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K.T. Alligood - E. Sander - J.A. Yorke*

EXPLOSIONS IN DIMENSIONS ONE THROUGH THREE

Abstract. Crises are discontinuous changes in the size of a chaotic attractor as a parameter is varied. A special type of crisis is an explosion, in which the new points of the attractor form far from any previously recurrent points. This article summarizes new results in explosions in dimension one, and surveys previous results in dimensions two and three. Explosions can be the result of homoclinic and heteroclinic bifurcations. In dimensions one and two, homoclinic and heteroclinic bifurcations occur at tangencies. We give a classification of one-dimensional explosions through homoclinic tangency. We describe our previous work on the classification of planar explosions through heteroclinic tangencies. Three-dimensional heteroclinic bifurcations can occur without tangencies. We describe our previous work, which gives an example of such a bifurcation and explains why three-dimensional crossing bifurcations exhibit unstable dimension variability, a type of non-hyperbolic behavior which results in a breakdown of shadowing. In addition, we give details for a new scaling law for the parameter-dependent variation of the density of the new part of the chaotic attractor.

1. Introduction

Crises of chaotic attractors are discontinuous changes in the size of an attractor as a parameter is varied. Crises are the most easily observed and most often described global bifurcations. A classic example of a crisis is the onset of the period three window in the bifurcation for the logistic map $f(x) = \mu x(1-x)$ [31], in which the attractor changes discontinuously from consisting of an uncountable collection of points to containing only one period three orbit.

A specific type of crisis is an explosion, which is a bifurcation in which new recurrent points form discontinuously far from any previously recurrent points. An explosion is stronger than a crisis, since a crisis can be caused by the mere merging of an attractor and a pre-existing chaotic saddle.

In this article, we review results on explosions in one, two, and three dimensions, as well as stating open problems on explosions. We proceed as follows: Section 2 outlines classifications of explosions at homoclinic and heteroclinic tangencies in one and two dimensions. It is also possible to have explosions far from tangency but as a result of a tangency. We give a simple example of this in one dimension. In two dimensions, we give a topological description of the well-studied two-dimensional example of this phenomenon for the Ikeda attractor. The ultimate goal is a complete classification of one- and two-dimensional explosions. Section 3 describes a type of three-dimensional bifurcation in which explosions occur without tangencies. The section includes a numerical example of such a bifurcation. Section 4 presents a new scaling law for the parameter-dependent change in the density of the newly formed piece of the chaotic attractor. Section 4.3 describes some interesting numerical implications of tangency-free explosions as the onset of *unstable dimension variability*, a

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specific type of non-hyperbolic behavior.

2. Explosions via tangency

Let f_λ be a one-parameter family of functions. An explosion occurs at a parameter value λ_0 and a point x which is not recurrent prior to the bifurcation but is recurrent at the bifurcation. Clearly a saddle node bifurcation for either a fixed point or periodic orbit gives rise to an explosion point. In addition, the existence of explosion points is often the result of bifurcations involving tangencies between the stable and unstable manifolds of fixed or periodic points. In this section, we describe classes of explosions arising from homoclinic and heteroclinic tangency. The ultimate goal is a full understanding and classification of all types of explosions. In this direction, Palis and Takens made the following conjecture for planar one-parameter families of diffeomorphisms. We have extended this conjecture to include one-parameter families of one-dimensional maps as well.

CONJECTURE 1 (Palis and Takens [25]). Explosions within generic one-parameter families of smooth one-dimensional maps or smooth two-dimensional diffeomorphisms are the result of either a tangency between stable and unstable manifolds of fixed or periodic points or a saddle node bifurcation of a fixed or periodic point.

Newhouse, Palis, and Takens have shown that this statement is true in the case when the limit set is still finite at the bifurcation point [24]. Bonatti, Diaz, and Viana point out that this question can also be posed from a probabilistic point of view, in which case they conjecture the opposite conclusion [6]. In one dimension, we believe this problem to be tractable. In two dimensions an answer depends on a detailed classification of the accessibility of periodic points and their manifolds on the boundary of a basic set [4].

