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CONTENTS

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Preface

On September 19-21, 2005, the Department of Mathematics of the University of Turin guested a “WORKSHOP ON DYNAMICS”. We invited six among the speakers to write a survey for this special issue of the Journal. Song Jiang, Huseyin Koçak, Jean Mawhin, James Yorke and Fabio Zanolin wrote a paper with coauthors, moreover Matteo Franca’s survey is based on the seminar delivered by Russell Johnson at the Workshop.

Some of the articles present new results in a more general framework, so they also have the character of a survey even if they are research papers.

The following topics in Dynamics are studied: Explosion in dimensions 1–3, Periodic solutions of difference equations, Compressible viscous flows, A Dynamical approach to the p-Laplace equation, Shadowing in ODEs, Periodic points and chaos for nonlinear Hill equations.

We are grateful to the authors and to all the other speakers of the Workshop for their enthusiastic participation.

Hisao Fujita Yashima, Gaetano Zampieri

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Subalpine Rhapsody in Dynamics

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EXPLOSIONS IN DIMENSIONS ONE THROUGH THREE

Abstract. Crises are discontinuous changes in the size of a chaotic attractor as a parameter is varied. A special type of crisis is an explosion, in which the new points of the attractor form far from any previously recurrent points. This article summarizes new results in explosions in dimension one, and surveys previous results in dimensions two and three. Explosions can be the result of homoclinic and heteroclinic bifurcations. In dimensions one and two, homoclinic and heteroclinic bifurcations occur at tangencies. We give a classification of one-dimensional explosions through homoclinic tangency. We describe our previous work on the classification of planar explosions through heteroclinic tangencies. Three-dimensional heteroclinic bifurcations can occur without tangencies. We describe our previous work, which gives an example of such a bifurcation and explains why three-dimensional crossing bifurcations exhibit unstable dimension variability, a type of non-hyperbolic behavior which results in a breakdown of shadowing. In addition, we give details for a new scaling law for the parameter-dependent variation of the density of the new part of the chaotic attractor.

1. Introduction

Crises of chaotic attractors are discontinuous changes in the size of an attractor as a parameter is varied. Crises are the most easily observed and most often described global bifurcations. A classic example of a crisis is the onset of the period three window in the bifurcation for the logistic map $f(x) = \mu x(1-x)$ [31], in which the attractor changes discontinuously from consisting of an uncountable collection of points to containing only one period three orbit.

A specific type of crisis is an explosion, which is a bifurcation in which new recurrent points form discontinuously far from any previously recurrent points. An explosion is stronger than a crisis, since a crisis can be caused by the mere merging of an attractor and a pre-existing chaotic saddle.

In this article, we review results on explosions in one, two, and three dimensions, as well as stating open problems on explosions. We proceed as follows: Section 2 outlines classifications of explosions at homoclinic and heteroclinic tangencies in one and two dimensions. It is also possible to have explosions far from tangency but as a result of a tangency. We give a simple example of this in one dimension. In two dimensions, we give a topological description of the well-studied two-dimensional example of this phenomenon for the Ikeda attractor. The ultimate goal is a complete classification of one- and two-dimensional explosions. Section 3 describes a type of three-dimensional bifurcation in which explosions occur without tangencies. The section includes a numerical example of such a bifurcation. Section 4 presents a new scaling law for the parameter-dependent change in the density of the newly formed piece of the chaotic attractor. Section 4.3 describes some interesting numerical implications of tangency-free explosions as the onset of *unstable dimension variability*, a

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specific type of non-hyperbolic behavior.

2. Explosions via tangency

Let f_λ be a one-parameter family of functions. An explosion occurs at a parameter value λ_0 and a point x which is not recurrent prior to the bifurcation but is recurrent at the bifurcation. Clearly a saddle node bifurcation for either a fixed point or periodic orbit gives rise to an explosion point. In addition, the existence of explosion points is often the result of bifurcations involving tangencies between the stable and unstable manifolds of fixed or periodic points. In this section, we describe classes of explosions arising from homoclinic and heteroclinic tangency. The ultimate goal is a full understanding and classification of all types of explosions. In this direction, Palis and Takens made the following conjecture for planar one-parameter families of diffeomorphisms. We have extended this conjecture to include one-parameter families of one-dimensional maps as well.

CONJECTURE 1 (Palis and Takens [25]). Explosions within generic one-parameter families of smooth one-dimensional maps or smooth two-dimensional diffeomorphisms are the result of either a tangency between stable and unstable manifolds of fixed or periodic points or a saddle node bifurcation of a fixed or periodic point.

Newhouse, Palis, and Takens have shown that this statement is true in the case when the limit set is still finite at the bifurcation point [24]. Bonatti, Diaz, and Viana point out that this question can also be posed from a probabilistic point of view, in which case they conjecture the opposite conclusion [6]. In one dimension, we believe this problem to be tractable. In two dimensions an answer depends on a detailed classification of the accessibility of periodic points and their manifolds on the boundary of a basic set [4].

We now give some basic definitions. Let $f : R^k \times R \to R^k$ be a C^1-smooth one-parameter family of C^2 diffeomorphisms, $k = 1, 2, 3$ where we use two notations interchangeably: $f(x, \lambda) = f_\lambda(x)$. For the definition of an explosion it is more natural to use the concept of chain recurrence rather than recurrence:

DEFINITION 1. *For an iterated function g, there is an ϵ-**chain** from x to y when there is a finite sequence $\{z_0, z_1, \ldots, z_N\}$ such that $z_0 = x$, $z_N = y$, and $d(g(z_{n-1}), z_n) < \epsilon$ for all n.*

If there is an ϵ-chain from x to itself for every $\epsilon > 0$ (where $N > 0$), then x is said to be **chain recurrent** *[7, 8]. The* **chain recurrent set** *is the set of all chain recurrent points. For a one-parameter family f_λ, we say (x, λ) is chain recurrent if x is chain recurrent for f_λ.*

If for every $\epsilon > 0$, there is an ϵ-chain from x to y and an ϵ-chain from y to x, then x and y are said to be in the same **chain component** *of the chain recurrent set.*

Note that the chain recurrent set and the chain components are invariant under forward iteration.

DEFINITION 2 (Chain explosions). *A* **chain explosion point** (x, λ_0) *is a point such that x is chain recurrent for f_{λ_0}, but there is a neighborhood N of x such that on one side of λ_0 (i.e. either for all $\lambda < \lambda_0$ or for all $\lambda > \lambda_0$), no point in N is chain recurrent for f_λ.*

Note that in the above definition, at f_{λ_0}, x is not necessarily an isolated point of the chain recurrent set. A well studied example of this is the explosion that occurs at a saddle node bifurcation on an invariant circle. The chain recurrent set consists of two fixed points prior to bifurcation and the whole circle at and in many cases after bifurcation. In subsequent usage, if the distinction is not important, we will refer to recurrent points rather than always saying chain recurrent.

2.1. One dimension

This section describes a classification of explosions via homoclinic tangencies in one dimension which appears in [2]. Although one dimension would seem to be the easiest case, there are some key differences between one- and two-dimensional explosions which are not simplifications in one dimension. For example, for a diffeomorphism, homoclinic and heteroclinic orbits require the existence of saddle points with stable and unstable manifolds of dimension at least one. However, since a one-dimensional map is in general noninvertible, it is possible to have fixed or periodic points with one-dimensional unstable manifolds and a non-trivial zero-dimensional stable manifolds. Marotto terms such points snap-back repellers [22]. It is not possible to reverse the dimensions of the stable and unstable manifolds; the existence of a homoclinic orbit to an attracting fixed point requires a multivalued map [29]. In addition, the chain recurrent set is not invariant under backwards iteration of a noninvertible map, so the discussion of explosions in one dimension includes cases in which a point is not an explosion point, but the preimages are explosion points. As this is a broad survey, the statements and proofs of the results below are only sketches. The full details appear in [2].

Let f be a one variable function with a repelling fixed point x_0. Let y and k be such that y is a k^{th} preimage of x_0, and assume that a sequence of preimages of y converge to x_0. Then y is contained in an orbit which limits both forwards and backwards to x_0. That is, y is a homoclinic point for x_0. Homoclinic points for periodic orbits are defined by replacing f with some appropriate iterate f^m. Notice that for diffeomorphisms, all orbits through homoclinic points are homoclinic orbits. For one-dimensional maps, there may be many non-homoclinic orbits through a homoclinic point.

Since the stable manifold of a homoclinic point is zero-dimensional, a homoclinic tangency is a tangency of the graph of the map at a homoclinic point. That is, a homoclinic tangency occurs if the graph of f has a horizontal tangent at a homoclinic point.

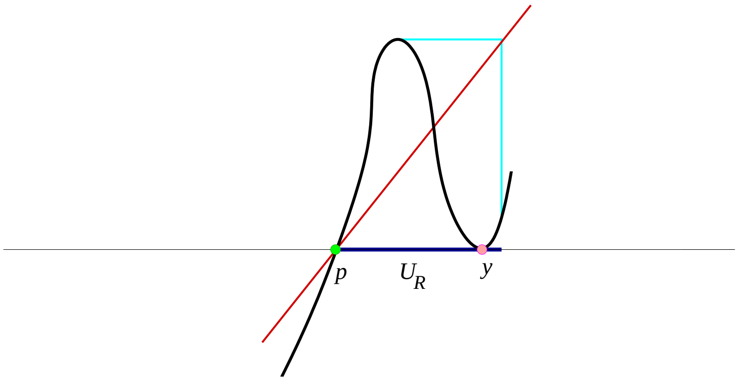

Figure 1: A one-dimensional map with a repelling fixed point p with positive derivative. The point y is a homoclinic tangency point. Since y is contained in a non-crossing orbit, by Theorem 2, y is not an explosion point. The unstable manifold branch U_R is an interval shown by a thick line on the x-axis.

Assume that f_λ is a C^1 smooth family of C^2 maps and that the point $x_0(\lambda)$ (which we write as x_0) is a repelling fixed point.

THEOREM 1 (No explosions with negative derivative at x_0, Alligood, Sander, Yorke [2]). *If f has a negative derivative at x_0, and y is homoclinic to x_0, at λ_0, then (y, λ_0) is not an explosion point.*

The idea is that images of a neighborhood of y map across preimages of a neighborhood of y, implying that there is a periodic point near y, which persists under perturbation of the parameter. Therefore y is arbitrarily close to recurrent points prior to the tangency bifurcation.

Now consider the case of a positive derivative at x_0. Assume that a homoclinic orbit limits to the local right (resp. left) branch of the unstable manifold of x_0. Let y be a homoclinic tangency point within this orbit. Then for some k, $f^k(y) = x_0$. If the graph of the map f^{k-1} near $f^{k-1}(y)$ is below (resp. above) the horizontal line, we call the homoclinic orbit a *crossing orbit*. A homoclinic orbit that is not crossing is called a *non-crossing orbit*.

THEOREM 2 (No explosions for non-crossing bifurcations [2]). *Assume that y is contained in a non-crossing homoclinic orbit, then y is not an explosion point.*

The proof is very similar to the case of negative derivative at x_0. Namely, an image of a neighborhood of y covers a preimage of the same neighborhood, implying that y is not an explosion point. This situation is depicted in Figure 1.

Closely related to crossing and non-crossing orbits, we can distinguish the two

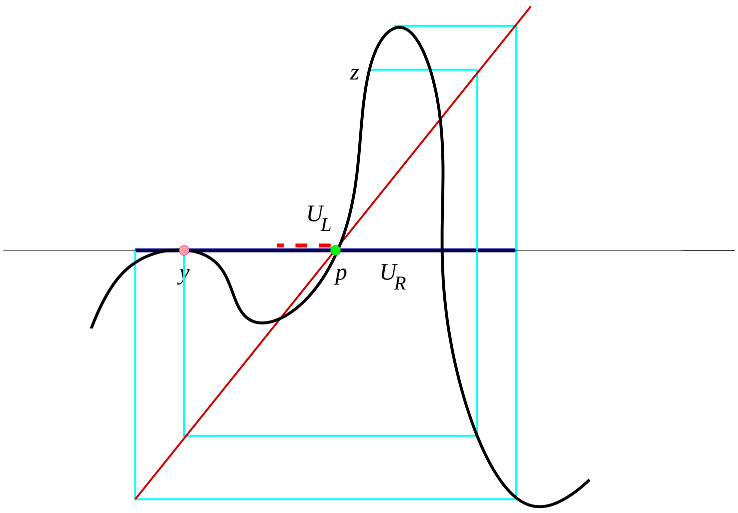

Figure 2: In this figure, y is a homoclinic tangency point in a crossing orbit, contained in U_R, but not U_L. Therefore y is an explosion point. The preimage z of y is also an explosion point. The interval U_R is shown by a thick line as in Figure 1. U_L is denoted by a dashed line slightly above the x-axis.

manifold branches U_R and U_L, being the iterates of the local right and lefthand branches of the unstable manifold of x_0. The union of U_L and U_R is the entire unstable manifold of x_0. If $\{(U_L \cap U_R) \setminus x_0\}$ is not empty, then the intersection must contain either all of U_L or all of U_R. For example, U_R may contain points both to the left and to the right of U_L. See Figures 2 and 3. We can show that if $y \in U_R$, and the k^{th} image of every neighborhood of y contains points in $\{U_R \setminus x_0\}$, then y is not an explosion point.

Under certain generic conditions, we can show a converse to the non-crossing orbit theorem: If y is in a crossing orbit, then y is an explosion point [2].

We are interested not only in explosion points which are themselves tangency points, but also in points which are explosion points far from tangencies, but are caused by a tangency. Since the chain recurrent set is invariant under forwards iteration, the image of non-crossing tangency point is a non-explosion point. However, there may be explosion points with iterates that are non-explosion points. For example, Figure 3 shows points of which are not tangency points but are preimages of tangency points. We can prove that for a generic one-parameter family, there are explosion points at U_R (such as i and j depicted in the figure) if there are points in U_R which map to a tangency (such as y) on the other side, even though the tangency point is not an explosion point [2].

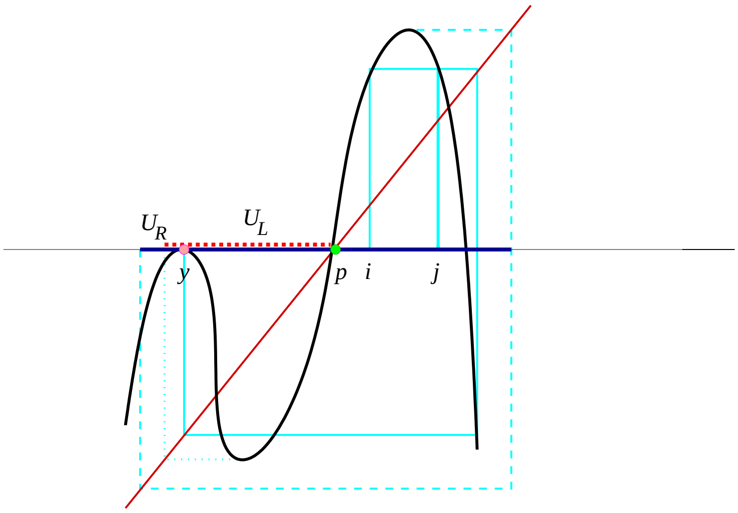

Figure 3: The tangency point y is contained in $U_L \cap U_R$, and is thus not an explosion point. However, the preimages i and j of y are explosion points. U_R and U_L are depicted again with a solid and dashed line respectively.

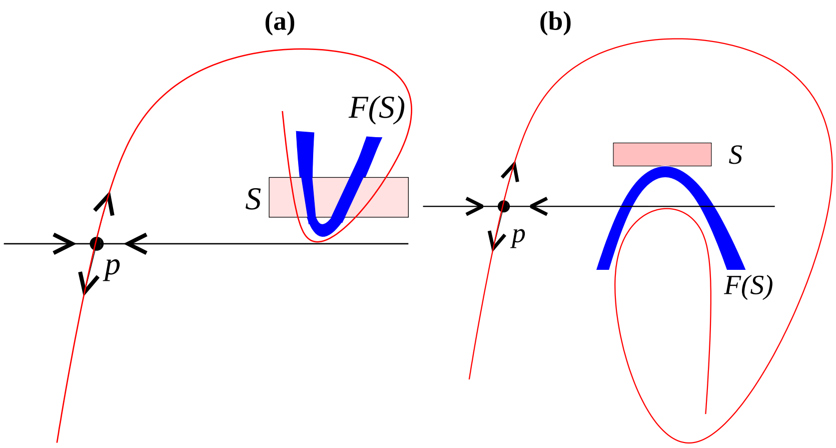

Figure 4: Homoclinic tangency points in the plane may or may not be explosion points. A set S and a large iterate $f^K(S) = F(S)$. In (a), it is never possible for the homoclinic tangency point to be an explosion point. In part (b), it depends on the eigenvalues.

2.2. Tangencies in two dimensions

In two dimensions, a heteroclinic intersection is a hallmark for chaotic behavior in the form of a Smale horseshoe. Thus a homoclinic tangency is a bifurcation point. It is not always an explosion point. Figure 4(a) shows a configuration of stable and unstable manifolds at a homoclinic tangency which can never be an explosion, since there is a horseshoe forming locally prior to tangency. In Figure 4(a), the situation is not so clear, since a large iterate of a rectangular set S may or may not map across itself under a large iterate. In fact, whether this is a possible explosion point depends on the relative strength of contraction and expansion at the fixed point p. Palis and Takens [25] classified homoclinic explosion bifurcations, as shown in Figure 4 by describing which planar homoclinic tangencies can be explosion points as a function of the placement of the tangency point, sign of the eigenvalues, and area contraction or expansion of the map.

An explosion can also occur as a result of tangencies between stable and unstable manifolds of different fixed or periodic points: heteroclinic tangencies. In this case, in order for a heteroclinic tangency to result in any sort of recurrence, there needs to be a means of return. A natural way is another heteroclinic intersection, resulting in a heteroclinic cycle. These ideas are given in the following definitions.

DEFINITION 3 (n-connection). *An n-**connection** is a sequence of points $\{\rho_1, t_1, \rho_2, t_2, \ldots, \rho_n\}$ such that for all i, ρ_i is a fixed point, and for each $i < n$, t_i is a heteroclinic point such that $t_i \in W^u(\rho_i) \cap W^s(\rho_{i+1})$.*

DEFINITION 4 (Cycle). *A **cycle** is an n-connection $\{\rho_1, t_1, \rho_2, t_2, \ldots, \rho_n\}$ such that $\rho_n = \rho_1$.*

Generically, we can assume a unique tangency in a cycle, with all other heteroclinic intersections being transverse. Thus any heteroclinic cycle can be reduced to a heteroclinic cycle containing only two periodic or fixed points. In a previous paper [1], we gave necessary and sufficient conditions for planar heteroclinic bifurcations to result in explosions. The classification does not involve eigenvalue conditions and area contraction/expansion conditions. Rather, the classification is a set of necessary and sufficient conditions for explosions at heteroclinic tangency based on the configuration of the manifolds, in addition to another "chain" condition on the behavior. Figure 5 shows two heteroclinic cycles with a tangency. In (a), any curve which is arbitrarily close to the heteroclinic cycle necessarily intersects both the unstable manifold of p and the stable manifold of q, whereas in (b), this is not the case. The tangency point in (a) is an explosion point and in (b) it is not; just as in the one-dimensional case, the key to an explosion is that different points of the cycle are on opposite sides of a dividing manifold. Therefore, any ϵ-chain must cross the cycle. Such heteroclinic tangency explosion bifurcations are known as crossing bifurcations.

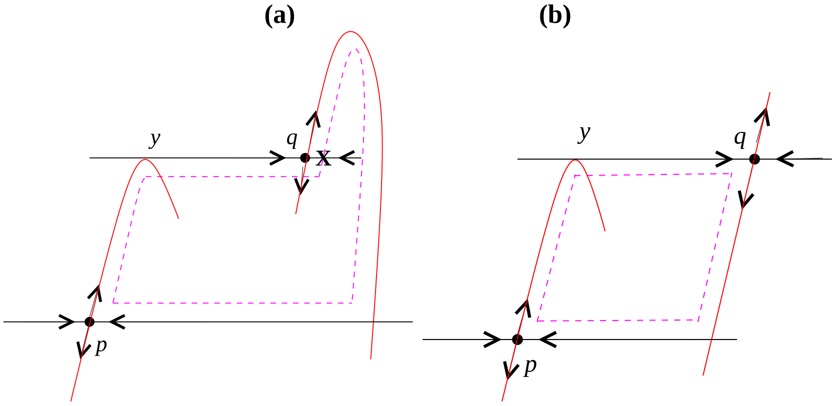

Figure 5: Heteroclinic cycles with a tangency at y. In (a), all curves close to the cycle intersect both the unstable manifold of p and the stable manifold of q (intersection shown with an X). Under the chain condition, the point y is an explosion point. In (b), it is possible to draw a curve arbitrarily close to the cycle which does not intersect the stable manifold of q. The point y is never an explosion point.

2.3. Explosions far from tangencies and planar gap filling

A planar explosion far from tangency but through a tangency occurs at a crisis bifurcation of the Ikeda map. This example has been studied by a number of authors, such as [17, 26]. Prior to bifurcation, there is a small attractor surrounded by an unstable chaotic saddle set. The set has noticable gaps, in which there are no recurrent points. After the bifurcation, the new attractor includes both the old attractor and the saddle set. In addition, the gaps in the recurrent set are filled in. This occurs in a discontinuous manner at the bifurcation value. Thus the points in the gaps at the bifurcation parameter are all explosion points. This discontinuous change in the recurrent set is a result of a heteroclinic tangency.

Figure 6 depicts the geometry of manifolds of a heteroclinic cycle that give rise to gap filling. The tangency point y is contained simultaneously in a crossing and a non-crossing orbit. Thus y is not an explosion point. However, there are points on the non-crossing orbit which are explosion points. There is an attractor with its boundary being the unstable manifold branch of p including the tangency. The basin of attraction includes all points below the stable manifold branch of q including the tangency. Therefore all points of the upper unstable manifold branch of q (in green in the figure for the color online version) are in the basin of attraction. After the bifurcation, these points become recurrent. Robert et al. [27] showed that this topological construction occurs during gap filling as a result of a heteroclinic cycle with a tangency for a pair of period five orbits of the Ikeda map.

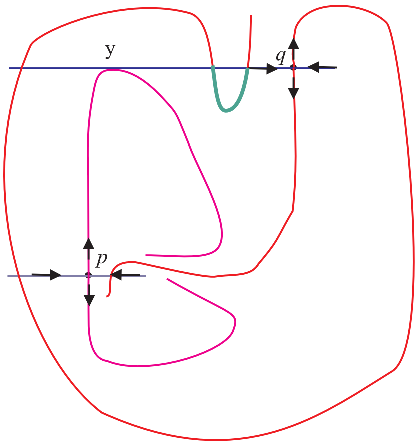

Figure 6: A geometric description of gap filling. The unstable manifold branch of p with the tangency encloses an attractor. This attractor only includes points "inside" the non-crossing inner cycle prior to tangency. When the tangency pushes through, the attractor also includes points "on the outside" – that is, on the crossing cycle.

3. Three-dimensional crises

There are many theoretical results and analyses of planar examples of crises, whereas relatively little is known about three-dimensional examples. In three dimensions, Diaz and Rocha [12, 13] described explosions that occur as the result of a non-tangency heteroclinic bifurcation. In [3], we have adapted these results to show that a crisis in a three-dimensional attractor can occur at a heteroclinic bifurcation without tangencies. After the crisis, the attractor contains two fixed points with different numbers of unstable directions. The existence of such an attractor is known as unstable dimension variability (UDV), and has been studied in the physics literature [5, 9, 10, 18, 20, 21, 19, 23, 28, 30, 32]. It is of particular interest, as unstable dimension variability results in nonshadowability.

We start by describing the topological dynamics of the example. To understand the role of the heteroclinic orbit in an attractor crisis, we need to be able to describe when there is a transverse heteroclinic orbit connecting two periodic points. We then need to know how a heteroclinic cycle can form. The dynamics can be similar to the planar case. Namely, when two three-dimensional periodic points have the same number of unstable directions, then a bifurcation must occur through tangency. Furthermore, since surfaces divide three-dimensional space, the heteroclinic cycles with explosions in this case parallel the two-dimensional case.

We now consider the case in which the two periodic points have a different number of unstable directions (1 and 2). Assume that q is a fixed point with two unstable directions, whereas p is a fixed point which only has one unstable direction.

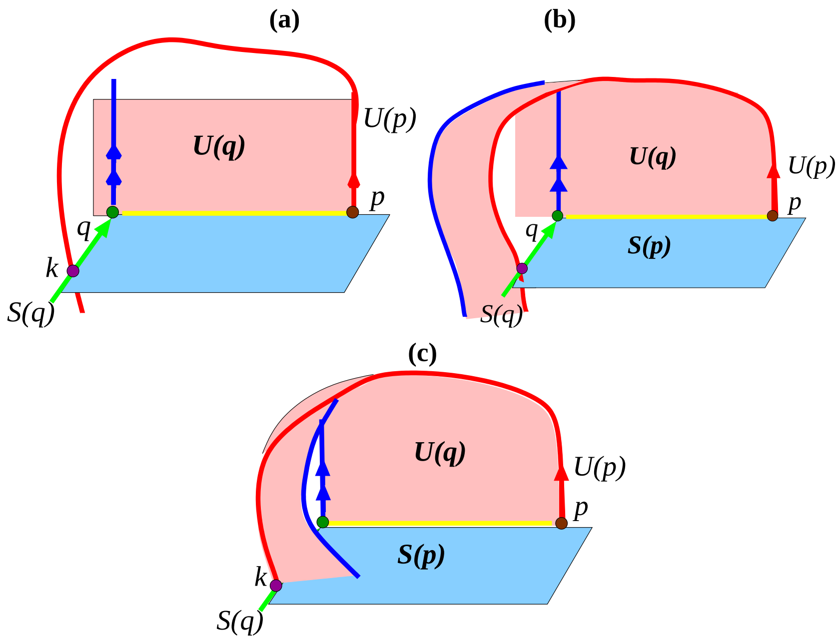

Figure 7: A heteroclinic bifurcation in three dimension without tangencies. (a) At the bifurcation, the one-dimensional manifolds intersect at k. The two-dimensional manifolds can either intersect with (b) or without (c) a twist.

(For simplicity, we have chosen fixed points, but the statements below also hold for periodic points.) For a heteroclinic cycle, we assume that a transverse intersection occurs between the manifolds of q and p. This must necessarily be an intersection of the two two-dimensional manifolds: the unstable manifold of q and the stable manifold of p. Either the transverse intersection of stable and unstable manifolds contains non-invariant components [11], or the intersection persistently connects the two periodic points. We assume the latter case, depicted in Figure 7. This implies that the one-dimension invariant manifolds form the boundaries of the respective two-dimensional manifolds. That is, the unstable manifold of q is bounded by the unstable manifold of p, and the stable manifold of q is the boundary of the stable manifold of p.

Consider a bifurcation parameter value for which the two one-dimensional manifolds intersect. This is not a tangency bifurcation, since generically these manifolds will not share a common tangent space, as in Figure 7(a). However, since the one-dimensional manifolds are the boundaries of two-dimensional manifolds, a bifurcation occurs at the intersection of the one-dimensional manifolds.

Diaz and Rocha use this construction with the assumption that the unstable strip and the two-dimensional stable manifold intersect without a twist, as shown in Figure 7(b). In this case, the intersection point between the one-dimensional manifolds is

(a) **(b)**

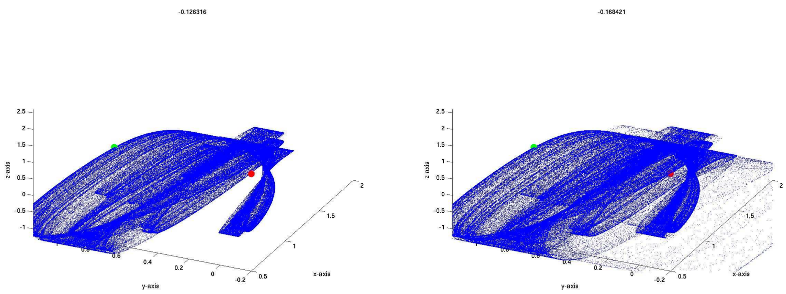

Figure 8: A numerical example of a heteroclinic bifurcation in three dimensions without tangencies. There are two saddle fixed points, p and q. (a) Prior to bifurcation, the attractor contains p but does not contain q. It is contained in the closure of a portion of $U(q)$. (b) At bifurcation, there is an explosion in which new points become part of the attractor. However, the density of the new part of the attractor is low.

an explosion point: Prior to bifurcation, p and q are isolated recurrent points; all points near $U(p)$ map far from $S(q)$, never returning near p. Likewise, no point in $U(q)$ intersects $S(q)$, making q isolated in the recurrent set as well. After bifurcation, there is a basic set containing transverse homoclinic points to both q and p. This implies that there is a basic set containing both q and p, fixed points with different numbers of unstable directions. The basic set in this example is in general unstable. This is a counterpart of the explosions at tangency points in the one- and two-dimensional cases. If there is an attractor involved, the crisis here would correspond to a blowout bifurcation, in which the entire attractor ceases to exist after the bifurcation point.

