mathcal{M}_i$*);*

(b_2) *For each one-sided infinite sequence* $\mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in \{1, \dots, k\}^{\mathbb{N}}$ *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{K}_{s_0}$ *with*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})^+_l \neq \emptyset, \quad \textit{and } \ \mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})^+_r \neq \emptyset,$$

such that for each point $w \in \mathcal{C}^{\mathbf{s}}$, *the sequence*

$$z_{j+1} = \psi(z_j), \quad z_0 = w, \quad \textit{for } \ j = 0, 1, \dots, n, \dots$$

satisfies

$$z_j \in \mathcal{K}_{s_j}, \ \forall \, j = 0, 1, \dots, n, \dots \quad ;$$

(b_3) ψ *has a fixed point in each set* $\mathcal{K}_i := \mathcal{M}_i \cap \mathcal{K}$ *and, for each finite sequence* $(s_0, s_1, \dots, s_m) \in \{1, \dots, k\}^{m+1}$, *with* $m \geq 1$, *there is at least one point* $z^* \in \mathcal{K}_{s_0}$ *such that the position*

$$z_{j+1} = \psi(z_j), \quad z_0 = z^*, \quad \textit{for } \ j = 0, 1, \dots, m$$

defines a sequence of points with

$$z_j \in \mathcal{K}_{s_j}, \ \forall \, j = 0, 1, \dots, m \quad \textit{and } \ z_{m+1} = z^*.$$

As a comment to this result, we look again at Figure 8 and observe that, besides having a coin-tossing dynamics on three symbols, we have also the existence of fixed points in each of the three darker regions and, moreover, once we have labelled these

Figure 9: (compare to Figure 6). Now the worm crosses nicely the cheese for two times and we obtain a complete dynamics on two symbols (as well as periodic points of any period).

three regions with corresponding symbols, say 1, 2, 3, we have also that to any periodic sequence in $\{1, 2, 3\}$ we can find a corresponding periodic point for the map ψ which follows (along ψ) the itinerary described by the periodic sequence. Figure 8 illustrates the case in which ψ is a homeomorphism and the three good intersections are pairwise disjoint. Actually, our Theorem 3 is more flexible (cf. Corollary 2 below) and it allows to come to the same conclusion by a careful selection of disjoint subsets of the domain of the map (which is not necessarily a homeomorphism). As an illustration of this remark, let us consider Figure 9.

In this direction, two possible corollaries of Theorem 3 are the following. They correspond, respectively: (a) to the property $(H_{\pm})$ which holds with respect to the solutions of system (1) and the two conical shells $W(+)$ and $W(-)$, and (b) to the example depicted in Figure 9. We refer to [59] for the proof of both the corollaries, as well as for the proof of a more general result from which they both come.

COROLLARY 1. *Let* $\widehat{\mathcal{R}}_0 = (\mathcal{R}_0, \mathcal{R}_0^-)$ *and* $\widehat{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$, *be two oriented rectangles with* $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$ *and such that*

$$\psi : \widehat{\mathcal{R}}_i \,\triangleleft\!\!\leftrightsquigarrow\, \widehat{\mathcal{R}}_j\,, \quad \forall\, i, j \in \{0, 1\}.$$

Then the following conclusions hold:

(c_1) ψ *has a dynamics of coin-tossing type with respect to the pair* $(\mathcal{R}_0, \mathcal{R}_1)$.;

(c_2) *For every sequence* $\mathbf{s} = (s_n)_n$, *with* $s_n \in \{0, 1\}$ *for each* $n \geq 0$, *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{R}_{s_0}$ *satisfying*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{R}_{s_0})^+_b \neq \emptyset, \quad \mathcal{C}^{\mathbf{s}} \cap (\mathcal{R}_{s_0})^+_t \neq \emptyset$$

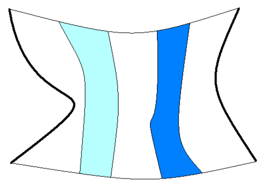

Figure 10: (compare to Figure 9). The two good crossings come from two disjoint subregions (those painted with a darker color) of the domain.

and such that for each $w \in \mathcal{C}^{\mathbf{s}}$ *there is a sequence* $(y_n)_n$, *with* $y_n \in \mathcal{R}_{s_n}$ *and* $y_0 = w$, $y_{n+1} = \psi(y_n)$ *for each* $n \geq 0$;

(c_3) *For each finite sequence* $(s_0, s_1, \ldots, s_k) \in \{0,1\}^{k+1}$ *with* $s_k = s_0$, *there is a* $w_0 \in \mathcal{R}_{s_0}$ *which generates a finite sequence* $(w_\ell)_{0\leq \ell \leq k}$ *such that*

$$\psi(w_\ell) = w_{\ell+1} \in \mathcal{R}_{s_\ell}\,, \quad \forall\, \ell = 0, \ldots, k-1$$

and $w_k = w_0$.

COROLLARY 2. *Let* $\widehat{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented cell and suppose that there are two disjoint compact sets* $\mathcal{D}_0$, $\mathcal{D}_1 \subseteq \mathcal{R} \cap D_\psi$ *such that*

$$(\mathcal{D}_i\,, \psi) : \widehat{\mathcal{R}} \Leftarrow\!\!\!\rightsquigarrow \widehat{\mathcal{R}}\,, \quad \forall\, i \in \{0,1\}.$$

Then the following conclusions hold:

(d_1) ψ *has a dynamics of coin-tossing type with respect to the pair* $(\mathcal{D}_0, \mathcal{D}_1)$;

(d_2) *For every sequence* $\mathbf{s} = (s_n)_n$, *with* $s_n \in \{0,1\}$ *for each* $n \geq 0$, *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{D}_{s_0}$ *satisfying*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{R})^+_b \neq \emptyset, \quad \mathcal{C}^{\mathbf{s}} \cap (\mathcal{R})^+_t \neq \emptyset$$

and such that for each $w \in \mathcal{C}^{\mathbf{s}}$ *there is a sequence* $(y_n)_n$, *with* $y_n \in \mathcal{D}_{s_n}$ *and* $y_0 = w$, $y_{n+1} = \psi(y_n)$ *for each* $n \geq 0$;

(d_3) *For each finite sequence* $(s_0, s_1, \ldots, s_k) \in \{0,1\}^{k+1}$ *with* $s_k = s_0$, *there is a* $w \in \mathcal{D}_{s_0}$ *which generates a finite sequence* $(w_\ell)_{0\leq \ell \leq k}$ *such that*

$$\psi(w_\ell) = w_{\ell+1} \in \mathcal{D}_{s_\ell}\,, \quad \forall\, \ell = 0, \ldots, k-1$$

and $w_k = w_0$.

1.3. Extensions to higher dimensions

Suppose now that we have a (non-autonomous) nonlinear differential system in $\mathbb{R}^{\mathrm{N}}$ (with N possibly strictly larger than 2)

$$x' = F(t, x) \tag{5}$$

with $F : \mathbb{R} \times \Omega \to \mathbb{R}^{\mathrm{N}}$ a continuous vector field (more general Carathéodory hypothesis could be considered as well) which satisfies a local Lipschitz condition with respect to $x \in \Omega$, where Ω is an open subset of $\mathbb{R}^{\mathrm{N}}$. Assume also that there is $T > 0$ such that $F(t + T, x) = F(t, x)$ for every $(t, x) \in \mathbb{R} \times \Omega$. Then we can define the Poincaré's operator

$$\mathcal{Q} : \Omega \supseteq D_{\mathcal{Q}} \to \mathbb{R}^{\mathrm{N}}, \qquad z_0 \mapsto \zeta(t_0 + T; t_0, z_0),$$

where $\zeta(\cdot\,; t_0, z_0)$ is the solution of (5) satisfying the initial condition $x(t_0) = z_0$. From the fundamental theory of ODEs we know that $D_{\mathcal{Q}}$ is an open subset of Ω and $\mathcal{Q}$ is a homeomorphism of $D_{\mathcal{Q}}$ onto its image $\mathcal{Q}(D_{\mathcal{Q}})$. The problems that we want to discuss concern:

(A) the search of T-periodic solutions of equation (5),

(B) the existence of “true” subharmonic solutions (that is, mT-periodic solutions of (5), for some $m \geq 2$, that are not jT-periodic for every $j = 1, \dots, m - 1$),

(C) the evidence of a complex behavior of the solutions of (5). For instance, the existence of two sets like the sets $W(+)$ and $W(-)$ found in [55] for the systems $x_1' = x_2$, $x_2' = -q(t)g(x_1)$, where the trajectories can arbitrarily get in and out respecting any a priori fixed coin-tossing sequence of indexes (say 0 or 1 or “left” and “right”) labelling the two sets.

For these goals, following a classical method [40] we rely on the study of the Poincaré's map $\mathcal{Q}$ and its iterates, or, in other words, we investigate the discrete dynamical system associated to $\mathcal{Q}$. Actually, we study general maps ψ which are not necessarily homeomorphisms (even continuity on their whole domain will be not assumed; of course, we'll need continuity on some “interesting” subsets of their domain) for which we discuss the existence of fixed points, periodic points and chaotic-like dynamics. Our main assumption is a *stretching condition along the paths* that extends to a broader setting the property $(H_{\pm})$ recalled above as well as it generalizes to higher dimension the previous treatment for the two-dimensional case considered in [56, 59, 60].

Maps which act on a topological rectangle as an expansion along some directions and a compression along the remaining ones have been widely studied in the literature. They appear, for instance, in the construction of Markov partitions (cf. [51, Appendix 2, pp.169–177]) and therefore they are crucial in the study of the multidimensional chaos. Another area in which such maps are involved concerns the search of periodic solutions for periodic non-autonomous differential systems which are partially dissipative (cf. [2, 3, 41]). In order to set a suitable list of hypotheses for a fixed point theorem concerning a continuous mapping $\psi = (\psi_u, \psi_s)$ defined on a N-dimensional rectangle $\mathcal{R} = B_u[0, 1] \times B_s[0, 1] \subseteq \mathbb{R}^{\mathrm{N}} = \mathbb{R}^u \times \mathbb{R}^s$ (where we think at the u-components and the s-components as the unstable-expansive and the stable-compressive ones, respectively), a reasonable choice of assumptions to put on the map ψ along the s-component will be that of taking conditions that reduce to those of the Brouwer or of the Rothe fixed point theorems (or to analogous ones) in the special case when $u = 0$ and $s = \mathrm{N}$. On the other hand, it seems perhaps less evident which could be the best choice of assumptions to express the expansive effect along the s-components. With this respect, both conditions on the norm (like in [2, 3]) and componentwise conditions (like in [80, 81]) have been assumed. As we have already explained with some details in the first part of this Introduction, motivated by the stretching property $(H_{\pm})$ discovered in [55] for equation (2) we obtained in [56] a fixed point theorem for planar mappings where the main hypothesis requires that the map expands the paths connecting two opposite sides of a topological rectangle. Further generalizations were then given in [59, 60], but still for a setting which is basically two-dimensional in nature. We recall that an expansive condition for paths connecting the opposite faces of a N-dimensional rectangle was also considered by Kampen in [32], allowing an arbitrary number of expansive directions (see [32, Corollary 4]). However, when reduced to the special case $\mathrm{N} = 2$, Kampen's result and ours seem to differ in some relevant points. In particular, a crucial assumption of our fixed point theorem in [56] allows the map to be defined only on some subsets of the rectangle and, moreover, even when the the mapping is defined on the whole rectangle, the assumptions in [32] and those in [56] about the compressing direction are basically different. One of the main features that we ask to a fixed point theorem for expansive-compressive mappings is to depend on hypotheses that can be easily reproduced for compositions of maps. This, in turns, permits to apply the theorem to the iterates of ψ and thus obtain results about the existence of nontrivial periodic points. Since our path-stretching property well fits also with respect to this requirement (of course, it is not the only one; in fact, nice alternative approaches are available in literature), we want to address our investigations toward a suitable extension of such property to the case $\mathrm{N} > 2$.