We now give some basic definitions. Let $f : R^k \times R \to R^k$ be a C^1-smooth one-parameter family of C^2 diffeomorphisms, $k = 1, 2, 3$ where we use two notations interchangeably: $f(x, \lambda) = f_\lambda(x)$. For the definition of an explosion it is more natural to use the concept of chain recurrence rather than recurrence:

DEFINITION 1. *For an iterated function g, there is an ϵ-***chain** *from x to y when there is a finite sequence $\{z_0, z_1, \ldots, z_N\}$ such that $z_0 = x$, $z_N = y$, and $d(g(z_{n-1}), z_n) < \epsilon$ for all n.*

If there is an ϵ-chain from x to itself for every $\epsilon > 0$ (where $N > 0$), then x is said to be **chain recurrent** *[7, 8]. The* **chain recurrent set** *is the set of all chain recurrent points. For a one-parameter family f_λ, we say (x, λ) is chain recurrent if x is chain recurrent for f_λ.*

If for every $\epsilon > 0$, there is an ϵ-chain from x to y and an ϵ-chain from y to x, then x and y are said to be in the same **chain component** *of the chain recurrent set.*

Note that the chain recurrent set and the chain components are invariant under forward iteration.

DEFINITION 2 (Chain explosions). *A* **chain explosion point** (x, λ_0) *is a point such that* x *is chain recurrent for* f_{λ_0}*, but there is a neighborhood* N *of* x *such that on one side of* λ_0 *(i.e. either for all* $\lambda < \lambda_0$ *or for all* $\lambda > \lambda_0$*), no point in* N *is chain recurrent for* f_λ*.*

Note that in the above definition, at f_{λ_0}, x is not necessarily an isolated point of the chain recurrent set. A well studied example of this is the explosion that occurs at a saddle node bifurcation on an invariant circle. The chain recurrent set consists of two fixed points prior to bifurcation and the whole circle at and in many cases after bifurcation. In subsequent usage, if the distinction is not important, we will refer to recurrent points rather than always saying chain recurrent.

2.1. One dimension

This section describes a classification of explosions via homoclinic tangencies in one dimension which appears in [2]. Although one dimension would seem to be the easiest case, there are some key differences between one- and two-dimensional explosions which are not simplifications in one dimension. For example, for a diffeomorphism, homoclinic and heteroclinic orbits require the existence of saddle points with stable and unstable manifolds of dimension at least one. However, since a one-dimensional map is in general noninvertible, it is possible to have fixed or periodic points with one-dimensional unstable manifolds and a non-trivial zero-dimensional stable manifolds. Marotto terms such points snap-back repellers [22]. It is not possible to reverse the dimensions of the stable and unstable manifolds; the existence of a homoclinic orbit to an attracting fixed point requires a multivalued map [29]. In addition, the chain recurrent set is not invariant under backwards iteration of a noninvertible map, so the discussion of explosions in one dimension includes cases in which a point is not an explosion point, but the preimages are explosion points. As this is a broad survey, the statements and proofs of the results below are only sketches. The full details appear in [2].

Let f be a one variable function with a repelling fixed point x_0. Let y and k be such that y is a k^{th} preimage of x_0, and assume that a sequence of preimages of y converge to x_0. Then y is contained in an orbit which limits both forwards and backwards to x_0. That is, y is a homoclinic point for x_0. Homoclinic points for periodic orbits are defined by replacing f with some appropriate iterate f^m. Notice that for diffeomorphisms, all orbits through homoclinic points are homoclinic orbits. For one-dimensional maps, there may be many non-homoclinic orbits through a homoclinic point.

Since the stable manifold of a homoclinic point is zero-dimensional, a homoclinic tangency is a tangency of the graph of the map at a homoclinic point. That is, a homoclinic tangency occurs if the graph of f has a horizontal tangent at a homoclinic point.

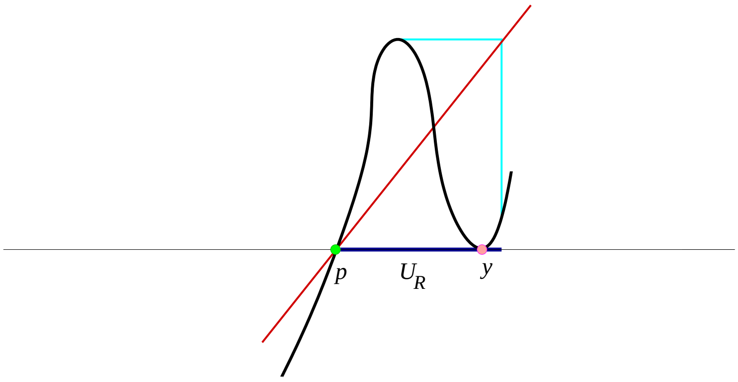

Figure 1: A one-dimensional map with a repelling fixed point p with positive derivative. The point y is a homoclinic tangency point. Since y is contained in a non-crossing orbit, by Theorem 2, y is not an explosion point. The unstable manifold branch U_R is an interval shown by a thick line on the x-axis.