We are interested in the three-dimensional counterpart of planar gap filling. That is, the case in which the intersection between the one-dimensional manifolds is not and explosion point, but there are explosions occuring through this point. Gap filling corresponds to the case when the manifold $U(q)$ twists at the bifurcation, as in Figure 7(c). Thus, prior to bifurcation there are homoclinic points to p, and after bifurcation there are homoclinic points to q. The intersection is not an explosion point, but there may be other explosion points occuring when the dynamics change. We have constructed the first numerical example of this type of three-dimensional crossing bifurcation, as depicted in Figure 8. Prior to bifurcation, there is an attractor contained in the unstable manifold strip bounded by $U(p)$ and the strong unstable manifold of p. Points feed into the attractor from the inaccessible side of $U(q)$. After bifurcation, this region becomes accessible. As in the planar case, points are now able to return to this newly accessible side, resulting in an explosion in the size in the attractor. It also turns out that after bifurcation, the attractor displays unstable dimension variability. Numerically computed stable and unstable manifolds for the example appearing in [3] are displayed in Figure 8.

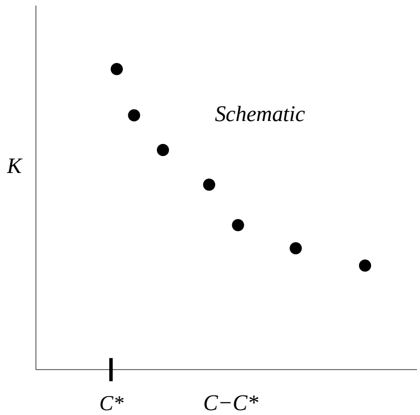

Figure 9: Scaling of the iterate length in the old part of the attractor as a function of the parameter.

4. Scaling laws for attractor density

Immediately after a crisis, there is a very low density of points in the new attractor. This low density is known as intermittency in the case of the period three window of the logistic map [31]. For planar homoclinic and heteroclinic tangency crises, the scaling of the low density of the new points as a function of the parameter was analyzed in the 1980's, as we describe in the section below. We have used similar methods to write down a scaling law for three-dimensional bifurcations without tangencies.

Consider a crisis in which the attractor before tangency is denoted A_{old}, and the new part of the attractor appearing at tangency is denoted A_{new}. Consider any dense orbit. Define τ as the orbit length in A_{old} between visits to A_{new}, and K as the mean of τ. For large τ, $P(\tau) \propto K^{-1} exp(\tau/K)$. The density of A_{new} can be approximated by the reciprocal of K. The variation in density can be analyzed using scaling near the bifurcation point $c = c^*$. Notice that after the bifurcation $\lim_{|c-c^*|\to 0} K = \infty$. We expect K to increase exponentially as $|c - c^*|$ approaches zero, as depicted schematically in Figure 9.

4.1. Scaling in two dimensions

Grebogi, Ott, Yorke [14, 15, 16] stated scaling laws for a planar homoclinic or heteroclinic bifurcation with a quadratic tangency. In the heteroclinic case such as depicted in Figure 6, we briefly describe the calculation used in order to illustrate its heavy reliance on the existence of a tangency. Let L be the lobe formed at y after passing through tangency. Then the following formula holds:

$$A(L) = \ell \cdot w = m(c - c*) \cdot \sqrt{c - c*}.$$

The density of the new part of the attractor depends on this area, in that this is the only way for points in A_{old} to enter A_{new}. The square root in this formula is due to the quadratic tangency. Let $L_n = f^n(L)$. Then $A(L_m) = MA(L)$, where M depends on the eigenvalues of q. These are the key ingredients giving rise to the scaling law.

4.2. A new scaling law in three dimensions

For the three-dimensional non-tangency bifurcation described in Section 3, we have demonstrated numerically that there is a linear relationship between the logarithm of the mean transient length K and the logarithm of the distance from the bifurcation parameter (*cf.* [3]).

Denote the eigenvalues of q by $|\lambda_1| > |\lambda_2| > 1 > |\mu|$, and those of p by $|\beta_1| < |\beta_2| < 1 < |\alpha|$. The set of points in the attractor (and thus on the two-dimensional unstable manifold $U(q)$) which exit A_{old} must do so by coming very close to the unstable manifold of p. Starting near p, the area of $U(q)$ within ϵ of $U(p)$ is approximated using α and β_2. We also need to know the fraction of points which exit A_{old} near $U(p)$ which re-enter the attractor. This is done using the two unstable eigenvalues for the linearization at q. The estimate leads to the new scaling law, which give good agreement with numerical calculation. It states that the mean transient length is $K(\eta) = \eta^\gamma$, where

$$\gamma = 1 + \frac{\log|\lambda_1|}{\log|\lambda_2|} + \frac{\log|\alpha|}{\log|\beta_2|}.$$

4.3. Unstable dimension variability

The low density of the new part of the attractor has interesting numerical implications in terms of testing for unstable dimension variability. Precisely, near the parameter at which a crisis occurs, the standard test for UDV is not applicable.

In an attractor with a dense orbit which exhibits UDV, we know that the dense orbit comes arbitrarily close to the stable manifold of each fixed or periodic saddle point. Thus it is possible to find a sequence within the dense orbit which stays close to a fixed or periodic point for any prescribed number of iterates after any finite transient is removed. For simplicity, assume p and q are fixed points with different numbers of unstable directions in an attractor which exhibits UDV. Since the middle Lyapunov exponent of the fixed points have opposite signs, the orbit must have arbitrarily long finite time sequences with the middle Lyapunov exponent being negative, and arbitrarily long finite time sequences with the middle Lyapunov exponent being positive. The standard test for UDV uses this fluctuation of Lyapunov exponents around zero [10].

In the case of UDV after a crossing bifurcation, although it is theoretically correct that the Lyapunov exponents fluctuate around zero, the density of the attractor is quite low near the newly added fixed point. Therefore, it is computationally infeasible to use the Lyapunov exponent test for UDV. See [3] for detailed numerical calculations illustrated using the example depicted in Figure 8.

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PERIODIC SOLUTIONS OF FIRST ORDER NONLINEAR DIFFERENCE EQUATIONS

Abstract. This paper surveys some recent results on the existence and multiplicity of periodic solutions of nonlinear difference equations of the first order under Ambrosetti-Prodi or Landesman-Lazer type conditions.

1. Introduction

Periodic solutions of first and second order nonlinear difference equations have been widely studied, and the reader can consult [1, 9] for references. In some recent work with C. Bereanu, we have adapted the topological approach to the upper and lower solutions method to this class of problems and used it, together with Brouwer degree, to obtain new existence and multiplicity results of the Ambrosetti-Prodi and Landesman-Lazer type [2, 3]. In [4], we have used the same methodology to prove similar results for second order nonlinear difference equations with Dirichlet boundary conditions. The present paper surveys some of those results and is restricted, for the sake of simplicity, to the case of periodic solutions of first order difference equations. Some of the arguments of [2, 3] are simplified, and some of the conclusions are sharpened.

2. Periodic solutions

Let $n \geq 2$ be a fixed integer. For $(x_1, \ldots, x_n) \in \mathbb{R}^n$, define the first order difference operator $(Dx_1, \ldots, Dx_{n-1}) \in \mathbb{R}^{n-1}$ by

$$Dx_m := x_{m+1} - x_m \quad (1 \leq m \leq n-1).$$

Let $f_m : \mathbb{R}^n \to \mathbb{R} \quad (1 \leq m \leq n-1)$ be continuous functions. We study the existence of solutions for the periodic boundary value problem

$$(1) \qquad Dx_m + f_m(x_1, \ldots, x_n) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

Let

$$(2) \qquad U^{n-1} = \{x \in \mathbb{R}^n : x_1 = x_n\},$$

so that $U^{n-1} \simeq \mathbb{R}^{n-1}$ because an element of U^{n-1} can be characterized by the coordinates $x_1, \ldots, x_{n-1}$. The restriction $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ of D to $\mathbb{R}^{n-1}$ is given by

$$(3) \qquad (Lx)_m = x_{m+1} - x_m \quad (1 \leq m \leq n-2), \quad (Lx)_{n-1} = x_1 - x_{n-1},$$

or, in matrix form, by the circulant matrix [6]

$$
\begin{pmatrix}
-1 & 1 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & -1 & 1 \\
1 & 0 & \cdots & \cdots & 0 & -1
\end{pmatrix}. \tag{4}
$$

If we define $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_m(x_1, \cdots, x_{n-1}) = f_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \le m \le n-1),$$

problem (1) is equivalent to study the zeros of the continuous mapping $H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$H_m(x) = (Lx)_m + F_m(x) \quad (1 \le m \le n-1). \tag{5}$$

3. Bounded nonlinearities

Let us first consider the linear periodic problem

$$Dx_m + \alpha x_m = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n, \tag{6}$$

where $\alpha \in \mathbb{R}$. The solutions of the corresponding difference system are given by

$$x_m = (1-\alpha)^{m-1} x_1 \quad (1 \le m \le n),$$

and hence (6) has a solution if and only if

$$x_1 = (1-\alpha)^{n-1} x_1.$$

This immediately implies the following

LEMMA 1. *Problem (6) has only the trivial solution if n is even and $\alpha \neq 0$ or if n is odd and $\alpha \notin \{0, 2\}$. When $\alpha = 0$, the solutions are of the form $x_m = c \quad (1 \le m \le n)$, and when n is odd and $\alpha = 2$, they have the form $x_m = (-1)^{m-1} c \quad (1 \le m \le n)$, with $c \in \mathbb{R}$ arbitrary.*

Let

$$b_m : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto b_m(x_1, \ldots, x_n) \quad (1 \le m \le n-1) \tag{7}$$

be continuous and bounded, and consider the semilinear periodic problem

$$Dx_m + \alpha x_m + b_m(x_1, \ldots, x_n) = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n. \tag{8}$$

If L is defined like above and $B : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$B_m(x_1, \ldots, x_{n-1}) = b_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \le m \le n-1),$$

then problem (8) is equivalent to the semilinear problem in $\mathbb{R}^{n-1}$

$$Lx + \alpha x + B(x) = 0. \tag{9}$$

We have the following existence result.

THEOREM 1. *Assume n odd and $\alpha \neq 0$ or n even and $\alpha \notin \{0, 2\}$ and assume that the functions b_m in (7) are continuous and bounded. Then problem (8) has at least one solution and, for all sufficiently large R,*

$$d_B[L + \alpha I + B, B(R), 0] = \pm 1.$$

Proof. Let $M > 0$ is such that $\|B(v)\| \leq M$ for all $v \in \mathbb{R}^{n-1}$. For each $\lambda \in [0, 1]$, each possible zero u of $L + \alpha I + \lambda B$ is such that, using Lemma 1,

$$\|u\| = \lambda \|(L + \alpha I)^{-1} B(u)\| \leq \|(L + \alpha I)^{-1}\| M.$$

Hence, if we take any $R > \|(L + \alpha I)^{-1}\| M$, and denote the Brouwer degree by d_B (see [7]), the homotopy invariance of the degree implies that

$$d_B[L + \alpha I + B, B(R), 0] = d_B[L + \alpha I, B(R), 0] = \pm 1,$$

and the existence follows from the existence property of Brouwer degree [7]. □

4. Upper and lower solutions

Let $f_m : \mathbb{R} \to \mathbb{R}$ $(1 \leq m \leq n-1)$ be continuous functions, and let us consider the periodic problem

$$Dx_m + f_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n. \tag{10}$$

DEFINITION 1. *$\alpha = (\alpha_1, \ldots, \alpha_n)$ (resp. $\beta = (\beta_1, \ldots, \beta_n)$) is called a* lower solution *(resp.* upper solution*) for (10) if*

$$\alpha_1 \geq \alpha_n \quad (\textit{resp. } \beta_1 \leq \beta_n),$$

and the inequalities

$$D\alpha_m + f_m(\alpha_m) \geq 0 \quad (\textit{resp. } D\beta_m + f_m(\beta_m) \leq 0) \tag{11}$$

hold for all $1 \leq m \leq n-1$. Such a lower or upper solution will be called strict *if the inequality (11) is strict for all $1 \leq m \leq n-1$.*

The basic theorem for the method of upper and lower solutions goes as follows. The proof given here is a simplification of that given in [2], which is modeled on the corresponding one for differential equations in [11].

THEOREM 2. *If (10) has a lower solution $\alpha = (\alpha_1, \ldots, \alpha_n)$ and an upper solution $\beta = (\beta_1, \ldots, \beta_n)$ such that $\alpha_m \leq \beta_m$ $(1 \leq m \leq n)$, then (10) has a solution $x = (x_1, \ldots, x_n)$ such that $\alpha_m \leq x_m \leq \beta_m$ $(1 \leq m \leq n)$. Moreover, if α and β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leq m \leq n-1)$.*

Proof. I. A modified problem.
Let $\gamma_m : \mathbb{R} \to \mathbb{R}$ $(1 \leq m \leq n-1)$ be the continuous functions defined by

$$\gamma_m(x) = \begin{cases} \beta_m & \text{if} \quad x > \beta_m \\ x & \text{if} \quad \alpha_m \leq x \leq \beta_m \\ \alpha_m & \text{if} \quad x < \alpha_m. \end{cases} \tag{12}$$

We consider the modified problem

$$Dx_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n, \tag{13}$$

and show that if $x = (x_1, \ldots, x_n)$ is a solution of (13) then $\alpha_m \leq x_m \leq \beta_m$ $(1 \leq m \leq n)$, and hence x is a solution of (10). Suppose by contradiction that there is some $1 \leq i \leq n$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max_{1 \leq j \leq n}(\alpha_j - x_j) > 0$. If $1 \leq m \leq n-1$, then

$$\alpha_{m+1} - x_{m+1} \leq \alpha_m - x_m,$$

which gives

$$D\alpha_m \leq Dx_m = x_m - \alpha_m - f_m(\alpha_m) \leq x_m - \alpha_m + D\alpha_m < D\alpha_m,$$

a contradiction. Now the condition $\alpha_1 \geq \alpha_n$ shows that the maximum is reached at $m = n$ only if it is reached also at $m = 1$, a case already excluded. Analogously we can show that $x_m \leq \beta_m$ $(1 \leq m \leq n)$. We remark that if α, β are strict, the same reasoning gives $\alpha_m < x_m < \beta_m$ $(1 \leq m \leq n-1)$.
II. Solution of the modified problem.
We use Brouwer degree to study the zeros of the continuous mapping $G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$G_m(x) = (Lx)_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) \quad (1 \leq m \leq n-1). \tag{14}$$

By Lemma 1, $L - I : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is invertible. On the other hand the mapping with components $f_m \circ \gamma_m + \gamma_m$ $(1 \leq m \leq n-1)$ is bounded on $\mathbb{R}^{n-1}$. Consequently, Theorem 1 implies the existence of $R > 0$ such that, for all $\rho > R$, one has

$$|d_B[G, B(\rho), 0]| = 1, \tag{15}$$

and, in particular, G has a zero $\widetilde{x} \in B(\rho)$. Hence, $x = (\widetilde{x}, x_1)$ is a solution of (13), which means that $\alpha_m \leq x_m \leq \beta_m$ $(1 \leq m \leq n)$ and x is a solution of (10). Moreover if α, β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leq m \leq n-1)$. $\square$

Suppose now that α (resp. β) is a strict lower (resp. upper) solution of (10). Define the open set

$$\Omega_{\alpha\beta} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \alpha_m < x_m < \beta_m \quad (1 \le m \le n-1)\}, \tag{16}$$

and the continuous mapping $H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$H_m(x) = (Lx)_m + f_m(x_m) \quad (1 \le m \le n-1). \tag{17}$$

COROLLARY 1. *Assume that the conditions of Theorem 2 hold with strict lower and upper solutions. Then*

$$|d_B[H, \Omega_{\alpha\beta}, 0]| = 1, \tag{18}$$

with $\Omega_{\alpha\beta}$ defined in (16).

Proof. If ρ is large enough, then, using the additivity-excision property of Brouwer degree [7], we have

$$|d_B[G, \Omega_{\alpha\beta}, 0]| = |d_B[G, B(\rho), 0]| = 1.$$

On the other hand, H is equal to G on $\overline{\Omega_{\alpha\beta}}$, and then

$$|d_B[G, \Omega_{\alpha\beta}, 0]| = |d_B[H, \Omega_{\alpha\beta}, 0]|.$$

□

A simple but useful consequence of Theorem 2, goes as follows.

COROLLARY 2. *Assume that there exists numbers $\alpha \le \beta$ such that*

$$f_m(\alpha) \ge 0 \ge f_m(\beta) \quad (1 \le m \le n-1).$$

Then problem (10) has at least one solution with $\alpha \le x_m \le \beta$ $(1 \le m \le n-1)$.

Proof. Just observe that $(\alpha, \ldots, \alpha)$ is a lower solution and $(\beta, \ldots, \beta)$ an upper solution for (10). □

COROLLARY 3. *For each $p > 0$, $a_m > 0$ and $b_m \in \mathbb{R}$ $(1 \le m \le n-1)$ the problem*

$$Dx_m - a_m|x_m|^{p-1}x_m = b_m \quad (1 \le m \le n-1), \quad x_1 = x_n$$

has at least one solution.

Proof. If $R \ge \left(\max_{1 \le m \le n-1} \frac{|b_m|}{a_m}\right)^{1/p}$, then $(-R, \ldots, -R)$ is a lower solution and $(R, \ldots, R)$ an upper solution. □

REMARK 1. *When* $\beta_m \leq \alpha_m$ $(1 \leq m \leq n-1)$, *one can try to repeat the argument of Theorem 2 by defining*

$$\delta_m(x) = \begin{cases} \alpha_m & \text{if} \quad x > \alpha_m \\ x & \text{if} \quad \beta_m \leq x \leq \alpha_m \\ \beta_m & \text{if} \quad x < \beta_m, \end{cases}$$

and considering the modified problem

$$(19) \qquad Dx_m + x_m + f_m \circ \delta_m(x_m) - \delta_m(x_m) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.$$

As $L + I$ *is invertible, the degree argument still gives the existence of at least one solution for (19). If one tries to show that, say,* $\beta_m \leq x_m$ $(1 \leq m \leq n-1)$, *and assume by contradiction that* $\beta_i - x_i > 0$ *for some* $1 \leq i \leq n$, *one gets no contradiction with* $D\beta_m + f_m(\beta_m) \leq 0$. *This is in contrast with the ordinary differential equation case, for which the argument works independently of their order [13]. The reason of this difference comes from the fact that a local extremum is characterized by an equality (vanishing of the first derivative) in the differential case and by two inequalities (with only one usable in the argument) in the difference case. This raises the question of the validity of the method of upper and lower solutions with reversed upper and lower solutions in the difference case. This question is solved by the negative in the next two sections.*

5. Spectrum of the linear part

The construction of the counter-example proving the last assertion above is clarified by analyzing the spectral properties of the first order difference operator with periodic boundary conditions.

DEFINITION 2. *An* eigenvalue *of the first order difference operator with periodic boundary conditions is any* $\lambda \in \mathbb{C}$ *such that the problem*

$$(20) \qquad Dx_m = \lambda x_m \quad (1 \leq m \leq n-1), \quad x_1 = x_n$$

has a nontrivial solution.

Explicitly, system (20) can be written as

$$(21) \qquad \begin{array}{rcl} x_1 - x_n & = & 0 \\ x_2 - (1+\lambda)x_1 & = & 0 \\ \cdots\ \cdots\ \cdots\ \cdots & & \cdots \\ x_n - (1+\lambda)x_{n-1} & = & 0 \end{array}$$

and is equivalent to the matrix eigenvalue problem

$$(22)\qquad \begin{pmatrix} -1-\lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1-\lambda & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -1-\lambda & 1 \\ 1 & 0 & \cdots & \cdots & 0 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = 0.$$

Hence the eigenvalues λ_k are $\lambda_k = -1 + \mu_k$ $(0 \leq k \leq n-2)$, where the μ_k are the eigenvalues of the (permutation, unitary, circulant) matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

namely (see e.g. [6]),

$$(23)\qquad \lambda_k = -1 + e^{\frac{2k\pi i}{n-1}} \quad (0 \leq k \leq n-2).$$

The corresponding eigenvectors φ^k $(0 \leq k \leq n-2)$ have components

$$\varphi^k_m = e^{\frac{2km\pi i}{n-1}} \quad (1 \leq m \leq n-1).$$

In particular, $\lambda_0 = 0$ is always a real eigenvalue, and all the other eigenvalues have negative real part. If $n = 2$, 0 is the unique eigenvalue; if $n > 2$ is even, 0 is the unique real eigenvalue; if n is odd, $\lambda_{\frac{n-1}{2}} = -2$ is the unique nonzero real eigenvalue.

6. Reversing the order of upper and lower solutions

For $n \geq 2$ odd and $\lambda = -2$, system (21) becomes

$$(24)\qquad \begin{aligned} x_1 - x_n &= 0 \\ x_2 + x_1 &= 0 \\ \cdots\ \cdots & \ \cdots \\ x_n + x_{n-1} &= 0 \end{aligned}$$

and has the solution φ associated to $\varphi^{(n-1)/2}$ with components

$$\varphi_m = (-1)^{m-1} \quad (1 \leq m \leq n).$$

The adjoint system

$$(25)\qquad \begin{aligned} x_1 + x_2 &= 0 \\ \cdots\ \cdots & \ \cdots \\ x_{n-1} + x_n &= 0 \\ -x_1 + x_n &= 0 \end{aligned}$$

has the same nontrivial solution φ. As $b_m = \delta_{nm}$ $(1 \le m \le n)$ (Kronecker symbol) is not orthogonal to the kernel of the adjoint system (25), the problem

$$\begin{aligned} x_1 - x_n &= 0 \\ x_2 + x_1 &= 0 \\ \dots \ \dots &\ \dots \\ x_{n-1} + x_{n-2} &= 0 \\ x_n + x_{n-1} &= 1 \end{aligned}$$

has no solution, or, equivalently *the problem*

(26) $$Dx_m + 2x_m = 0 \quad (1 \le m \le n-2), \quad Dx_{n-1} + 2x_{n-1} = 1, \quad x_1 = x_n$$

has no solution. However, $\alpha = (1, \dots, 1)$ *is a lower solution and* $\beta = (0, \dots, 0)$ *is an upper solution of (26) such that* $\beta_m \le \alpha_m$ $(1 \le m \le n)$.

If now $n > 2$ is even, the problem

(27) $$\begin{aligned} Dx_m + 2x_m &= 0 \quad (1 \le m \le n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \\ Dx_{n-1} &= 0, \quad x_1 = x_n \end{aligned}$$

is of course equivalent to the problem

$$Dx_m + 2x_m = 0 \quad (1 \le m \le n-3), \quad Dx_{n-2} + 2x_{n-2} = 1, \quad x_1 = x_{n-1}.$$

As $n-1$ is odd, it follows from the counter-example (26) that *problem (27) has no solution. However* $\alpha = (1, \dots, 1)$ *is a lower solution and* $\beta = (0, \dots, 0)$ *is an upper solution of (27) such that* $\beta_m \le \alpha_m$ $(1 \le m \le n)$. Those counter-examples were first given in [3].

For $n = 2$, problem (10) is equivalent to the unique scalar equation

$$f_1(x_1) = 0$$

and, in this case, the validity of the method of upper and lower solutions, independently of their order, follows from its equivalence with Bolzano's theorem applied to the real function f_1.

REMARK 2. *Notice that, in contrast to the periodic problem for difference equations, whose eigenvalues are in the left half-plane, all the eigenvalues* $\lambda_k = \frac{2k\pi i}{T}$ $(k \in \mathbb{Z})$ *of the differential operator* $\frac{d}{dt}$ *with periodic boundary conditions on* $[0, T]$ *are on the imaginary axis. This explains that the method of upper and lower solutions works irrespectively to the order of the lower and the upper solution.*

7. Ambrosetti-Prodi type multiplicity result

Let $f_1, \dots, f_{n-1} : \mathbb{R} \to \mathbb{R}$ be continuous functions, $s \in \mathbb{R}$. Consider the problem, with $n \ge 2$,

(28) $$Dx_m + f_m(x_m) = s \quad (1 \le m \le n-1), \quad x_1 = x_n,$$

with the *coercivity condition*

$$f_m(u) \to \infty \quad \text{as} \quad |u| \to \infty \quad (1 \le m \le n-1). \tag{29}$$

When $n = 2$, problem (28) is equivalent to the scalar equation

$$f_1(x_1) = s \tag{30}$$

and, under condition (29) with $m = 1$, it is clear that there exists s_1 $(= \min_{\mathbb{R}} f_1)$ such that for $s < s_1$, equation (30) has no solution, for $s = s_1$, equation (30) has at least one solution, and for $s > s_1$, equation (30) has at least two solutions. We show that a similar result holds for any $n \ge 2$. Problems of this type were initiated by Ambrosetti-Prodi for second order semilinear Dirichlet problems and the approach given here slightly simplifies the one given in [2], modeled on the method introduced in [12, 13] for periodic solutions of first and second order ordinary differential equations.

LEMMA 2. *If condition (29) holds, then*

$$\sum_{m=1}^{n-1} f_m(x_m) \to +\infty \quad \textit{if} \quad \|x\| \to \infty.$$

Proof. From (29),

$$(\exists\, c \in \mathbb{R})(\forall\, u \in \mathbb{R})(\forall\, m \in \{1, \dots, n-1\}) : f_m(u) \ge c, \tag{31}$$

and

$$\begin{aligned} &(\forall\, R > 0)(\exists\, r' > 0)(\forall u \in \mathbb{R} : |u| \ge r')(\forall\, m \in \{1, \dots, n-1\}) : \\ &\qquad\qquad\qquad f_m(u) \ge R - (n-2)c. \end{aligned} \tag{32}$$

If $r = \sqrt{n-1}r'$ and if $x \in \mathbb{R}^{m-1}$ is such that $\|x\| \ge r$, then, for at least one $j \in \{1, \dots, n-1\}$, one has $|x_j| \ge r'$, so that, using (31) and (32),

$$\begin{aligned} \sum_{m=1}^{n-1} f_m(x_m) &= \sum_{m=1}^{n-1} [f_m(x_m) - c] + (n-1)c \ge f_j(x_j) - c + (n-1)c \\ &\ge R - (n-2)c - c + (n-1)c = R. \end{aligned}$$

Consequently, $\sum_{m=1}^{n-1} f_m(x_m) \to +\infty$ if $\|x\| \to \infty$. □

LEMMA 3. *Let $b \in \mathbb{R}$. If condition (29) holds, there is $\rho = \rho(b) > 0$ such that each possible solution x of (28) with $s \le b$ is such that $\|x\| < \rho$.*

Proof. Let $s \le b$ and $(x_1, \dots, x_n)$ be a solution of (28). We see that

$$\sum_{m=1}^{n-1} f_m(x_m) = (n-1)s \le (n-1)b. \tag{33}$$

and the result follows from Lemma 2. □

THEOREM 3. *If the functions* f_m $(1 \leq m \leq n-1)$ *satisfy (29), there exists* $s_1 \in \mathbb{R}$ *such that* (28) *has zero, at least one or at least two solutions according to* $s < s_1, s = s_1, s > s_1$.

Proof. Let

$$S_j = \{s \in \mathbb{R} : (28) \text{ has at least j solutions } \} \quad (j \geq 1).$$

(a) $S_1 \neq \varnothing$.
Take $s^* > \max_{1 \leq m \leq n-1} f_m(0)$ and use (29) to find $R_-^* < 0$ such that

$$\min_{1 \leq m \leq n-1} f_m(R_-^*) > s^*.$$

Then α with $\alpha_j = R_-^* < 0$ $(1 \leq j \leq n)$ is a strict lower solution and β with $\beta_j = 0$ $(1 \leq j \leq n)$ is a strict upper solution for (28) with $s = s^*$. Hence, using Theorem 2, $s^* \in S_1$.
(b) If $\widetilde{s} \in S_1$ *and* $s > \widetilde{s}$ *then* $s \in S_1$.
Let $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a solution of (28) with $s = \widetilde{s}$, and let $s > \widetilde{s}$. Then $\widetilde{x}$ is a strict upper solution for (28). Take now $R_- < \min_{1 \leq m \leq n} \widetilde{x}_m$ such that $\min_{1 \leq m \leq n-1} f_m(R_-)$ $> s$ It follows that α with $\alpha_j = R_-$ $(1 \leq j \leq n)$ is a strict lower solution for (28), and hence, using Theorem 2, $s \in S_1$.
(c) $s_1 = \inf S_1$ *is finite and* $S_1 \supset]s_1, \infty[$.
Let $s \in \mathbb{R}$ and suppose that (28) has a solution $(x_1, \ldots, x_n)$. Then (33) holds, from where we deduce that $s \geq c$, with $c \in \mathbb{R}$ given in (31). To obtain the second part of claim (c) $S_1 \supset]s_1, \infty[$ we apply (b).
(d) $S_2 \supset]s_1, \infty[$.
We reformulate (28) to apply Brouwer degree theory. Consider the continuous mapping $\mathcal{G} : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$\mathcal{G}_m(s, x) = (Lx)_m + f_m(x_m) - s \quad (1 \leq m \leq n-1).$$

Then $(x_1, \ldots, x_{n-1}, x_1)$ is a solution of (28) if and only if $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ is a zero of $\mathcal{G}(s, \cdot)$. Let $s_3 < s_1 < s_2$. Using Lemma 3 we find $\rho > 0$ such that each possible zero of $\mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $\max_{1 \leq m \leq n-1} |x_m| < \rho$. Consequently, $d_B[\mathcal{G}(s, \cdot), B(\rho), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $\mathcal{G}(s_3, x) \neq 0$ for all $x \in \mathbb{R}^{n-1}$. This implies that $d_B[\mathcal{G}(s_3, \cdot), B(\rho), 0] = 0$, so that $d_B[\mathcal{G}(s_2, \cdot), B(\rho), 0] = 0$ and, by excision property, $d_B[\mathcal{G}(s_2, \cdot), B(\rho'), 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ be a solution of (28) (using (c)). Then $\widehat{x}$ is a strict upper solution of (28) with $s = s_2$. Let $R < \min_{1 \leq j \leq n} \widehat{x}_j$ be such that $\min_{1 \leq m \leq n-1} f_m(R) > s_2$. Then $(R, \ldots, R) \in \mathbb{R}^n$ is a strict lower solution of (28) with $s = s_2$. Consequently, using Corollary 1, (28) with $s = s_2$ has a solution in $\Omega_{R\widehat{x}}$ and

$$|d_B[\mathcal{G}(s_2, \cdot), \Omega_{R\widehat{x}}, 0]| = 1.$$

Taking ρ' sufficiently large, we deduce from the additivity property of Brouwer degree that

$$\begin{aligned}|d_B[\mathcal{G}(s_2,\cdot), B(\rho')\setminus\overline{\Omega}_{R\widehat{x}}, 0]| &= |d_B[\mathcal{G}(s_2,\cdot), B(\rho'), 0] - d_B[\mathcal{G}(s_2,\cdot), \Omega_{R\widehat{x}}, 0]|\\ &= |d_B[\mathcal{G}(s_2,\cdot), \Omega_{R\widehat{x}}, 0]| = 1,\end{aligned}$$

and (28) with $s = s_2$ has a second solution in $B(\rho')\setminus\overline{\Omega}_{R\widehat{x}}$.
(e) $s_1 \in S_1$.
Taking a decreasing sequence $(\sigma_k)_{k\in\mathbb{N}}$ in $]s_1,\infty[$ converging to s_1, a corresponding sequence $(x_1^k,\ldots,x_n^k)$ of solutions of (28) with $s=\sigma_k$ and using Lemma 3, we obtain a subsequence $(x_1^{j_k},\ldots,x_n^{j_k})$ which converges to a solution $(x_1,\ldots,x_n)$ of (28) with $s = s_1$. □

COROLLARY 4. *If $p>0$, $a_m>0$ and $b_m\in\mathbb{R}$ $(1\le m\le n-1)$, there exists $s_1\in\mathbb{R}$ such that the periodic problem*

$$Dx_m + a_m|x_m|^p = s + b_m \quad (1\le m\le n-1), \quad x_1 = x_n$$

has no solution if $s<s_1$, at least one solution if $s=s_1$ and at least two solutions if $s>s_1$.