1.4. Contents

After such a long introduction in which we surveyed some of our preceding results for the two-dimensional case, we are ready to present some new developments in the higher dimensional setting. Then the rest of this paper is organized as follows. In Section 2 we present our main result (Theorem 6) which is a fixed point for a compact map defined on a subset of a cylinder in a normed space. In order to simplify the exposition, we confine ourselves to the idealized situation in which we split our space as a product $\mathbb{R} \times X$ and indicate its elements as pairs (t, x), so that we can easily express our main assumption as an hypothesis of expansion of the paths contained in the cylinder $\mathcal{B}[a, R] = [-a, a] \times B[0, R]$ along the t-direction. The principal tool for the proof of our basic fixed point theorem is the Leray$-$Schauder continuation theorem in its strongest form asserting the existence of a continuum of solution-pairs for a nonlinear operator equation depending on a real parameter (Théorème Fondamental [42]). Such result, with its variants and extensions, is one of the main theorems of the Leray$-$Schauder topological degree theory and it has found several important applications to bifurcation (and co-bifurcation) theory [20, 21, 22, 62], to the investigation of the structure of the solution set for parameter dependent equations [19, 31, 43] and to the study of nonlinear problems in absence of a priori bounds [7, 10, 44, 46]. Thus, as a byproduct of our proof of Theorem 6 we also provide a new proof of the main fixed point result for planar maps in [57], without the need to rely on properties of plane topology.

Then, we give some variants of Theorem 6 which are analogous to the different forms in which the Schauder fixed point theorem is usually presented. In Section 3 we investigate an abstract fixed point property for topological spaces which express in a more abstract fashion the content of Theorem 6 and its variants. An analysis of such a new fixed point property allows (like in the case of the classical fixed point property) to prove that it is invariant under homeomorphisms as well as it is preserved under continuous retractions. This in turns, permits to obtain some general results in which we produce fixed points for maps defined on topological cylinders. By topological cylinders we mean sets which are obtained from a cylinder like $\mathcal{B}[a, R]$ after a deformation given by a homeomorphism. For instance, the following result (see Corollary 3 of Section 3.1) is obtained.

THEOREM 5. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[0,1] \times K$, via a homeomorphism $h : Z \to [0,1] \times K$. Define*

$$Z_l^- := h^{-1}(\{0\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Suppose that $\psi : Z \supseteq D_\psi \to Z$ is a map which is continuous on a set $\mathcal{D} \subseteq D_\psi$ and assume the following property is satisfied:

> *there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\gamma) \cap Z_l^- \neq \emptyset$, $\phi(\gamma) \cap Z_r^- \neq \emptyset$.*

Then there exists a fixed point $\tilde{z}$ of ψ with $\tilde{z} \in \mathcal{D}$ (actually, $\tilde{z} \in \mathcal{W}$).

The possibility of studying topological cylinders (instead of topological rectangles like in [59, 60]) open the way toward an extension of the results about oriented rectangles presented in Section 1.3 to higher dimensional objects possessing a privileged direction. Thus we conclude the paper with a list of possible applications to maps which stretch the paths along a direction in a $(1, \mathrm{N}-1)$-*rectangular cell* (see Section 4.1 for the corresponding definition).

1.5. Notation

Throughout the paper, the following notation is used. Let Z be a topological space and let $A \subseteq B \subseteq Z$. By $\mathrm{cl}_B A$ and $\mathrm{int}_B A$ we mean, respectively, the closure and the interior of A relatively to B (that is, as a subset of the topological space B with the topology inherited by Z). When no confusion may occur, we also set cl$A$ and int$A$ for $\mathrm{cl}_Z A$ and $\mathrm{int}_Z A$, respectively.

For a metric space (X, d), we denote by $B(x_0, R) := \{x \in X : d(x, x_0) < r\}$ the open ball of center $x_0 \in X$ and radius $r > 0$ and by $B[x_0, R] := \{x \in X : d(x, x_0) \leq r\}$ the corresponding closed ball. Given a map $\psi : X \supseteq D_\psi \to Y$, with X, Y metric spaces and a given subset $\mathcal{D}$ of the domain D_ψ of ψ, we say that ψ is compact on $\mathcal{D}$ if it is continuous on $\mathcal{D}$ and $\psi(\mathcal{D})$ is relatively compact in Y, that is, $\mathrm{cl}(\psi(\mathcal{D}))$ is compact.

Let Z be a topological space, let $\theta_1 : [a_1, b_1] \to Z$ and $\theta_2 : [a_2, b_2] \to Z$ be two continuous mappings (parameterized curves). We write $\theta_1 \sim \theta_2$ if there is a homeomorphism h of $[a_1, b_1]$ onto $[a_2, b_2]$ (a change of variable in the parameter) such that $\theta_2(h(t)) = \theta_1(t),\ \forall\, t \in [a_1, b_1]$. It is easy to check that $\sim$ is in fact an equivalence relation and that $\theta_1([a_1, b_1]) = \theta_2([a_2, b_2])$ whenever $\theta_1 \sim \theta_2$. By a *path* γ *in* Z we mean (formally) the equivalence class $\gamma = [\theta]$ of a continuous parameterized curve $\theta : [a, b] \to Z$. In this case, with small abuse in the notation, we write $\gamma \subseteq Z$. Since the image set $\theta([a, b])$ is the same for each $\theta : [a, b] \to Z$ with $\gamma = [\theta]$, the set

$$\bar{\gamma} := \{\theta([a, b]) : \theta \in \gamma\}$$

is well defined. Given a set $A \subseteq Z$ and a path $\gamma \subseteq Z$, we write $\gamma \cap A \neq \emptyset$ to mean that $\bar{\gamma} \cap A \neq \emptyset$, that is, for every parameterized curve θ representing γ we have that $\theta(t) \in A$ for some t in the interval-domain of θ. Given a path $\sigma \subseteq Z$, we say that $\gamma \subseteq Z$ is a *sub-path* of σ and write $\gamma \subseteq \sigma$ if there is $\theta : [a, b] \to Z$ with $[\theta] = \sigma$ such that the restriction $\theta|_{[c,d]}$, for some $[c, d] \subseteq [a, b]$, represents γ. According to these positions, given the paths $\gamma, \sigma \subseteq Z$ and a set $W \subset Z$, the condition $\gamma \subseteq \sigma \cap W$, means that γ *is a sub-path of* σ *with values in* W. If Z, Y are topological spaces and $\phi : Z \supseteq D_\phi \to Y$ is a continuous map, then for any path $\gamma \subseteq D_\phi$ and $\theta : [a, b] \to D_\phi$ such that $[\theta] = \gamma$, we have that $\phi \circ \theta : [a, b] \to Y$ is a continuous map. It is easy to check that $\phi \circ \theta_1 \sim \phi \circ \theta_2$ when $\theta_1 \sim \theta_2$ and therefore $\phi(\gamma) := [\phi \circ \theta]$ is well defined.

At last we recall a known definition. Let Z be a topological space. We say that Z is *arcwise connected* if, given any two points $P, Q \in Z$ with $P \neq Q$, there is a continuous map $\theta : [a, b] \to Z$ such that $\theta(a) = P$ and $\theta(b) = Q$. In such a situation, we'll also write $P, Q \in \gamma$, where $\gamma = [\theta]$. In the case of a Hausdorff topological space Z, the image set $\theta([a, b])$ turns out to be a locally connected metric continuum (a Peano space according to [29]). Then, the above definition of arcwise connectedness is equivalent to the fact that, given any two points $P, Q \in Z$ with $P \neq Q$, there exists an *arc* (that is the homeomorphic image of a compact interval) contained in Z and having P and Q as extreme points (see, e.g., [18, p.29], [29, pp.115–131] or [71]).

2. A fixed point theorem in normed spaces and its variants

2.1. Main results

Let $(X, \|\cdot\|)$ be a normed space and suppose that

$$\phi = (\phi_1, \phi_2) : \mathbb{R} \times X \supseteq D_\phi \to \mathbb{R} \times X$$

is a map (not necessarily continuous on its whole domain D_ϕ even if, in the sequel, we assume the continuity of ϕ on some relevant subset $\mathcal{D}$ of D_ϕ).

Let $\mathcal{D} \subseteq D_\phi$ be a given set (in our applications we'll usually take $\mathcal{D}$ closed, for instance, $\mathcal{D} = \mathcal{W}$ of Theorem 6 below, but such an assumption for the moment is not required). We are looking for fixed points of ϕ belonging to $\mathcal{D}$, i.e., we want to prove the existence of a pair $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{D}$ which solves the equation

$$\begin{cases} t = \phi_1(t, x) \\ x = \phi_2(t, x). \end{cases}$$

Our first result is the following.