Assume that f_λ is a C^1 smooth family of C^2 maps and that the point $x_0(\lambda)$ (which we write as x_0) is a repelling fixed point.

THEOREM 1 (No explosions with negative derivative at x_0, Alligood, Sander, Yorke [2]). *If f has a negative derivative at x_0, and y is homoclinic to x_0, at λ_0, then (y, λ_0) is not an explosion point.*

The idea is that images of a neighborhood of y map across preimages of a neighborhood of y, implying that there is a periodic point near y, which persists under perturbation of the parameter. Therefore y is arbitrarily close to recurrent points prior to the tangency bifurcation.

Now consider the case of a positive derivative at x_0. Assume that a homoclinic orbit limits to the local right (resp. left) branch of the unstable manifold of x_0. Let y be a homoclinic tangency point within this orbit. Then for some k, $f^k(y) = x_0$. If the graph of the map f^{k-1} near $f^{k-1}(y)$ is below (resp. above) the horizontal line, we call the homoclinic orbit a *crossing orbit*. A homoclinic orbit that is not crossing is called a *non-crossing orbit*.

THEOREM 2 (No explosions for non-crossing bifurcations [2]). *Assume that y is contained in a non-crossing homoclinic orbit, then y is not an explosion point.*

The proof is very similar to the case of negative derivative at x_0. Namely, an image of a neighborhood of y covers a preimage of the same neighborhood, implying that y is not an explosion point. This situation is depicted in Figure 1.

Closely related to crossing and non-crossing orbits, we can distinguish the two

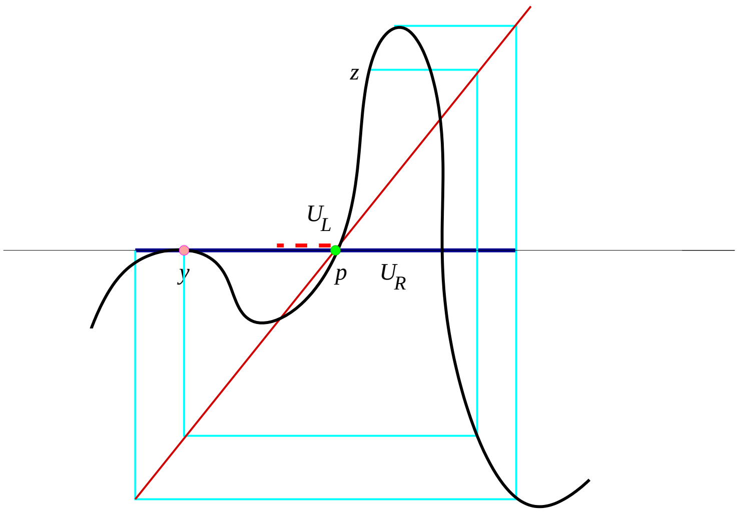

Figure 2: In this figure, y is a homoclinic tangency point in a crossing orbit, contained in U_R, but not U_L. Therefore y is an explosion point. The preimage z of y is also an explosion point. The interval U_R is shown by a thick line as in Figure 1. U_L is denoted by a dashed line slightly above the x-axis.

manifold branches U_R and U_L, being the iterates of the local right and lefthand branches of the unstable manifold of x_0. The union of U_L and U_R is the entire unstable manifold of x_0. If $\{(U_L \cap U_R) \setminus x_0\}$ is not empty, then the intersection must contain either all of U_L or all of U_R. For example, U_R may contain points both to the left and to the right of U_L. See Figures 2 and 3. We can show that if $y \in U_R$, and the k^{th} image of every neighborhood of y contains points in $\{U_R \setminus x_0\}$, then y is not an explosion point.

Under certain generic conditions, we can show a converse to the non-crossing orbit theorem: If y is in a crossing orbit, then y is an explosion point [2].

We are interested not only in explosion points which are themselves tangency points, but also in points which are explosion points far from tangencies, but are caused by a tangency. Since the chain recurrent set is invariant under forwards iteration, the image of non-crossing tangency point is a non-explosion point. However, there may be explosion points with iterates that are non-explosion points. For example, Figure 3 shows points of which are not tangency points but are preimages of tangency points. We can prove that for a generic one-parameter family, there are explosion points at U_R (such as i and j depicted in the figure) if there are points in U_R which map to a tangency (such as y) on the other side, even though the tangency point is not an explosion point [2].