Similar arguments allow to prove the following result.

THEOREM 4. *If the functions f_m satisfy condition*

$$f_m(x)\to-\infty \quad \text{as} \quad |x|\to\infty \quad (1\le m\le n-1). \tag{34}$$

then there is $s_1\in\mathbb{R}$ such that (28) has zero, at least one or at least two solutions according to $s>s_1$, $s=s_1$ or $s<s_1$.

COROLLARY 5. *If $p>0$, $a_m>0$ and $b_m\in\mathbb{R}$ $(1\le m\le n-1)$, there exists $s_1\in\mathbb{R}$ such that the periodic problem*

$$Dx_m - a_m|x_m|^p = s + b_m \quad (1\le m\le n-1), \quad x_1 = x_n$$

has no solution if $s>s_1$, at least one solution if $s=s_1$ and at least two solutions if $s<s_1$.

8. One-side bounded nonlinearities

The nonlinearity in Ambrosetti-Prodi type problems is bounded from below and coercive or bounded from above and anticoercive. In this section, we consider nonlinearities which are bounded from below or above but have different limits at $+\infty$ and $-\infty$.

Let $n\ge 2$ be an integer and $f_m:\mathbb{R}\to\mathbb{R}$ continuous functions $(1\le m\le n-1)$. Consider the problem

$$Dx_m + f_m(x_m) = 0 \quad (1\le m\le n-1), \quad x_1 = x_n. \tag{35}$$

It is easy to check that the linear mapping L defined in (3) is such that

$$\begin{aligned} N(L) &= \{(c,\dots,c)\in\mathbb{R}^{n-1} : c\in\mathbb{R}\},\\ R(L) &= \{(y_1,\dots,y_{n-1})\in\mathbb{R}^{n-1} : \sum_{m=1}^{n-1} y_m = 0\}. \end{aligned}$$

The projector $P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$

$$\begin{aligned} P(x_1,\dots,x_{n-1}) &= \left(\frac{1}{n-1}\sum_{m=1}^{n-1} x_m,\dots,\frac{1}{n-1}\sum_{m=1}^{n-1} x_m\right)\\ &= \left(\frac{1}{n-1}\sum_{m=1}^{n-1} x_m\right)(1,\dots,1) \end{aligned}$$

is such that $N(P) = R(L)$, $R(P) = N(L)$. Let us finally define $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(x_1,\dots,x_{n-1}) = (f_1(x_1),\dots,f_{n-1}(x_{n-1})),$$

and let $H = L + F$, so that the solutions of (35) correspond to the zeros of H. To study them using Brouwer degree, we introduce, like in Theorem IV.13 of [10] the family of equations

$$Lx + (1-\lambda)PF(x) + \lambda F(x) = 0, \quad \lambda\in[0,1]. \tag{36}$$

LEMMA 4. *For each* $\lambda \in]0,1]$, *equation (36) is equivalent to equation*

$$Lx + \lambda F(x) = 0. \tag{37}$$

For $\lambda = 0$, *equation (36) is equivalent to equation*

$$PF(x) = 0, \quad x \in N(L). \tag{38}$$

Proof. We first notice that, applying P to both members of equation (36), we get

$$PF(x) = 0$$

and hence, for $\lambda \in]0,1]$, equation (36) implies equation (37), and, for $\lambda = 0$, implies equation (38). Conversely, if equation (37) holds and $\lambda \in]0,1]$, then, applying P to both members, we get $PF(x) = 0$ and we may add $(1-\lambda)PF(x)$ to the left-hand member to obtain (36). If equation (38) holds, then

$$PF(x) = 0, \quad Lx = 0,$$

and hence (36) with $\lambda = 0$ follows by addition. □

The following Lemma, taken from [2], adapts to difference equations an argument of Ward [14] for ordinary differential equations.

LEMMA 5. *If the functions* f_m $(1 \leq m \leq n-1)$, *are all bounded from below or all bounded from above, say by* c, *and if for some* $R > 0$

$$\sum_{m=1}^{n-1} f_m(x_m) \neq 0 \quad \textit{whenever} \quad \min_{1 \leq j \leq n-1} x_j \geq R \quad \textit{or} \quad \max_{1 \leq j \leq n-1} x_j \leq -R, \tag{39}$$

then, for each $\lambda \in]0,1]$ *each possible zero* x *of* $L + \lambda F$ *is such that*

$$\max_{1 \leq j \leq n-1} |x_j| < R + 2(n-1)|c|. \tag{40}$$

Proof. Let $(\lambda, x) \in]0,1] \times \mathbb{R}^{n-1}$ be a possible zero of $L + \lambda N$. It is a solution of the equivalent system

$$\sum_{m=1}^{n-1} f_m(x_m) = 0, \quad Dx_m + \lambda f_m(x_m) = 0, \quad x_1 = x_n, \quad (1 \leq m \leq n-1). \tag{41}$$

On the other hand, if we assume, say, that each f_m $(1 \leq m \leq n-1)$ is bounded from below, say by c, we have, for all $1 \leq m \leq n-1$, and all $u \in \mathbb{R}$,

$$|f_m(u)| - |c| \leq |f_m(u) - c| = f_m(u) - c,$$

and hence

$$|f_m(u)| \leq f_m(u) + 2|c|. \tag{42}$$

Consequently, using (41) and (42), we obtain

$$\begin{aligned} \sum_{m=1}^{n-1} |Dx_m| &= \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \leq \sum_{m=1}^{n-1} |f_m(x_m)| \\ &\leq \sum_{m=1}^{n-1} f_m(x_m) + 2(n-1)|c| = 2(n-1)|c|. \end{aligned} \tag{43}$$

We deduce

$$\begin{aligned} \max_{1 \leq m \leq n-1} x_m &\leq \min_{1 \leq m \leq n-1} x_m + \sum_{m=1}^{n-1} |Dx_m| \\ &\leq \min_{1 \leq m \leq n-1} x_m + 2(n-1)|c|. \end{aligned} \tag{44}$$

Using (41) and assumption (39), we obtain $\min_{1 \leq m \leq n-1} x_m < R$ and $-R < \max_{1 \leq m \leq n-1} x_m$. Combined with (44), this gives

$$-[R + 2(n-1)|c|] < \min_{1 \leq m \leq n-1} x_m \leq \max_{1 \leq m \leq n-1} x_m < R + 2(n-1)|c|.$$

If the f_m are bounded from above, it suffices to consider the equivalent problem $-Lx - F(x) = 0$ with all function $-f_m$ bounded from below, as $-L$ has the same null-space and range as L. □

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{n-1}\left(\sum_{m=1}^{n-1} f_m(u)\right),$$

so that, for $u(1,\ldots,1) \in N(L)$,

$$PF(u(1,\ldots,1)) = \varphi(u)(1,\ldots,1).$$

The following theorem slightly sharpens a result of [2].

THEOREM 5. *Suppose that the functions f_m $(1 \le m \le n-1)$ are all bounded from below or all bounded from above, and that for some $R > 0$ and $\epsilon \in \{-1,1\}$,*

$$\begin{aligned} \epsilon \sum_{m=1}^{n-1} f_m(x_m) &\ge 0 \quad \textit{whenever} \quad \min_{1\le j\le n-1} x_j \ge R \\ \epsilon \sum_{m=1}^{n-1} f_m(x_m) &\le 0 \quad \textit{whenever} \quad \max_{1\le j\le n-1} x_j \le -R. \end{aligned} \tag{45}$$

Then, problem (35) has at least one solution.

Proof. For definiteness, assume that each f_m is bounded from below by c. For each $k \ge 1$, let us define

$$f_m^{(k)}(x_m) = f_m(x_m) + \frac{\epsilon x_m}{k(1+|x_m|)} \quad (1 \le m \le n-1),$$

so that each $f_m^{(k)}$ is bounded from below by $c-1$ and, using assumption (45),

$$\begin{aligned} \epsilon \sum_{m=1}^{n-1} f_m^{(k)}(x_m) &> 0 \quad \text{whenever} \quad \min_{1\le j\le n-1} x_j \ge R \\ \epsilon \sum_{m=1}^{n-1} f_m^{(k)}(x_m) &< 0 \quad \text{whenever} \quad \max_{1\le j\le n-1} x_j \le -R. \end{aligned} \tag{46}$$

Define $F^{(k)} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_m^{(k)}(x_1,\ldots,x_{n-1}) = f_m^{(k)}(x_m) \quad (1 \le m \le n-1).$$

Lemma 5 implies that each possible solution $x^{(k)}$ of each equation

$$Lx + \lambda F^{(k)}(x) = 0 \quad (k = 1,2,\ldots), \quad (\lambda \in]0,1]) \tag{47}$$

is such that

$$\max_{1\le m\le n-1} |x_m^{(k)}| < R + 2(n-1)(|c|+1) := \rho \quad (k = 1,2,\ldots).$$

Furthermore, condition (46) with $x_1 = \ldots = x_{n-1} = \pm\rho$ implies that

$$PF^{(k)}(\pm\rho(1,\ldots,1)) = \frac{1}{n-1}\sum_{m=1}^{n-1} f_m^{(k)}(\pm\rho) \neq 0.$$

If $C(\rho) = (]-\rho,\rho[)^{n-1}$, it follows then from Lemma 4 and the homotopy invariance of Brouwer degree, that, for each $k = 1, 2, \ldots,$

$$d_B[L + F^{(k)}, C(\rho)), 0] = d_B[L + PF^{(k)}, C(\rho), 0].$$

Now, $L + P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is an isomorphism and, for $z \in R(P)$, we have $(L+P)^{-1}z = z$, so that, using the multiplication property of the Brouwer degree and denoting the Brouwer index by i_B (see e.g. [7]), we obtain

$$\begin{aligned} d_B[L + PF^{(k)}, C(\rho), 0] &= d_B[(L+P)[I + (L+P)^{-1}(PF^{(k)} - P)], C(\rho), 0] \\ &= i_B(L+P, 0)\cdot d_B[I - P + PF^{(k)}, C(\rho), 0] \\ &= \pm d_B[I - P + PF^{(k)}, C(\rho), 0]. \end{aligned}$$

Now, the Leray-Schauder reduction formula (see e.g. [7]) implies that

$$\begin{aligned} d_B[I - P + PF^{(k)}, C(\rho), 0] &= d_B[PF^{(k)}|_{N(L)}, C(\rho)\cap N(L), 0] \\ &= d_B[\varphi^{(k)},]-\rho,\rho[, 0], \end{aligned}$$

where

$$\varphi^{(k)}(u) = \varphi(u) + \frac{\epsilon u}{k(1+|u|)}.$$

Now assumption (45) with $x_1 = \ldots = x_{n-1} = \pm\rho$ implies that $\varphi^{(k)}(-\rho)\varphi^{(k)}(\rho) < 0$ for all $k = 1, 2, \ldots,$ so that

$$d_B[\varphi^{(k)},]-\rho,\rho[, 0] = \pm 1.$$

Thus it follows from the existence property of Brouwer degree that equation (47) has at least one solution $x^{(k)}$ such that $x^{(k)} \in C(\rho)$ for all $k = 1, 2, \ldots$. Going if necessary to a subsequence, we can assume that $x^{(k)} \to x \in \overline{C(\rho)}$ which is a zero of $L + F$ and hence a solution of (35) □

Let $u^+ = \max\{u, 0\}$.

COROLLARY 6. *For $p > 0$, $a_m > 0$, $b_m \in \mathbb{R}$ $(1 \le m \le n-1)$, the periodic problem*

(48) $$Dx_m + a_m(x_m^+)^p - b_m = 0 \quad (1 \le m \le n-1), \quad x_1 = x_n,$$

has at least one solution if and only if

$$\sum_{m=1}^{n-1} b_m \ge 0.$$

When $a_m < 0$ and $b_m \in \mathbb{R}$ $(1 \leq m \leq n-1)$, problem (48) has at least one solution if and only if

$$\sum_{m=1}^{n-1} b_m \leq 0.$$

Proof. For the necessity, if problem (48) has a solution x, then

$$\sum_{m=1}^{n-1} b_m = \sum_{m=1}^{n-1} a_m (x_m^+)^p \geq 0.$$

For the sufficiency, each function $f_m(x_m) = a_m(x_m^+)^p - b_m$ is bounded from below by $-b_m$. Furthermore, if

$$R \geq \left(\frac{\sum_{m=1}^{n-1} b_m}{\sum_{m=1}^{n-1} a_m} \right)^{1/p},$$

then $\sum_{m=1}^{n-1} f_m(x_m) \geq 0$ when $\min_{1 \leq m \leq n-1} x_m \geq R$. On the other hand, $\sum_{m=1}^{n-1} f_m(x_m) = -\sum_{m=1}^{n-1} b_m \leq 0$ when $\max_{1 \leq m \leq n-1} x_m \leq 0$. Hence the result follows from Theorem 5. The proof of the other case is similar. □

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ZERO SHEAR VISCOSITY LIMIT FOR THE NAVIER-STOKES EQUATIONS OF COMPRESSIBLE ISENTROPIC FLUIDS WITH CYLINDRIC SYMMETRY*

Abstract. We study the problem of the limit process as the shear viscosity goes to zero for global weak solutions to the Navier-Stokes equations of compressible isentropic fluids with cylindric symmetry between two circular cylinders. We prove that the limit of the global weak solutions is a weak solution of the corresponding system with zero shear viscosity.

1. Introduction

We shall study the convergence of solutions of the the Navier-Stokes equations for a compressible isentropic fluid with cylindric symmetry, as the shear viscosity goes to zero. In this paper we restrict ourselves to isentropic flows between two circular coaxial cylinders and assume that the motion of the flows depends only on the radial variable and the time variable. The corresponding symmetric form of the compressible isentropic Navier-Stokes equations, which express the conservation of mass and the balance of momentum, can be written as [35]

$$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0, \tag{1}$$

$$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\rho v^2}{x} + P_x = (\lambda + 2\epsilon)\left(u_x + \frac{u}{x}\right)_x, \tag{2}$$

$$(\rho v)_t + (\rho u v)_x + \frac{2\rho u v}{x} = \epsilon\left(v_x + \frac{v}{x}\right)_x, \tag{3}$$

$$(\rho w)_t + (\rho u w)_x + \frac{\rho u w}{x} = \epsilon\left(w_{xx} + \frac{w_x}{x}\right). \tag{4}$$

Here ρ is the density, u, v and w are the radial, angular and axial components of the velocity vector $\vec{v}$, respectively, x is the radial variable;

$$P \equiv P(\rho) = a\rho^{\gamma}, \quad \gamma > 1$$

denotes the pressure, and λ, ϵ, a and γ are positive constants which stand for the bulk (expansion) viscosity coefficient, shear viscosity coefficient, gas constant and specific heat ratio, respectively.

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We shall consider the following initial boundary value problem for (1)–(4) in the domain $Q_T := (0,T)\times\Omega$ with $\Omega := \{0 < r_1 < x < r_2 < \infty\}$:

$$u = 0,\quad v = v_i(t),\quad w = w_i(t)\quad \text{for } x = r_i,\quad i = 1,2, \tag{5}$$

$$(\rho, u, v, w)|_{t=0} = (\rho_0, u_0, v_0, w_0). \tag{6}$$

The boundary conditions (5) imply that the fluid sticks at the bounding cylinders which move in such a way that the axis of symmetry is fixed. For simplicity, we take here $v_i(t) = w_i(t) \equiv 0$, since otherwise we can use

$$v - \Big(\frac{r_2 - x}{r_2 - r_1}v_1 + \frac{x - r_1}{r_2 - r_1}v_2\Big) \ \text{ and } \ w - \Big(\frac{r_2 - x}{r_2 - r_1}w_1 + \frac{x - r_1}{r_2 - r_1}w_2\Big)$$

to replace v and w, respectively, and thus the proof only needs minor modifications.

The asymptotic behavior of viscous flows, as the viscosity vanishes, is one of the important topics in the theory of compressible flows, and the problem of small viscosity finds many applications, for example, in the boundary layer theory [43].

Assuming that

$$\rho_0 \in H^1(\Omega),\quad \inf_{\Omega}\rho_0 > 0,\quad u_0,\ v_0,\ w_0 \in H_0^1(\Omega), \tag{7}$$

Shelukhin studies the zero shear viscosity limit for flows with heat-conducting between two parallel plates [47, 48], while in [49] he investigates the passage to the limit for a free-boundary problem of describing a joint motion of two compressible fluids with different viscosities, as the shear viscosity of one of the fluids vanishes.

In [17] Frid and Shelukhin investigate the cylinder symmetric isentropic problem (1)–(6). For the cylinder symmetric case, the velocity components influence each other through the momentum equations. This is one of the main differences between the systems considered in [47, 48, 17]. Under the conditions (7), Frid and Shelukhin prove that the problem (1)–(6) possesses a unique strong solution $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ satisfying

$$\rho_\epsilon \in L^\infty(0,T;H^1),\quad \partial_t\rho_\epsilon \in L^\infty(0,T;L^2),\quad \inf_{Q_T}\rho_\epsilon > 0,$$
$$(u_\epsilon, v_\epsilon, w_\epsilon) \in L^\infty(0,T;H_0^1)\cap L^2(0,T;H^2),\quad (\partial_t u_\epsilon, \partial_t v_\epsilon, \partial_t w_\epsilon) \in L^2(Q_T).$$

Furthermore, they use the method developed in [47] to obtain the following uniform in ϵ estimates:

$$\int_\Omega \Big[\rho_\epsilon(u_\epsilon^2 + v_\epsilon^2 + w_\epsilon^2) + \rho_\epsilon^\gamma\Big]dx + \int_0^T\int_\Omega \Big[(\lambda + 2\epsilon)(\partial_x u_\epsilon)^2 + \epsilon(\partial_x v_\epsilon)^2 + \epsilon(\partial_x w_\epsilon)^2\Big]dxdt \le C,$$

$$C^{-1} \le \rho_\epsilon \le C,\quad \|(v_\epsilon, w_\epsilon)\|_{L^\infty(Q_T)} \le C,$$

$$\|\partial_x\rho_\epsilon\|_{L^\infty(0,T;L^2)} \le C,\quad \|\partial_t\rho_\epsilon\|_{L^\infty(0,T;L^2)} \le C,$$

$$\|(\partial_t u_\epsilon, \partial_x^2 u_\epsilon)\|_{L^2(Q_T)} + \|\partial_x u_\epsilon\|_{L^\infty(0,T;L^2)} \le C,$$

where C is a positive constant independent of ϵ. Thus, using these uniform estimates, they can prove by the standard compactness imbedding arguments that as $\epsilon \to 0$,

$$\begin{aligned} &(\rho_\epsilon, u_\epsilon) \to (\rho, u) \quad \text{strongly in} \quad C(\bar{Q}_T), \\ &(v_\epsilon, w_\epsilon) \rightharpoonup (v, w) \quad \text{weak-}* \text{ in } L^\infty(Q_T). \end{aligned} \tag{8}$$

Then, with the help of (8), they utilize a framework suitable for transport equations which allows one to improve the weak convergence to the strong one by analyzing and comparing the equations deduced for $\overline{\Phi(z)}$ and $\Phi(z)$, where z is any of the two velocity components v or w, $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function, and $\overline{\Phi(z)}$ denotes the weak limit of $\Phi(z_\epsilon)$ with $z_\epsilon = v_\epsilon$ or $z_\epsilon = w_\epsilon$. This idea of improvement of weak convergence goes back to the notion of the renormalized solutions introduced by Diperna and Lions [11], and applied and further developed in [25, 38, 39, 33, 34, 29, 14, 16, 48, 18, 30, 15, 50] and among others.

The problem of vanishing both shear and bulk (expansion) viscosity coefficients (λ, μ) is much more complex and the situation becomes delicate. It is expected that a general weak entropy solution to the Euler equations should be (strong) limit of solutions to the corresponding compressible Navier-Stokes equations with the same initial data as the viscosity and heat conductivity tend to zero. Indeed, the vanishing viscosity limit for the Cauchy problem for the compressible Navier-Stokes equations has been studied by several researchers.

For the one-dimensional isentropic compressible Navier-Stokes equations, DiPerna [9] uses the method of compensated compactness and establishes a.e. convergence as viscosity goes to zero of admissible solutions of the Navier-Stokes equations to an admissible solution of the corresponding Euler equations, provided that solutions of the Navier-Stokes equations are bounded and the density is bounded away from zero uniformly with respect to the viscosity coefficients. However, this uniform boundedness is difficult to verify in general, and the abstract analysis in [9] gets little information on the qualitative nature of the viscous solutions. In [27] Hoff and Liu investigate the inviscid limit problem in the case that the underlying inviscid flow is a single weak shock wave, and they show that solutions of the compressible Navier-Stokes equations with shock data exist and converge to the inviscid shocks, as viscosity vanishes, uniformly away from the shocks. Based on [19, 27], Xin in [52] shows that the solution to the Cauchy problem of the one-dimensional compressible isentropic Navier-Stokes equations with weak centered rarefaction wave data exists for all time and converges to the weak centered rarefaction wave solution of the corresponding Euler equations, as viscosity tends to zero, uniformly away from the initial discontinuity. Moreover, for a given centered rarefaction wave to the Euler equations with finite strength, he constructs a viscous solution to the compressible Navier-Stokes system with initial data depending on the viscosity, such that the viscous solution approaches the centered rarefaction wave at the rate $1/4$ of viscosity uniformly for all time away from $t = 0$. Recently, Jiang, Ni and Sun [32] extended this result to the non-isentropic case by using some ideas from the stability study of rarefaction waves [40, 41] and the time-decay property of initial discontinuities [24].

In the vanishing viscosity limit, the existence and stability of multidimensional shock fronts for the multidimensional compressible Navier-Stokes equations are proved in [23] and the Prandtl boundary layers (characteristic boundaries) are studied for the linearized case in [53, 54, 51] by using asymptotic analysis, while the boundary layer stability in the case of non-characteristic boundaries and one spatial dimension is discussed in [46, 42]. We also mention that there is an extensive literature on the vanishing artificial viscosity limit for hyperbolic systems of conservation laws, see, for example, [9, 10, 19, 37, 36, 55, 20, 45, 5, 21, 22, 3], also cf. the monographs [4, 8, 44] and the references therein.

Concerning the zero shear viscosity limit for (1)–(6), to our best knowledge, the known results are concerned with strong solutions under the conditions (7). The aim of this paper is to prove a similar vanishing shear viscosity limit result under weaker regularity assumptions on the initial data. Namely, we will study the limit as $\epsilon \to 0$ of (1)–(6) under the following conditions on the initial data:

$$\inf_{\Omega} \rho_0 > 0, \quad \rho_0 \in L^\infty(\Omega), \quad u_0 \in L^2(\Omega), \quad (v_0, w_0) \in L^\infty(\Omega). \tag{9}$$

Under (9), it is not difficult to prove that there exists at least one global weak solution $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ with $\rho_\epsilon > 0$ to the problem (1)–(6) by using arguments similar to those in, for example, [1, 2, 6, 39, 56, 26, 31, 28]. Moreover, $(\rho_\epsilon, u_\epsilon)$ is a renormalized solution of the equation (1), see [39]. On the other hand, for the vanishing shear viscosity limit for the weak solutions here, compared with the strong solutions dealt with in [17], the main difficulty lies in the derivation of the strong convergence of the density ρ_ϵ, due to lack of uniform a priori estimates on derivatives of ρ_ϵ. To overcome such difficulties, we use the techniques in the study of the global existence of weak solutions to the multidimensional compressible Navier-Stokes equations (see, e.g., [38, 14, 16, 29]), and exploit the feature of the equation (2).

Before stating our main result, we introduce the definition of weak solutions.

DEFINITION 1. *(i) We call $(\rho, u, v, w)(x,t)$ a global weak solution of (1)–(6), if for any $T > 0$, $\rho(x,t) \geq 0$ on $[0,T] \times \Omega$, and*

$$\rho, v, w \in L^\infty(Q_T), \quad u, v, w \in L^2(0,T; H_0^1), \quad u \in L^\infty(0,T; L^2),$$

and the following equations hold:

$$\int_0^T \int_\Omega \rho(\varphi_t + u\varphi_x) x\,dx\,dt + \int_\Omega \rho_0 \varphi(x,0) x\,dx = 0, \tag{10}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big\{ x\rho u \phi_t + x\rho u^2 \phi_x + \rho v^2 \phi + \Big[P(\rho) - (\lambda + 2\epsilon)\Big(u_x + \frac{u}{x}\Big)\Big](x\phi)_x \Big\} dx\,dt \\ &\qquad + \int_\Omega x\rho_0 u_0 \phi(x,0)\,dx = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\int_0^T \int_\Omega \Big\{ x\rho v \phi_t + x\rho u v \phi_x - \rho u v \phi - \epsilon\Big(v_x + \frac{v}{x}\Big)(x\phi)_x \Big\} dx\,dt \\ &\qquad + \int_\Omega x\rho_0 v_0 \phi(x,0)\,dx = 0, \end{aligned} \tag{12}$$

$$(13)\qquad \int_0^T\int_\Omega \Big\{x\rho w\phi_t + x\rho u w\phi_x - \epsilon x w_x\phi_x\Big\}dxdt + \int_\Omega x\rho_0 w_0\phi(x,0)dx = 0,$$

for any $\varphi, \phi \in C^1(\bar{Q}_T)$, $\phi \in C([0,T], H_0^1)$ *and* $\varphi(\cdot,T) = \phi(\cdot,T) = 0$.

ii) We call $(\rho, u, v, w)(x,t)$ *a global weak solution of (1)–(6) with* $\epsilon = 0$, *if for any* $T > 0$, $\rho(x,t) \geq 0$ *on* $[0,T]\times\Omega$, *and*

$$\rho, v, w \in L^\infty(Q_T), \quad u \in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1),$$

and ρ, u, v, w *satisfy the equations (10)–(13) with* $\epsilon = 0$.

Thus, the main result of this paper reads:

THEOREM 1. *Assume that the initial data satisfy (9). Then there exists a global weak solution* $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ *of the problem (1)–(6). Moreover, there is a sequence* $\epsilon_n \downarrow 0$, *such that as* $\epsilon_n \to 0$,

$$(\rho_{\epsilon_n}, v_{\epsilon_n}, w_{\epsilon_n}) \to (\rho, v, w) \text{ strongly in } L^p(Q_T), \quad u_{\epsilon_n} \to u \text{ strongly in } L^s(Q_T),$$
$$\partial_x u_{\epsilon_n} \to u_x \text{ strongly in } L^2(Q_T)$$

for any $p \in [1,\infty)$ *and* $s \in [1,6)$. *In addition, the limit* (ρ, u, v, w) *is a global weak solution of (1)–(6) with* $\epsilon = 0$.

REMARK 1. (i) If $\inf \rho_0 = 0$ and (ρ_0, u_0) satisfies a natural compatibility condition, then we can prove that the problem (1)–(6) has a unique global smooth solution. For the proof, see [7] when $\gamma \geq 2$ and [13] when $1 < \gamma \leq 2$.

(ii) A similar result has been obtained recently for the magnetohydrodynamic equations by Fan [12].

The next section gives the uniform estimates which will be used in the final section to complete the proof of Theorem 1.