THEOREM 6. *Let* $\mathcal{B}[a, R] := [-a, a] \times B[0, R]$ *and define*

$$\mathcal{B}_l := \{(-a, x) : \|x\| \le R\}, \quad \mathcal{B}_r := \{(a, x) : \|x\| \le R\}$$

the left and the right bases of the cylinder $\mathcal{B}[a, R]$. *Assume that*

$$\phi \text{ is compact on } \mathcal{D} \cap \mathcal{B}[a, R]$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{B}[a, R]$ *such that the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{B}[a, R]$ *with* $\sigma \cap \mathcal{B}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{B}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{B}[a, R]$ *and* $\phi(\gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{B}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. First of all we observe that, as a consequence of Dugundji Extension Theorem and Mazur's Lemma (see, e.g., [64, p.22] in the case of Banach spaces or [16, Th.2.5, p.56] for a general situation), there exists a *compact* operator $\tilde{\phi}$ defined on $\mathbb{R} \times X$ which extends ϕ restricted to $\mathcal{W}$, i.e.

$$\tilde{\phi} : \mathbb{R} \times X \to \mathbb{R} \times X, \quad \tilde{\phi}|_{\mathcal{W}} = \phi|_{\mathcal{W}}.$$

Consider also the projection

$$P_R : X \to B[0, R], \quad P_R(x) := x \, \min\{1, R\,\|x\|^{-1}\}$$

and define the compact operator

$$\psi = (\psi_1, \psi_2), \quad \psi_1(t, x) := \tilde{\phi}_1(t, x), \quad \psi_2(t, x) := P_R(\tilde{\phi}_2(t, x))$$

Note that if $\bar{z} = (\bar{t}, \bar{x})$ is a fixed point of ψ with

$$\bar{z} \in \mathcal{W} \quad \text{and} \quad \phi_2(\bar{t}, \bar{x}) \in B[0, R], \tag{6}$$

then $\bar{t} = \psi_1(\bar{z}) = \tilde{\phi}_1(\bar{z}) = \phi_1(\bar{z})$ and $\bar{x} = \psi_2(\bar{z}) = P_R(\tilde{\phi}_2(\bar{z})) = P_R(\phi_2(\bar{z})) = \phi_2(\bar{t}, \bar{x})$, so that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ.

We study now the auxiliary fixed point problem

(7) $$x = \psi_2(t,x), \quad x \in X$$

where we take, for a moment, $t \in [-a,a]$ as a parameter.

Observe that, by definition, $\psi_2(t,x) \in B[0,R]$ for every (t,x) and therefore, for any $r > R$, it follows that

$$x - \psi_2(t,x) \neq 0, \quad \forall\, t \in [-a,a],\ \forall\, x \in \partial B(0,r).$$

Thus the Leray−Schauder topological degree

$$d_0 := \deg(I - \psi_2(t,\cdot), B(0,r), 0)$$

is well defined and is constant with respect to $t \in [-a,a]$. Using the compact homotopy $h_\lambda(x)$ defined by

$$(\lambda, x) \mapsto x - \lambda\psi_2(t,x), \quad \text{with } \lambda \in [0,1] \text{ and } x \in B[0,r]$$

we find that $h_\lambda(x) \neq 0$ for every $\lambda \in [0,1]$ and $x \in \partial B(0,r)$ and therefore $d_0 = \deg(I, B(0,r), 0) = 1$. Hence, the Leray−Schauder Théorème Fondamental [42] implies that the solution set

$$\Sigma := \{(t,x) \in [-a,a] \times B(0,r) : x = \psi_2(t,x)\}$$

is nonempty and contains a continuum (compact and connected set) $\mathcal{S}$ such that

$$p_1(\mathcal{S}) = [-a,a],$$

where we have denoted by $p_1 : \mathbb{R} \times X \to \mathbb{R}$, $p_1(t,x) = t$, the projection of the product space onto its first factor (see also [44] for more information about this fundamental result). Since the projection of $\mathcal{S}$ onto the t-axis covers the interval $[-a,a]$, we obtain

$$\mathcal{S} \cap \mathcal{B}_l \neq \emptyset, \quad \mathcal{S} \cap \mathcal{B}_r \neq \emptyset.$$

By the definition of P_R it is clear also that

(8) $$p_2(\mathcal{S}) \subseteq B[0,R] \subseteq B(0,r),$$

where we have denoted by $p_2 : \mathbb{R} \times X \to X$, $p_2(t,x) = x$, the projection of the product space onto its second factor.

Let now $\varepsilon \in]0, r-R[$ be a fixed number and consider a covering of $\mathcal{S}$ by a finite number of open balls of the form $]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \varepsilon)$, with $(t_i, x_i) \in \mathcal{S}$. Without loss of generality, we can suppose that $-a \le t_1 < t_2 \dots t_{i-1} < t_i \dots t_N \le a$, where N is the number of the balls required for the covering. The set

$$\mathcal{U}_\varepsilon := \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \varepsilon) \subseteq]-a-\varepsilon, a+\varepsilon[\times B(0,r),$$

is open and connected. Hence, it is arcwise connected as well. Therefore, there is a continuous map $\theta : [0,1] \to \mathcal{U}_\varepsilon$ with $\theta(0) \in \mathcal{B}_l$ and $\theta(1) \in \mathcal{B}_r$ and, without loss of generality (i.e., possibly cutting off some points of the interval and changing the parameter for the curve) we can also assume that

$$\theta_1(s) := p_1(\theta(s)) \in [-a, a], \quad \forall s \in [0,1].$$

Next, we define the new curve $\zeta(s) = (\zeta_1(s), \zeta_2(s))$, with

$$\zeta_1(s) := p_1(\theta(s)) = \theta_1(s), \quad \zeta_2(s) := P_R(p_2(\theta(s))) = P_R(\theta_2(s))$$

and observe that $\zeta(\cdot)$ satisfies the following properties:

(I_1) $\zeta(s) \in \mathcal{V}_\varepsilon \cap \mathcal{B}[a, R], \quad \forall s \in [0,1]$;

(I_2) $\zeta(0) \in \mathcal{B}_l$ and $\zeta(1) \in \mathcal{B}_r$;

where we have set

$$\mathcal{V}_\varepsilon := \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, 2\varepsilon).$$

To check (I_1), let us set $x := \zeta_2(s)$ and assume that $\|x\| > R$ as well as $x \in B(x_i, \varepsilon)$, for some i. Then,

$$\begin{aligned} \left\| \frac{Rx}{\|x\|} - x_i \right\| &= \|Rx - \|x\|\, x_i\| / \|x\| \\ &\le \frac{R}{\|x\|} \|x - x_i\| + (\|x\| - R) \frac{\|x_i\|}{\|x\|} < 2\varepsilon. \end{aligned}$$

The proofs of all the remaining cases for the verification of (I_1) are obvious.

From (I_1) and (I_2), it follows that the path $\sigma := [\zeta]$ is contained in the cylinder $\mathcal{B}[a, R]$ and it has a nonempty intersection with the left and the right bases of $\mathcal{B}[a, R]$. Then, by hypothesis (H), we know that there exists a sub-path γ of σ, such that $\gamma \subseteq \mathcal{W}$ with $\phi(\gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{B}_r \neq \emptyset$. Let $\xi = (\xi_1, \xi_2) : [0,1] \to \mathbb{R} \times X$ be a continuous map such that $[\xi] = \gamma$. By the above assumptions, we have that

(J_1) $\xi(s) \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \forall s \in [0,1]$;

(J_2) $\phi(\xi(0)) \in \mathcal{B}_l$ and $\phi(\xi(1)) \in \mathcal{B}_r$;

(J_3) $\phi(\xi(s)) \in \mathcal{B}[a, R], \quad \forall s \in [0,1]$;

are satisfied.

We consider now the continuous map $g : [0, 1] \ni s \mapsto \xi_1(s) - \phi_1(\xi(s))$, where $\xi_1(s) = p_1(\xi(s))$. Since $\xi(s) \in \bar{\gamma} \subseteq \bar{\sigma}$, we have that $\xi_1(0) \geq -a$ and therefore $g(0) \geq -a - (-a) = 0$. Similarly, one can check that $g(1) \leq a - a = 0$. By Bolzano's Theorem, we conclude that there exists some $\hat{s} = \hat{s}_\varepsilon \in [0, 1]$ such that, setting

$$\hat{t} = \hat{t}_\varepsilon := \xi_1(\hat{s}), \quad \hat{x} = \hat{x}_\varepsilon := \xi_2(\hat{s}), \quad \hat{z} = \hat{z}_\varepsilon := (\hat{t}, \hat{x}),$$

we find that

$$\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \hat{t} = \phi_1(\hat{z}), \quad \phi_2(\hat{z}) \in B[0, R].$$

By the definition of $\tilde{\phi}$ and ψ, it is clear that

$$\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \hat{t} = \psi_1(\hat{z}), \quad \phi_2(\hat{z}) \in B[0, R].$$

Moreover, for each $\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}$, there is $\hat{z}_i \in \mathcal{S}$ such that $||\hat{z} - \hat{z}_i|| < 2\varepsilon$.
Then, letting $\varepsilon = \varepsilon_n \searrow 0$ and passing to a subsequence on the corresponding $\hat{z}_n$'s (thanks to the compactness of $\mathcal{S}$), we can find a point

$$\bar{z} = (\bar{t}, \bar{x}) \in \mathcal{S}$$

such that (by the continuity of ϕ and ψ and the closure of $\mathcal{W}$)

$$\bar{z} \in \mathcal{S} \cap \mathcal{W}, \quad \bar{t} = \psi_1(\bar{z}), \quad \phi_2(\bar{z}) \in B[0, R],$$

follows. The fact that $\mathcal{S}$ is contained in the solution set Σ of (7) implies that

$$\bar{x} = \psi_2(\bar{t}, \bar{x})$$

and therefore $\bar{z} \in \mathcal{W}$ is a fixed point of ψ (since we have already proved that $\bar{t} = \psi_1(\bar{t}, \bar{x})$). The fact that $\phi_2(\bar{t}, \bar{x}) \in B[0, R]$, implies (in view of (6) and the remarks at the beginning of the proof) that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ. □

REMARK 1. The first part of the proof of Theorem 6 can be used to obtain the following result where we use a slightly different expansive condition which is inspired from [33, 36, 37]. We recall that here by a *continuum* we mean a compact and connected set.

THEOREM 7. *With the notation of Theorem 6, assume that ϕ is compact on $\mathcal{D} \cap \mathcal{B}[a, R]$ and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{B}[a, R]$ such that the assumption*

(H') *for every continuum $\sigma \subseteq \mathcal{B}[a, R]$ with $\sigma \cap \mathcal{B}_l \neq \emptyset$ and $\sigma \cap \mathcal{B}_r \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\Gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\Gamma) \cap \mathcal{B}_r \neq \emptyset$,*

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D}$, with $\phi(\tilde{z}) = \tilde{z}$.