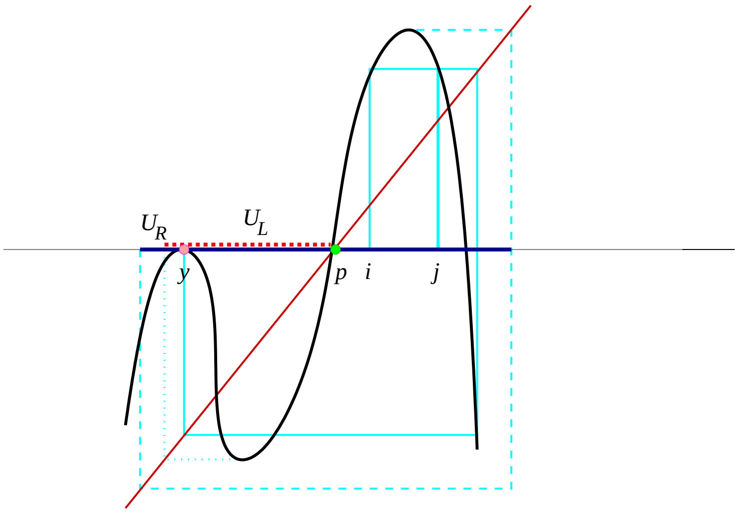

Figure 3: The tangency point y is contained in $U_L \cap U_R$, and is thus not an explosion point. However, the preimages i and j of y are explosion points. U_R and U_L are depicted again with a solid and dashed line respectively.

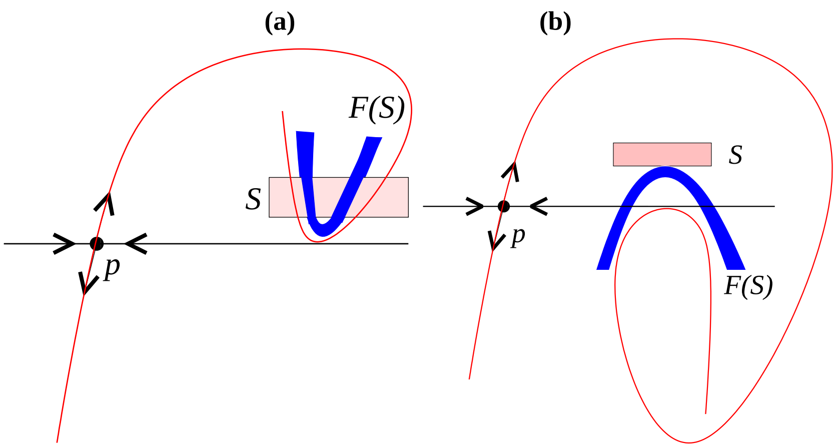

Figure 4: Homoclinic tangency points in the plane may or may not be explosion points. A set S and a large iterate $f^K(S) = F(S)$. In (a), it is never possible for the homoclinic tangency point to be an explosion point. In part (b), it depends on the eigenvalues.

2.2. Tangencies in two dimensions

In two dimensions, a heteroclinic intersection is a hallmark for chaotic behavior in the form of a Smale horseshoe. Thus a homoclinic tangency is a bifurcation point. It is not always an explosion point. Figure 4(a) shows a configuration of stable and unstable manifolds at a homoclinic tangency which can never be an explosion, since there is a horseshoe forming locally prior to tangency. In Figure 4(a), the situation is not so clear, since a large iterate of a rectangular set S may or may not map across itself under a large iterate. In fact, whether this is a possible explosion point depends on the relative strength of contraction and expansion at the fixed point p. Palis and Takens [25] classified homoclinic explosion bifurcations, as shown in Figure 4 by describing which planar homoclinic tangencies can be explosion points as a function of the placement of the tangency point, sign of the eigenvalues, and area contraction or expansion of the map.

An explosion can also occur as a result of tangencies between stable and unstable manifolds of different fixed or periodic points: heteroclinic tangencies. In this case, in order for a heteroclinic tangency to result in any sort of recurrence, there needs to be a means of return. A natural way is another heteroclinic intersection, resulting in a heteroclinic cycle. These ideas are given in the following definitions.