As the end of this section, we introduce the notation used throughout this paper. $L^p(I,B)$ respectively $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B respectively its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. $C(I, B-w)$ is the space of all functions which are in $L^\infty(I,B)$ and continuous in t with values in B endowed with the weak topology. We will use the abbreviation:

$$L^q(0,T;W^{m,p}) \equiv L^q(0,T;W^{m,p}(\Omega)),$$
$$\|\cdot\|_{L^q(0,T;W^{m,p})} \equiv \|\cdot\|_{L^q(0,T;W^{m,p}(\Omega))}, \quad \|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\Omega)}.$$

The same letter C will denote various positive constants which do not depend on ϵ.

2. Uniform a priori estimates

We denote the weak solution of (1)–(6) by $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ throughout the rest of this paper. This section is devoted to the derivation of a priori estimates of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$

which are independent of ϵ .

We start with the following conservation identity which is obtained by multiplying (1) by x, integrating the resulting equation over $(0,t)\times\Omega$ and using the boundary conditions (5):

$$\int_\Omega x\rho_\epsilon(x,t)dx=\int_\Omega x\rho_0(x)dx. \tag{14}$$

The following lemma gives an elementary energy estimate which is proved in [17] by multiplying the system (2)–(4) by $(u_\epsilon, v_\epsilon, w_\epsilon)$ in $L^2((0,t)\times\Omega)$ and using (1).

LEMMA 1. *The following energy estimate holds.*

$$\begin{aligned}\sup_{t\in[0,T]}\int_\Omega\Big[\frac{x}{2}\rho_\epsilon(u_\epsilon^2+v_\epsilon^2+w_\epsilon^2)+\frac{ax}{\gamma-1}\rho_\epsilon^\gamma\Big](x,t)dx\\ +\int_0^T\int_\Omega x\Big[(\lambda+2\epsilon)(\partial_x u_\epsilon)^2+\epsilon(\partial_x v_\epsilon)^2+\epsilon(\partial_x w_\epsilon)^2\\ +(\lambda+2\epsilon)\frac{u_\epsilon^2}{x^2}+\epsilon\frac{v_\epsilon^2}{x^2}\Big]dxdt\le C.\end{aligned} \tag{15}$$

As in [17], we rewrite the equation (2) in the form

$$(\rho_\epsilon u_\epsilon)_t+\Big[\rho_\epsilon u_\epsilon^2+P_\epsilon-(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)+\sigma_\epsilon\Big]_x=0,$$

where

$$\sigma_\epsilon:=\int_{r_1}^x\frac{\rho_\epsilon(u_\epsilon^2-v_\epsilon^2)}{z}dz,\qquad P_\epsilon:=a\rho_\epsilon^\gamma.$$

It is easy to see that by (15),

$$\|\sigma_\epsilon\|_{L^\infty(Q_T)}\le C. \tag{16}$$

Introducing the function

$$\varphi_\epsilon(t,x):=\int_0^t\Big\{(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)-\rho u_\epsilon^2-P_\epsilon-\sigma_\epsilon\Big\}(x,\tau)d\tau+\int_{r_1}^x\rho_0u_0d\xi,$$

one has

$$\partial_x\varphi_\epsilon=\rho_\epsilon u_\epsilon,\quad \partial_t\varphi_\epsilon=(\lambda+2\epsilon)\Big(\partial_x u_\epsilon+\frac{u_\epsilon}{x}\Big)-\rho_\epsilon u_\epsilon^2-P_\epsilon-\sigma_\epsilon. \tag{17}$$

Observe that by virtue of (15), (17), the Cauchy-Schwarz inequality and (14),

$$\begin{aligned}\|\partial_x\varphi_\epsilon\|_{L^\infty(0,T;L^1)}&\le\|\rho_\epsilon u_\epsilon\|_{L^\infty(0,T;L^1)}\\ &\le C\|\sqrt{x\rho_\epsilon}u_\epsilon\|_{L^\infty(0,T;L^2)}\|\sqrt{x\rho_\epsilon}\|_{L^\infty(0,T;L^2)}\\ &\le C\end{aligned}$$

and

$$\sup_{t\in[0,T]}\left|\int_\Omega \varphi(x,t)dx\right| \le C.$$

Hence, the generalized Poincaré inequality implies

$$\|\varphi_\epsilon\|_{L^\infty(Q_T)} \le C. \tag{18}$$

Utilizing the equations (17), and the estimates (16) and (18), following the same arguments as in [17, Lemma 2.2], we obtain the following lemma, the proof of which is therefore omitted.

LEMMA 2. *There are positive constants $\underline{\rho}$, $\overline{\rho}$ independent of ϵ, such that*

$$\underline{\rho} \le \rho_\epsilon(x,t) \le \overline{\rho} \qquad \forall\, x \in \bar{\Omega},\ t \ge 0. \tag{19}$$

As a consequence of Lemmas 1 and 2, one has by the Cauchy-Schwarz inequality that

$$\begin{aligned}\int_0^T \|u_\epsilon\|_{L^6}^6 dt &\le \|u_\epsilon\|^2_{L^\infty(0,T;L^2)}\int_0^T \|u_\epsilon\|^4_{L^\infty} dt\\ &\le C\int_0^T \left(\int_{r_1}^{r_2} |u_\epsilon \partial_x u_\epsilon| d\xi\right)^2 dt\\ &\le C\int_0^T \|u_\epsilon\|^2_{L^2}\|\partial_x u_\epsilon\|^2_{L^2} dt\\ &\le C.\end{aligned} \tag{20}$$

Now, one can apply Lemma 1 and (19) to the parabolic equations (2)–(4) to obtain bounds on the time derivative of $(\rho_\epsilon, \rho_\epsilon \vec{v}_\epsilon)$:

LEMMA 3.

$$\|\partial_t \rho_\epsilon\|_{L^\infty(0,T;H^{-1})} + \|\partial_t(\rho_\epsilon \vec{v}_\epsilon)\|_{L^2(0,T;H^{-1})} \le C, \tag{21}$$

where $\vec{v}_\epsilon = (u_\epsilon, v_\epsilon, w_\epsilon)$.

The following lemma gives us uniform bounds of (w_ϵ, v_ϵ) in L^∞-norm, the proof of which is based on using the properties of transport equations and Lemmas 1 and 2, and can be found in [17].

LEMMA 4.

$$\begin{aligned}\|v_\epsilon\|_{L^\infty(Q_T)} &\le \|v_0\|_{L^\infty(\Omega)} \exp\left(C\int_0^T \|u_\epsilon\|_{L^\infty(\Omega)} dt\right)\\ &\le C\|v_0\|_{L^\infty(\Omega)},\\ \|w_\epsilon\|_{L^\infty(Q_T)} &\le \|w_0\|_{L^\infty(\Omega)}.\end{aligned}$$

3. Proof of Theorem 1

In this section we pass to the limit for $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ as $\epsilon \to 0$ in (1)–(6). First, it is easy to see by the uniform a priori estimates established in the last section and Lemma C.1 in [38] that one can extract a subsequence of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$, still denoted by $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)$ for simplicity, such that as $\epsilon \to 0$,

$$\rho_\epsilon \rightharpoonup \rho \quad \text{weak-}* \text{ in } L^\infty(Q_T), \quad \underline{\rho} \le \rho(x,t) \le \overline{\rho}, \quad \text{a.e.,} \tag{22}$$

$$\rho_\epsilon \to \rho \ \text{ in } \ C([0,T], L^\gamma(\Omega) - w) \ \text{ for any } \gamma > 1, \tag{23}$$

$$\begin{aligned} &u_\epsilon \rightharpoonup u \quad \text{weak-}* \text{ in } L^\infty(0,T; L^2(\Omega)) \\ &\qquad \text{and weakly in } L^2(0,T; H_0^1(\Omega)) \cap L^6(Q_T), \end{aligned} \tag{24}$$

$$(v_\epsilon, w_\epsilon) \rightharpoonup (v, w) \text{ weak-}* \text{ in } L^\infty(Q_T), \tag{25}$$

$$(\epsilon \partial_x u_\epsilon, \epsilon \partial_x v_\epsilon, \epsilon \partial_x w_\epsilon) \to (0,0,0) \text{ strongly in } L^2(Q_T), \tag{26}$$

and from (23) and the Sobolev compact imbedding theorem, one gets

$$\rho_\epsilon \to \rho \quad \text{in } C([0,T], H^{-1}(\Omega)). \tag{27}$$

Using (22)–(24), (21) and Lemma 5.1 in [39], we find that

$$\begin{aligned} &\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ weak-}* \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ and} \\ &\qquad \text{weakly in } L^2(0,T; L^p(\Omega)) \text{ for all } p > 1, \end{aligned} \tag{28}$$

and by Lemma C.1 in [38],

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ in } C([0,T], L^2(\Omega) - w),$$

from which and the Sobolev compact imbedding theorem, it follows that

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \text{ in } C([0,T], H^{-1}(\Omega)).$$

Hence, the above weak convergence together with (24) results in

$$\rho_\epsilon u_\epsilon^2 \rightharpoonup \rho u^2 \text{ weakly in } L^2(Q_T). \tag{29}$$

On the other hand, noticing that by virtue of Lemma 1 and the Sobolev imbedding theorem, $\int_0^T \|u_\epsilon(t)\|_{L^\infty}^2 dt \le C$. Therefore,

$$\int_0^T \|u_\epsilon^2(t)\|_{H^1} dt \le C \quad \text{uniformly in } \epsilon,$$

which together with (27) implies

$$\left| \int_0^T \int_\Omega (\rho_\epsilon - \rho) u_\epsilon^2 \phi \, dx dt \right| \to 0 \quad \text{as } \epsilon \to 0, \quad \phi \in C_0^\infty(Q_T). \tag{30}$$

So, recalling the pointwise boundedness of ρ, one immediately gets from (29) and (30) that $u_\epsilon^2 \rightharpoonup u^2$ weakly in $L^2(Q_T)$, which combined with (24) shows that

$$u_\epsilon \to u \text{ strongly in } L^2(Q_T).$$

Therefore, by interpolation and (20), we infer that

$$u_\epsilon \to u \text{ strongly in } L^s(Q_T), \quad \forall\, s < 6. \tag{31}$$

Now, we use and adapt the techniques in [39, 14, 16] (also cf. [29]) to prove the following strong convergence of ρ_ϵ.

LEMMA 5.

$$\rho_\epsilon \to \rho \textit{ strongly in } L^1(Q_T), \quad \textit{as } \epsilon \to 0.$$

Proof. First, multiplying (2) by $\phi \in C_0^\infty(\mathbf{\Omega})$ and integrating over (r_1, x), then multiplying the resulting equation by $\psi(t)\rho_\epsilon$, $\psi(t) \in C_0^\infty(0, T)$, and integrating over $(0, T) \times \mathbf{\Omega}$, we obtain after a straightforward calculation that

$$\begin{aligned}
&\int_0^T \psi(t) \int_\Omega \rho_\epsilon \left[a\rho_\epsilon^\gamma - (\lambda + 2\epsilon)\left(\partial_x u_\epsilon + \frac{u_\epsilon}{x}\right)\right]\phi dx dt \\
&= \int_0^T \psi'(t) \int_\Omega \rho_\epsilon \int_{r_1}^x \rho_\epsilon u_\epsilon \phi d\xi dx dt - \int_0^T \psi \int_\Omega \frac{1}{x}\rho_\epsilon u_\epsilon \int_{r_1}^x \rho_\epsilon u_\epsilon \phi d\xi dx dt \\
&\quad + \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x (a\rho_\epsilon^\gamma + \rho_\epsilon u_\epsilon^2)\phi_\xi d\xi dx dt - \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \frac{\rho_\epsilon u_\epsilon^2}{\xi}\phi d\xi dx dt \\
&\quad + \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \frac{\rho_\epsilon v_\epsilon^2}{\xi}\phi d\xi dx dt \\
&\quad -(\lambda + 2\epsilon) \int_0^T \psi \int_\Omega \rho_\epsilon \int_{r_1}^x \left(\partial_\xi u_\epsilon + \frac{u_\epsilon}{\xi}\right)\phi_\xi d\xi dx dt,
\end{aligned} \tag{32}$$

where $H := a\rho_\epsilon^\gamma - (\lambda + 2\epsilon)(\partial_x u_\epsilon + u_\epsilon/x)$ is so-called the effective viscous pressure which possesses some smoothing property and plays an important role in the existence proof of global weak solutions to the multidimensional compressible Navier-Stokes equations, cf. [39, 29, 14, 15].

Now, passing to the limit in (32) as $\epsilon \to 0$, and making use of (22)–(29) and (31), we see that

$$\begin{aligned}
&\int_0^T \psi(t) \int_\Omega \left[a\overline{\rho^{\gamma+1}} - \lambda\left(\overline{\rho u_x} + \frac{\rho u}{x}\right)\right]\phi dx dt \\
&= \int_0^T \psi'(t) \int_\Omega \rho \int_{r_1}^x \rho u \phi d\xi dx dt - \int_0^T \psi \int_\Omega \frac{1}{x}\rho u \int_{r_1}^x \rho u \phi d\xi dx dt \\
&\quad + \int_0^T \psi \int_\Omega \rho \int_{r_1}^x (a\overline{\rho^\gamma} + \rho u^2)\phi_\xi d\xi dx dt - \int_0^T \psi \int_\Omega \rho \int_{r_1}^x \frac{\rho u^2}{\xi}\phi d\xi dx dt \\
&\quad + \int_0^T \psi \int_\Omega \rho \int_{r_1}^x \frac{\overline{\rho v^2}}{\xi}\phi d\xi dx dt - \lambda \int_0^T \psi \int_\Omega \rho \int_{r_1}^x \left(u_\xi + \frac{u}{\xi}\right)\phi_\xi d\xi dx dt.
\end{aligned} \tag{33}$$

Here and in what follows, $\overline{f(\eta)}$ denotes again the weak limit of $f(\eta_\epsilon)$ as $\epsilon \to 0$.

On the other hand, with the help of (22)–(29) and (31), one has by taking $\epsilon \to 0$ in (1) and (2) that

$$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0, \tag{34}$$

$$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\overline{\rho v^2}}{x} + \left(a\overline{\rho^\gamma}\right)_x - \lambda\left(u_x + \frac{u}{x}\right)_x = 0 \tag{35}$$

in $\mathcal{D}'((0,T)\times\mathbf{\Omega})$, where $\overline{\rho v^2}$ and $\overline{\rho^\gamma}$ denote the weak limits of $\rho_\epsilon v_\epsilon^2$ and ρ_ϵ^γ, respectively.

Now, multiplying (35) by $\phi \in C_0^\infty(\mathbf{\Omega})$, integrating then over (r_1, x), and multiplying the resulting equation by $\psi(t)\rho_\epsilon$, $\psi(t) \in C_0^\infty(0,T)$, and integrating over $(0,T)\times\mathbf{\Omega}$, we deduce by the same arguments as in the derivation of (32) that

$$\begin{aligned}
&\int_0^T \psi(t)\int_\Omega \rho\left[a\overline{\rho^\gamma} - \lambda\left(u_x + \frac{u}{x}\right)\right]\phi\, dx\, dt \\
&= \int_0^T \psi'(t)\int_\Omega \rho\int_{r_1}^x \rho u\phi\, d\xi\, dx\, dt - \int_0^T \psi\int_\Omega \frac{1}{x}\rho u\int_{r_1}^x \rho u\phi\, d\xi\, dx\, dt \\
&\quad + \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \left(a\overline{\rho^\gamma} + \rho u^2\right)\phi_\xi\, d\xi\, dx\, dt - \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \frac{\rho u^2}{\xi}\phi\, d\xi\, dx\, dt \\
&\quad + \int_0^T \psi\int_\Omega \rho\int_{r_1}^x \frac{\overline{\rho v^2}}{\xi}\phi\, d\xi\, dx\, dt - \lambda\int_0^T \psi\int_\Omega \rho\int_{r_1}^x \left(u_\xi + \frac{u}{\xi}\right)\phi_\xi\, d\xi\, dx\, dt.
\end{aligned} \tag{36}$$

Comparing the right hand side of (33) with that of (36), we infer that

$$\begin{aligned}
&\int_0^T \psi\int_\Omega \left[a\overline{\rho^{\gamma+1}} - \lambda\left(\overline{\rho u_x} + \frac{\rho u}{x}\right)\right]\phi\, dx\, dt \\
&\qquad = \int_0^T \psi\int_\Omega \rho\left[a\overline{\rho^\gamma} - \lambda\left(u_x + \frac{u}{x}\right)\right]\phi\, dx\, dt,
\end{aligned} \tag{37}$$

whence

$$a\overline{\rho^{1+\gamma}} - \lambda\overline{\rho u_x} = a\rho\overline{\rho^\gamma} - \lambda\rho u_x. \tag{38}$$

In the sequel, we apply the idea of the renormalized solutions to the equation (1) introduced by DiPerna and Lions [39] to show that ρ satisfies

$$\left(\overline{\rho\log\rho}\right)_t + \left(u\,\overline{\rho\log\rho}\right)_x + \overline{\rho u_x} + \frac{\rho u}{x} + \frac{\overline{\rho\log\rho}}{x}u = 0, \tag{39}$$

$$(\rho\log\rho)_t + (u\,\rho\log\rho)_x + \rho u_x + \frac{\rho u}{x} + \frac{\rho\log\rho}{x}u = 0 \tag{40}$$

in the sense of distributions. In fact, since ρ_ϵ is uniformly bounded in $L^\infty(Q_T)$ and $u \in L^2(0,T;H_0^1(\mathbf{\Omega}))$, we get from Proposition 4.2 in [15] that ρ_ϵ is a renormalized solution of (1), i.e., ρ_ϵ satisfies

$$\partial_t b(\rho_\epsilon) + [b(\rho_\epsilon)u_\epsilon]_x + [b'(\rho_\epsilon)\rho_\epsilon - b(\rho_\epsilon)]\partial_x u_\epsilon + b'(\rho_\epsilon)\frac{\rho_\epsilon u_\epsilon}{x} = 0 \tag{41}$$

for any $b \in C^1(\mathbb{R})$, $b'(z) = 0$ for z large enough. It is not difficult to verify that one can take $b(z) = z \log z$ in (41) by an approximate argument and the uniform a priori estimates established for ρ_ϵ and u_ϵ. Thus, we have

$$(\rho_\epsilon \log \rho_\epsilon)_t + (u_\epsilon \rho_\epsilon \log \rho_\epsilon)_x + \rho_\epsilon \partial_x u_\epsilon + \frac{\rho_\epsilon u_\epsilon}{x} + \frac{u_\epsilon}{x}\rho_\epsilon \log \rho_\epsilon = 0.$$

Letting $\epsilon \to 0$ in the above equation and making use of (22), (28) and (31), we obtain (39) immediately.

Similarly, the limit functions ρ, u are still a renormalized solution to (34). That is, the equation (41) with $(\rho_\epsilon, u_\epsilon)$ replaced by (ρ, u) is still valid, and by an approximation one can take $b(z) = z \log z$, and hence, the equation (40) holds.

Subtraction of (40) from (39) leads to

$$[\overline{\rho \log \rho} - \rho \log \rho]_t + [u(\overline{\rho \log \rho} - \rho \log \rho)]_x + \frac{\overline{\rho \log \rho} - \rho \log \rho}{x} u = \rho u_x - \overline{\rho u_x}. \tag{42}$$

On the other hand, from (38) and the weak lower semicontinuity of convex functions, we find that

$$\overline{\rho u_x} - \rho u_x = \frac{a}{\lambda}\left(\overline{\rho^{1+\gamma}} - \rho \overline{\rho^\gamma}\right) \geq 0, \quad \text{a.e.} \tag{43}$$

and

$$\overline{\rho \log \rho} \geq \rho \log \rho, \quad \text{a.e.} \tag{44}$$

As in [14, 16], consider a sequence of functions $\phi_m \in C_0^\infty(\Omega)$, such that

$$\begin{aligned}
&0 \leq \phi_m \leq 1,\ \ \phi_m(x) = 1 \text{ for all } x \text{ such that } \operatorname{dist}(x, \partial\Omega) \geq m^{-1},\\
&|\partial_x \phi_m(x)| \leq 2m \ \text{ and } \ \operatorname{dist}(x, \partial\Omega)|\partial_x \phi_m(x)| \leq 2 \ \text{ for all } x \in \Omega,\\
&\phi_m(x) \to 1 \text{ as } m \to \infty \text{ for all } x \in \Omega.
\end{aligned}$$

Notice that by virtue of $u \in L^2(0, T; H_0^1)$, $|u|[\operatorname{dist}(x, \partial\Omega)]^{-1} \in L^2(0, T; L^2)$. Therefore, multiplying (42) by $x\phi_m(x)$ and integrating over $(0, t) \times \Omega$, then taking $m \to \infty$, and using (43) and (23), we infer

$$\int_\Omega x(\overline{\rho \log \rho} - \rho \log \rho)(x, t)dx \leq \int_\Omega x(\overline{\rho \log \rho} - \rho \log \rho)(x, 0) = 0, \quad \text{a.e. } t \in [0, T],$$

which combined with (44) gives $\overline{\rho \log \rho} = \rho \log \rho$ a.e. on Q_T. This identity, together with, for example, Theorem 2.11 in [15], yields $\rho_\epsilon \to \rho$ a.e. on Q_T, which combined with the Egorov theorem proves the lemma. □

By Lemma 5 and interpolation, it is easy to see that

$$\rho_\epsilon \to \rho \quad \text{strongly in } L^p(Q_T) \quad \text{for any } 1 \le p < \infty. \tag{45}$$

Now, passing to the limit in (1)–(4) as $\epsilon \to 0$, and utilizing (22)–(29), (31) and (45), we see that the limit functions ρ, u, v, w satisfy the following equations in the sense of distributions:

$$\rho_t + (\rho u)_x + \frac{\rho u}{x} = 0, \tag{46}$$

$$(\rho u)_t + (\rho u^2)_x + \frac{\rho u^2}{x} - \frac{\overline{\rho v^2}}{x} + a(\rho^\gamma)_x = \lambda \left(u_x + \frac{u}{x}\right)_x, \tag{47}$$

$$(\rho v)_t + (\rho u v)_x + \frac{2\rho u v}{x} = 0, \tag{48}$$

$$(\rho w)_t + (\rho u w)_x + \frac{\rho u w}{x} = 0. \tag{49}$$

In the sequel, we prove the strong convergence of $(v_\epsilon, w_\epsilon, \partial_x u_\epsilon)$ to (v, w, u_x) in $L^2(Q_T)$. To this end, we introduce the Lagrangian coordinates (y, t) or (z, t) which are connected to the Eulerian coordinates (x, t) by

$$y \equiv y(x,t) := r_1 + \int_{r_1}^x s\rho_\epsilon(s,t)ds$$

and

$$z \equiv z(x,t) := r_1 + \int_{r_1}^x s\rho(s,t)ds.$$

Without loss of generality, we may assume that

$$\int_\Omega x\rho_0(x)dx = r_2 - r_1. \tag{50}$$

From (50) and the mass conservation, we find that $y, z \in \Omega$. Since ρ_ϵ and ρ are bounded and strictly away from 0, the mappings (the inverse mappings of $y(x,t)$ and $z(x,t)$) $x(y,t)$ and $x(z,t)$: $\Omega \to \Omega$ are surjective. Moreover,

$$y_t = -x\rho_\epsilon u_\epsilon, \quad y_x = x\rho_\epsilon, \quad dydt = x\rho_\epsilon dxdt.$$

Thus, the equations (1)–(4) in the new variables (y, t) read:

$$(x\rho_\epsilon)_t + x^2\rho_\epsilon^2\partial_y u_\epsilon = 0, \tag{51}$$

$$\partial_t u_\epsilon - \frac{v_\epsilon^2}{x} = (\lambda + 2\epsilon)x\left(x\rho_\epsilon\partial_y u_\epsilon + \frac{u_\epsilon}{x}\right)_y - x\left[P(\rho_\epsilon)\right]_y, \tag{52}$$

$$\partial_t v_\epsilon + \frac{u_\epsilon v_\epsilon}{x} - \epsilon x\left(x\rho_\epsilon\partial_y v_\epsilon + \frac{v_\epsilon}{x}\right)_y = 0, \tag{53}$$

$$\partial_t w_\epsilon - \epsilon(x^2\rho_\epsilon\partial_y w_\epsilon)_y = 0, \tag{54}$$

$$x_t = u_\epsilon, \quad \rho_\epsilon x x_y = 1, \tag{55}$$

$$(u_\epsilon, v_\epsilon, w_\epsilon)|_{\partial\Omega} = (0,0,0), \quad (\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon)|_{t=0} = (\rho_0, u_0, v_0, w_0). \tag{56}$$

First, we show the strong convergence of w_ϵ. Multiplying (54) by $2w_\epsilon\phi$ with $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ and $\phi(\cdot, T) = 0$, then integrating over $(0, T) \times \Omega$, integrating by parts and using the boundary conditions (56), we obtain

$$\int_0^T \int_\Omega w_\epsilon^2 \phi_t dydt + \int_\Omega w_0^2 \phi(y, 0)dy = 2\epsilon \int_0^T \int_\Omega \Big[x^2 \rho_\epsilon (\partial_y w_\epsilon)^2 \phi + x^2 \rho_\epsilon \partial_y w_\epsilon \phi_y w_\epsilon\Big] dydt.$$

Transforming this identity into the Eulerian coordinates, we see that

$$\int_0^T \int_\Omega x\rho_\epsilon w_\epsilon^2 (\phi_t + u_\epsilon \phi_x) dxdt + \int_\Omega x\rho_0 w_0^2 \phi(x, 0)dx = 2\epsilon \int_0^T \int_\Omega \Big[x(\partial_x w_\epsilon)^2 \phi + x\partial_x w_\epsilon \phi_x w_\epsilon\Big] dxdt.$$

Letting $\epsilon \to 0$ in the above equation, and using (26), (31) and (45), we find that

$$\int_0^T \int_\Omega x\rho \overline{w^2} (\phi_t + u\phi_x) dxdt + \int_\Omega x\rho_0 w_0^2 \phi(x, 0)dx = 2\langle \overline{\epsilon x w_x^2}, \phi\rangle \geq 0, \tag{57}$$

where $\overline{\epsilon x w_x^2}$, the weak limit of $\epsilon x(\partial_x w_\epsilon)^2$ in the space of signed Radon measures on Q_T, is a nonnegative Radon measure on Q_T. By transforming (57) into the Lagrangian coordinates (z, t), the inequality (57) in the variables (z, t) reads

$$\int_0^T \int_\Omega \overline{w^2} \phi_t dzdt + \int_\Omega w_0^2 \phi(z, 0)dz \geq 0. \tag{58}$$

On the other hand, transforming (49) into the Lagrangian coordinates (calculated in the weak form), testing then the resulting equation with $2w\phi$, we get

$$\int_0^T \int_\Omega w^2 \phi_t dzdt + \int_\Omega w_0^2 \phi(z, 0)dz = 0. \tag{59}$$

Subtracting (59) from (58) and noticing that ϕ_t can be nonpositive and arbitrary, we find that $\overline{w^2}(z, t) \leq w^2(z, t)$, a.e. in Q_T. Hence, $\overline{w^2}(z, t) = w^2(z, t)$ a.e. in Q_T, which implies

$$w_\epsilon \to w \quad \text{strongly in } L^2(Q_T). \tag{60}$$

As a consequence of (60) and (59), we see that the left hand side of (57) in the Lagrangian coordinates (z, t) is equal to zero, therefore,

$$\langle \overline{\epsilon x w_x^2}, \phi\rangle = 0, \qquad \forall\, \phi \in C^\infty(Q_T),\ \phi \geq 0 \text{ in } Q_T,\ \phi(\cdot, T) = 0.$$

Next, we show the strong convergence of v_ϵ in $L^2(Q_T)$ by similar arguments. Multiplying (53) by $2v_\epsilon\phi$ in $L^2(Q_T)$ with $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ in Q_T and $\phi(\cdot, T) = 0$, integrating by parts, and using (55) and (56), we infer

$$\int_0^T \int_\Omega v_\epsilon^2 \phi_t dydt + \int_\Omega v_0^2 \phi(y,0)dy = 2\epsilon \int_0^T \int_\Omega \Big\{ x^2\rho_\epsilon(\partial_y v_\epsilon)^2\phi + 2v_\epsilon \partial_y v_\epsilon \phi \\ + x^2\rho_\epsilon v_\epsilon \partial_y v_\epsilon \phi_y + v_\epsilon^2\phi_y + \frac{v_\epsilon^2\phi}{x^2\rho_\epsilon} \Big\} dydt + 2\int_0^T \int_\Omega \frac{u_\epsilon v_\epsilon^2}{x}\phi dydt,$$

which, in the Eulerian coordinates, turns out

$$\int_0^T \int_\Omega x\rho_\epsilon v_\epsilon^2(\phi_t + u_\epsilon\phi_x)dxdt + \int_\Omega x\rho_0 v_0^2\phi(x,0)dx = 2\epsilon \int_0^T \int_\Omega \Big\{ x(\partial_x v_\epsilon)^2\phi \\ +2v_\epsilon \partial_x v_\epsilon \phi + x v_\epsilon \partial_x v_\epsilon \phi_x + v_\epsilon^2\phi_x + \frac{v_\epsilon^2}{x}\phi \Big\} dxdt + 2\int_0^T \int_\Omega \rho_\epsilon u_\epsilon v_\epsilon^2 \phi dxdt.$$

Passing to the limit as $\epsilon \to 0$ in the above equation, similarly to (57), we deduce that

$$\begin{aligned} &\int_0^T \int_\Omega x\rho\overline{v^2}(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 v_0^2 \phi(x,0)dx \\ &= 2\langle \overline{\epsilon x v_x^2}, \phi\rangle + 2\int_0^T \int_\Omega \rho u \overline{v^2}\phi dxdt \\ &\geq 2\int_0^T \int_\Omega \rho u\overline{v^2}\phi dxdt, \end{aligned} \tag{61}$$

where, $\overline{\epsilon x v_x^2}$, the weak limit of $\epsilon x(\partial_x v_\epsilon)^2$ in the space of signed Radon measures on Q_T, is a nonnegative Radon measure. Transformation of (61) into the Lagrangian coordinates (z,t) results in

$$\int_0^T \int_\Omega \overline{v^2}\phi_t dzdt + \int_\Omega v_0^2\phi(z,0)dz \geq 2\int_0^T \int_\Omega \frac{u\overline{v^2}}{x}\phi dzdt. \tag{62}$$

On the other hand, transforming (48) into the Lagrangian coordinates (z,t), multiplying then the resulting equation by $2v\phi$ in $L^2(Q_T)$, $\phi \in C^\infty(Q_T)$, $\phi \geq 0$ and $\phi(\cdot, T) = 0$, one obtains

$$\int_0^T \int_\Omega v^2\phi_t dzdt + \int_\Omega v_0^2\phi(z,0)dz = 2\int_0^T \int_\Omega \frac{uv^2}{x}\phi dzdt. \tag{63}$$

Subtracting (63) from (62), we have

$$\int_0^T \int_\Omega \left(\overline{v^2} - v^2\right)\left(\phi_t - 2\frac{u}{x}\phi\right) dzdt \geq 0, \quad \forall \phi \in C^\infty(Q_T),\ \phi \geq 0,\ \phi(\cdot, T) = 0,$$

which implies $\overline{v^2} \le v^2$ a.e. in Q_T, since ϕ_t can be nonpositive and arbitrary. Consequently,

$$v_\epsilon \to v \quad \text{strongly in } L^2(Q_T). \tag{64}$$

Hence, it follows from (63) and the form of (61) in Lagrangian coordinates that

$$\langle \overline{\epsilon x v_x^2}, \phi \rangle = 0.$$

Again multiplying (2) by $2xu_\epsilon\phi$ with $\phi \in C^\infty(Q_T)$, $\phi \ge 0$ and $\phi(\cdot, T) = 0$, then integrating over $(0, T) \times \Omega$, integrating by parts and employing the boundary condition for u, one gets after a straightforward calculation that

$$\begin{aligned}
&\int_0^T \int_\Omega x\rho_\epsilon u_\epsilon^2(\phi_t + u_\epsilon\phi_x)dxdt + \int_\Omega x\rho_0 u_0^2\phi(x,0)dx \\
&= 2(\lambda + 2\epsilon)\int_0^T \int_\Omega \Big\{x(\partial_x u_\epsilon)^2\phi + 2u_\epsilon\partial_x u_\epsilon\phi + xu_\epsilon\partial_x u_\epsilon\phi_x \\
&\quad + \frac{u_\epsilon^2}{x}\phi + u_\epsilon^2\phi_x\Big\}dxdt - 2\int_0^T \int_\Omega \Big\{\rho_\epsilon u_\epsilon v_\epsilon^2\phi + P(\rho_\epsilon)(xu_\epsilon\phi)_x\Big\}dxdt.
\end{aligned} \tag{65}$$

Letting $\epsilon \to 0$ in (65) and making use of (24), (31), (45) and (64), we arrive at

$$\begin{aligned}
&\int_0^T \int_\Omega x\rho u^2(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 u_0^2\phi(x,0)dx \\
&= 2\langle \overline{(\lambda + 2\epsilon)xu_x^2}, \phi\rangle + 2\lambda\int_0^T \int_\Omega \Big\{2uu_x\phi + xuu_x\phi_x \\
&\quad + \frac{u^2}{x}\phi + u^2\phi_x\Big\}dxdt - 2\int_0^T \int_\Omega \Big\{\rho u v^2\phi + P(\rho)(xu\phi)_x\Big\}dxdt,
\end{aligned} \tag{66}$$

where $\overline{(\lambda + 2\epsilon)xu_x^2}$, the weak limit of $(\lambda + 2\epsilon)x(\partial_x u_\epsilon)^2$, is a nonnegative Radon measure.