Proof. We follow the proof of Theorem 6 till to (8). Now, assumption (H') guarantees the existence of a continuum $\Gamma \subseteq \mathcal{S} \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\Gamma) \cap \mathcal{B}_l \neq \emptyset$,

$\phi(\Gamma) \cap \mathcal{B}_r \neq \emptyset$. This means that there are points $Q_1 = (q_1^1, q_2^1),\ Q_2 = (q_1^2, q_2^2) \in \Gamma$ such that $\phi_1(Q_1) = -a$ and $\phi_1(Q_2) = a$. This implies that $p_1(Q_1) - \phi_1(Q_1) = q_1^1 + a \geq -a + a = 0$ and $p_1(Q_2) - \phi_1(Q_2) = q_1^2 - a \leq a - a = 0$. By the Bolzano's Theorem we can conclude that there exists a point $\bar{z} = (\bar{t}, \bar{x}) \in \Gamma \subseteq \mathcal{S} \cap \mathcal{W}$ such that $\bar{t} = \psi_1(\bar{z})$ and, by (8), $\phi_2(\bar{z}) \in B[0, R]$. At this point, we can complete our argument as in the proof of Theorem 6. Indeed, the fact that $\mathcal{S}$ is contained in the solution set Σ of (7) implies that

$$\bar{x} = \psi_2(\bar{t}, \bar{x})$$

and therefore $\bar{z} \in \mathcal{W}$ is a fixed point of ψ. The fact that $\phi_2(\bar{t}, \bar{x}) \in B[0, R]$, implies (in view of (6) and the remarks at the beginning of the proof of Theorem 6) that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ. □

REMARK 2. In our theorems we have confined ourselves to the case of compact maps. Extensions can be given to more general operators like, locally compact, k-contractive, etc., provided that a decent degree theory is available (see [15, 27, 50] for the corresponding definitions).

2.2. Results related to Theorem 6

Like in the case of the Schauder fixed point theorem, we give now some variants of Theorem 6 (see Theorem 8 and Theorem 9 below). As in Theorem 6 we assume that X is a normed space and

$$\phi : \mathbb{R} \times X \supseteq D_\phi \to \mathbb{R} \times X$$

is a map which is continuous on a set $\mathcal{D} \subseteq D_\phi$. For the subsequent proofs, we systematically check condition (H) by taking as a representation of the path σ a continuous curve $\theta(s)$ which is parameterized on the interval $[0, 1]$ and look for a suitable restriction of $\theta(s)$ with $s \in [s_0, s_1] \subseteq [0, 1]$, as a representation of a sub-path $\gamma \subseteq \sigma$.
We start with a preliminary lemma.

LEMMA 1. *Let $\rho = (\rho_1, \rho_2) : \mathcal{B}[a, R] := [-a, a] \times B[0, R] \to \mathcal{B}[a, R]$ be a continuous map such that, for each $t \in [-a, a]$, $\rho_1(t, x) = t$ and $\rho_2(t, \cdot)$ is a retraction of $B[0, R]$ onto its image. Define*

$$\mathcal{R} := \cup_{t \in [-a,a]} \{\rho(t, x) : x \in B[0, R]\} = \rho(\mathcal{B}[a, R])$$

and

$$\mathcal{R}_l := \rho(\mathcal{B}_l),\ \mathcal{R}_r := \rho(\mathcal{B}_r).$$

Assume that

$$\phi \textit{ is compact on } \mathcal{D} \cap \mathcal{R}$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$ such that the assumption

(H) *for every path $\sigma \subseteq \mathcal{R}$ with $\sigma \cap \mathcal{R}_l \neq \emptyset$ and $\sigma \cap \mathcal{R}_r \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\gamma) \subseteq \mathcal{R}$ and $\phi(\gamma) \cap \mathcal{R}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{R}_r \neq \emptyset$,* *holds. Then there exists* $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. Given the compact operator $\phi : \mathcal{D} \cap \mathcal{R} \to \mathbb{R} \times X$ we define

$$\psi(t, x) := \phi(\rho(t, x)) = \phi(t, \rho_2(t, x)), \quad \text{for } (t, x) \in \mathcal{D}' := \rho^{-1}(\mathcal{D} \cap \mathcal{R}).$$

Clearly, ψ is compact on $\mathcal{D}' = \mathcal{D}' \cap \mathcal{B}[a, R]$. We also define

$$\mathcal{W}' := \rho^{-1}(\mathcal{W}) \subseteq \mathcal{B}[a, R] =: \mathcal{B}.$$

Consider now a continuous parameterized curve $\theta = (\theta_1, \theta_2) : [0, 1] \to \mathcal{B}[a, R]$ such that $\theta(0) \in \mathcal{B}_l$ (that is, $\theta_1(0) = -a$) and $\theta(1) \in \mathcal{B}_r$ (that is, $\theta_1(1) = a$). Then for the curve $\vartheta : [0, 1] \ni s \mapsto \rho(\theta(s))$, it holds that

$$\vartheta(s) \in \mathcal{R}, \ \forall s \in [0, 1] \quad \text{and} \quad \vartheta(0) \in \mathcal{R}_l\,, \ \vartheta(1) \in \mathcal{R}_r\,.$$

By assumption (H) referred to $\mathcal{R}$, there exists a restriction of ϑ to an interval $[s_0, s_1] \subseteq [0, 1]$ such that $\vartheta(s) \in \mathcal{W}$ for every $s \in [s_0, s_1]$ and, moreover,

$$\phi(\vartheta(s)) \in \mathcal{R}, \ \forall s \in [s_0, s_1],$$

as well as

$$\phi(\vartheta(s_0)) \in \mathcal{R}_l \quad \text{and} \quad \phi(\vartheta(s_1)) \in \mathcal{R}_r\,, \quad \text{or} \quad \phi(\vartheta(s_1)) \in \mathcal{R}_l \quad \text{and} \quad \phi(\vartheta(s_0)) \in \mathcal{R}_r\,.$$

Just to fix one of the two possible cases for the rest of the proof, suppose that the first possibility occurs (the treatment of the other case is exactly the same, modulo minor changes in the role of s_0 and s_1). Then, by the definition of ϑ, $\mathcal{W}'$ and ψ, we can also write that

$$\theta(s) \in \mathcal{W}', \quad \forall s \in [s_0, s_1],$$

and

$$\psi(\theta(s)) \in \mathcal{R} \subseteq \mathcal{B}, \ \forall s \in [s_0, s_1]$$

$$\psi(\theta(s_0)) \in \mathcal{R}_l \subseteq \mathcal{B}_l\,, \ \psi(\theta(s_1)) \in \mathcal{R}_l \subseteq \mathcal{B}_r\,.$$

We have thus proved that assumption (H) of Theorem 6 is satisfied with respect to the operator ψ and the cylinder $\mathcal{B}[a, R]$ and therefore Theorem 6 guarantees that there exists for ψ a fixed point $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W}' \subseteq \mathcal{D}'$, with $\psi(\tilde{z}) = \tilde{z}$. As a last step, we just recall that the range of ψ coincides with the range of ϕ and that ρ (as a retraction) is the identity on $\mathcal{R}$. This implies that $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$ with $\phi(\tilde{z}) = \tilde{z}$. □

THEOREM 8. *Let* $C \neq \emptyset$ *be a closed convex subset of the normed space* X. *Let* $\mathcal{C} := [-a, a] \times C$ *and define*

$$\mathcal{C}_l := \{(-a, x) : x \in C\}, \quad \mathcal{C}_r := \{(a, x) : x \in C\}.$$

Assume that

$$\phi \text{ is compact on } \mathcal{D} \cap \mathcal{C}$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{C}$ *such that the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{C}$ *with* $\sigma \cap \mathcal{C}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{C}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{C}$ *and* $\phi(\gamma) \cap \mathcal{C}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{C}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{C}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. By assumption, the operator $\phi = (\phi_1, \phi_2)$ is compact on $\mathcal{D} \cap \mathcal{C}$, hence, $\phi(\mathcal{D} \cap \mathcal{C})$ is bounded in $\mathbb{R} \times X$. In particular, there is $R > 0$ such that,

$$||\phi_2(t, x)|| < R, \quad \forall z = (t, x) \in \mathcal{D} \cap \mathcal{C}.$$

As a consequence of the Dugundji Extension Theorem, the closed convex set $C' := C \cap B[0, R]$ is a retract of $B[0, R]$ (actually, it is a retract of the whole space X, but for us it is more convenient to restrict the retraction to $B[0, R]$). We denote by $\varrho : B[0, R] \to C'$ such a continuous retraction.

If we define now $\mathcal{C}' := [-a, a] \times C'$ and

$$\mathcal{C}'_l := \{(-a, x) : x \in C'\}, \quad \mathcal{C}'_r := \{(a, x) : x \in C'\},$$

as well as $\mathcal{D}' := \mathcal{D} \cap \mathcal{C}'$ and $\mathcal{W}' := \mathcal{W} \cap \mathcal{C}'$, we find that

(H) for every path $\sigma \subseteq \mathcal{C}'$ with $\sigma \cap \mathcal{C}'_l \neq \emptyset$ and $\sigma \cap \mathcal{C}'_r \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}'$ with $\phi(\gamma) \subseteq \mathcal{C}'$ and $\phi(\gamma) \cap \mathcal{C}'_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{C}'_r \neq \emptyset$,

holds. At last, we define the continuous retraction

$$\rho = (\rho_1, \rho_2) : \mathcal{B}[a, R] := [-a, a] \times B[0, R] \to \mathcal{B}[a, R],$$

$$\rho_1(t, x) = t, \; \rho_2(t, x) = \varrho(x)$$

and easily check that now the set $\mathcal{C}'$ plays here the same role as the set $\mathcal{R}$ in Lemma 1. Then, according to Lemma 1, there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W}' \subseteq \mathcal{W}$, with $\phi(\tilde{z}) = \tilde{z}$. The proof is complete. □

THEOREM 9. *Let* $K \neq \emptyset$ *be a compact convex subset of the normed space* X. *Let* $\mathcal{K} := [-a, a] \times K$ *and define*

$$\mathcal{K}_l := \{(-a, x) : x \in K\}, \quad \mathcal{K}_r := \{(a, x) : x \in K\}.$$

Suppose that there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$ *such that* ϕ *is continuous on* $\mathcal{W}$ *and the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{K}$ *with* $\sigma \cap \mathcal{K}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{K}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{K}$ *and* $\phi(\gamma) \cap \mathcal{K}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{K}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. This result is an immediate consequence of Theorem 8, with the position $C := K$. □

REMARK 3. Variants of Theorem 8 and Theorem 9 can be obtained, as a consequence of Theorem 7, using condition (H') instead of condition (H). For instance, Theorem 9 could be accompanied by the following.

THEOREM 10. *Under the same positions of Theorem 9, suppose that there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$ such that ϕ is continuous on $\mathcal{W}$ and the assumption*

(H') *for every continuum $\sigma \subseteq \mathcal{K}$ with $\sigma \cap \mathcal{K}_l \neq \emptyset$ and $\sigma \cap \mathcal{K}_r \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{K}$ and $\phi(\Gamma) \cap \mathcal{K}_l \neq \emptyset$, $\phi(\Gamma) \cap \mathcal{K}_r \neq \emptyset$,*

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$, with $\phi(\tilde{z}) = \tilde{z}$.

3. Extensions, remarks and consequences

3.1. The "stretching along the paths" fixed point property

Let Z be a topological space. According to a well known definition, *Z has the fixed point property* (FPP) if every continuous map of Z into itself has at least a fixed point. The FPP is invariant by homeomorphisms and it is preserved under continuous retractions. Thus, by the Brouwer fixed point theorem, we know that any topological space which is homeomorphic to (a retract of) a closed ball of a finite dimensional normed space has the FPP. It is the aim of this section to show that something similar (even if not exactly the same) holds with respect to the assumption of "stretching along the paths" (H) and the corresponding fixed point result in Theorem 6.