DEFINITION 3 (n-connection). *An n-**connection** is a sequence of points $\{\rho_1, t_1, \rho_2, t_2, \dots, \rho_n\}$ such that for all i, ρ_i is a fixed point, and for each $i < n$, t_i is a heteroclinic point such that $t_i \in W^u(\rho_i) \cap W^s(\rho_{i+1})$.*

DEFINITION 4 (Cycle). *A **cycle** is an n-connection $\{\rho_1, t_1, \rho_2, t_2, \dots, \rho_n\}$ such that $\rho_n = \rho_1$.*

Generically, we can assume a unique tangency in a cycle, with all other heteroclinic intersections being transverse. Thus any heteroclinic cycle can be reduced to a heteroclinic cycle containing only two periodic or fixed points. In a previous paper [1], we gave necessary and sufficient conditions for planar heteroclinic bifurcations to result in explosions. The classification does not involve eigenvalue conditions and area contraction/expansion conditions. Rather, the classification is a set of necessary and sufficient conditions for explosions at heteroclinic tangency based on the configuration of the manifolds, in addition to another “chain” condition on the behavior. Figure 5 shows two heteroclinic cycles with a tangency. In (a), any curve which is arbitrarily close to the heteroclinic cycle necessarily intersects both the unstable manifold of p and the stable manifold of q, whereas in (b), this is not the case. The tangency point in (a) is an explosion point and in (b) it is not; just as in the one-dimensional case, the key to an explosion is that different points of the cycle are on opposite sides of a dividing manifold. Therefore, any ϵ-chain must cross the cycle. Such heteroclinic tangency explosion bifurcations are known as crossing bifurcations.

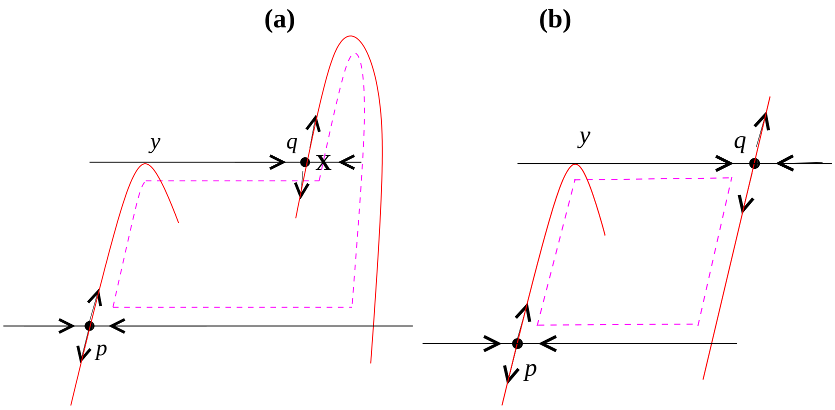

Figure 5: Heteroclinic cycles with a tangency at y. In (a), all curves close to the cycle intersect both the unstable manifold of p and the stable manifold of q (intersection shown with an X). Under the chain condition, the point y is an explosion point. In (b), it is possible to draw a curve arbitrarily close to the cycle which does not intersect the stable manifold of q. The point y is never an explosion point.

2.3. Explosions far from tangencies and planar gap filling

A planar explosion far from tangency but through a tangency occurs at a crisis bifurcation of the Ikeda map. This example has been studied by a number of authors, such as [17, 26]. Prior to bifurcation, there is a small attractor surrounded by an unstable chaotic saddle set. The set has noticable gaps, in which there are no recurrent points. After the bifurcation, the new attractor includes both the old attractor and the saddle set. In addition, the gaps in the recurrent set are filled in. This occurs in a discontinuous manner at the bifurcation value. Thus the points in the gaps at the bifurcation parameter are all explosion points. This discontinuous change in the recurrent set is a result of a heteroclinic tangency.

Figure 6 depicts the geometry of manifolds of a heteroclinic cycle that give rise to gap filling. The tangency point y is contained simultaneously in a crossing and a non-crossing orbit. Thus y is not an explosion point. However, there are points on the non-crossing orbit which are explosion points. There is an attractor with its boundary being the unstable manifold branch of p including the tangency. The basin of attraction includes all points below the stable manifold branch of q including the tangency. Therefore all points of the upper unstable manifold branch of q (in green in the figure for the color online version) are in the basin of attraction. After the bifurcation, these points become recurrent. Robert et al. [27] showed that this topological construction occurs during gap filling as a result of a heteroclinic cycle with a tangency for a pair of period five orbits of the Ikeda map.