Now, multiplying (47) by $2xu\phi$ in $L^2(Q_T)$ with the same ϕ as in (65), and recalling $\overline{v^2} = v^2$, we find, in the same manner as in (65), that

$$\begin{aligned}
&\int_0^T \int_\Omega x\rho u^2(\phi_t + u\phi_x)dxdt + \int_\Omega x\rho_0 u_0^2\phi(x,0)dx \\
&= 2\langle \lambda xu_x^2, \phi\rangle + 2\lambda\int_0^T \int_\Omega \Big\{2uu_x\phi + xuu_x\phi_x \\
&\quad + \frac{u^2}{x}\phi + u^2\phi_x\Big\}dxdt - 2\int_0^T \int_\Omega \Big\{\rho u v^2\phi + P(\rho)(xu\phi)_x\Big\}dxdt.
\end{aligned} \tag{67}$$

Combining (66) with (67), we conclude

$$\langle \lambda xu_x^2, \phi\rangle = \langle \overline{(\lambda + \epsilon)xu_x^2}, \phi\rangle \ge \langle \overline{\lambda xu_x^2}, \phi\rangle, \quad \forall\, \phi \in C^\infty(Q_T),\ \phi \ge 0,\ \phi(\cdot, T) = 0,$$

which implies $u_x^2 = \overline{u_x^2}$. Therefore, $\partial_x u_\epsilon \to u_x$ strongly in $L^2(Q_T)$.

Finally, by interpolation, (60), (64) and Lemma 4, we easily conclude $(v_\epsilon, w_\epsilon) \to (v, w)$ in $L^p(Q_T)$ for all $1 \leq p < \infty$.

Having had the strong convergence of $(\rho_\epsilon, u_\epsilon, v_\epsilon, w_\epsilon, \partial_x u_\epsilon)$, we easily see, by testing (46)–(49) with C_0^∞-functions and employing a density argument, that the limit functions ρ, u, v, w are indeed a weak solution of the initial boundary value problem (1)–(6) with $\epsilon = 0$ in the sense of Definition 1. This completes the proof of Theorem 1.

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A DYNAMICAL APPROACH TO THE STUDY OF RADIAL SOLUTIONS FOR P-LAPLACE EQUATION

Abstract. In this paper we give a survey of the results concerning the existence of ground states and singular ground states for equations of the following form:

$$\Delta_p u + f(u, |\mathbf{x}|) = 0$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$ is the p-Laplace operator, $\mathbf{x} \in \mathbb{R}^n$ and f is continuous, and locally Lipschitz in the u variable. We focus our attention mainly on radial solutions.

The main purpose is to illustrate a dynamical approach, which involves the introduction of the so called Fowler transformation. This technique turns to be particularly useful to analyze the problem, when f is spatial dependent, critical or supercritical and to detect singular ground states.

1. Introduction

Let $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$ denote the p-Laplace operator. The aim of this paper is to discuss the existence and the asymptotic behavior of positive solutions of equation of the following family

$$\Delta_p u + f(u, |\mathbf{x}|) = 0 \tag{1}$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $p > 1$, denotes the p-Laplace operator, $\mathbf{x} \in \mathbb{R}^n$ and $f(u, |\mathbf{x}|)$ is a continuous nonlinearity such that $f(0, |\mathbf{x}|) = 0$. The interest in equation of this type started from the classical Laplacian that is $p = 2$:

$$\Delta u + f(u, |\mathbf{x}|) = 0 \tag{2}$$

and is motivated by mathematical reasons, but also by the relevance of some equations of this type as model to describe phenomena coming from applied area of research. In particular Eq. (2) is important in quantum mechanic, astronomy and chemistry, while (1) is connected to problems arising in theory of elasticity, see e.g. [26]. Our purpose is to give a short, and not exhaustive, survey of the results which can be found in the wide literature concerning this argument, and in particular to discuss a method which is suitable to study radial solutions.

We think is worthwhile to stress that Eq. (2) can be regarded as the Euler equation of the following energy functional $E : \mathbb{R} \times W^{1,2}(\mathbb{R}^n) \to \mathbb{R}$,

$$E(\mathbf{x}, u, \nabla u) = \int_\Omega \Big(\frac{|\nabla u|^2}{2} - F(u, |\mathbf{x}|)\Big) d\mathbf{x}$$

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where $F(u, |\mathbf{x}|) = \int_0^u f(s, |\mathbf{x}|)ds$. The p-Laplace operator arises naturally when we want to extend this functional to $W^{1,p}(\mathbb{R}^n)$ functions. In fact (1) is the Euler equation for the functional $E_p : \mathbb{R} \times W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$,

$$E_p(\mathbf{x}, u, \nabla u) = \int_\Omega \big(\frac{|\nabla u|^p}{p} - F(u, |\mathbf{x}|)\big) d\mathbf{x}.$$

We will focus our attention mainly on radial solutions, hence we will reduce (1) to the following singular O.D.E.

$$(u'|u'|^{p-2}r^{n-1})' + f(u, r)r^{n-1} = 0 \tag{3}$$

where $r = |\mathbf{x}|$ and we commit the following abuse of notation: we write $u(r)$ for $u(\mathbf{x})$ when $|\mathbf{x}| = r$ and u has radial symmetry; here and later $'$ denotes derivation with respect to r. Observe that (3) is singular when $r = 0$ and when $u' = 0$, unless $p = 2$.

We introduce now some notation that will be in force throughout all the paper. We will use the term "regular solution" to refer to a solution $u(r)$ of Eq. (3) satisfying $u(0) = u_0 > 0$ and $u'(0) = 0$. We will use the term "singular solution" to refer to a solution $\upsilon(r)$ of Eq. (3) such that $\lim_{r\to 0} \upsilon(r) = +\infty$.

A basic question in this kind of PDE is the existence and the asymptotic behaviour of ground states (G.S.), that are solutions $u(\mathbf{x})$ of (1) which are nonnegative for any $\mathbf{x} \in \mathbb{R}^n$ and such that $\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$. We are also interested in detecting singular ground states (S.G.S.), that is solutions $\upsilon(\mathbf{x})$ which are well defined and nonnegative for any $\mathbf{x} \in \mathbb{R}^n\backslash\{0\}$ and such that $\lim_{|\mathbf{x}|\to\infty} \upsilon(\mathbf{x}) = 0$ and $\lim_{|\mathbf{x}|\to 0} u(\mathbf{x}) = +\infty$. Other interesting family of solutions for the radial equation (3) is the one of crossing solutions, that is regular solutions $u(r)$ which are positive for r smaller than a certain value $R > 0$ and become null with nonzero slope at $r = R$. So they can also be regarded as solutions of the Dirichlet problem in the ball of radius R. Finally we individuate solutions $u(r)$ of the Dirichlet problem in the exterior of the ball of radius R, that is $u(R) = 0$, $u(r) > 0$ for $r > R$, and $u(r)$ has fast decay. We say that a positive solution $u(r)$ of (3) has fast decay if $\lim_{r\to\infty} u(r)r^{(n-p)/(p-1)} < +\infty$ and that it has slow decay if $\lim_{r\to\infty} u(r)r^{(n-p)/(p-1)} = +\infty$.

This article has the following structure: in section 1 we introduce the generalized Fowler transformation, and we apply it to a toy example, mainly for illustrative purpose. In sections 2 and 3 we introduce the Pohozaev function, that is one of the main tool for the analysis of equation of type (1), and we consider the case where respectively $f(u, r) = k(r)u|u|^{q-1}$ and $f(u, r) = k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1}$ where $q > p$, $q_2 > q_1 > p$, the functions $k(r)$, $k_1(r)$, $k_2(r)$ are positive and continuous for $r > 0$. In both the cases we assume that the corresponding Pohozaev functions have constant sign. In section 4 we discuss the case $f(u, r) = k(r)u|u|^{q-1}$ when the Pohozaev function changes sign, stressing in particular the case $q = p^*$. In section 5 we explain briefly few results concerning Eq. (2) when $f(u, r) = u|u|^{q_1-1} + u|u|^{q_2-1}$, when $p_* < q_1 < p^* < q_2$ and $p = 2$. We remark that in this case there are still many open problems. In section 6 we discuss the case $f(u, r) = -k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1}$, where $q_1 < q_2$, and the functions k_1 and k_2 are positive and continuous for $r > 0$.

Finally in the appendix we show how some more general equations can be reduced to (3), and we explain the concept of natural dimension, introduced in [20].

2. Preliminary results and autonomous case

The main purpose of this paper is to explain the method of investigation of positive solution of (3) which has been used in [2], [3], [4], [11], [1], [12], [13], [14], [15], [16], [17]. The advantage in the use of this method lies essentially on the fact that we can benefit of a phase portrait, and of the use of techniques typical of dynamical systems theory, such as invariant manifold theory and Mel'nikov functions. Moreover, restricting ourselves to the study of radial solutions, we overcome the difficulties deriving from the lack of compactness of the critical and supercritical case. With our method we can also naturally detect and classify singular solutions, which are not easily found by variational techniques or by standard shooting arguments. The main fault of the method is that it can just give information on radial solutions. However we wish to stress that, when the domain has radial symmetry (e.g. it is the whole $\mathbb{R}^n$), G.S. and solutions of the Dirichlet problem, if they exist, are radial in many different situations, which will be discussed in details in the following sections, see [6], [9], [42], [44].

Furthermore radial solutions play a key role also for many parabolic equations associated to (2). In fact in many cases the ω-limit set is made up of the union of radial solutions, see e. g. [39], [23].

The first step in this analysis consists in applying the following change of coordinates

$$\begin{gathered}\alpha_l = \tfrac{p}{l-p}, \quad \beta_l = \tfrac{p(l-1)}{l-p} - 1, \quad \gamma_l = \beta_l - (n-1), \quad l > p \\ x_l = u(r) r^{\alpha_l} \quad y_l = u'(r)|u'(r)|^{p-2} r^{\beta_l} \quad r = e^t\end{gathered} \tag{4}$$

where $l > p$ is a parameter. This tool allows us to pass from (3) to the following dynamical system:

$$\begin{pmatrix}\dot{x}_l \\ \dot{y}_l\end{pmatrix} = \begin{pmatrix}\alpha_l & 0 \\ 0 & \gamma_l\end{pmatrix}\begin{pmatrix}x_l \\ y_l\end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g(x_l, t)\end{pmatrix} \tag{5}$$

Here and later "$\cdot$" stands for $\frac{d}{dt}$, and

$$g_l(x_l, t) := f(x_l \exp(-\alpha_l t), \exp(t)) e^{\alpha_l (l-1)t}. \tag{6}$$

This transformation was introduced by Fowler in the 30s for the case $p = 2$, and we generalized it to the case $p > 1$ just recently in [12], [13], [15] [14], [16], [17]. It will be useful to embed system (5), and in general all the dynamical systems that will be introduced in the paper, in a one-parameter family as follows:

$$\begin{pmatrix}\dot{x}_l \\ \dot{y}_l\end{pmatrix} = \begin{pmatrix}\alpha_l & 0 \\ 0 & \gamma_l\end{pmatrix}\begin{pmatrix}x_l \\ y_l\end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g_l(x_l, t+\tau)\end{pmatrix} \tag{7}$$

We start from the special nonlinearity $f(u,r) = k(r)u|u|^{q-2}$. In such a case, setting $l = q$ and $\phi(t) = k(e^t)$, system (5) reduces to the following:

$$\begin{pmatrix} \dot{x}_q \\ \dot{y}_q \end{pmatrix} = \begin{pmatrix} \alpha_q & 0 \\ 0 & \gamma_q \end{pmatrix} \begin{pmatrix} x_q \\ y_q \end{pmatrix} + \begin{pmatrix} y_q |y_q|^{\frac{2-p}{p-1}} \\ -\phi(t) x_q |x_q|^{q-2} \end{pmatrix} \tag{8}$$

At the beginning of this section we will also assume that $k = \phi > 0$ is a constant, both for illustrative purpose and because the results will be useful later on in more difficult situations. We think it is worthwhile to recall that most of the results are well known in this rather trivial situation; however our method gives a new point of view on the problem and allows to to clarify and complete some aspects concerning singular solutions even in this easy setting. A first advantage in this change of coordinates consist in the fact that it allows us to pass from a singular non-autonomous ODE to an autonomous dynamical system from which the singularity has been removed (obviously this is not the case for every type of nonlinearity). Moreover now we can apply to the problem techniques typical of dynamical system theory, thus exploiting a different point of view.

We recall the value of two exponents that are critical for this equation. When $n > p$, we denote by $p^* = np/(n-p)$ the Sobolev critical exponent and by $p_* = p\frac{n-1}{n-p}$; when $p \geq n$ we set both p^* and p_* equal to $+\infty$. Let Ω be an open bounded domain with non-empty smooth boundary $\partial\Omega$, then p^* is the largest $q > p$ such that the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ holds, while p_* is the largest q such that the trace operator $\gamma : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is continuous.

REMARK 1. We stress that, whenever $q > p$, $\alpha_q > 0$, $\gamma_q = -\frac{(p_*-q)(n-p)}{q-p}$ has the same sign as $p_* - q$, and $\alpha_q + \gamma_q = \frac{p^*-q}{(n-p)(q-p)}$, has the same sign as $p^* - q$. Also observe that (8) is C^1 if and only if $1 < p \leq 2$ and $q \geq 2$.

Notation.

In the whole paper we will use bold letters for vectorial objects. We denote by $u(d,r)$ a regular solution of (3) such that $u(d,0) = d$ and $u'(d,0) = 0$. Moreover if $\bar{u}(r)$ is solution of (3) we denote by $\bar{\mathbf{x}}_l(t) = (\bar{x}_l(t), \bar{y}_l(t))$ the corresponding trajectories of (5). For any $\mathbf{Q} \in \mathbb{R}^2$ we denote by $\mathbf{x}_l(\mathbf{Q},t) = (x_l(\mathbf{Q},t), y_l(\mathbf{Q},t))$ the trajectories of (5) passing through $\mathbf{Q}$ at $t = 0$, and by $\mathbf{x}_l^\tau(\mathbf{Q},t) = (x_l^\tau(\mathbf{Q},t), y_l^\tau(\mathbf{Q},t))$ the trajectory of (5) passing through $\mathbf{Q}$ at $t = \tau$ or equivalently the trajectory of (7) passing through $\mathbf{Q}$ at $t = 0$. Finally we denote by $\mathbb{R}^2_+$ the subset $\{(x,y), |x \geq 0\}$.

In this section we will always assume $q > p$. From a straightforward computation it is easy to observe that the system (8) admits three critical points whenever $q > p_*$: the origin $\mathbf{O}$, $\mathbf{P}= (P_x, P_y)$ and $-\mathbf{P}$, where $P_x = |\gamma_q \alpha_q^{p-1}/k|^{1/(q-p)}$, and $P_y = -|\gamma_q/k\alpha_q^{q-1}|^{(q-1)/(q-p)}$. Note that the critical point $\mathbf{P}$ is a center when $q = p^*$, it is asymptotically stable for $q > p^*$ and it is asymptotically unstable for $q < p^*$.

Positive and decreasing solutions $u(r)$ of (3) correspond to trajectories such that $y_q(t) \leq 0 < x_q(t)$. Moreover, trajectories $\mathbf{x}_q(t)$ which are bounded and such that $x_q(t)$ is uniformly positive for $t > 0$ (resp. for $t < 0$) correspond to solutions $u(r)$ which have slow decay (resp. are singular for $r = 0$), that is $u(r)r^{\alpha_q}$ is uniformly positive and bounded as $r \to \infty$ (resp. as $r \to 0$). Now we want to give a rough picture of the phase portrait of (8), in the autonomous case $\phi \equiv k > 0$. For this purpose we need to introduce a function which plays a key role in all our analysis. Let us denote by

$$H_q(x, y, t) := \frac{n-p}{p} xy + \frac{p-1}{p} |y|^{\frac{p}{p-1}} + \phi(t) \frac{|x|^q}{q}. \tag{9}$$

This function is a translation in this dynamical context of the well known Pohozaev function

$$P(u, u', r) = r^n \left[\frac{n-p}{p} \frac{uu'|u'|^{p-2}}{r} + \frac{p-1}{p} |u'|^p + k(r) \frac{|u|^q}{q} \right]$$

which is one of the main tool in the analysis of equations of these type, see e.g. [38], [34], [35]. Observe in fact that, if $\mathbf{x}_{p^*}(t) = (x_{p^*}(t), y_{p^*}(t))$ is the trajectory of (8) corresponding to $u(r)$, then

$$P(u(r), u'(r), r) = H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = H_q(x_q(t), y_q(t), t) e^{-(\alpha_q + \gamma_q)t} .$$

When k is differentiable from a simple computation we get the following

$$\frac{d}{dt} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) := \frac{d}{dt} \left[e^{\alpha_{p^*}(q-p^*)t} \phi(t) \right] \frac{|x_{p^*}|^q(t)}{q} . \tag{10}$$

Note that the function H_{p^*} does not depend explicitly on t, when k is a constant and $q = p^*$; so in this case it is a first integral for the system. Therefore, using some elementary argument, it is possible to draw each trajectory of the system, see Lemma in [12], and to give a picture of the phase portrait see fig. 1.

Then we easily get a lot of information on the original equation (3).

We stress that in this easy situation we have an explicit formula for all the regular solutions, that is

$$u(d, r) = d \left[1 + \left(\frac{p-1}{n-p}\right)^{\frac{p}{p-1}} \left(\frac{1}{2n}\right)^{\frac{1}{p-1}} d^{\frac{p^2}{(p-1)(n-p)}} r^{\frac{p}{p-1}} \right]^{-\frac{n-p}{p}} k^{-\frac{n-p}{p^2}} . \tag{11}$$

It can be shown easily that system (3) with $q = p^*$ and $\phi(t) \equiv k > 0$ and with $q = s$ and $\phi(t) \equiv e^{\alpha_{p^*}(p^*-s)t} k$, are topologically equivalent. In fact we can push much further this kind of identification. This is done in the appendix where the concept of natural dimension is introduced, see [20], [33].

We will see that an unstable set for (8) exists for any $q > p$, while a stable set exists just when $p_* < q < p^*$. It can be shown that the former existence result is equivalent to the existence of regular solutions of (3), while the latter is equivalent to the existence of solutions $u(r)$ with fast decay.

From now on we will commit the following abuse of notation: we will call stable and unstable sets (or manifolds) the branches which depart from the origin and

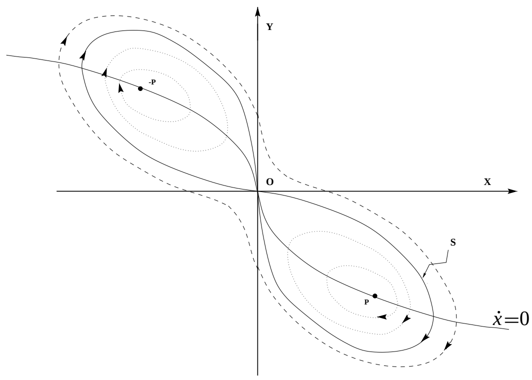

Figure 1: A sketch of the phase portrait for the autonomous system (8) when $\phi \equiv k > 0$, and $q = p^*$. The lines also represents the level curves for the function H_q for t fixed, when $g_q(x_q, t) = \phi(t)x_q|x_q|^{q-2}$.

gets into $\mathbb{R}^2_+$, which correspond to the positive solutions $u(r)$ of (3) we are interested in. The existence of trajectories converging to the origin either in the past or in the future can be inferred from invariant manifold theory, whenever $1 < p \leq 2$ and $q \geq 2$. In such a case we directly prove the existence of a stable and an unstable manifold, denoted respectively by W^s and by W^u, see [12].

When these regularity hypotheses are not satisfied the proofs become more difficult, due to the lack of local uniqueness of the trajectories crossing the coordinate axes. But using Wazewski's principle and the fact that the trajectories we are interested in do not cross the coordinate axes, it is possible to obtain a similar result. However, with this different proof, a priori W^u and W^s are just compact and connected sets. But in the autonomous case $k \equiv \text{const} > 0$, we can exploit the invariance of the system with respect to t, to conclude that W^s and W^u are in fact graph of a trajectory having the origin respectively as ω-limit set and α-limit set. Therefore, even in this case, they are 1 dimensional manifolds, see [15], [17]. We think it is worth mentioning the fact that, when the system is not Lipschitz, a priori the trajectories could reach the origin at some $t = T$ finite, either in the past or in the future. However it is easy to show that this possibility cannot take place when $q \geq p$, see [17] for a detailed proof.
Note that, if $k > 0$ is a constant, we also have that $H_q(x_q(t), y_q(t), t)$ is increasing along the trajectories if and only if $p_* < q < p^*$, and it is decreasing if and only if $q > p^*$. Moreover for any trajectory converging to the origin as $t \to \pm\infty$, we have $\lim_{t\to\pm\infty} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = 0$. Putting together all these results, we can draw fig. 2, and classify positive solutions in one of the following structures.

A All the regular solutions are monotone decreasing G.S. with slow decay. There are uncountably many solutions of the Dirichlet problem in the exterior of the ball. More precisely, for any $R > 0$ there is a solution $v(r)$ such that $v(R) = 0$, $v(r)$ is positive for any $r > R$ and it has fast decay. There is at least one S.G.S. with slow decay.

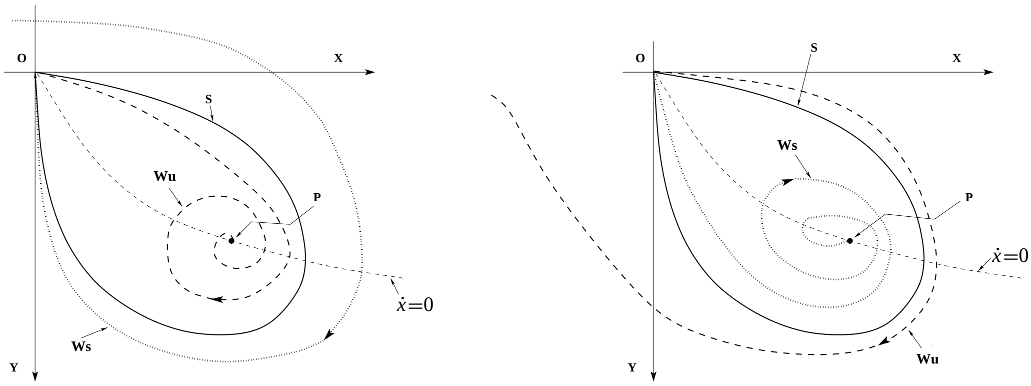

Figure 2: A sketch of the phase portrait for the autonomous system (8) when $\phi \equiv k > 0$, $1 < p \leq 2$ and $q \geq 2$. The figures show the stable manifold W^s (dotted line) and the unstable manifold W^u (dashed line). The solid curve S indicates the set $\{(x_q, y_q) \mid x_q \geq 0 \; H_q(x_q, y_q) = 0\}$. Figure 2A refers to the case $q \geq p_*$ while 2B to the case $p_* < q < p^*$.

B All the regular solutions $u(d,r)$ are crossing solutions, and there are uncountably many S.G.S. with fast decay $\upsilon(r)$. There is at least one S.G.S. with slow decay.

Namely, if $q > p^*$ positive solutions have structure **A**, while if $p_* < q < p^*$ they have structure **B**. In both the cases the S.G.S. with slow decay is unique and can be explicitly computed. If $q = p^*$ we are in the border situation, so all the regular solutions are G.S. with fast decay, see (11), there are uncountably many S.G.S. with slow decay, and uncountably many oscillatory solutions, see [12]. When $q \leq p_*$, it is easy to show that all the regular solutions $u(r)$ of (3) are crossing solutions.

We conclude this section with some basic results concerning the existence of regular solutions and positive fast decay solutions for (3) and a wide class of functions $f(u,r)$. First of all we recall that, if $f(u,r)$ is continuous and locally Lipschitz continuous in the u variable, the existence of regular solution is ensured, and if $f(d,0) > 0$ we also have local uniqueness of $u(d,r)$. The proof of this standard result can be found in [19] for the spatially independent case, but the argument can be easily adapted to the general case, see [16], [17]. We give now a result concerning the asymptotic behaviour of positive solutions. The proof of this result can be found in [19], [13], [17].

PROPOSITION 1. *Consider a solution $u(r)$ of (3) such that $u'(r) \leq 0 \leq u(r)$ for any $r > R$ for a certain $R > 0$, and $\lim_{r\to\infty} u(r) = 0$.*

A *Assume that there are $U > 0$ and $g(u) \in \mathcal{L}^1_{loc}$ such that $|f(u,r)| < g(u)$ for $r \geq 0$ and $0 \leq u \leq U$, and denote by $G(u) = \int_0^u g(s)ds$. Moreover assume that $\int_0 |G(s)|^{-1/p} ds < \infty$. Then the support of $u(r)$ is bounded.*

B *Assume that there are $C > 0$, $U > 0$ and $q_1 \geq p$ such that $|f(u,r)| < Cu^{q_1-1}$ for*

$0 \le u \le U$ and $r \ge 0$. Then $u(r) > 0$ for $r > R$ and the limit $\lim_{r\to\infty} u(r) r^{\frac{n-p}{p-1}} = \lambda$ exists. Moreover, if $f(u,r) > 0$ for u small and r large, then $\lambda > 0$, while if $f(u,r) < 0$ for u small and r large, then $\lambda < \infty$.

When Hypothesis B is satisfied we can go a bit further. Now we distinguish between the case in which $f(u,r)$ is always positive and the case in which it is negative for u small.