We consider now the following situation.

DEFINITION 1. *Assume that Z is a topological space and Z_l^-, Z_r^- are two nonempty disjoint subsets of Z. We set*

$$Z^- := Z_l^- \cup Z_r^-$$

and define

$$\widetilde{Z} := (Z, Z^-).$$

We call $\widetilde{Z}$ a two-sided oriented space or simply an oriented space. In view of our applications below which concern the case of arcwise connected spaces and where the family of paths connecting Z_l^- to Z_r^- is involved, we call $\widetilde{Z}$ a path-oriented space when Z is arcwise connected.
We say that $\widetilde{Z}$ has the fixed point property for maps stretching along the paths (in the sequel referred as FPP-γ) if Z is arcwise connected and, for every pair $(\mathcal{D}, \psi)$, satisfying the following conditions:

- *(i_1)* $\mathcal{D} \subseteq Z$;
- *(i_2)* $\psi : \mathcal{D} \to Z$ *is continuous;*
- *(i_3) there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \cap Z_l^- \neq \emptyset$ and $\phi(\gamma) \cap Z_r^- \neq \emptyset$;*

there exists at least a fixed point of ψ in $\mathcal{D}$.

DEFINITION 2. *Suppose we have two arcwise connected topological spaces Z, Y and assume that Z_l^-, Z_r^- are two nonempty disjoint subsets of Z with $Z^- = Z_l^- \cup Z_r^-$, as well as Y_l^-, Y_r^- are two nonempty disjoint subsets of Y, with $Y^- = Y_l^- \cup Y_r^-$. Defining, as above $\widetilde{Z} = (Z, Z^-)$ and $\widetilde{Y} = (Y, Y^-)$ the corresponding path-oriented spaces, we say that the pair $(\mathcal{D}, \psi)$ stretches $\widetilde{Z}$ to $\widetilde{Y}$ along the paths and write*

$$(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\!\rightsquigarrow \widetilde{Y},$$

if the conditions

- *(j_1)* $\mathcal{D} \subseteq Z$;
- *(j_2)* $\psi : \mathcal{D} \to Y$ *is continuous;*
- *(j_3) there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \cap Y_l^- \neq \emptyset$ and $\phi(\gamma) \cap Y_r^- \neq \emptyset$;*

hold.
Accordingly, we have that $\widetilde{Z}$ has the FPP-γ if and only if for every pair $(\mathcal{D}, \psi)$ with $(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\!\rightsquigarrow \widetilde{Z}$, there is at least a fixed point of ψ in $\mathcal{D}$.

REMARK 4. The definition of a map stretching along the paths was introduced in [56, 57] and refined in [59, 60] in the case of two-dimensional oriented cells. Our Definition 2 above is a generalization of the previous cited one as it reduces to [60] in the situation considered therein. We note that in [60], as well as in the other preceding papers, the map ψ was allowed to be defined possibly on some larger domains. However, up to a restriction, we can always enter in the case of Definition 2 when we consider the situation described in [60].
In the definition of path-oriented space, as well as in the subsequent stretching condition, the order in which we label the two sets Z_l^- and Z_r^- (or Y_l^- and Y_r^-) has no effect at all.
We also point out that given two nonempty disjoint sets W_l and W_r of a topological space W, the condition that there exists a path $\sigma \subseteq W$ with $\sigma \cap W_l \neq \emptyset$ and $\sigma \cap W_r \neq \emptyset$ is equivalent to the existence of a continuous map $\theta : [0, 1] \to W$ with $\theta(0) \in W_l$ and $\theta(1) \in W_r$.
The choice of the notation $\widetilde{Z}$ instead of $\widehat{Z}$ (previously considered in Section 1.2 and next again in Section 4.1) comes from the fact that, even if all the applications we present here are for the $\widehat{[\cdot]}$-sets, nonetheless, the oriented spaces $\widetilde{[\cdot]}$ are, in principle, more general. Thus we prefer to think to the $\widehat{[\cdot]}$-sets as some particular cases of the $\widetilde{[\cdot]}$-sets.
Finally, we mention that some analogous definitions, previously introduced in the literature (see, for instance, the concept of quadrilateral set given in [36]) also fit with our definition of oriented space.

The next two lemmas extend to the case of the FPP-γ two corresponding classical results about the usual fixed point property. Their proof is quite standard and therefore it is omitted.

LEMMA 2. *Let Z, Y be two arcwise connected topological spaces and let $h : Z \to Y$ be a homeomorphism. Suppose Z_l^- and Z_r^- are nonempty disjoint subsets of Z and set $Y_l^- = h(Z_l^-)$, $Y_r^- = h(Z_r^-)$. Then $\widetilde{Z}$ has the FPP-γ if and only if $\widetilde{Y}$ has the FPP-γ.*

Proof. We leave the proof as an exercise. □

LEMMA 3. *Let Z, Y be two arcwise connected topological spaces with $Y \subseteq Z$ and let $r : Z \to Y$ be a continuous retraction. Suppose Y_l^- and Y_r^- are nonempty disjoint subsets of Y and set $Z_l^- = r^{-1}(Y_l^-)$, $Z_r^- = r^{-1}(Y_r^-)$. Then $\widetilde{Y}$ has the FPP-γ if $\widetilde{Z}$ has the FPP-γ.*

Proof. The proof follows the same argument (mutatis mutandis) of that of Lemma 1 and therefore it is omitted. □

COROLLARY 3. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[-1, 1] \times K$, via a homeomorphism $h : Z \to [-1, 1] \times K$. Define*

$$Z_l^- := h^{-1}(\{-1\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Then $\widetilde{Z}$ has the FPP-γ.

Lemma 2 and Lemma 3 together with Theorem 6 (or its variants) permit to give some straightforward examples with some geometrical meaning. For simplicity, we confine ourselves to subsets of a finite dimensional space E.

EXAMPLE 1. Let $Z \subseteq E$ be a compact set which is homeomorphic to the closed unit ball $B[0, 1] \subseteq \mathbb{R}^{\mathrm{N}}$, with $\mathrm{N} \geq 1$. Let $P, Q \in Z$ with $P \neq Q$ be two given points. Let $\psi : E \supseteq D_\psi \to E$ be a continuous map and suppose that $\mathcal{E} \subseteq D_\psi$ is a closed set such that *for every path $\sigma \subseteq Z$, with $P, Q \in \sigma$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{E}$ with $\psi(\sigma) \subseteq Z$ and $P, Q \in \psi(\gamma)$*. Then ψ has at least a fixed point in $\mathcal{E} \cap Z$.

Proof. We discuss only the case $\mathrm{N} \geq 2$, since for $\mathrm{N} = 1$, the result is obvious. Let us consider the cylinder

$$\mathcal{C} := \{(x_1, \ldots, x_{\mathrm{N}-1}, x_{\mathrm{N}}) : ||(x_1, \ldots, x_{\mathrm{N}-1})|| \leq 1, \ |x_{\mathrm{N}}| \leq 1\}$$

on which we select as a right and left sides the south and the north bases respectively:

$$\mathcal{C}_l := \{x = (x_1, \ldots, , x_{\mathrm{N}}) \in \mathcal{C} : x_{\mathrm{N}} = -1\},$$

$$\mathcal{C}_r := \{x = (x_1, \ldots, , x_{\mathrm{N}}) \in \mathcal{C} : x_{\mathrm{N}} = 1\}$$

and we also set $\mathcal{C}^- := \mathcal{C}_l \cup \mathcal{C}_r$. Theorem 6 implies that the path-oriented space $\widetilde{\mathcal{C}} = (\mathcal{C}, \mathcal{C}^-)$ has the FPP-γ and therefore Lemma 3 ensures that the same fixed point property holds also with respect to retracts of $\widetilde{\mathcal{C}}$. The closed unit ball $B[0,1]$ is a retract of the cylinder $\mathcal{C}$ through the continuous map ϱ defined by

$$\varrho(x) = (x_1 \min\{1, \delta(x)\}, \dots, x_{N-1} \min\{1, \delta(x)\}, x_N),$$

where

$$\delta(x) = \delta(x_1, \dots, x_{N-1}, x_N) := \frac{\sqrt{1 - x_N^2}}{\sqrt{x_1^2 + \cdots + x_{N-1}^2}}.$$

Hence, the path oriented space $\widetilde{B} = (B, B^-)$, with $B = B[0,1]$, $B^- = B_l^- \cup B_r^-$, $B_l^- = \{\text{South pole}\}$ and $B_r^- = \{\text{North pole}\}$ has the FPP-γ. Finally, Lemma 2 implies the the FFP-γ holds for every oriented space in which the base space Z is homeomorphic to a closed ball and we select as Z_l^- and Z_r^- two different points of Z. This concludes the proof. □

EXAMPLE 2. Consider the cone

$$K = \{(x_1, \dots, x_k, x_{k+1}) : ||(x_1, \dots, x_k)|| \le x_{k+1} \le 1\} \subseteq \mathbb{R}^{k+1}$$

and select the point $0 = (0, \dots, 0, 0) \in K$ and the base $K_l = \{(x_1, \dots, x_k, 1) : ||(x_1, \dots, x_k)|| \le 1\} \subseteq K$. Let $Z \subseteq E$ be a compact set which is homeomorphic to K, by a homeomorphism $h : Z \to K$. Define $Z_r^- = h^{-1}(\{0\})$ and $Z_l^- = h^{-1}(K_l)$. Then $\widetilde{Z}$ has the FPP-γ.

Proof. It is possible to obtain our claim by suitably adapting the argument employed in the proof of Example 1. We omit the details. □

REMARK 5. We observe that one could define a fixed point property (say FPP-Γ) for maps satisfying a condition which extends property (H') to general (oriented) topological spaces. Then, after having obtained from Theorem 7 a result analogous to Lemma 2, the following corollary can be proved.

COROLLARY 4. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[-1,1] \times K$, via a homeomorphism $h : Z \to [-1,1] \times K$. Define*

$$Z_l^- := h^{-1}(\{-1\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Then, for every pair $(\mathcal{D}, \psi)$, satisfying the following conditions:

(i_1) $\mathcal{D} \subseteq Z$;

(i_2) $\psi : \mathcal{D} \to Z$ *is continuous;*

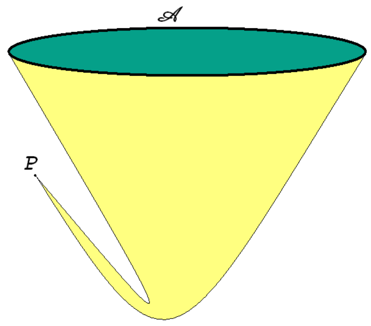

Figure 11: A possible illustration of Example 2 in $\mathbb{R}^3$, where we have denoted by P the point $h^{-1}(\{0\})$ and by $\mathcal{A}$ the surface $h^{-1}(K_l)$ of the deformed cone Z. According to our result there exists at least a fixed point $z = \psi(z) \in \mathcal{W}$, for any continuous map ψ defined on a closed subset $\mathcal{W}$ of Z and with values in Z having the property that any path σ in Z and joining P to $\mathcal{A}$ contains a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \subseteq Z$ and $\psi(\gamma)$ joining P to $\mathcal{A}$.