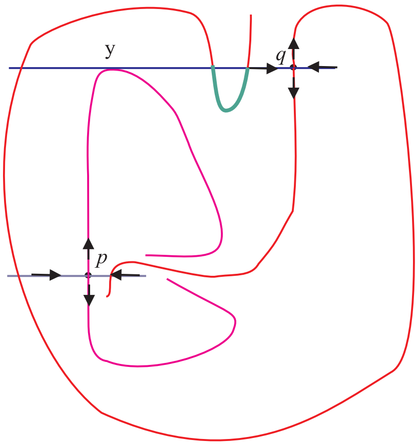

Figure 6: A geometric description of gap filling. The unstable manifold branch of p with the tangency encloses an attractor. This attractor only includes points “inside” the non-crossing inner cycle prior to tangency. When the tangency pushes through, the attractor also includes points “on the outside” – that is, on the crossing cycle.

3. Three-dimensional crises

There are many theoretical results and analyses of planar examples of crises, whereas relatively little is known about three-dimensional examples. In three dimensions, Diaz and Rocha [12, 13] described explosions that occur as the result of a non-tangency heteroclinic bifurcation. In [3], we have adapted these results to show that a crisis in a three-dimensional attractor can occur at a heteroclinic bifurcation without tangencies. After the crisis, the attractor contains two fixed points with different numbers of unstable directions. The existence of such an attractor is known as unstable dimension variability (UDV), and has been studied in the physics literature [5, 9, 10, 18, 20, 21, 19, 23, 28, 30, 32]. It is of particular interest, as unstable dimension variability results in nonshadowability.

We start by describing the topological dynamics of the example. To understand the role of the heteroclinic orbit in an attractor crisis, we need to be able to describe when there is a transverse heteroclinic orbit connecting two periodic points. We then need to know how a heteroclinic cycle can form. The dynamics can be similar to the planar case. Namely, when two three-dimensional periodic points have the same number of unstable directions, then a bifurcation must occur through tangency. Furthermore, since surfaces divide three-dimensional space, the heteroclinic cycles with explosions in this case parallel the two-dimensional case.

We now consider the case in which the two periodic points have a different number of unstable directions (1 and 2). Assume that q is a fixed point with two unstable directions, whereas p is a fixed point which only has one unstable direction.

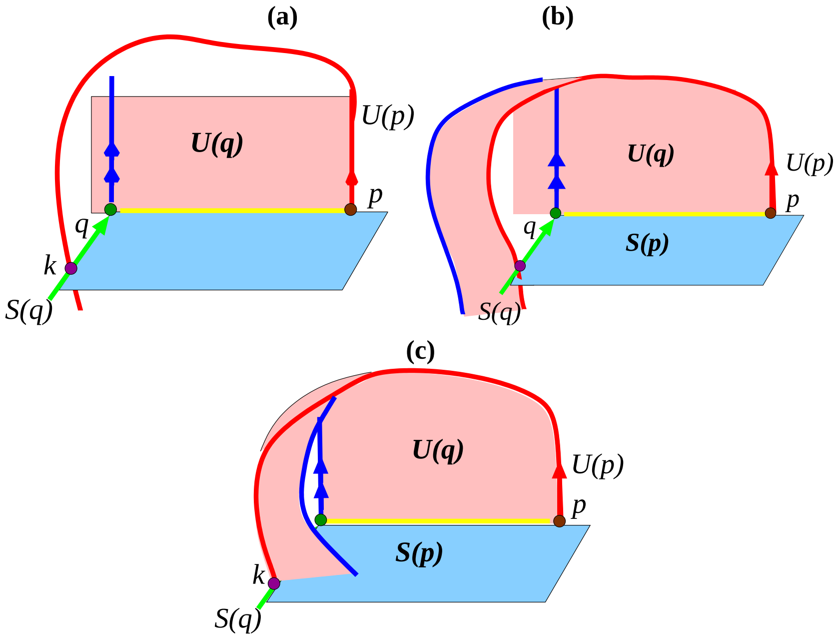

Figure 7: A heteroclinic bifurcation in three dimension without tangencies. (a) At the bifurcation, the one-dimensional manifolds intersect at k. The two-dimensional manifolds can either intersect with (b) or without (c) a twist.

(For simplicity, we have chosen fixed points, but the statements below also hold for periodic points.) For a heteroclinic cycle, we assume that a transverse intersection occurs between the manifolds of q and p. This must necessarily be an intersection of the two two-dimensional manifolds: the unstable manifold of q and the stable manifold of p. Either the transverse intersection of stable and unstable manifolds contains non-invariant components [11], or the intersection persistently connects the two periodic points. We assume the latter case, depicted in Figure 7. This implies that the one-dimension invariant manifolds form the boundaries of the respective two-dimensional manifolds. That is, the unstable manifold of q is bounded by the unstable manifold of p, and the stable manifold of q is the boundary of the stable manifold of p.