COROLLARY 1. *Assume that Hypothesis **B** of the previous Proposition is satisfied. First assume that $f(u,r) > 0$ for u small and r large.*

1 *If $q_1 \le p_*$, and there are $U > 0$, $c > 0$ and $Q_1 \in (p, q_1]$ such that $f(u,r) > cu^{Q_1-1}$ for r large and $0 \le u < U$. Then $\lambda = \infty$.*

Assume now that $f(u,r) < 0$ for u small and r large.

2 *If $q_1 > p_*$, then $\lambda > 0$.*

3 *If $q_1 \le p_*$, and there are $U > 0$, $c > 0$ and $Q_1 \in (p, q_1]$ such that $-f(u,r) > cu^{Q_1-1}$ for r large and $0 \le u < U$. Then $\lambda = 0$ and $\limsup_{r\to\infty} u(r) r^{-\frac{p}{Q_1-p}} < \infty$. Furthermore if $Q_1 = p_*$ we also have $\limsup_{r\to 0} u(r) r^{\frac{n-p}{p-1}} |\ln(r)|^{-\frac{n-p}{p(p-1)}} < \infty$.*

4 *Assume that the following limit exists is bounded and negative:*

$$\lim_{r\to\infty} \frac{f(ur^{-\frac{p}{Q_1-p}}, r)}{\left|ur^{-\frac{p}{Q_1-p}}\right|^{Q_1-1}} = -k(\infty).$$

If $Q_1 < p_*$, then $\lim_{r\to\infty} u(r) r^{-\frac{p}{Q_1-p}} = P_x > 0$ where $\boldsymbol{P} = (P_x, P_y)$ is the critical point of system (5) where $l = q$ and $g \equiv k(\infty) x|x|^{q-2}$. If $Q_1 = p_*$ then $u(r) r^{\frac{n-p}{p-1}} |\ln(r)|^{-\frac{n-p}{p(p-1)}}$ is uniformly positive and bounded for r large.

Exploiting the knowledge of the autonomous case (8) with $\phi \equiv k > 0$, it is possible to prove the existence of a local stable and unstable manifold also for the non-autonomous system (5), under suitable hypotheses on $g_l(x_l, t)$, or equivalently on $f(u,r)$.

PROPOSITION 2. *Assume that $f(u,r)$ is continuous for $r = 0$ and consider system (5) where $l > p$; then there is a local unstable set*

$$\tilde{W}^u(\tau) := \{\boldsymbol{Q} \in \mathbb{R}^2_+ \mid \boldsymbol{x}^\tau_l(\boldsymbol{Q}, t) \in \mathbb{R}^2_+ \text{ for any } t \le 0 \text{ and } \lim_{t\to-\infty} \boldsymbol{x}^\tau_l(\boldsymbol{Q}, t) = \boldsymbol{O}\}.$$

*This sets contains a closed connected component to which **O** belongs and whose diameter is positive, uniformly in τ.*

Assume that there are $\nu > 0$ and $q_2 > p_*$ such that, for any $r \in [0, \nu]$, we have $\limsup_{u\to\infty} \frac{f(u,r)}{u^{q_2-1}} < a(r)$ where $0 < a(r) < \infty$. Moreover assume that one of the following hypotheses are satisfied

- *$f(u,r) > 0$ for r large and $u > 0$; moreover there is $q_1 > p_*$ such that $\frac{f(u,r)}{u^{q_1-1}}$ is bounded for u positive and small and r large.*
- *$f(u,r) < 0$ for r large and $u > 0$.*

Then there is a local stable set

$$\tilde{W}^s(\tau) := \{\boldsymbol{Q} \in \mathbb{R}^2_+ \mid \boldsymbol{x}^\tau_{q_2}(\boldsymbol{Q}, t) \in \mathbb{R}^2_+ \text{ for any } t \le 0 \text{ and} \lim_{t\to-\infty} \boldsymbol{x}^\tau_{q_2}(\boldsymbol{Q}, t) = \boldsymbol{O}\}\,.$$

*This sets contains a closed connected component to which **O** belongs and whose diameter is positive, uniformly in τ.*

Proof. Consider first the case $f(u,r) = k(r)u|u|^{q-2}$, and assume that $k(r)$ is uniformly continuous. Then the existence of these stable and unstable sets follows from invariant manifold theory for non-autonomous system, see [28], [30]. Moreover in such a case we also know that these sets are indeed smooth manifolds which depend smoothly on τ. In the general case the existence of the unstable set easily follows from the existence of regular solutions for (3). The existence of the stable set is more complicated and can be proved through Wazewski's principle, see [15] and [17]. In [17] the proof is given for the case $f(u,r) < 0$ for u small and r large, but the argument can be easily extended also to the case $f(u,r) < ku|u|^{q_1-2}$ with $q_1 > p_*$, for u small and r large. □

We give now a result proved in [13] and [17] which explains the relationship between stable and unstable sets of (5) and solutions of (3).

PROPOSITION 3. *Consider system (5) and assume that $g_l(x_l, t)$ is bounded as $t \to -\infty$, for any $x_l > 0$. Then each regular solution $u(r)$ of (3) corresponds to a trajectory $\boldsymbol{x}^\tau_l(\boldsymbol{Q}^u, t)$ such that $\boldsymbol{Q}^u \in \tilde{W}^u(\tau)$, and viceversa. Moreover any trajectory $\boldsymbol{x}^\tau(\boldsymbol{Q}^s, t)$ where $\boldsymbol{Q}^s \in \tilde{W}^s(\tau)$, corresponds to a solution $u(r)$ of (3) with fast decay.*

REMARK 2. Take $f(u,r) = -k_1(r)u|u|^{q_1-2} + k_2(r)u|u|^{q_1-2}$, where $q_1 < q_2$ and the functions $k_i(r)$ are continuous, uniformly positive and bounded as $r \to \infty$. Then if $q_1 < p$ we are in the Hypotheses of claim A of Proposition 1, while if $q_1 \ge p$ Hyp. B is satisfied. Moreover if $q_1 > p_*$, then we are in Hyp. 2 of Corollary 1, while if $p < q_1 \le p_*$ Hyp. 3 of Corollary 1 holds. To satisfy Hyp. 4 we need to assume that $p < q_1 \le p_*$ and $\lim_{r\to\infty} k_1(r) = k(\infty) > 0$.

We recall that, roughly speaking, positive solutions of (3) can have two asymptotic behaviours, both as $r \to 0$ and as $r \to \infty$. Obviously the asymptotic behavior as $r \to 0$ is influenced by the behaviour of f for u large and r small, while their behavior as $r \to \infty$ depends on the behaviour of f for u small and r large. Generally speaking, when f is positive for u small, we have seen that solutions with fast and slow decay may coexist, while when it is negative we can have either solutions with fast decay or oscillatory solutions. Analogously when $f(u,r)$ is positive and supercritical with respect to p_*, for u large and r small, we can have regular solutions $u(d,r)$ such that $u(d,0)=d$ and $u'(d,0)=0$, and singular solutions $v(r)$ that are such that $\lim_{r\to 0}v(r)=+\infty$. More precisely

PROPOSITION 4. *Assume that there are $s > p_*$, $\rho > 0$ and positive functions $b(r)\geq a(r)$ such that, for any $0\leq r\leq \rho$ we have*

$$0 < a(r) \leq \liminf_{u\to+\infty} \frac{f(u,r)}{u^{s-1}} \leq \limsup_{u\to+\infty} \frac{f(u,r)}{u^{s-1}} \leq b(r) < \infty\,.$$

If $\boldsymbol{Q}\in W^u(\tau)$ then the solution $u(r)$ corresponding to $\boldsymbol{x}_s^\tau(\boldsymbol{Q},t)$ is a regular solution. Moreover any singular solution, if it exists, is such that $u(r)r^{p/(s-p)}$ is bounded for r small and, if $s\neq p^*$, $u(r)r^{p/(s-p)}$ is uniformly positive, too.

Assume further that $s\neq p^*$ and that the limit $\lim_{u\to+\infty}\frac{f(u,0)}{u^{s-1}}=k(\infty)>0$ exists and is finite. Then $\lim_{r\to\infty}u(r)r^{p/(s-p)}=P_x>0$ where $\boldsymbol{P}=(P_x,P_y)$ is the critical point of system (5) where $l=q$ and $g\equiv k(\infty)x|x|^{s-2}$.

Assume that there are $q>p_*$, $R>0$ and positive functions $B(r)\geq A(r)$ such that, for any $r>R$ we have

$$0 < A(r) \leq \liminf_{u\to 0} \frac{f(u,r)}{u^{q-1}} \leq \limsup_{u\to 0} \frac{f(u,r)}{u^{q-1}} \leq B(r) < \infty\,.$$

Then, if $\boldsymbol{Q}\in\tilde{W}^s(\tau)$, the solution $u(r)$ corresponding to $\boldsymbol{x}_q^\tau(\boldsymbol{Q},t)$ has fast decay, that is the limit $\lim_{r\to\infty}u(r)r^{(n-p)/(p-1)}>0$ exists and is finite. A slow decay solution (if it exists), is such that $u(r)r^{p/(q-p)}$ is bounded for r large; moreover if $q\neq p^*$, $u(r)r^{p/(q-p)}$ is uniformly positive, too.

Assume further that the limit $\lim_{u\to 0}\frac{\lim_{r\to\infty}f(u,r)}{u^{q-1}}=k(\infty)>0$ exists and is finite. Then $\lim_{r\to\infty}u(r)r^{p/(q-p)}=P_x>0$ where $\boldsymbol{P}=(P_x,P_y)$ is the critical point of system (5) where $l=q$ and $g\equiv k(\infty)x|x|^{q-2}$.

These results are proved in [12], [13], [17] using dynamical arguments.

3. When the Pohozaev function does not change sign

3.1. The case $f(u,r)=k(r)u|u|^{q-2}$

In this subsection we discuss positive solutions of equation (3) in the case $f(u,r)=k(r)u|u|^{q-2}$ and $q>p_*$. This problem has been subject to rather deep investigations in the '90s also for the relevance it has in different applied areas. First of all, when $p=2$ eq. (1) can be regarded as a nonlinear Schroedinger equation. Moreover, when $q=p^*$ and again $p=2$, this equation is known with the name of scalar curvature equation. In fact the existence of a G.S. $u(\mathbf{x})$ amounts to the existence of a metric g conformal to a standard metric g_0 on $\mathbb{R}^n$ ($g = u^{\frac{4}{n-2}} g_0$), whose scalar curvature is $k(|\mathbf{x}|)$. Furthermore, if the G.S. has fast decay, the metric g gives rise, via the stereographic projection, to a metric on the sphere deprived of a point $S^n \setminus$ {a point} which is equivalent to the standard metric.

Moreover when $q > 1$ and $k(r)$ takes the form $k(r) = \frac{r^\alpha}{1+r^\beta}$ eq. (2) is also known as Matukuma equation and it was proposed as a model in astrophysics. This problem will be investigated in details also in section 4, where we will assume that the Pohozaev functions change their sign, so that positive solutions have a richer structure.

We begin by some preliminary results concerning forward and backward continuability and long time behaviour for positive solutions, in relation with the Pohozaev function. These results can be proved using directly the Pohozaev identity, or through a dynamical argument exploiting our knowledge of the level sets of the function $H(\mathbf{x}, t)$, see [34] and [13].

LEMMA 1. *Let $u(r)$ be a solution of (3), and $\boldsymbol{x}_{p^*}(t)$ the corresponding trajectory. Assume that* $\liminf_{t\to\pm\infty} H(\boldsymbol{x}_{p^*}(t), t) > 0$*, then $\boldsymbol{x}_{p^*}(t)$ has to cross the coordinate axes indefinitely as $t \to \pm\infty$, respectively.*

Assume that $\limsup_{t\to\pm\infty} H(\boldsymbol{x}_{p^*}(t), t) < 0$*, then $\boldsymbol{x}_{p^*}(t)$ cannot converge to the origin or cross the coordinate axes.*

When $p = 2$, the standard tool to understand the behaviour of solutions with fast decay is the Kelvin transformation. Let us set

$$s = r^{-1} \quad \tilde{u}(s) = r^{n-2} u(r) \quad \tilde{K}(s) = r^{2\lambda} K(r^{-1}) \quad \lambda = \frac{(n+2)(q-2^*)}{2}\,; \tag{12}$$

Then (3) is transformed into

$$[\tilde{u}_s(s)s^{n-1}]_s + \tilde{K}(s)\tilde{u}|\tilde{u}|^{q-2}(s)s^{n-1} = 0\,. \tag{13}$$

Note that a regular solution $u(d, r)$ of (3) is transformed into a fast decay solutions $\tilde{u}(s)$ of (13) such that $\lim_{r\to\infty} \tilde{u}(r) r^{\frac{n-p}{p-1}} = d$, and viceversa. So we can reduce the problem of discussing fast decay solutions to an analysis of regular solutions for the transformed problem.

However we do not have an analogous result for the case $p \neq 2$, so we need the following Lemma, that, when $p = 2$, is a trivial consequence of the existence of the Kelvin inversion.

LEMMA 2. *Assume that $f(u, r) > 0$ for any $u > 0$ and consider a solution $u(r)$ which is positive and decreasing for any $r > R$. Then $u(r) r^{\frac{n-p}{p-1}}$ is increasing for any $r > R$.*

Proof. Consider system (8) where $l = p_*$ and the trajectory $\mathbf{x}_{p_*}(t)$ corresponding to $u(r)$. Note that $x_{p_*}(t) = u(r) r^{\frac{n-p}{p-1}}$ and that $\gamma_l = 0$; hence $\dot{y}_{p_*}(t) < 0$ whenever $x_{p_*}(t) > 0$. Assume for contradiction that there is $t_1 > T = \ln(R)$ such that $\dot{x}_{p_*}(t_1) <$

0, then, from an elementary analysis on the phase portrait either there is $t_2 > t_1$ such that $x_{p_*}(t_2) < 0$, or $\lim_{t\to\infty} x_{p_*}(t) = 0$. Assume the latter, then $\lim_{r\to\infty} u(r) r^{\frac{n-p}{p-1}} = \lim_{t\to\infty} x_{p_*}(t) = 0$; but from (3) it follows that $u'(r)^{\frac{n-1}{p-1}}$ is decreasing and admits limit $\lambda \le 0$. Using de l'Hospital rule we find that $u'(r)^{\frac{n-1}{p-1}} \to \frac{p-1}{n-p}\lambda$; so we get $\lambda = 0$ and $u'(r) \equiv u(r) \equiv 0$ for $r > R$, so the claim is proved. □

Recall that, when $k(r)$ is differentiable, the Pohozaev identity can be reformulated in this dynamical context as (10). Therefore we can think of H_{p^*} as an energy function, which is increasing along the trajectories when $\phi(s)e^{\alpha_{p^*}(p^*-q)t}$ is increasing and decreasing when $\phi(s)e^{\alpha_{p^*}(p^*-q)t}$ is decreasing. This observation can be refined combining it with the fact that all the regular solutions $u(r)$ are decreasing, and fast decay solutions are such that $x_{p_*}(t)$ is increasing, whenever they are positive. For this purpose we define two auxiliary functions, which are closely related to the Pohozaev identity, and which were first introduced in [35]. In this subsection we will always assume (without mentioning) that $e^{nt}\phi(t) \in \mathcal{L}^1\big((-\infty, 0]\big)$ and $e^{(n-q\frac{n-p}{p-1})s}\phi(t) \in \mathcal{L}^1\big([0, +\infty)\big)$, so that we can define the following functions:

$$
(14)\qquad
\begin{aligned}
J^+(t) &:= \frac{\phi(t)e^{nt}}{q} - \frac{n-p}{p}\int_{-\infty}^{t} \phi(s)e^{ns}ds\\
J^-(t) &:= \frac{\phi(t)e^{(n-q\frac{n-p}{p-1})t}}{q} - \frac{n-p}{p(p-1)}\int_{-\infty}^{t} \phi(s)e^{(n-q\frac{n-p}{p-1})s}ds
\end{aligned}
$$

We will see, that the sign of these functions play a key role in determining the structure of positive solutions for (3). When ϕ is differentiable we can rewrite J^+ and J^- in this form, from which we can more easily guess the sign:

$$
(15)\qquad
\begin{aligned}
J^+(t) &:= \frac{1}{q}\int_{-\infty}^{t} \frac{d}{ds}[\phi(s)e^{\alpha_{p^*}(p^*-q)s}]e^{\alpha_{p^*}qs}ds\\
J^-(t) &:= \frac{1}{q}\int_{t}^{+\infty} \frac{d}{ds}[\phi(s)e^{\alpha_{p^*}(p^*-q)s}]e^{-\frac{(n-p)q}{p(p-1)}s}ds
\end{aligned}
$$

Let $u(r)$ be a solution of (3) and let $\mathbf{x}(t)$ be the corresponding trajectory of (8). Using (10) and integrating by parts we easily find the following

$$
(16)\qquad
\begin{aligned}
&H_{p^*}(\mathbf{x}_{p^*}(t), t) + \lim_{t\to-\infty} H(\mathbf{x}_{p^*}(t), t)\\
&\qquad = J^+(t)\frac{|u|^q(e^t)}{q} - \int_{-\infty}^{t} J^+(s)u'(e^s)u|u|^{q-2}(e^s)ds\\
&H_{p^*}(\mathbf{x}_{p^*}(t), t) - \lim_{t\to\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t)\\
&\qquad == J^-(t)\frac{|x_{p_*}|^q(t)}{q} + \int_{t}^{+\infty} J^-(s)\dot{x}_{p_*}(s)x_{p_*}|x_{p_*}|^{q-2}(s)ds
\end{aligned}
$$

From (16) we easily deduce the following useful result.

REMARK 3. Assume that there is T such that $J^+(t) \geq 0$ (resp. $J^+(t) \leq 0$), but $J^+(t) \not\equiv 0$ for any $t \leq T$, and consider a regular solution $u(r)$ which is positive and decreasing for any $0 < r < R = \ln(T)$. Then $H_{p^*}(\mathbf{x}_{p^*}(t), t) \geq 0$ (resp. $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$) for any $t \leq T$.

Analogously assume that $J^-(t) \geq 0$ (resp. $J^-(t) \leq 0$) but $J^-(t) \not\equiv 0$ for any $t \geq T$, and consider a solution $u(r)$ which is positive and decreasing for any $r > R = \ln(T)$ and has fast decay. Then $H_{p^*}(\mathbf{x}_{p^*}(t), t) \geq 0$ (resp. $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$) for any $t \geq T$.

Using Remark 3 and Lemma 1 we obtain the following result.

THEOREM 1. *Assume that either $J^+(r) \geq 0$ and $J^+(r) \not\equiv 0$, or $J^-(r) \geq 0$ and $J^-(r) \not\equiv 0$ for any $r > 0$. Then all the regular solutions are crossing solutions and there exists uncountably many S.G.S. with fast decay.*

Assume that either $J^+(r) \leq 0$ and $J^+(r) \not\equiv 0$ or $J^-(r) \leq 0$ and $J^-(r) \not\equiv 0$ for any $r > 0$. Then all the regular solutions are G.S. with slow decay. Moreover there are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of a ball.

The proof of the result concerning regular solutions can be find in [34], and involves just a shooting argument and the use of J^+ and J^- in relation with the Pohozaev identity. Translating this argument in this dynamical context we easily get a classification also of singular solutions, see also [12].

Proof. Assume that $J^+(r) \leq 0$ for any $r > 0$, but $J^+ \not\equiv 0$; consider a regular solution $u(r)$ which is positive and decreasing in the interval $[0, R)$ and the corresponding trajectory $\mathbf{x}_{p^*}(t)$. Using (16) we easily deduce that $H_{p^*}(\mathbf{x}_{p^*}(t), t) \leq 0$ for any $t < T = \ln(R)$. From our assumption we easily get that there is $l < p^*$ such that $g(x_l, t)$ is uniformly positive for t large and $\lim_{t\to\infty} H_l(\mathbf{x}_l(t), t) < 0$. It follows that $\mathbf{x}_l(t)$ is forced to stay in a compact subset of the open 4^{th} quadrant for t large, so $u(r)$ is a G.S. with slow decay.

Analogously consider a trajectory $\bar{\mathbf{x}}_l(t)$ converging to $\mathbf{O}$ as $t \to +\infty$. Then the corresponding solution $\bar{u}(r)$ has fast decay, is positive and decreasing for any $r > R$ where $R > 0$ is a constant. Assume for contradiction that $R = 0$; then from (16) we find that $\liminf_{t\to-\infty} H_{p^*}(\bar{\mathbf{x}}_{p^*}(t), t) > \lim_{t\to\infty} H_{p^*}(\bar{\mathbf{x}}_{p^*}(t), t) = 0$. Hence, from Lemma 1, we deduce that $\bar{\mathbf{x}}_{p^*}(t)$ has to cross the coordinate axes indefinitely as $t \to -\infty$. Thus $R > 0$ and $\bar{u}(r)$ is a solution of the Dirichlet problem in the exterior of a ball.

The other claims can be proved reasoning in the same way, see again [12]. $\square$

Reasoning similarly we can complete the previous result proving the existence of S.G.S. with slow decay, to obtain the following Corollary.

COROLLARY 2. *Assume that $J^-(r) \geq 0$ for any r but $J^-(r) \not\equiv 0$, and that there is $p_* < m \leq p^*$ such that the limit $\lim_{t\to\infty}\phi(t)e^{\alpha_m(m-q)t} = k(\infty)$ exists is positive and finite. Then positive solutions have a structure of type **A**.*

*Analogously assume that $J^+(r) \leq 0$ for any r but $J^+(r) \not\equiv 0$, and that there is $s \geq p^*$ such that the limit $\lim_{t\to-\infty}\phi(t)e^{\alpha_s(s-q)} = k(0)$ exists is positive and finite. Then positive solutions have a structure of type **B**.*

Note that if the limits $\lim_{r\to\infty}k(r) = k(\infty)$ and $\lim_{r\to 0}k(r) = k(0)$ exist are positive and finite we can simply set $m = q = s$. In [12] there is a condition sufficient to obtain the uniqueness of the S.G.S. with slow decay. Roughly speaking this result is achieved respectively when $s \neq p^*$ and $m \neq p^*$.

We give some examples of application of Theorem 1 and Corollary 2.

REMARK 4. Assume that $k(r)$ is uniformly positive and bounded and that the limit $\lim_{r\to\infty}k(r)$ exists. Then, if $q < p^*$ and $k(r)$ is nondecreasing, positive solutions have a structure of type **A**, while if $q > p^*$ and $k(r)$ is nonincreasing, positive solutions have a structure of type **B**.

Consider the generalized Matukuma equation, that is (3) where $f(u,r) = \frac{1}{1+r^\tau}u|u|^{q-2}$. Then, if $\tau \leq p$ positive solutions have a structure of type **A** when $p < q < p(n-\tau)/(n-p)$, and of type **B** when $q > p^*$, see also [35]. The remaining cases will be analyzed in section 4.

3.2. The generic case: $f(u,r) = k_1(r)f_1(u) + k_1(r)f_2(u)$

Now we try to extend the results of the previous subsection to a wider class of functions $f(u,r)$:

$$f(u,r) = \sum_{i=1}^{N} k_i(r)u|u|^{q_i-2}, \qquad p < q_1 < q_2 < \cdots < q_N \tag{17}$$

where $N \geq 1$ and the functions $k_i(r)$, are continuous and positive. We will see that, under natural conditions on the functions $k_i(r)$, when $p_* < q_1 < q_N \leq p^*$ positive solutions have a structure of type **A**, while when $q_1 \geq p^*$ they have a structure of type **B**. The behavior of regular solutions have been classified directly using Pohozaev identity in [35]. In [13] we have completed the results by classifying the behaviour of singular solutions, using dynamical methods. In fact we have followed the path paved by Johnson and Pan in [29], for the analogous problem in the case $p = 2$.

In this paper we generalize slightly the techniques used in [34] and [13], combining them with some ideas of [17], to obtain more general results. As usual we set $\phi_i(t) = k_i(e^t)$, and we always assume (without mentioning) that $e^{nt}\phi_i(t) \in \mathcal{L}^1\big((-\infty,0]\big)$ and $e^{(n-q_i\frac{n-p}{p-1})s}\phi_i(t) \in \mathcal{L}^1\big([0,+\infty)\big)$ for any $i = 1,\dots,N$, so that we can define functions similar to $J^\pm$ of the previous subsections:

$$J_i^+(t) := \frac{\phi_i(t)e^{nt}}{q_i} - \frac{n-p}{p}\int_{-\infty}^{t}\phi_i(s)e^{ns}ds$$

$$J_i^-(t) := \frac{\phi_i(t)e^{(n-q_i\frac{n-p}{p-1})t}}{q} - \frac{n-p}{p(p-1)}\int_{t}^{+\infty}\phi_i(s)e^{(n-q_i\frac{n-p}{p-1})s}ds$$

Observe that, if $f_1(u) = u|u|^{q-1}$ and $k = 1$, the functions $J_1^{\pm}(t)$ defined in (3.2) coincide with the functions $J^{\pm}(t)$ defined in section 3.1. As we did in section 3.1, if $\phi_i \in C^1$ we can rewrite the functions $J_i^{\pm}$ in a form similar to (15) from which we can more easily guess the sign. So we find the analogous of (16):

$$\begin{aligned}
&H_{p^*}(\mathbf{x}_{p^*}(t),t) + \lim_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t)\\
&\qquad = \sum_{i=1}^{N}\Big[J_i^+(t)\frac{u^{q_i}(e^t)}{q_i} - \int_{-\infty}^{t} J_i^+(s)u'(e^s)u^{q_i-1}(e^s)ds\Big]\\
&H_{p^*}(\mathbf{x}_{p^*}(t),t) - \lim_{t\to\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t)\\
&\qquad = \sum_{i=1}^{N}\Big[J_i^-(t)\frac{x_{p_*}^{q_i}(t)}{q_i} + \int_{t}^{\infty} J_i^-(s)\dot{x}_{p_*}(t)x_{p_*}^{q_i-1}(s)\Big]ds
\end{aligned}$$

Therefore we have a result analogous to Remark 3 and repeating the argument of the proof of Theorem 1, we obtain the following generalization.

THEOREM 2. *Assume that either $J_i^+(t) \geq 0$ for any i and $\sum_{i=1}^N J_i^+(t) \not\equiv 0$ or $J_i^-(t) \geq 0$ for any i and $\sum_{i=1}^N J_i^-(t) \not\equiv 0$. Then all the regular solutions are crossing solutions; moreover if $q_1 > p_*$, and $k_i(r)$ is uniformly positive for r large, there are uncountably many S.G.S. with fast decay.*

Assume that either $J_i^+(t) \leq 0$ for any i and $\sum_{i=1}^N J_i^+(t) \not\equiv 0$ or $J_i^-(t) \leq 0$ for any i and $\sum_{i=1}^N J_i^-(t) \not\equiv 0$. Then all the regular solutions are G.S. with slow decay. Moreover there are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of a ball.

This result is proved in [13] for the case $1 < p \leq 2$. However it can be easily extended to the case $p > 2$ putting together the construction of a stable set $\tilde{W}^s(\tau)$ developed in [17] (and quoted in Theorem 2), and the argument of [13] concerning the function H_{p^*} (that we have sketched in this section). In fact the minimal requirement for the fast decay solution to exist, is that there are $c > 0$ and $m > p_*$ such that $g(x_m(t),t) > cx_m(t)|x_m(t)|^{m-2}$ for t large.

Repeating the argument in Corollary 2 we easily obtain also this result:

COROLLARY 3. *Assume that all the functions $J_i^-(t) \geq 0$ for any r but $\sum_{i=1}^N J_i^-(t) \not\equiv 0$, and that there is $p_* < m \leq p^*$ such that the limit $\lim_{t\to\infty} g(x_m(t),t)$ $/|x_m(t)|^{m-1} = k_m(\infty)$ exists, is positive and finite. Then positive solutions have a structure of type **A**.*

*Analogously assume that all the functions $J_i^+(t) \leq 0$ for any r but $\sum_{i=1}^N J_i^+(t) \not\equiv 0$, and that there is $s \geq p^*$ such that the limit $g(x_s(t),t)/|x_s(t)|^{s-1} = k_s(0)$ exists, is is positive and finite. Then positive solutions have a structure of type **B**.*

Once again if $m \neq p^*$ and $s \neq p^*$ respectively, and a further technical condition is satisfied the S.G.S. with slow decay is unique, see [13]. From Theorem 2 and

Corollary 3 we easily get the following.

REMARK 5. Consider (3) where f is as in (17) and assume that the functions $k_i(r)$ are uniformly positive and bounded. Then, if $p_* < q_1 < q_N \leq p^*$ and the functions $k_i(r)$ are nondecreasing, positive solutions have a structure of type **A**, while if $q_1 \geq p^*$ and the functions $k_i(r)$ are nonincreasing, positive solutions have a structure of type **B**.

4. When the Pohozaev function changes sign

In this section we discuss equation (3) when $f(u,r) = k(r)u|u|^{q-2}$, and we assume that $e^{nt}\phi(t) \in \mathcal{L}^1\big((-\infty,0]\big)$ and $e^{(n-q\frac{n-p}{p-1})s}\phi(t) \in \mathcal{L}^1\big([0,+\infty)\big)$, so that the functions $J^{\pm}(r)$ are well defined. We discuss now the case when $J^+(r)$ and $J^-(r)$ change sign. In such a case positive solutions may exhibit the following rich structure:

C There are uncountably many G.S. with slow decay and crossing solutions, and at least one G.S. with fast decay. There are uncountably many S.G.S. with fast decay, S.G.S. with slow decay, and solutions of the Dirichlet problem in the exterior of the ball.