(i_3') there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every continuum $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\Gamma) \cap Z_l^- \neq \emptyset$ and $\phi(\Gamma) \cap Z_r^- \neq \emptyset$;

there exists at least a fixed point of ψ in $\mathcal{D}$.

In the present paper we do not further pursue the research in this direction and confine ourselves to the study of the stretching condition along the paths. Investigations toward the fixed point properties for maps satisfying an expansive conditions with respect to other kind of connected sets will be considered elsewhere.

3.2. Further definitions and consequences

As a next step, we give now some simple (but nevertheless useful) properties about the stretching along the paths condition. Unless otherwise specified, all the spaces involved are arcwise connected topological spaces. When we consider a triple (Z, Z_l^-, Z_r^-), we always assume that Z_l^- and Z_r^- are nonempty disjoint subsets of Z. First of all, we consider two further definitions.

DEFINITION 3. *Let $\widetilde{Z} = (Z, Z^-)$ and $\widetilde{Y} = (Y, Y^-)$ be two path-oriented spaces with Y a subspace of Z. We say that $\widetilde{Y}$ is a horizontal slab of $\widetilde{Z}$ and write*

$$\widetilde{Y} \subseteq_h \widetilde{Z},$$

if every path $\gamma \subseteq Y$ with $\gamma \cap Y_l^- \neq \emptyset$ and $\gamma \cap Y_r^- \neq \emptyset$ is such that $\gamma \cap Z_l^- \neq \emptyset$ and

$\gamma \cap Z_r^- \neq \emptyset$.
Similarly, we say that $\widetilde{Y}$ is a vertical slab of $\widetilde{Z}$ and write

$$\widetilde{Y} \subseteq_v \widetilde{Z},$$

if every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$ contains a sub-path $\gamma \subseteq Y$ such that $\gamma \cap Y_l^- \neq \emptyset$ and $\gamma \cap Y_r^- \neq \emptyset$.

REMARK 6. The definition of slabs generalizes the case of rectangles with horizontal and vertical sides parallel to the contracting and expanding directions in the Smale horseshoe (see, for instance, [72, Section 2.3]). In our general setting of a topological space Z oriented by the paths connecting two disjoint subsets Z_l^- and Z_r^- and in view of Theorem 6, we consider as horizontal-expanding the "direction" along (Z_l^-, Z_r^-) (of course, in a very vague sense and taking also into account the fact that in our setting "horizontal" and "vertical" are merely conventional terms). Definition 3 generalizes the analogous concepts of "slices" considered in [60] in the setting of oriented two-dimensional cells and recalled in Section 1.2 as well as some possibilities considered in [61] for N-dimensional cells (namely, the case in which there is a one-dimensional expansive direction). Note that our definitions are purely topological in nature and therefore we do not need (like in [72, Section 2.3]) the slabs to be described by means of graphs of Lipschitz functions. We refer to Figure 14 as a possible picture of horizontal and vertical slabs in a simple situation.

Having available in the general setting the definition of slabs, we can now borrow from [60] and [61] the next definition (compare also to the corresponding definition in Section 1.2).

DEFINITION 4. *Let $\widetilde{Z} = (Z, Z^-)$, $\widetilde{Y} = (Y, Y^-)$ and $\widetilde{X} = (X, X^-)$ be three path-oriented spaces with X, Y, Z subspaces of the same topological space W and $X \subseteq Y \cap Z$.*
We say that $\widetilde{Y}$ crosses $\widetilde{Z}$ in $\widetilde{X}$ and write

$$\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\},$$

if

$$\widetilde{X} \subseteq_h \widetilde{Z} \quad \text{and} \quad \widetilde{X} \subseteq_v \widetilde{Y}.$$

REMARK 7. As already remarked in [60] and [61], in our setting, the definition of $\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\}$, covers very general situations, in particular also when there is no way to define any kind of transversal intersection. A possible illustration is given in Figure 15 of Section 4.

LEMMA 4. *The following properties hold:*

(e_1) *if $(\mathcal{D}, \psi) : \widetilde{Z} \Leftrightarrow\!\!\!\rightsquigarrow \widetilde{Y}$ and $(\mathcal{E}, \phi) : \widetilde{Y} \Leftrightarrow\!\!\!\rightsquigarrow \widetilde{X}$, then $(\mathcal{F}, \phi \circ \psi) : \widetilde{Z} \Leftrightarrow\!\!\!\rightsquigarrow \widetilde{X}$ for $\mathcal{F} = \mathcal{D} \cap \psi^{-1}(\mathcal{E})$;*

(e_2) *if* $\phi : Z \to Y$ *is a homeomorphism such that* $\phi(Z_l^-) = Y_l^-$ *and* $\phi(Z_r^-) = Y_r^-$ *(or* $\phi(Z_l^-) = Y_r^-$ *and* $\phi(Z_r^-) = Y_l^-$*), then* $(Z, \phi) : \widetilde{Z} \Leftrightarrow \widetilde{Y}$ *;*

(e_3) *if* $(\mathcal{D}, \psi) : \widetilde{Z} \Leftrightarrow \widetilde{Y}$*, then* $(\mathcal{D} \cap X \cap \psi^{-1}(W), \psi) : \widetilde{X} \Leftrightarrow \widetilde{W}$*, for every* $\widetilde{X} \subseteq_h \widetilde{Z}$ *and every* $\widetilde{W} \subseteq_v \widetilde{Y}$ *;*

(e_4) *if* $(\mathcal{D}, \psi) : \widetilde{Z} \Leftrightarrow \widetilde{Y}$*, then* ψ *has a fixed point in* $\mathcal{D} \cap X$*, for every* $\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\}$*, having the FPP-*$\gamma$ *.*

Proof. The above properties follow immediately by the corresponding definitions. $\square$

Now we are in position to consider a sequence of spaces and maps and obtain a result which is in line with [34] and [73] and extend to a general setting some results [59, Theorem 2.2], [60, Theorem 4.2] previously obtained in the two-dimensional setting. For simplicity, we confine ourselves to the framework of compact metric spaces. This simplifies somehow our proofs. We point out, however, that some of the properties exposed in the next Theorem 11 would be still true in some more general situations.

THEOREM 11. *Suppose that there is a (double) sequence of path-oriented spaces*

$$(\widetilde{X}_k)_{k\in\mathbb{Z}} = ((X_k, X_k^-))_{k\in\mathbb{Z}} ,$$

where, for each $k \in \mathbb{Z}$, X_k *is a compact and arcwise connected metric space. Denote by* $(X_k^-)_l$ *and* $(X_k^-)_r$ *the two sides of* X_k^- *. Assume that there is a sequence of maps* $((\mathcal{D}_k, \psi_k))_{k\in\mathbb{Z}}$ *, such that*

$$(\mathcal{D}_k, \psi_k) : \widetilde{X}_k \Leftrightarrow \widetilde{X}_{k+1} , \quad \forall\, k \in \mathbb{Z}.$$

Then the following conclusions hold:

(a_1) *There is a sequence* $(w_k)_{k\in\mathbb{Z}}$ *with* $w_k \in \mathcal{D}_k$ *and* $\psi_k(w_k) = w_{k+1}$ *for all* $k \in \mathbb{Z}$*;*

(a_2) *For each* $j \in \mathbb{Z}$ *there is a compact set* $\mathcal{C}_j \subseteq \mathcal{D}_j$ *such that for each* $w \in \mathcal{C}_j$ *there exists a sequence* $(y_\ell)_{\ell\geq j}$ *, with* $y_\ell \in \mathcal{D}_\ell$ *and* $y_j = w$, $y_{\ell+1} = \psi_\ell(y_\ell)$ *for each* $\ell \geq j$*.*
The compact set $\mathcal{C}_j$ *satisfies the following separation property:*

$$\mathcal{C}_j \cap \sigma \neq \emptyset, \text{ for each path } \sigma \subseteq X_j \text{ with } \sigma \cap (X_k^-)_l \neq \emptyset \text{ and } \sigma \cap (X_k^-)_r \neq \emptyset;$$

(a_3) *If there are integers* h, k *with* $h < k$ *such that* $\widetilde{X}_h = \widetilde{X}_k$ *and* $\widetilde{X}_h$ *possesses the FPP-*γ*, then there is a finite sequence* $(z_i)_{h\leq i\leq k}$ *, with* $z_i \in \mathcal{D}_i$ *and* $\psi_i(z_i) = z_{i+1}$ *for each* $i = h, \ldots, k-1$*, such that* $z_h = z_k$ *.*

Proof. As in [59, Theorem 2.2] we prove the three properties in the reverse order. First of all we observe that, for each $i \in \mathbb{Z}$, there is a closed (and hence compact) set $W_i \subseteq \mathcal{D}_i \subseteq X_i$ such that, for every path $\sigma \subseteq X_i$ with $\sigma \cap (X_i)_l^- \neq \emptyset$ and $\sigma \cap (X_i)_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap W_i$ with $\psi_i(\gamma) \cap (X_{i+1})_l^- \neq \emptyset$ and $\psi_i(\gamma) \cap (X_{i+1})_r^- \neq \emptyset$.

Proof of (a_3). Let $k = h + s$ for some $s \geq 1$. Let us also set $\widetilde{\mathcal{Z}} = \widetilde{X}_h = \widetilde{X}_k$, and $\psi = \psi_{k-1} \circ \cdots \circ \psi_h$. By property (e_1) of Lemma 4 we have that $(\mathcal{D}, \psi) : \widetilde{\mathcal{Z}} \Leftarrow\!\!\Rightarrow \widetilde{\mathcal{Z}}$, with

$$\mathcal{D} := \{z \in W_h \,:\, \psi_j \circ \cdots \circ \psi_{h+1} \circ \psi_h(z) \in W_{j+1}\,, \text{for } j = h, \ldots, k-1\}.$$

Then the assumption about the FPP-γ for X_h implies the existence of a fixed point $z^* \in \mathcal{D}$ for ψ. Moreover, $z_h := z^* \in \mathcal{D}_h$ and, setting $z_i = \psi_{i-1} \circ \cdots \circ \psi_h(z_h)$, for $i = h+1, \ldots, k$, the verification of the properties in (a_3) is straightforward.