Consider a bifurcation parameter value for which the two one-dimensional manifolds intersect. This is not a tangency bifurcation, since generically these manifolds will not share a common tangent space, as in Figure 7(a). However, since the one-dimensional manifolds are the boundaries of two-dimensional manifolds, a bifurcation occurs at the intersection of the one-dimensional manifolds.

Diaz and Rocha use this construction with the assumption that the unstable strip and the two-dimensional stable manifold intersect without a twist, as shown in Figure 7(b). In this case, the intersection point between the one-dimensional manifolds is

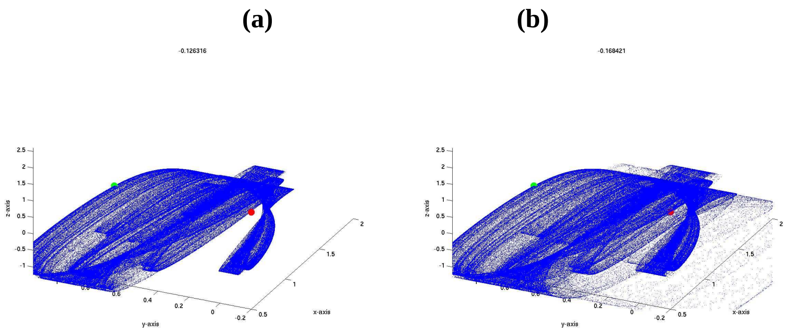

Figure 8: A numerical example of a heteroclinic bifurcation in three dimensions without tangencies. There are two saddle fixed points, p and q. (a) Prior to bifurcation, the attractor contains p but does not contain q. It is contained in the closure of a portion of $U(q)$. (b) At bifurcation, there is an explosion in which new points become part of the attractor. However, the density of the new part of the attractor is low.

an explosion point: Prior to bifurcation, p and q are isolated recurrent points; all points near $U(p)$ map far from $S(q)$, never returning near p. Likewise, no point in $U(q)$ intersects $S(q)$, making q isolated in the recurrent set as well. After bifurcation, there is a basic set containing transverse homoclinic points to both q and p. This implies that there is a basic set containing both q and p, fixed points with different numbers of unstable directions. The basic set in this example is in general unstable. This is a counterpart of the explosions at tangency points in the one- and two-dimensional cases. If there is an attractor involved, the crisis here would correspond to a blowout bifurcation, in which the entire attractor ceases to exist after the bifurcation point.

We are interested in the three-dimensional counterpart of planar gap filling. That is, the case in which the intersection between the one-dimensional manifolds is not and explosion point, but there are explosions occuring through this point. Gap filling corresponds to the case when the manifold $U(q)$ twists at the bifurcation, as in Figure 7(c). Thus, prior to bifurcation there are homoclinic points to p, and after bifurcation there are homoclinic points to q. The intersection is not an explosion point, but there may be other explosion points occuring when the dynamics change. We have constructed the first numerical example of this type of three-dimensional crossing bifurcation, as depicted in Figure 8. Prior to bifurcation, there is an attractor contained in the unstable manifold strip bounded by $U(p)$ and the strong unstable manifold of p. Points feed into the attractor from the inaccessible side of $U(q)$. After bifurcation, this region becomes accessible. As in the planar case, points are now able to return to this newly accessible side, resulting in an explosion in the size in the attractor. It also turns out that after bifurcation, the attractor displays unstable dimension variability. Numerically computed stable and unstable manifolds for the example appearing in [3] are displayed in Figure 8.

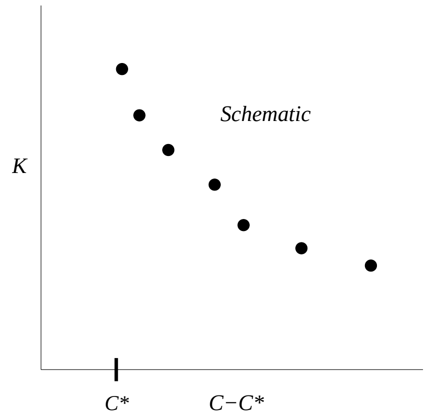

Figure 9: Scaling of the iterate length in the old part of the attractor as a function of the parameter.

4. Scaling laws for attractor density

Immediately after a crisis, there is a very low density of points in the new attractor. This low density is known as intermittency in the case of the period three window of the logistic map [31]. For planar homoclinic and heteroclinic tangency crises, the scaling of the low density of the new points as a function of the parameter was analyzed in the 1980's, as we describe in the section below. We have used similar methods to write down a scaling law for three-dimensional bifurcations without tangencies.