Let us recall that, for $n > 2$ we denote by $2^* = 2n/(n-2)$ and by $2_* = 2(n-1)/(n-2)$. We start from this interesting result proved by Bianchi in [6].

THEOREM 3. *Consider (2) and define $g(r) = k(r)|r^2 - c^2|^{\frac{2^*-q}{2(n-2)}}$; assume that there is $c > 0$ such that $g(r)$ is non-increasing for $0 < r < c$ and non-decreasing for $r > c$. Then all the G.S. and the S.G.S. are radial.*

This fact gives more relevance to the study of radial solutions. Note that, if $q = 2^*$ then the Theorem simply requires that there is c such that $k(r)$ is non-increasing for $r < c$ and non-decreasing for $r > c$. In fact we think that these results may be extended also to the case $p \neq 2$; but it cannot be extended to any kind of potential $k(r)$ in fact, modifying the nonnegative potential $K_0 = \big(1-(r/\delta)^{\rho_1}\big)_+ + \big(1-(\delta r)^{\rho_2}\big)_+$, where $\delta, \rho_1, \rho_2 > 0$, Bianchi in [6] constructed a positive potential $k = \bar{k}(r)$ so that (2) admits no radial G.S. with fast decay, but it admits non-radial G.S. with fast decay. The potential $\bar{k}(r)$ is obtained from a potential satisfying the hypotheses of Theorem 3 and subtracting an arbitrarily small bump at $r = 0$ and at $r = \infty$.

For completeness we also quote the following result, borrowed from [5], concerning potential $k(\mathbf{x})$ which are not necessarily radial.

THEOREM 4. *Consider (2) where $q = 2^*$ and assume $n \geq 4$. Choose two points $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and two numbers $C_{\boldsymbol{y}}, C_{\boldsymbol{z}} > 0$. Then there is a positive potential $k = \tilde{k}(\boldsymbol{x})$ of the form*

$$\tilde{k}(\boldsymbol{x}) = C_{\boldsymbol{y}} - \epsilon|\boldsymbol{x}-\boldsymbol{y}|^{\rho} \quad \textit{for } \boldsymbol{x} \textit{ in a neighborhood of } \boldsymbol{y}$$
$$\tilde{k}(\boldsymbol{x}) = C_{\boldsymbol{z}} - \epsilon|\boldsymbol{x}-\boldsymbol{z}|^{\rho} \quad \textit{for } \boldsymbol{x} \textit{ in a neighborhood of } \boldsymbol{z} \tag{18}$$

where $\epsilon > 0$ is small enough and $\rho = n-2$, such that (2) admits no G.S. with fast decay.

In the same article Bianchi has also proved that a potential satisfying condition (18), with $\rho > n-2$ and ϵ arbitrary chosen, and some further sufficient conditions necessarily admits a G.S. with fast decay. This result shows how sensitive to small changes in the potential k the behaviour of positive solutions is.

Now we turn again to the case $p \neq 2$, and we focus our attention on radial solutions. In order to find a G.S. with fast decay we need to find a balance between the gain of energy, due to the values for which $k(r)r^{p^*-q}$ is increasing, and the loss of energy, due to the values for which it is decreasing. A first important result concerning the structure of positive solutions is the following:

THEOREM 5. *Consider (3) and assume that there is $R > 0$ such that one of the following conditions is satisfied*

$J^+(r) \geq 0$ *for any* $0 \leq r \leq R$ *and it is decreasing for* $r \geq R$

$J^-(r) \leq 0$ *for any* $r \geq R$ *and it is increasing for* $0 \leq r \leq R$.

Then regular solutions have one of the following structure.

1. *They are all crossing solutions*
2. *They are all G.S. with slow decay*
3. *There is $D > 0$ such that $u(d,r)$ is a crossing solution for $d > D$, it is a G.S. with slow decay for $d < D$, and a G.S. with fast decay for $d = D$.*

The result concerning $J^+(r)$ has been proved in [35], evaluating the Pohozaev function on regular solutions. The part concerning $J^-(r)$ is not explicitly stated in [35], however it can be easily obtained as follows, see also [18]. We can construct a stable set $\tilde{W}^s(\tau)$ through Proposition 2, and then deduce the existence of solutions with fast decay. Then, applying the argument of [35] to these solutions, we conclude. We think that one could easily reach a classification result also for S.G.S. in this situation, combining the argument in [35] with a dynamical argument. In fact, if we restrict to regular solutions, structure **A** and **B** give back structure 1 and 2 respectively, and structure 3 is a special case of **C**.

We have already seen that, when either $J^+(r)$ or $J^-(r)$ are positive for any $r > 0$ we have structure **A** (so we are in the first case), while when they are negative we have structure **B** (so we are in the second case). In order to derive a sufficient condition for structure **C** to exist, we start from the case $p = 2$ and following [43] we introduce the function:

$$Z(t) := e^{-\frac{(n-2)p}{2}t} J^+(t) - e^{\frac{(n-2)p}{2}t} J^-(t) \tag{19}$$

Then we define $\rho_+ = \inf\{r \in (0,\infty) \mid J^+(r) < 0\}$, and $\rho_- = \sup\{r \in (0,\infty) \mid J^-(r) < 0\}$, setting $\rho_+ = \infty$ if $J^+(r) \geq 0$ for any $r > 0$ and $\rho_- = 0$ if $J^-(r) \geq 0$ for any

$r > 0$. Now we can state the following result proved in [43], using the Pohozaev identity and the Kelvin transformation.

THEOREM 6. *Consider (3) where $p = 2$ and assume $\rho_+ > 0$ and $\rho_- < \infty$. If $Z(r_1) > 0$ for some $r_1 \in (0, \rho_+]$ and $Z(r_2) > 0$ for some $r_2 \in [\rho_-, \infty)$, there is $D > 0$ such that $u(D, r)$ is a G.S. with fast decay.*

Let us set $\lambda := \frac{(n-p)(q-p^*)}{p}$; following [43], we get the following more explicit result.

COROLLARY 4. *Consider (3) where $p = 2$ and suppose $q \neq p^*$ and that $k(r)$ is nonnegative and satisfies:*

$$k(r) = Ar^\sigma + o(r^\sigma) \text{ at } r = 0 \qquad k(r) = Br^l + o(r^l) \text{ at } r = \infty,$$

where $A, B > 0$ and $l < \lambda < \sigma$, then there is a G.S. with fast decay.

Now assume $q = p^*$ and that $k(r)$ satisfies

$$k(r) = A_0 + A_1 r^\sigma + o(r^\sigma) \text{ at } r = 0 \qquad k(r) = B_0 + B_1 r^l + o(r^l) \text{ at } r = \infty,$$

where $A_1, B_1 > 0$, $A_0, B_0 > 0$, $-n < l < 0 < \sigma < n$. Then there is a G.S. with fast decay.

Note that the case $q = p^*$ is more delicate; we stress that the restriction $|l|, |\sigma|$ smaller than n is needed even if it was not required in [43]. However when $A_0 = 0$ we do not need the restriction on $|l|$ and when $B_0 = 0$ we do not need the restriction on $|\sigma|$.

Using a similar argument Kabeya, Yanagida and Yotsutani, in [32] found an analogous result for the case $p \neq 2$.

THEOREM 7. *Consider (3) where $k(r) \geq 0$ for any r. Assume that either $\liminf_{r\to 0} \frac{rk'(r)}{k(r)} > \lambda$ or $k(r) = Ar^\sigma + o(r^\sigma)$ at $r = 0$ for some $A > 0$ and $\sigma > \lambda$. Moreover assume that either $\limsup_{r\to\infty} \frac{rk'(r)}{k(r)} < \lambda$ or $k(r) = Br^l + o(r^l)$ at $r = 0$ for some $B > 0, l < \lambda$. Then there is a strictly increasing sequence $d_j > 0$, $j = 0, \ldots, \infty$, such that $u(d_j, r)$ has exactly j zeroes and has fast decay. So in particular $u(d_0, r)$ is a G.S. with fast decay.*

Note that the conditions of the previous Theorem at $r = 0$ (and at $r = \infty$) are similar, but they do not imply each other. In fact the former is useful when $k(r)$ has a logarithmic term, e. g. $k(r) = |\ln(r)|r^\sigma$, and the latter when $k(r)$ behaves like a power at $r = 0$ (and at $r = \infty$). However, in both the cases, when $q = p^*$ we have $k(0) = k(\infty) = 0$.

We wish to mention that Bianchi and Egnell in [5], [7] have some other sufficient conditions for the existence of G.S. with fast decay. Each condition, as the ones of Corollary 4 and of Theorem 7, in some sense, requires a change in the sign of the function $J^+(r)$ which is "sufficiently large to be detected". We stress that the situation becomes more delicate when $q = p^*$ and $k(r)$ is uniformly positive and bounded, see [5], [7], for a careful analysis. In fact, as suggested from Theorem 8 stated below and borrowed from [7], we are convinced that the condition on the smallness of $|l|$, $|\sigma|$ of Corollary 4 is not technical.

THEOREM 8. *Consider (2) where $q = 2^*$ and take two numbers $\rho_1, \rho_2 > n(n-2)/(n+2)$, such that $1/\rho_1 + 1/\rho_2 \geq 2/(n-2)$. Then there is a function $k(r)$ such that $k(r) = 1 - M_1 r^{\rho_1}$ near the origin and $k(r) = 1 - M_2 r^{-\rho_2}$ near ∞, where M_1, M_2 are positive large constants so that (2) admits no radial G.S. with fast decay.*

We also stress that the previous result shows that the situation is much more clear when $J^+(r)$ is positive for r small and negative for r large, than when we are in the opposite situation.

We introduce now some perturbative results, proved with dynamical techniques, that help us to understand better also what happens to singular solutions, and also which is the difference between the case in which we have a subcritical behaviour for r small and supercritical for r large (easier situation), and the opposite case (difficult situation). We focus on the case $q = p^*$: in the autonomous case this is a border situation between a structure of type **A** and **B**. So it is the best setting in order to have new phenomena as the existence of G.S. with fast decay.

Let us assume that $k(r)$ has one of the following two form:

$k(r) = 1 + \epsilon K(r)$, where K is a bounded smooth function,

$k(r) = K(r^\epsilon)$, where K is a bounded smooth function, positive in some interval,

where $\epsilon > 0$ is a small parameter and we assume $K \in C^2$. In the former case we say that $k(r)$ is a regular perturbation of a constant (k changes little), in the latter we say that it is a singular perturbation of a constant (k changes slowly). It is worthwhile to note that, in the latter case k may change sign.

This problem was studied in the case $p = 2$ by Johnson, Pan and Yi in [30] using the Fowler transformation, invariant manifold theory for non-autonomous system and Mel'nikov theory. In both the cases they found a non-degeneracy condition of Mel'nikov type, related to some kind of expansion in ϵ of the Pohozaev function, which is sufficient for the existence of G.S. with fast decay. They also proved that when $K(e^t)$ is periodic and the Mel'nikov condition is satisfied, there is a Smale horseshoe for the associated dynamical system. Then they inferred the existence of a Cantor set of S.G.S. with slow decay.

In the singular perturbation case the condition is easy to compute: there is a G.S. with fast decay for each non-degenerate positive critical point of $K(r)$. These results have been completed by Battelli and Johnson in [2], [3], [4], and eventually they proved the existence of a Smale horseshoe also in this case. Thus they inferred again the existence of a Cantor set of S.G.S. with slow decay, assuming that $K(e^t)$ is periodic.

These results have been extended to the case $2n/(n+2) \leq p \leq 2$ in [14], and completed to obtain a structure result for positive solutions. First we have introduced a dynamical system of the form (8) through (4) with $l = q = p^*$. In this section we will always set $l = q = p^*$ in (4) so we will leave the subscript unsaid, to simplify the notation. Since (8) is C^1 and uniformly continuous in the t variable, $\mathbf{O}$ admits local unstable and stable manifolds, denoted respectively by $W^u_{\epsilon,loc}(\tau)$ and $W^s_{\epsilon,loc}(\tau)$, see [30], [14]. From Proposition 3 we know that, if $\mathbf{Q}^u \in W^u_{\epsilon,loc}(\tau)$, then $\lim_{t\to-\infty}\mathbf{x}^\tau(\mathbf{Q}^u, t) = \mathbf{O}$ and the corresponding solution $u(r)$ of (3) is a regular solution, while if $\mathbf{Q}^s \in W^s_{\epsilon,loc}(\tau)$, then $\lim_{t\to\infty}\mathbf{x}^\tau(\mathbf{Q}^s, t) = \mathbf{O}$ and the corresponding solution $v(r)$ of (3) is a solution with fast decay. Using the flow it is possible to extend the local manifolds to global manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. As usual we commit the following abuse of notation: we denote by $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ just the branches of the manifolds that depart from the origin and get into $\mathbb{R}^2_+$. From [30] we also know that the leaves are C^1 and vary continuously in the C^1 topology with respect to τ and ϵ. Observe that for $\epsilon = 0$, both in the regular and in the singular perturbation case, the manifold $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ coincide and are the image of the homoclinic trajectory. We fix a segment L which is transversal to $W^u_0(\tau) \equiv W^s_0(\tau)$ and which intersects it in a point, say $\mathbf{U}$. Using a continuity argument, we deduce that, for $\epsilon > 0$ small enough, $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ continue to cross L transversally in points $\xi^s(\tau,\epsilon)$ and $\xi^u(\tau,\epsilon)$ close to $\mathbf{U}$. We want to find intersections $\mathbf{Q}$ between $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$; then the trajectory $\mathbf{x}^\tau(\mathbf{Q}, t)$ corresponds to a regular solution $u(r)$ having fast decay. Then it is easily proved that $\mathbf{x}^\tau(\mathbf{Q}, t) \in \mathbb{R}^2_+$ for any t so it is a monotone decreasing G.S. with fast decay.

Let us rewrite (8) as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \tau + t, \epsilon)$. From now on we restrict our attention to the singularly perturbed system since the other can be treated similarly, see [30] and [14]. We define a Melnikov function which measures the distance with sign between $\xi^s(\tau,\epsilon)$ and $\xi^u(\tau,\epsilon)$ along L.

$$M(\tau) = \frac{d}{d\epsilon}\left[\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)\right]\lfloor_{\epsilon=0} \wedge \mathbf{f}(\mathbf{U},\tau)$$

where "$\wedge$" denotes the standard wedge product in $\mathbb{R}^2$. Then define

$$h(\tau,\epsilon) = \begin{cases} M(\tau) & \text{for} \quad \epsilon = 0 \\ \frac{\xi^-(\tau,\epsilon)-\xi^+(\tau,\epsilon)}{\epsilon} \wedge \mathbf{f}(\mathbf{U},\tau) & \text{for} \quad \epsilon \neq 0. \end{cases}$$

We point out that the vector $\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)$ belongs to the transversal segment L, so we have $h(\tau,\epsilon) = 0 \iff \xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon) = 0$ for $\epsilon \neq 0$.

Suppose $M(\tau_0) = 0$ and $M'(\tau_0) \neq 0$, then, using the implicit function theorem, we construct a C^1 function $\epsilon \to \tau(\epsilon)$ defined on a neighborhood of $\epsilon = 0$, such that $\tau(0) = \tau_0$, for which we have $\xi^-(\tau(\epsilon),\epsilon) = \xi^+(\tau(\epsilon),\epsilon)$. Therefore we have a homoclinic solution of the system (8).

Following [30] and [14] we find that

$$M(\tau) = -\phi'(\tau)\phi(\tau)^{-\frac{n}{p}} \int_{-\infty}^{+\infty} \frac{|x_1(t)|^\sigma}{\sigma} dt = -C\phi'(\tau)\phi(\tau)^{-\frac{n}{p}} \tag{20}$$

where $\mathbf{x}_1(t) = (x_1(t), y_1(t))$ is a homoclinic trajectory of (8) where $\phi \equiv 1$, so $C > 0$ is a computable positive constant. Note that $M(\tau)$ is closely related to the first term in the expansion in ϵ of the function $Z(t)$ defined in (19). It follows that for any positive non degenerate critical point of $k(r)$ there is a crossing between $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$, so we have a G.S. with fast decay.

Introducing a further Mel'nikov function depending on two parameters, it can be proved that such a crossing is transversal, see [30], [2], [14]. In order to use the Smale construction of the horseshoe, we need to prove that the functions $\xi^\pm(\epsilon,\tau)$ are C^2 even if the system is just C^1. This has been done in [4], using some fixed point theorems in weighted spaces, and observing that the first branch of $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ cannot cross the coordinate axes, where part of the regularity is lost.

Now we assume that ϕ is periodic and admits a non-degenerate positive critical point. Using the previous Lemma we find a point $\mathbf{Q}(\epsilon) \in W^u_\epsilon(\tau(\epsilon)) \cap W^s_\epsilon(\tau(\epsilon))$. Then, using the Smale construction, we find a Cantor set Λ close to the transversal crossing $\mathbf{Q}(\epsilon)$, such that the trajectories $\mathbf{x}^\tau(\mathbf{P},t)$, where $\mathbf{P} \in \Lambda$ are bounded, and do not converge to the origin. With some elementary analysis on the phase portrait we can also show that $\mathbf{x}^\tau(\mathbf{P},t) \in \mathbb{R}^2_+$ for any $t \in \mathbb{R}$. So we find the following, see [30], [2], [3], [4] [14] for the proof.

THEOREM 9. *Consider (3) where $q = p^*$, $2n/(n+2) \le p \le 2$, and $k \in C^2$ is a singular perturbation of a constant. Then there is a monotone decreasing G.S. with fast decay for each positive non-degenerate critical point of $k(r)$.*

Moreover assume that $k(e^t)$ is a periodic function and it admits a non degenerate positive extremum. Then there is a Cantor-like set of monotone decreasing S.G.S. with slow decay $\upsilon(r)$. Moreover if $k(r)$ is strictly positive, the S.G.S. are monotone decreasing.

When k is a regular perturbation of a constant, we proceed in the same way but we find a different Mel'nikov function:

$$\bar{M}(\tau) = \int_{-\infty}^{+\infty} \phi'(t+\tau)\frac{|x_1|^{p^*}}{p^*}dt\,, \qquad \bar{M}'(\tau) = \int_{-\infty}^{+\infty} \phi''(t+\tau)\frac{|x_1|^{p^*}}{p^*}dt$$

Then, arguing as above we find the following.

THEOREM 10. *Assume that $k(r) = 1 + \epsilon K(r)$ is a C^2 function and $\epsilon > 0$ is a sufficiently small parameter. Then equation (3) admits a G.S. with fast decay for each non degenerate zero of $M(\tau)$. Assume in addition that $K(e^t)$ is a periodic function. Then equation (3) admits a Cantor-like set of monotone decreasing S.G.S. with slow decay.*

Following [14], we point out that now it is possible to get further information on the structure of positive solutions, both regular and singular, with a careful analysis of the phase portrait. The idea is to construct a barrier set made up of branches of the manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. We illustrate it with an example, remanding to [14] for a detailed discussion. Let us assume that $k(r)$ admits 9 positive non degenerate critical points for $r > 0$, 5 maxima and 4 minima , see figure 3.

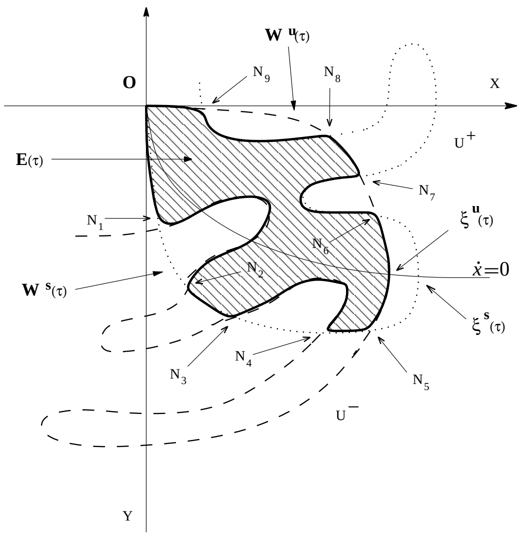

Figure 3: A sketch of the set $E(\tau)$, when $k(r) = K(r^{\epsilon})$ has 5 maxima and 4 minima. The solid line represents $B(\tau)$, and it is obtained joining segments of $W^u(\tau)$ (dotted line), and of $W^s(\tau)$ (dashed line).

First observe that if $\mathbf{Q}^{\tau} \in W^u_{\epsilon}(\tau) \cap W^s_{\epsilon}(\tau)$ then $\mathbf{x}^{\tau}(\mathbf{Q}^{\tau}, t) \in W^u_{\epsilon}(\tau + t) \cap W^s_{\epsilon}(\tau + t)$ for any t. So the number of intersection between $W^u_{\epsilon}(\tau)$ and $W^s_{\epsilon}(\tau)$ does not depend on τ. Therefore there are 9 functions $\tau_i(\epsilon)$ such that $\xi^u(\tau(\epsilon), \epsilon) = \xi^s(\tau(\epsilon), \epsilon)$ for $i = 1, \cdots, 9$ and 9 points $\mathbf{N}^i(\tau)$ of intersection between stable and unstable manifolds. We denote by $\mathbf{N}^1(\tau)$, the first point met following $W^s_{\epsilon}(\tau)$ from the origin towards $\mathbb{R}^2_+$, by $\mathbf{N}^2(\tau)$ the second, and so on. Let us denote by $B^0(\tau)$ the branch of $W^s_{\epsilon}(\tau)$ between the origin and $\mathbf{N}^1(\tau)$, by $B^1(\tau)$ the branch of $W^u_{\epsilon}(\tau)$ between $\mathbf{N}^1(\tau)$ and $\mathbf{N}^2(\tau)$, by $B^2(\tau)$ the branch of $W^s_{\epsilon}(\tau)$ between $\mathbf{N}^2(\tau)$ and $\mathbf{N}^3(\tau)$, and so on till the branch of $W^u_{\epsilon}(\tau)$ between $\mathbf{N}^9(\tau)$ and the origin which is denoted by $B^9(\tau)$. Finally we denote by $B(\tau) = \cup_{i=0}^{9} B^i(\tau)$, and by $E(\tau)$ the bounded open subset enclosed by $B(\tau)$. The key observation is that $B(\tau)$ is contained in $\mathbb{R}^2_+$ for any τ, and in $\{\mathbf{x} \mid y < 0 < x\}$ when ϕ is uniformly positive, see [14] for a detailed proof.

Observe that $E(\tau) \setminus (W^u_{\epsilon}(\tau) \cup W^s_{\epsilon}(\tau))$ contains uncountably many points and take $\mathbf{Q}$ in it. The trajectory $\mathbf{x}^{\tau}(\mathbf{Q}, t)$ is forced to stay in the interior of $E(\tau + t)$ for any t, therefore it corresponds to a S.G.S. with slow decay. With a careful analysis on the phase portrait it is possible to find points $\mathbf{Q} \in B(\tau) \setminus W^s_{\epsilon}(\tau)$ such that $\mathbf{x}^{\tau}(\mathbf{Q}, t)$ is forced to stay in the interior of $E(\tau + t)$ for any $t > 0$, and $\mathbf{P} \in B(\tau) \setminus W^s_{\epsilon}(\tau)$ such that $\mathbf{x}^{\tau}(\mathbf{P}, t)$ has to cross the y axis for some $t > 0$. Therefore they correspond respectively to G.S. with slow decay and to crossing solutions. Analogously we find $\mathbf{Q}, \mathbf{P} \in B(\tau) \setminus W^u_{\epsilon}(\tau)$ such that $\mathbf{x}^{\tau}(\mathbf{Q}, t) \in E(\tau + t)$ for any $t < 0$ and $\mathbf{x}^{\tau}(\mathbf{P}, t)$ has to cross the y axis for some $t < 0$, which correspond respectively to S.G.S. with fast decay and to solutions of the Dirichlet problem in the exterior of a ball, see [14] for more details. The results can be summed up as follows. Let us introduce the following hypotheses:

M$_1$ there exists $\rho > 0$ such that $k(\rho) > 0$ is a non degenerate maximum and $k(r)$ is uniformly positive and monotone increasing for $0 \leq r \leq \rho$.

M$_2$ there exists $R > 0$ such that $k(R) > 0$ is a non degenerate maximum and $k(r) > 0$ is uniformly positive and monotone decreasing for $r \geq R$.

O$_1$ $k(r)$ is oscillatory as $r \to 0$ and admits infinitely many positive non degenerate critical points.

O$_2$ $k(r)$ is oscillatory as $r \to \infty$ and admits infinitely many positive non degenerate critical points.

Then we have the following result:

THEOREM 11. *Consider equation (3) and assume that $k(r) = K(r^\epsilon)$ is bounded. Then, for $\epsilon > 0$ small enough, we have at least as many G.S. with fast decay as the non degenerate critical points of $k(r)$. Moreover*

1. *Assume that either* $\mathbf{M_2}$ *or* $\mathbf{O_2}$ *is satisfied. Then the there are uncountably many G.S. with slow decay and uncountably many crossing solutions.*
2. *Assume that either* $\mathbf{M_1}$ *or* $\mathbf{O_1}$ *is satisfied. Then there are uncountably many S.G.S. with fast decay and uncountably many solutions $v(r)$ of Dirichlet problem in the exterior of a ball.*
3. *Assume that both Hypotheses 1 and 2 are satisfied. Then the positive solutions of equation (3) have a structure of type* **C**.

Furthermore, if $k(r)$ is uniformly positive, then G.S. and S.G.S. are decreasing.

REMARK 6. Note that when $k(r)$ is decreasing for r small and increasing for r large, we are not able to state the existence of S.G.S. and of G.S. with slow decay. This is due to the fact that, in such a case it is not possible to construct a set $B(\tau)$ which is contained in $\mathbb{R}^2_+$ for any τ, so our argument fails. However also in this case we are able to prove the existence of G.S. with fast decay.

Following [14] we can easily obtain an analogous result for the regularly perturbed problem. The difference lies in the fact that the Melnikov condition is a bit more complicated, so we have to replace the assumption that $k(r)$ has a positive critical point by the condition that $\bar{M}(\tau) = 0$ and $\bar{M}'(\tau) \neq 0$.

Now we want to extend some of these results to the "in the large" case, so we want to see what happens when $\epsilon \to 1$. This in fact will shed some light on the reason for which positive solutions exhibit the same structure, under two completely different types of perturbation. The idea is to use our knowledge of the autonomous case to understand the non-autonomous one, replacing the Melnikov function by the energy function H. We will discuss the following Hypotheses

$\bar{\mathbf{M}}_\mathbf{1}^+$ $k(r)$ is increasing for r small and $k'(r)r^{-n/(p-1)} \notin L^1(0,1]$.

$\bar{\mathbf{M}}_\mathbf{1}^-$ $k(r)$ is decreasing for r small and $k'(r)r^{-n/(p-1)} \notin L^1(0,1]$.

$\bar{\mathbf{M}}_\mathbf{2}^+$ $k(r)$ is increasing for r large and $k'(r)r^n \notin L^1[1,\infty)$.

$\bar{\mathbf{M}}_\mathbf{2}^-$ $k(r)$ is decreasing for r large and $k'(r)r^n \notin L^1[1,\infty)$.

Now we can state the following theorem, see [18], [15].

THEOREM 12. *Consider (3) where $q = p^*$ and $k(r) \in [a,b]$ for any $r \geq 0$, for some $b > a > 0$. Assume that either hypotheses $\bar{\mathbf{M}}_\mathbf{1}^+$ and $\bar{\mathbf{M}}_\mathbf{2}^-$, or $\bar{\mathbf{M}}_\mathbf{1}^-$ and $\bar{\mathbf{M}}_\mathbf{2}^+$ are satisfied. Then there is a G.S. with fast decay. Moreover*

1. *If $\bar{M}_-^2$ is satisfied there are uncountably many G.S. with slow decay and uncountably many crossing solutions.*
2. *If $\bar{\mathbf{M}}_\mathbf{1}^+$ is satisfied, there are uncountably many S.G.S. with fast decay and uncountably many solutions of Dirichlet problem in the exterior of a ball.*
3. *If $\bar{\mathbf{M}}_\mathbf{1}^+$ and $\bar{M}_-^2$ are satisfied positive solutions have structure **C**.*

Proof. Consider the autonomous system (8) where $q = p^*$ and $\phi \equiv a$, or $\phi \equiv b$ respectively. Denote by $\mathbf{x}_a(t)$ and $\mathbf{x}_b(t)$ the trajectories of the former and the latter system such that $\dot{x}_a(0) = 0 = \dot{x}_b(0)$. Denote by $A^+ = \{\mathbf{x}_a(t)\,|t \leq 0\}$, $A^- = \{\mathbf{x}_a(t)\,|t \geq 0\}$, $B^+ = \{\mathbf{x}_b(t)\,|t \leq 0\}$, $B^- = \{\mathbf{x}_b(t)\,|t \geq 0\}$, by $\mathbf{A} = (A_x, A_y) = \mathbf{x}_a(0)$ and by $\mathbf{B} = (B_x, B_y) = \mathbf{x}_b(0)$. Let us denote by E^+ (respectively E^-) the bounded subsets enclosed by A^+, B^+ (resp. A^-, B^-) and the isocline $\dot{x} = 0$.