Proof of (a_2). The situation described here is similar to that considered in [33, The Expander Lemma, p.417]. Without loss of generality, assume $j = 0$. Define the closed set

$$\mathcal{S} = \{x \in W_0 \,:\, \psi_\ell \circ \cdots \circ \psi_0(x) \in W_{\ell+1}\,,\ \forall\, \ell = 0, 1, 2, \ldots\}.$$

Let $\gamma_0 \subseteq X_0$ be a path intersecting both the components of X_0^-. By the stretching assumption between $\widetilde{X}_0$ and $\widetilde{X}_1$, there is a sub-path $\gamma_1 \subseteq W_0 \subseteq \mathcal{D}_0$ such that $\psi_0(\gamma_1) \subseteq X_1$ and with $\psi_0(\gamma_1)$ meeting both the components of X_1^-. On the other hand, the path $\psi_0(\gamma_1) = \sigma_1$ contains a sub-path $\sigma_2 \subseteq W_1 \subseteq \mathcal{D}_1$ such that $\psi_1(\sigma_2) \subseteq X_2$ and with $\psi_1(\sigma_2)$ intersecting both the components of X_2^-. We also define $\gamma_2 = \{x \in \gamma_1 : \psi_0(x) \in \sigma_2\}$. Then, by induction, we can find a sequence of nonempty compact sets contained in X_0

$$W_0 \supseteq \gamma_0 \supseteq \gamma_1 \supseteq \gamma_2 \supseteq \ldots \gamma_n \supseteq \gamma_{n+1} \supseteq \ldots$$

with $\psi_i(\gamma_i) \subseteq W_{i+1} \subseteq \mathcal{D}_{i+1}$ and such that $\psi_i \circ \cdots \circ \psi_0(\gamma_{i+1})$ is a path in X_{i+1} meeting both the components of X_{i+1}^-. Taking a point $w \in \bigcap_{n=0}^{\infty} \gamma_n$ we have that $\psi_\ell \circ \cdots \circ \psi_0(w) \in W_{\ell+1} \subseteq \mathcal{D}_{\ell+1}$ for each $\ell \geq 0$. Thus, any path $\gamma_0 \in X_0$ intersecting both the components of X_0^- contains a point of $\mathcal{S}$ which generates a sequence as in (a_2).

Proof of (a_1). A diagonal argument (see, e.g., [33, Proposition 5] or [56, Theorem 2, (w_4)]) allows to prove (a_1) as a consequence of (a_2). We give a sketch of it for the reader's convenience. By (a_2) we have that for each $n = 1, 2, \ldots$ there is a compact set $\mathcal{C}_{-n} \subseteq W_{-n} \subseteq \mathcal{D}_{-n}$ such that

$$K_{j+1,n} := \psi_j \circ \cdots \circ \psi_{-n}(\mathcal{C}_{-n}) \subseteq W_{j+1} \subseteq \mathcal{D}_{j+1}\,.$$

We take a point $y_{j,n} \in K_{j,n}$, for each $j \geq -n+1$, in order to form the infinite matrix

$$\begin{array}{cccccccccc}
 & & & & y_{0,1} & y_{1,1} & \cdots & y_{j,1} & \cdots \\
 & & & y_{-1,2} & y_{0,2} & y_{1,2} & \cdots & y_{j,2} & \cdots \\
 & & y_{-2,3} & y_{-1,3} & y_{0,3} & y_{1,3} & \cdots & y_{j,3} & \cdots \\
 & \cdots & & & & & & & \\
y_{-n+1,n} & \cdots & y_{-2,n} & y_{-1,n} & y_{0,n} & y_{1,n} & \cdots & y_{j,n} & \cdots \\
 & \cdots & & & & & & &
\end{array}$$

where, for each n and j, we have that $\psi_j(y_{j,n}) = y_{j+1,n}$. Now, a standard compactness and diagonal argument (or, from another point of view, the fact that the product of countably many sequentially compact spaces is sequentially compact) allows to pass to the limit on each "column" along a common subsequence of indexes in order to find, for each $j \in \mathbb{Z}$, a point $w_j \in W_j \subseteq \mathcal{D}_j$ and the continuity of ψ_j implies also that $\psi(w_j) = w_{j+1}$, $\forall\, j \in \mathbb{Z}$.

□

3.3. A final remark about sub-paths

We conclude this section with a remark about the stretching condition that we have chosen for property (H) and its variants and consequences. As pointed out in the Introduction, our definition is mainly motivated by our previous applications to ODEs and, more precisely, to our paper [55] where we obtained the stretching property $(H_{\pm})$ in a concrete example of the planar system (1). In that specific example, the proof was carried on by considering a continuous parameterized curve defined in the interval $[0, 1[$ and with an unbounded image in $\mathbb{R}^2$. Subsequently, in our search of fixed points for general continuous mappings defined on two-dimensional cells we used a definition of path as the continuous image of an interval. As shown in [60] as long as we are concerned with fixed points of a single mapping defined on a topological rectangle, there is no effect on the possible different choices in the definitions. With this respect, consider also Theorem 6 and Theorem 7 which show how, as long as we are looking for the existence of a fixed point, the stretching condition for paths and that for continua are both sufficient to obtain the desired result. Things, however, seem to be somehow more complicated when we focus our attention on the search of fixed points for the composition of maps (and thus, in particular, for the iterations of a given map). In such a case, the possibility of considering parameterized curves (modulo some equivalent relation like in Section 1.5) instead of images of curves as sets embedded in a space, looks simpler from the point of view of stating some hypotheses which are easily verifiable through the composition of maps. In the next example we try now to express better our point of view which lead to the choice of definition for a path considered in Section 1.5.

EXAMPLE 3. We define the function

$$g : [0, 3\pi] \to \mathbb{R}, \quad g(t) := \max\{t - 4\pi, \min\{t, \pi - t\}\}$$

and the continuous curve

$$\theta = (\theta_1, \theta_2) : [0, 3\pi] \to \mathbb{R}^2, \quad \theta_1(t) := \cos(g(t)), \;\; \theta_2(t) := \sin(g(t)) = \sin(t).$$

Observe that for the path $\gamma = [\theta]$, the image set is

$$\bar{\gamma} = \{\theta(t) : t \in [0, 3\pi]\} = S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

The motion of the point $\theta(t)$ along the circumference S^1 can be described as follows: we start for $t = 0$ at the point $P = (1, 0)$, we move on S^1 in the counterclockwise sense till to the point $P' = (0, 1)$ for $t = \pi/2$. At this moment, the point $\theta(t)$ starts moving to the reverse direction (clockwise sense) till it reaches again the point P' at

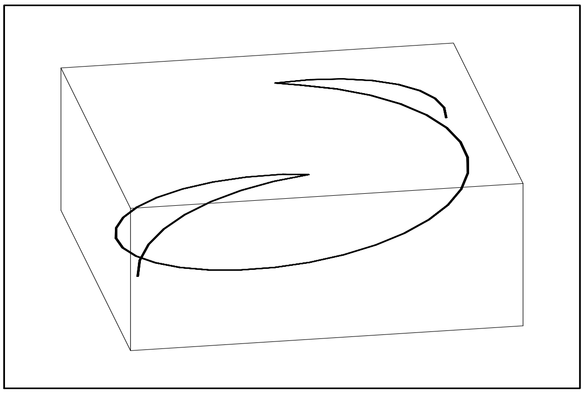

Figure 12: The bold line viewed from a suitably chosen point of perspective represents the arc in $\mathbb{R}^3$ parameterized by $t \mapsto (t, \theta_1(t), \theta_2(t))$, for $t \in [0, 3\pi]$. The plot has been performed by Maple software.

the time $t = 5\pi/2$. Finally, the motion switches again to the counterclockwise sense and the point $Q = (-1, 0)$ is reached at the time $t = 3\pi$.
Let us set now $Z = S^1$ and $Z_l^- = \{P\}$, $Z_r^- = \{Q\}$ (we intentionally take this choice to show that the terms "left" and "right" are merely conventional and their order is not important). According to our definitions, γ is a path in Z with $\gamma \cap Z_l^- \neq \emptyset$ and $\gamma \cap Z_r^- \neq \emptyset$. If we consider now the arcs $\Gamma^{\text{upper}} := \{(x, y) \in S^1 : y \geq 0\}$ and $\Gamma^{\text{lower}} := \{(x, y) \in S^1 : y \leq 0\}$ which are contained in $\bar{\gamma}$, we see that while Γ^{lower} is the image set of a sub-path of γ, Γ^{upper} is not the image of any sub-path of γ.

4. Applications to topological cells in finite dimensional spaces, periodic points and topological dynamics

In this section we propose an application of the results in Section 3 to the setting of [26, 56, 57, 59, 60, 61, 82].

4.1. Definitions

Let X be a Hausdorff topological space. We define a $(1, \text{N} - 1)$-*rectangular cell of X* as a pair

$$\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}),$$

where $\mathcal{N} \subseteq X$ is a compact set and $c_{\mathcal{N}} : \mathcal{N} \to [-1, 1]^{\text{N}} \subseteq \mathbb{R}^{\text{N}}$ is a homeomorphism of $\mathcal{N}$ onto its image $[-1, 1]^{\text{N}}$. Sometimes, it will be convenient to put in evidence the Hausdorff topological space X containing a given cell $\mathcal{N}$ and the dimension N of the codomain of the homeomorphism. In such a situation, we'll write

$$\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}; X, \text{N}).$$

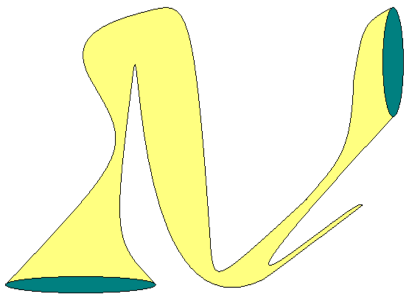

Figure 13: A possible picture of a $(1,2)$-rectangular cell $\widehat{\mathcal{N}}$, where we have put in evidence the two components of the $\mathcal{N}^-$ set which are painted with a darker color.

Our definition of $(1, \mathrm{N}-1)$-rectangular cell is borrowed from that of *h*-set given by Zgliczyński and Gidea [82, Definition 1] and considered also by Pireddu and Zanolin in [61]. However, we point out that, differently than in [82] and [61], we don't assume here $\mathcal{N}$ to be a subset of $\mathbb{R}^{\mathrm{N}}$ and moreover in the present case the homeomorphism $c_{\mathcal{N}}$ is defined only on $\mathcal{N}$ whence in the above cited articles $c_{\mathcal{N}}$ was defined on the whole space X. We also define the sets

$$\mathcal{N}_l^- := {c_{\mathcal{N}}}^{-1}(\{-1\} \times [-1,1]^{\mathrm{N}-1}), \quad \mathcal{N}_r^- := {c_{\mathcal{N}}}^{-1}(\{1\} \times [-1,1]^{\mathrm{N}-1}),$$

conventionally called *the left and the right faces of* $\widehat{\mathcal{N}}$, as well as the set

$$\mathcal{N}^- := \mathcal{N}_l^- \cup \mathcal{N}_r^- .$$

If we define now

$$\widetilde{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-),$$

we have that $\widetilde{\mathcal{N}}$ is a path-oriented spaces which possesses the FPP-γ.