Consider a crisis in which the attractor before tangency is denoted A_{old}, and the new part of the attractor appearing at tangency is denoted A_{new}. Consider any dense orbit. Define τ as the orbit length in A_{old} between visits to A_{new}, and K as the mean of τ. For large τ, $P(\tau) \propto K^{-1} exp(\tau/K)$. The density of A_{new} can be approximated by the reciprocal of K. The variation in density can be analyzed using scaling near the bifurcation point $c = c^*$. Notice that after the bifurcation $\lim_{|c-c^*|\to 0} K = \infty$. We expect K to increase exponentially as $|c - c^*|$ approaches zero, as depicted schematically in Figure 9.

4.1. Scaling in two dimensions

Grebogi, Ott, Yorke [14, 15, 16] stated scaling laws for a planar homoclinic or heteroclinic bifurcation with a quadratic tangency. In the heteroclinic case such as depicted in Figure 6, we briefly describe the calculation used in order to illustrate its heavy reliance on the existence of a tangency. Let L be the lobe formed at y after passing through tangency. Then the following formula holds:

$$A(L) = \ell \cdot w = m(c - c*) \cdot \sqrt{c - c*}.$$

The density of the new part of the attractor depends on this area, in that this is the only way for points in A_{old} to enter A_{new}. The square root in this formula is due to the quadratic tangency. Let $L_n = f^n(L)$. Then $A(L_m) = MA(L)$, where M depends on the eigenvalues of q. These are the key ingredients giving rise to the scaling law.

4.2. A new scaling law in three dimensions

For the three-dimensional non-tangency bifurcation described in Section 3, we have demonstrated numerically that there is a linear relationship between the logarithm of the mean transient length K and the logarithm of the distance from the bifurcation parameter (*cf.* [3]).

Denote the eigenvalues of q by $|\lambda_1| > |\lambda_2| > 1 > |\mu|$, and those of p by $|\beta_1| < |\beta_2| < 1 < |\alpha|$. The set of points in the attractor (and thus on the two-dimensional unstable manifold $U(q)$) which exit A_{old} must do so by coming very close to the unstable manifold of p. Starting near p, the area of $U(q)$ within ϵ of $U(p)$ is approximated using α and β_2. We also need to know the fraction of points which exit A_{old} near $U(p)$ which re-enter the attractor. This is done using the two unstable eigenvalues for the linearization at q. The estimate leads to the new scaling law, which give good agreement with numerical calculation. It states that the mean transient length is $K(\eta) = \eta^\gamma$, where

$$\gamma = 1 + \frac{\log|\lambda_1|}{\log|\lambda_2|} + \frac{\log|\alpha|}{\log|\beta_2|}.$$

4.3. Unstable dimension variability

The low density of the new part of the attractor has interesting numerical implications in terms of testing for unstable dimension variability. Precisely, near the parameter at which a crisis occurs, the standard test for UDV is not applicable.

In an attractor with a dense orbit which exhibits UDV, we know that the dense orbit comes arbitrarily close to the stable manifold of each fixed or periodic saddle point. Thus it is possible to find a sequence within the dense orbit which stays close to a fixed or periodic point for any prescribed number of iterates after any finite transient is removed. For simplicity, assume p and q are fixed points with different numbers of unstable directions in an attractor which exhibits UDV. Since the middle Lyapunov exponent of the fixed points have opposite signs, the orbit must have arbitrarily long finite time sequences with the middle Lyapunov exponent being negative, and arbitrarily long finite time sequences with the middle Lyapunov exponent being positive. The standard test for UDV uses this fluctuation of Lyapunov exponents around zero [10].

In the case of UDV after a crossing bifurcation, although it is theoretically correct that the Lyapunov exponents fluctuate around zero, the density of the attractor is quite low near the newly added fixed point. Therefore, it is computationally infeasible to use the Lyapunov exponent test for UDV. See [3] for detailed numerical calculations illustrated using the example depicted in Figure 8.

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Kathleen ALLIGOOD, Evelyn SANDER, Department of Mathematical Sciences, George Mason University, 4400 University Dr., Fairfax, VA, 22030, USA
e-mail: `alligood@gmu.edu, sander@math.gmu.edu`

James A. YORKE, IPST, University of Maryland, College Park, MD 20742, USA
e-mail: `yorke@ipst.umd.edu`