Our definition of oriented cell $\widehat{\mathcal{N}}$ fits with that of $(1, \mathrm{N}-1)$-*window* considered by Gidea and Robinson in [25] and, in the special case $\mathrm{N} = 2$, is equivalent to that of two-dimensional *oriented cell* by Papini and Zanolin in [60].

For completeness we also recall the form that the stretching condition takes with respect to the path-oriented spaces determined by the rectangular cells that we have just defined.

Let $\widehat{\mathcal{A}} = (\mathcal{A}, c_{\mathcal{A}}; X, \mathrm{N}_1)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, c_{\mathcal{B}}; Y, \mathrm{N}_2)$ be two rectangular cells contained in the Hausdorff topological spaces X and Y, respectively. Let $\phi : X \supseteq D_\phi \to Y$ be a map (not necessarily continuous on its whole domain D_ϕ) and let us consider a set $\mathcal{D} \subseteq D_\phi$.

DEFINITION 5. *We say that the pair* $(\mathcal{D}, \phi)$ *stretches* $\widehat{\mathcal{A}}$ *to* $\widehat{\mathcal{B}}$ *along the paths and write*

$$(\mathcal{D}, \phi) : \widehat{\mathcal{A}} \Leftrightarrow \widehat{\mathcal{B}},$$

if ϕ *is continuous on* $\mathcal{D} \cap \mathcal{A}$ *and, moreover, there is a compact set* $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{A}$ *such that, for every path* $\sigma \subseteq \mathcal{A}$ *with* $\sigma \cap \mathcal{A}_l^- \neq \emptyset$ *and* $\sigma \cap \mathcal{A}_r^- \neq \emptyset$*, there is a path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\psi(\gamma) \cap \mathcal{B}_l^- \neq \emptyset$ *and* $\phi(\gamma) \cap \mathcal{B}_r^- \neq \emptyset$*.*

Observe that this definition coincides with

$$(\mathcal{D}', \phi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{B}},$$

according to Definition 2, for

$$\mathcal{D}' = \mathcal{D} \cap \mathcal{A} \cap \phi^{-1}(\mathcal{B}).$$

As in [60] we introduce now some special subsets of a cell which are crucial for our applications.

DEFINITION 6. *Let* $\widehat{\mathcal{M}} = (\mathcal{M}, c_{\mathcal{M}}; X, d_1)$ *and* $\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}; X, d_2)$ *be two rectangular cells of the same topological space* X *and let* $\widetilde{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ *and* $\widetilde{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ *be the corresponding path-oriented spaces. We say that* $\widehat{\mathcal{M}}$ *is a horizontal slab of* $\widehat{\mathcal{N}}$ *and write*

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}},$$

if $\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}$*, that is, if* $\mathcal{M} \subseteq \mathcal{N}$ *and if every path* $\gamma \subseteq \mathcal{M}$ *with* $\gamma \cap \mathcal{M}_l^- \neq \emptyset$ *and* $\gamma \cap \mathcal{M}_r^- \neq \emptyset$ *is such that* $\gamma \cap \mathcal{N}_l^- \neq \emptyset$ *and* $\gamma \cap \mathcal{N}_r^- \neq \emptyset$*.*
Similarly, we say that $\widehat{\mathcal{M}}$ *is a vertical slab of* $\widehat{\mathcal{N}}$ *and write*

$$\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}},$$

if $\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}$*, that is, if* $\mathcal{M} \subseteq \mathcal{N}$ *and if every path* $\sigma \subseteq \mathcal{N}$ *with* $\sigma \cap \mathcal{N}_l^- \neq \emptyset$ *and* $\sigma \cap \mathcal{N}_r^- \neq \emptyset$ *contains a sub-path* $\gamma \subseteq \mathcal{M}$ *such that* $\gamma \cap \mathcal{M}_l^- \neq \emptyset$ *and* $\gamma \cap \mathcal{M}_r^- \neq \emptyset$*.*

REMARK 8. Note that in order to have $\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}}$ it is equivalent to require that $\mathcal{M} \subseteq \mathcal{N}$ and either

$$\widehat{\mathcal{M}}_l^- \subseteq \widehat{\mathcal{N}}_l^-, \quad \widehat{\mathcal{M}}_r^- \subseteq \widehat{\mathcal{N}}_r^-$$

or

$$\widehat{\mathcal{M}}_l^- \subseteq \widehat{\mathcal{N}}_r^-, \quad \widehat{\mathcal{M}}_r^- \subseteq \widehat{\mathcal{N}}_l^- .$$

In this manner, our definition of horizontal slab reduces to the one of horizontal slice in [60, Def.1.2]

As in Section 3, we can now borrow from [60] and [61] the next definition.

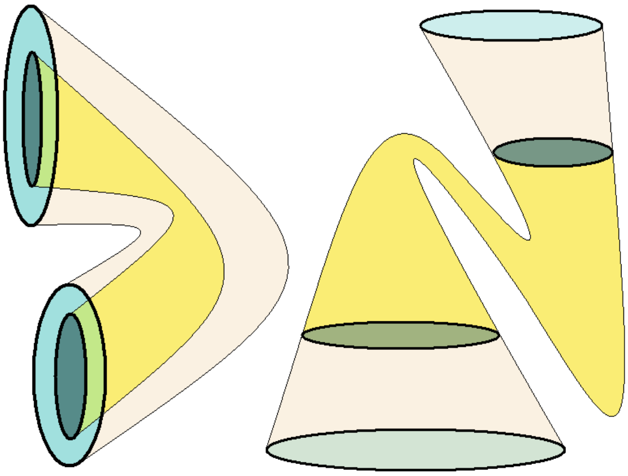

Figure 14: Examples of $\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}}$ and of $\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}}$ (the left and the right figures, respectively). The painted areas represent $\widehat{\mathcal{M}}$ as embedded in $\widehat{\mathcal{N}}$. The contours of $[\,\cdot\,]^-$-sets for the oriented cells $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are indicated with a bold line.

DEFINITION 7. *Let $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{M}}$ be three rectangular cells with $\mathcal{A}, \mathcal{B}, \mathcal{M}$ subspaces of the same topological space X and suppose that $\mathcal{M} \subseteq \mathcal{A} \cap \mathcal{B}$.*
We say that $\widehat{\mathcal{B}}$ crosses $\widehat{\mathcal{A}}$ in $\widehat{\mathcal{M}}$ and write

$$\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\},$$

if

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{A}} \quad \text{and} \quad \widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{B}}.$$

4.2. Applications

At this step, we can just reconsider the same main results from [59, 60] already proved for the stretching property in the case of generalized two-dimensional cells and extend them to $(1, \mathrm{N}-1)$-rectangular cells. For instance, we have the following (compare to Theorem 4).

THEOREM 12. *Suppose that $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ are oriented cells in X. If $(\mathcal{D}, \psi) : \widehat{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{B}}$ and there are $k \geq 2$ oriented cells $\widehat{\mathcal{M}}_1 \ldots, \widehat{\mathcal{M}}_k$ such that*

$$\widehat{\mathcal{M}}_i \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}, \quad \text{for } i = 1, \ldots, k,$$

with

$$\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{D} = \emptyset, \ \text{for all } i \neq j, \ \text{with } i, j \in \{1, \ldots, k\},$$

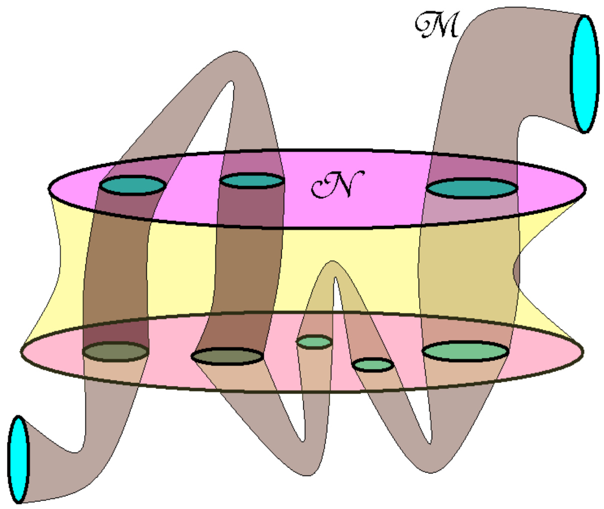

Figure 15: In $\mathbb{R}^3$, the $(1, 2)$-rectangular cell $\widehat{\mathcal{N}}$ (the cheese shaped set) is crossed by the $(1, 2)$-rectangular cell $\widehat{\mathcal{M}}$ (the snake-like set). Among the four intersections of $\mathcal{M}$ with $\mathcal{N}$, the first two (counting from the left and painted by a darker color) belong to $\{\widehat{\mathcal{N}} \pitchfork \widehat{\mathcal{M}}\}$.

then the following conclusion holds:

- ψ *has a chaotic dynamics of coin-tossing type on k symbols (with respect to the sets* $\mathcal{K}_i := \mathcal{D} \cap \mathcal{M}_i$*).*
- ψ *has a fixed point in each set* $\mathcal{K}_i := \mathcal{D} \cap \mathcal{M}_i$ *and, for each finite sequence* $(s_0, s_1, \ldots, s_m) \in \{1, \ldots, k\}^{m+1}$, *with* $m \geq 1$, *there is at least one point* $z^* \in \mathcal{K}_{s_0}$ *such that the position*

$$z_{j+1} = \psi(z_j), \quad z_0 = z^*, \quad \textit{for } j = 0, 1, \ldots, m$$

defines a sequence of points with

$$z_j \in \mathcal{K}_{s_j}, \ \forall\, j = 0, 1, \ldots, m \quad \textit{and } z_{m+1} = z^*.$$

REMARK 9. The two conclusions in Theorem 12 corresponds to (a_1) and (a_3) of Theorem 11. We could derive from (a_2) also a conclusion about the existence of a continuum of initial points which generate any (fixed) forward itinerary and thus obtain an extension of the conclusion (b_2) of Theorem 4. This one as well as some related topics, which require a more careful treatment, will be discussed elsewhere.

As shown by this example, from Theorem 11 and the definitions of stretching, slabs and crossings adapted to the case of $(1, \mathrm{N} - 1)$-rectangular cells, we have now available all the tools which are needed in order to achieve a full extension of the topological results contained in [59, 60] and partially recalled in Section 1.2, to maps which expand the arcs along one direction. A more complete investigation on this subject will appear in a future work.

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