Note that the flow of the non autonomous system (8) on $A^+ \cup B^+$ points towards the interior of E^+ while on $A^- \cup B^-$ points towards the exterior of E^-. So, using Wazewski's principle, we can construct compact connected sets as follows, see [15].

$$W^u(\tau) :=\{\mathbf{Q} \in E^+ \mid \lim_{t\to-\infty} \mathbf{x}^\tau(\mathbf{Q},t) = \mathbf{O} \text{ and } \mathbf{x}^\tau(\mathbf{Q},t) \in E^+ \text{ for } t \leq 0\},$$
$$W^s(\tau) :=\{\mathbf{Q} \in E^- \mid \lim_{t\to+\infty} \mathbf{x}^\tau(\mathbf{Q},t) = \mathbf{O} \text{ and } \mathbf{x}^\tau(\mathbf{Q},t) \in E^- \text{ for } t \geq 0\}.$$

We denote by $\xi^u(\tau)$ and $\xi^s(\tau)$ the intersection of the isocline $\dot{x} = 0$ respectively with $W^u(\tau)$ and $W^s(\tau)$. In analogy to what we have done in the perturbative case we want to measure the distance with sign of the compact non-empty sets $\xi^u(\tau)$ and $\xi^s(\tau)$ evaluating the energy function H on these sets.

We wish to stress that we have committed a mistake in [15] in such evaluation, but we can correct it as follows, see [18]. Let us denote by L the line $x = B_x$, and by $\mathbf{C}^+$ the intersection of L with A^+; finally let L^+ be the segment of L between $\mathbf{C}^+$ and $\mathbf{B}$. Denote by $\mathbf{x}_a^\tau(t)$, the trajectory of the autonomous system where $\phi \equiv a$ such that $\mathbf{x}_a^\tau(0) = \mathbf{C}^+$, and by $\mathbf{x}_b^\tau(t)$, the trajectory of the autonomous system where $\phi \equiv b$ such that $\mathbf{x}_b^\tau(0) = \mathbf{B}$. Recall that we have explicit formulas for $x_a^\tau(t)$ and $x_b^\tau(t)$ and that we can find $C > c$ such that $\sqrt[p^*]{c}e^{\frac{n-p}{p}t} < x_b^\tau(t) < x_a^\tau(t) < \sqrt[p^*]{C}e^{\frac{n-p}{p}t}$ for

$t \le 0$. Consider a trajectory $\mathbf{x}^\tau(\mathbf{Q}^u(\tau), t)$ of the non-autonomous system (8) such that $\mathbf{x}^\tau(\mathbf{Q}^u(\tau), 0) = \mathbf{Q}^u(\tau) \in L^+$. It can be proved that

$$ce^{nt} < |x_b^\tau(t)|^{p^*} \le |x^\tau(\mathbf{Q}^u(\tau), t)|^{p^*} \le |x_a^\tau(t)|^{p^*} < Ce^{nt}$$

for any $t \le 0$, see [18]. Denote by $\bar{W}^u(\tau)$ and $\bar{W}^s(\tau)$ respectively the subset of $W^u(\tau)$ and $W^s(\tau)$ contained in $\{\mathbf{x} \,|\, 0 < x < L_x\}$. It can be shown easily that for any point $\mathbf{Q} \in \bar{W}^u(\tau)$ we have $c(\mathbf{Q})e^{nt} \le |x^\tau(\mathbf{Q}^u(\tau), t)|^{p^*} \le C(\mathbf{Q})e^{nt}$, where $C(\mathbf{Q})/C = K(\mathbf{Q}) = c(\mathbf{Q})/c > 0$.

Now assume that hypothesis $\bar{M}_2^+$ is satisfied; then there is $T_0 > 0$ such that $\dot{\phi}(t) > 0$ for any $t > T_0$. Hence for any $\mathbf{Q} \in \bar{W}^u(\tau)$ we have

$$\begin{aligned} H_{p^*}(\mathbf{Q}, \tau) &= \int_{-\infty}^0 \dot{\phi}(\tau + t)\frac{|x^\tau(\mathbf{Q}; t)|^{p^*}}{p^*} dt \ge \\ &\ge \frac{e^{-n\tau} K(\mathbf{Q})}{\sigma}\Big[C(\phi(T_0) - b)e^{nT_0} + c\int_{T_0}^\tau \dot{\phi}(\zeta)e^{n\zeta} d\zeta\Big] \end{aligned} \tag{21}$$

Since $\dot{\phi}(\zeta)e^{n\zeta} \notin \mathcal{L}^1\big[[0, \infty)\big]$, we can find $N^+ > T_0$ such that $H_{p^*}(\mathbf{Q}, \tau) > 0$ for any $\mathbf{Q} \in \bar{W}^u(\tau)$ and $\tau > N^+$.

We denote by $\Phi_{\tau,t}(\mathbf{Q})$ the diffeomorphism defined by the flow of (8), precisely $\Phi_{\tau,t}(\mathbf{Q}) = x^\tau(\mathbf{Q}; t)$. Note that for any $\mathbf{Q} \in \bar{W}^u(\tau)$, where $\tau > N^+$, and any $t \ge 0$, we have $H(\Phi_{\tau,t}(Q), t + \tau) > H(Q, \tau) > 0$ since $\dot{\phi}(s) > 0$ for $s > \tau > N^+$.

Observe that there is a unique $t = T^u(\mathbf{Q}) > 0$ such that $\mathbf{x}^\tau(\mathbf{Q}; t) \in E^+$ for any $t < T^u(\mathbf{Q})$ and $\mathbf{x}^\tau(\mathbf{Q}; T^u(\mathbf{Q})) \in \xi^u(T^u(\mathbf{Q}) + \tau)$. We choose $T_\omega^+ = \min\{T^u(\mathbf{Q}) + N^+ \,|\, \mathbf{Q} \in \bar{W}^u(N^+)\}$; it follows that $\Phi_{N^+,t}[\bar{W}^u(N^+)] \supset \tilde{W}^u(N^+ + t)$, for any $t \ge T_\omega^+ - N^+$. Hence $H(Q, \tau) > 0$ for any $Q \in \tilde{W}^u(\tau)$ for any $\tau > T_\omega^+$.

Moreover, for any $\mathbf{P} \in \tilde{W}^s(\tau)$ we have

$$H_{p^*}(\mathbf{P}, \tau) = -\int_\tau^{+\infty} \dot{\phi}(t + \tau)\frac{|x^\tau(\mathbf{P}, t)|^{p^*}}{p^*} dt < 0\,,$$

since $\dot{\phi}(t) > 0$ for $t + \tau > T_0$. Therefore $H_{p^*}(\mathbf{P}, \tau) < 0 < H_{p^*}(\mathbf{Q}, \tau)$ for any $\mathbf{P} \in \tilde{W}^s(\tau)$ and any $\mathbf{Q} \in \tilde{W}^u(\tau)$. Analogously if $\bar{M}_1^-$ is satisfied, we can find $T_\alpha^- < 0$ such that $H_{p^*}(\mathbf{Q}, \tau) < 0 < H_{p^*}(\mathbf{P}, \tau)$ for any point $\mathbf{P} \in \tilde{W}^s(\tau)$ and $\mathbf{Q} \in \tilde{W}^u(\tau)$, for any $\tau < T_\alpha^-$. It follows that there is $\tau_0 \in (T_\alpha^-, T_\omega^+)$ such that $\xi^s(\tau_0) \cap \xi^u(\tau_0) \ne \emptyset$. So if $\mathbf{Q}^0 \in \xi^s(\tau_0) \cap \xi^u(\tau_0)$ we have that the solution $u(r)$ of (3) corresponding to $\mathbf{x}^{\tau_0}(\mathbf{Q}^0, t)$ is a G.S. with fast decay.

Then repeating the argument of the perturbative case we conclude the proof of the Theorem. $\square$

This way we have proved structure results for positive solutions also in the case $p > 2$ and corrected the corresponding results in [15]. However we cannot correct the proof of the results concerning the existence of multiple G.S. with fast decay, published in [15].

Note that with this approach it is possible to prove the existence of G.S. with fast decay also when $\bar{M}_1^-$ and $\bar{M}_2^+$ are satisfied, while the approach of [43], [32] fails in that case. However the latter article is able to deal also with the case $q \neq p^*$. We wish to stress that the condition on the integrability of $k'(r)r^n$ and $k'(r)r^{-n/(p-1)}$ is in some sense optimal, in view of Theorem 8. Moreover observe that we can combine the existence results for G.S. with fast decay given in Theorem 6, 11, 12 with the structure result of Theorem 5 to obtain uniqueness. Furthermore we have the following, see [18], [15].

REMARK 7. Assume that hypotheses $\bar{\mathbf{M}}_\mathbf{1}^+$ and $\bar{\mathbf{M}}_\mathbf{2}^-$ are satisfied. Then there are $B \geq A > 0$ such that $u(d,r)$ is a crossing solution for any $d > B$ and it is a G.S. with slow decay for $0 < d < A$.

Assume that hypotheses $\bar{\mathbf{M}}_\mathbf{1}^-$ and $\bar{\mathbf{M}}_\mathbf{2}^+$ are satisfied. Then there are $B \geq A > 0$ such that $u(d,r)$ is a crossing solution for any $d > B$ and any $0 < d < A$. Moreover there are $R \geq \rho > 0$ such that the Dirichlet problem in the ball of radius r admits 2 solutions for $r > R$ and 0 solutions for $0 < r < \rho$.

Roughly speaking, if $k(r) \in C^1$ is uniformly positive and bounded, admits just one critical point which is a maximum and it is not too flat for r small and r large, regular solutions have structure 3 of Theorem 5 (and positive solutions have structure **C**). But if the critical point is a minimum, the situation is more complicated. We know from Theorem 12 a sufficient condition to have a G.S. with fast decay. However we conjecture, that, in such a case, we may have multiple G.S. with fast decay, perhaps even infinitely many.

Theorem 12 also helps to understand what happens in the perturbative case. When we have a regular perturbation, the stripes E^+ and E^- are very narrow. So, when we approximate the trajectory of the perturbed system with a trajectory of the unperturbed one, we commit a small mistake. In the singular perturbation case we have that ϕ varies slowly, so $\dot{\phi}$ has constant sign in long intervals. Since the trajectory of the stable and unstable sets have an exponential decay, the sign of the energy function H mainly depends on the sign of $\phi(\epsilon t + \tau)x^{p*}(t)$ evaluated when $\mathbf{x}(t)$ is far from the origin. Choose $\mathbf{Q}$ either in $\xi^s(\tau)$ or in $\xi^u(\tau)$. The idea hidden in Theorem 11 is that, playing with the values of the parameters τ and ϵ, we can make the sign of $H_{p*}(\mathbf{Q}, \tau)$ depend just on the sign of $\dot{\phi}$ evaluated at $t = \tau$.

5. f subcritical for u small and supercritical for u large

In this section we collect few results about an equation for which even some basic questions are still unsolved. We consider Eq. (1) where $f(u) = u|u|^{q_1-2} + u|u|^{q_2-2}$, and $p_* < q_1 < p^* < q_2$. In fact as far as we are aware there are only two articles, [11] and [1], concerning the argument and they deal with the case $p = 2$. Recall that $2^* = 2n/(n-2)$ and $2_* = 2(n-1)/(n-2)$.

Zhou in [44] established that G.S. for (2), in this case have to be radial. So we can in fact consider directly an equation of the form (3) (with $p = 2$).

Flores et al. in [11] and [1] use the classical Fowler transformation and change equation (3) into a dynamical system of the form (5). Then they face the problem using dynamical techniques such as invariant manifold theory. They set $l = q_2$ in (4) and obtain a system of the form (5) such that $g_{q_2}(x_{q_2}, t)$ is bounded as $t \to -\infty$, for any fixed x_{q_2}. In fact they consider the 3-dimensional autonomous system obtained from (5) adding the extra variable $z = e^{\xi t}$, where $\xi > 0$. As usual this system admits 3 critical points: the origin **O**, $\mathbf{P}(-\infty) = (P_x(-\infty), P_y(-\infty)$ and $-\mathbf{P}(-\infty)$, where $P_y(-\infty) < 0 < P_x(-\infty)$. In such a case regular solutions of the original problem correspond to trajectories of the $2-$dimensional unstable manifold of the origin, while the singular solutions corresponds to the trajectory whose graph is the $1-$dimensional unstable manifold of $\mathbf{P}(-\infty)$. Then they consider the system obtained from (5) with $l = q_1$, adding the extra variable $z = e^{\xi t}$, where $\xi < 0$, which again have three critical points: **O**, $\mathbf{P}(+\infty) = (P_x(+\infty), P_y(+\infty)$ and $-\mathbf{P}(+\infty)$, where $P_y(+\infty) < 0 < P_x(+\infty)$. In this case **O** admits a 2 dimensional stable manifold whose trajectories correspond to solutions with fast decay of (3), and $\mathbf{P}(+\infty)$ admits a $1-$dimensional stable manifold made up of a trajectory corresponding to a solution with slow decay. Then they use dynamical arguments in order to find intersections between these objects, and this way in [11] they prove the following very interesting results.

THEOREM 13. **a)** *Let $q_2 > 2^*$ be fixed. Then, given an integer $k \geq 1$, there is a number $s_k < 2^*$ such that if $s_k < q_1 < 2^*$, then (2) has at least k radial G.S. with fast decay.*

b) *Let $2_* < q_1 < 2^*$ be fixed. Then, given an integer $k \geq 1$, there is a number $S_k < 2^*$ such that if $2^* < q_2 < S_k$, then (2) has at least k radial G.S. with fast decay.*

They have also found a non-existence counterpart, which shows how sensitive to the variations of the exponents these existence results are.

THEOREM 14. *Let $q_2 > 2^*$ be fixed. Then there is a number $Q > 2_*$ such that if $1 < q_1 < Q$, then (2) admits no G.S neither S.G.S.*

This non-existence result is in some sense optimal. In fact Lin and Ni in [36] have constructed explicitly a G.S. with slow decay of the form $u(r) = A(B+r^2)^{-1/(p-1)}$, where A and B are suitable positive constants, in the special case $q_2 = 2(q_1 - 1) > 2^*$ (note that $2^* = 2(2_* - 1)$). However the existence of G.S. with slow decay probably is not a generic phenomenon. In fact it corresponds to the existence of 1 dimensional intersection of a $2-$dimensional object with a $1-$dimensional object in 3 dimensions.

Finally we have this result concerning S.G.S. and G.S. with slow decay.

THEOREM 15. **a)** *Given $q_2 > 2^*$, there is an increasing sequence of numbers $Q_k \to 2^*$ such that if $q_1 = Q_k$ then there is a radial S.G.S. of (2) with either slow or fast decay.*

b) *Given $2_* < q_1 < 2^*$, there is a decreasing sequence of numbers $S_k \to 2^*$ such that if $q_2 = S_k$ then (2) admits either a radial S.G.S. with slow decay or a radial G.S.*

with slow decay.

Moreover, exploiting the existence of the G.S. with slow decay in the case $q_2 = 2(q_1-1) > 2^*$, Flores was able to prove the following result in [1].

THEOREM 16. *Assume that* $2_* < q_1 < 2^* < q_2$ *and* $q_1 > 2\frac{N+2\sqrt{N-1}-2}{N+2\sqrt{N-1}-4}$. *Then, given any integer* $k \geq 1$, *there is a number* $\epsilon_k > 0$ *such that, if* $|q_2 - 2(q_1-1)| < \epsilon_k$, *then there are at least k radial G.S. with fast decay for (2). In particular if* $q_2 = 2(q_1-1)$ *there are infinitely many G.S. with fast decay.*

The condition $q_1 > 2\frac{N+2\sqrt{N-1}-2}{N+2\sqrt{N-1}-4}$ guarantees that $\mathbf{P}(+\infty)$ is a focus and this point is crucial for the proof.

We think that all the Theorems of this section could be generalized to the case $p \neq 2$ using the new change of coordinates (4). Moreover we think that these techniques could be adapted to generalize Theorems 13, 14, 15 also to the spatial dependent case, that is when $f(u,r) = k_1(r)u|u|^{q_1-2} + k_k(r)u|u|^{q_2-2}$, where k_1 and k_2 are actually functions. The last Theorem 16 crucially depends on the existence of the G.S. with slow decay, that seems to be structurally unstable. However we have been able to compute this solution also for the corresponding equation (1). So perhaps also Theorem 16 can be extended to the case $p \neq 2$.

REMARK 8. Consider (1) where $f(u,r) = u|u|^{q_1-1} + u|u|^{q_2-1}$, where $q_2 = \frac{(q_1-1)p}{p-1}$ and $p_* < q_1 < p^* < q_2$. Then there is a radial G.S. with slow decay

$$u(r) = A\left(\frac{1}{B + r^{\frac{p}{p-1}}}\right)^{\frac{p-1}{q_1-p}}$$

where $A = \left[\left|\frac{p}{q_1-p}\right|^{p-1}\left(n - \frac{p(q_1-1)}{q_1-p}\right)\right]^{\frac{1}{q_1-p}}$ and $B = \left(n - \frac{p(q_1-1)}{q_1-p}\right)A^{\frac{q_1-1}{p-1}}$.

6. f negative for u small and positive for u large

In this section we will consider (3), assuming that $f(u,r)$ is negative for u small and positive for u large and r small. The prototypical non-linearity we are interested in is the following

$$f(u,r) = -k_1(r)u|u|^{q_1-2} + k_2(r)u|u|^{q_2-2} \tag{22}$$

where the functions $k_i(r)$ are nonnegative and continuous. When $p = q_1 = 2$ (2) describes a Bose-Einstein condensate, and the G.S., if it exists, is the least energy solution. In order to have G.S. we need to have a balance between the gain of energy due to the negative terms and the loss of energy due to the positive terms. The strength of the contribution is proportional to the corresponding value of $|J_i^+(r)|$, so it depends strongly on the exponent q_i. When f is as in (22) and $q_1 < p^* \leq q_2$ the contribution given by the positive term is not strong enough, while when $q_1 \geq p^*$ the contribution of the positive term is too strong. When $q_1 < q_2 < p^*$ we expect to find a richer scenario, similar to the one depicted in Theorem 5 (3), where G.S. with slow decay are replaced by oscillatory solutions.

Also in this situation, roughly speaking, solutions $u(r)$ which are positive for r large can have two different behaviour: either they converge to 0, usually with fast decay (see Proposition 1 and Corollary 1), or they are uniformly positive, and typically they oscillate indefinitely between two values c_2, c_1 where $0 < c_1 < c_2 < \infty$.

Also in this case radial solutions are particularly important, since in many cases G.S. in the whole $\mathbb{R}^n$, S.G.S. and solutions of the Dirichlet problem in the ball for (1) have to be radial. This fact was proved when $p = 2$ and f is as in (22) and the functions $-k_1(r)$ and $k_2(r)$ are decreasing by Gidas, Ni, Nirenberg in [24], [25] using the moving plane method and the maximum principle. Afterwards this results have been extended to the case $1 < p \leq 2$ in [9], [10], and finally in [42] to the case $p > 1$, and to more general spatial independent nonlinearities $f(u)$: they simply assume that $f(0) = 0$, f is negative in a right neighborhood of $u = 0$ and it is positive for u large.

Once again the Pohozaev identity proves to be an important tool to face the problem of looking for positive solutions. In fact it was used by Ni and Serrin in [38] to construct obstructions for the existence of G.S. in the spatial independent case.

THEOREM 17. *Consider (3) where f has the following form*

$$f(u) = -\sum_{i=1}^{N} k_i u|u|^{q_i-2} + \sum_{i=N+1}^{M} k_i u|u|^{q_i-2} \qquad q_i < q_{i+1} \tag{23}$$

where $k_i > 0$ are constants for any $i = 1, \ldots, M$, $q_{N+1} \geq p^*$, $M > N \geq 1$. Then there are no crossing solutions neither G.S.

Note that when $N = 1$ and $M = 2$ (23) reduces to (22). Recall that in such a case all the G.S. of (1) and also all the solutions of the Dirichlet problem in a ball have to be radial. Therefore the non existence result holds globally for the PDE (1).

When $k_i(r)$ behave like powers at $r = 0$ or at $r = \infty$, using the concept of natural dimension explained in the appendix (see [20] and [17]), it is possible to reduce the problem to an equivalent one in which the functions are uniformly positive and bounded either for r small, or for r large, or for both.

In [13] we have discussed a problem similar to the one of Theorem 17, but in the spatial dependent framework, using again dynamical techniques combined with the Pohozaev identity. In [13] we have analyzed functions f of the form (22), but the proofs work also when f is as in (23) and satisfies:

F0 $q_N \leq p^* \leq q_{N+1}$; $k_i(r)$ is a positive, continuous function for $r > 0$, for any $i \leq M$. $J_j^+(t) \leq 0 \leq J_i^+(t)$ for any t and $j \leq N < i$ and $\sum_{i=1}^{M} |J_i^+(t)| \not\equiv 0$.

REMARK 9. Assume that **F0** holds and that there is $s \geq p^*$ such that the limits $\lim_{t\to-\infty} \phi_i(t) e^{\alpha_s(p^*-q_i)t} = A_i \geq 0$ exists and are finite for $i \leq M$, and that

$\sum_{i=N+1}^{M} A_i > 0$. Then there is at least one singular solution $v(r)$ of (3).

THEOREM 18. *Consider (3) where f satisfies **F0**. Moreover assume that there are positive constants C_i and c_j such that $k_i(r) > C_i$ and $k_j(r) < C_j$ for r large and $i \leq N$ and $j > N$. Then all the regular and singular solutions are defined and positive for any $r \geq 0$ and $\limsup_{r\to\infty} u(d,r) > 0$ for any $d > 0$.*

Assume further that $-k_i(r)$ and $k_j(r)$ are decreasing and bounded for r large and $i \leq N$ and $j > N$, then there is is a computable constant b^*, such that all the regular solutions $u(r)$ (and the singular, if they exist) are such that

$$0 < \liminf_{r\to\infty} u(r) \leq \liminf_{r\to\infty} u(r) < b^*$$

REMARK 10. The Hypotheses of Theorem 18 are satisfied for example if we take f as in (22), $q_1 \leq p^* \leq q_2$, $q_1 < q_2$, and the functions $k_i(r)$ uniformly positive, bounded and $-k_1(r)$ and $k_2(r)$ are decreasing.

It is possible to give some ad hoc condition for the existence of G.S. even in the case $q_1 \leq p^* \leq q_2$ and $q_2 > q_1$. In fact we have to lower the contribution given by the negative term $-k_1(r)u|u|^{q_1-2}$, taking a strongly decreasing function $k_1(r)$, see [13]. More precisely

THEOREM 19. *Assume **F0**, and that the limits $\lim_{t\to\infty}\phi_i(t)e^{\alpha_{p^*}(p^*-q_i)t} = B_i \geq 0$ exist and are finite for $i \leq M$, and that $\sum_{i=N+1}^{M} B_i > 0$.*
Then all the regular solutions $u(r)$ are G.S. with slow decay. Finally, if there is a singular solution it is a S.G.S. with slow decay.

REMARK 11. The Hypotheses of Theorem 19 and Remark 9 are satisfied for example if we take f as in (22), $q_1 = p^* < q_2$, $k_1(r)$ uniformly positive, bounded and increasing, $k_2(r) = a + br^{\alpha_{p^*}(q_2-p^*)}$ where $a, b > 0$; or if we take $q_1 < p^* = q_2$, $k_2(r)$ uniformly positive, bounded and increasing, and $k_1(r) = a/(1 + br^{\alpha_{p^*}(p^*-q_1)})$, where $a, b > 0$.

As we said at the beginning of the section, the situation becomes more interesting when f is subcritical both as $u \to 0$ and as $u \to \infty$. A first important step to understand equation (2) in this setting was made in [25], where the authors proved the existence of a G.S. in the case $p = 2$ and assuming that $f(u,r)$ is as in (22), $q_1 = 2 < q_2 < 2^*$ and $-k_1(r)$ and $k_2(r)$ non-increasing. These results have been extended to more general operators, including the p-Laplacian for $p > 1$, in [19] and to a wider class of nonlinearities f. They just require that there is $A > 0$ such that $F(u) < 0$ for $0 < u < A$, $F(A) = 0$ and $f(A) > 0$, where $F(u) := \int_0^u f(s)ds$.

In [19] the non-linearity f is assumed to be spatially independent and sub-halflinear, namely either there is $b > A$ such that $f(b) = 0$, or $\liminf_{u\to\infty} \frac{F(u)}{u^p} < \infty$. If we consider the prototypical case (22) the assumptions of [19] reduce to $1 < q_1 < q_2 < p$, $k_1 \equiv 1 \equiv k_2$. They also proved the uniqueness of the G.S., and they have given good estimates of the asymptotic behaviour. The question of uniqueness has been discussed in many papers, see e.g. [19], [20], [8], but it is beyond the purpose of this survey. Roughly speaking radial G.S. for the spatial independent equation are unique. This is usually proved with an argument involving the moving plane method or the maximum principle. We think that G.S. are unique also when $-k_1(r)$ and $k_2(r)$ are decreasing, but we believe that a clever choice of the functions $k_i(r)$ could produce multiple G.S. However the question is still open as far as we are aware.

Gazzola, Serrin and Tang in [22] managed to extend the existence results to a wider class of spatial independent non-linearities. In particular, when f is as in (22), they proved that there is a G.S. when, either $n \le p$ and $q_1 > 0$, or $n > p$ and $0 < q_1 < q_2 < p^*$, moreover the G.S. is always positive if and only if $q_1 > p$. They have also found out that if $n = p$, there are functions f with exponential growth in the u variable for which (3) admits G.S.

However, also in [22], f does not depend explicitly on r. In [16] we have extended the existence result to the spatial dependent case, under suitable Hypotheses on the functions $k_i(r)$, and assuming $1 < p \le 2$, $q_1 \ge 2$ and $p_* < q_1 < q_2 < p^*$. But the main contribution of that paper was the proof of the existence of uncountably many S.G.S., which as far as we are aware had not been detected previously even in the original problem with $p = 2$ and $k_1 \equiv 1 \equiv k_2$. Then in [17] we have been able to discuss also the case $p > 2$, and to prove the existence of G.S. and S.G.S. for different type of non-linearity f, satisfying some of the following hypotheses:

F1 $\left\{\begin{array}{l} \bullet \text{ The function } f(u,r) \text{ is continuous in } \mathbb{R}^2 \text{ and locally Lipschitz in} \\ \text{the } u \text{ variable for any } u, r > 0;\ f(0,r) = 0 \text{ for any } r \ge 0. \\ \bullet \text{ There are } \nu > 0 \text{ and } p < q < p^* \text{ such that, for any } 0 \le r \le \nu \\ \lim_{u\to\infty} \frac{f(u,r)}{|u|^{q-1}} = a_0(r) > 0 \text{ and } a_0(r) \text{ is continuous.} \end{array}\right.$

F2 There are positive constants $A \ge a > 0$ and $\rho > 0$ such that

$$\begin{array}{ll} f(u,r) < 0 & \text{for } r > \rho \text{ and } 0 < u < a \\ F(A,0) = 0 \quad \text{and} \quad f(u,0) > 0 & \text{for} \quad u \ge A. \end{array}$$

F3 $f(u,0) \ge f(u,r)$ for any $0 < u \le A$ and any $r \ge 0$.

F4 The exponent q in Hyp. **F1** is such that $q > p_*$.

THEOREM 20. *Assume that Hyp.* **F1**, **F2**, **F3** *are satisfied. Then there exists $D > A$ such that $u(D,r)$ is a monotone decreasing G.S.*

REMARK 12. Note that if f is as in (22) and $q_1 \ge p$, G.S. and S.G.S. are positive for any $r > 0$, while if $q_1 < p$ their support is bounded, see Proposition 1. This means that there is $R > 0$ such that $u(r) > 0$ for $0 < r < R$, $u(R) = u'(R) = 0$ and $u(r) \equiv 0$ for $r > R$. Also note that if $\lim_{r\to\infty} u(r) = 0$, u has fast decay, in view of Corollary 1.

Using a standard continuity argument we can also prove the following.

COROLLARY 5. *Assume that Hyp.* **F1**, **F2**, **F3** *are satisfied. Then $u(d,r)$ is a crossing solution for any $d > D$ and its first zero $R_1(d)$ is such that* $\lim_{d\to\infty} R_1(d) =$

0. *Furthermore assume that we are in the Hypotheses of Proposition 1 B, then we also have that* $\lim_{d\to D} R_1(d) = \infty$. *Therefore the Dirichlet problem in the ball of radius* $R > 0$ *for equation (3) admits at least one solution for any* $R > 0$.

THEOREM 21. *Assume that Hyp.* **F1**, **F2** *and* **F4** *are satisfied, then (3) admits uncountably many S.G.S.*

REMARK 13. Assume that f is as in (23), that the functions $k_i(r)$ are uniformly positive and bounded for any $r \geq 0$ and $p < q_M < p^*$. Then hypotheses **F1** and **F2** are satisfied. Moreover if $-k_i(r)$ and $k_j(r)$ are decreasing for any $r > 0$, $1 \leq i \leq N$ and $N < j \leq M$, hypothesis **F3** is satisfied; finally if $q_M > p_*$ **F4** holds.

The proof of Theorem 20 can be found in [17] and follows, with some minor changes, the scheme introduced in [25] and then used in [19], [22] and [16].

When Hyp. **F1** and **F2** are satisfied the initial value problem (3), with $u(0) = d > 0$, $u'(0) = 0$ admits at least a solution. Moreover such a solution, denoted by $u(d,r)$ is unique for any $d \geq A$ and $u'(r) \leq 0$ for r small. All these solutions can be continued in $J(d) = (0, R_d) = \{r > 0 \,|\, u'(r) < 0 < u(r)\}$, where R_d can also be infinite. This was proved for the spatial independent problem in [19] and then adapted with some trivial changes to the spatial dependent problem in [17]. Since $u(d,r)$ is positive and decreasing for $r < R_d$, the limit $\lim_{r\to R_d} u(d,r)$ exists and is nonnegative, so we can define the following set:

$$I := \{d \geq A \mid \lim_{r\to R_d} u'(d,r) < 0\}$$

Using an energy analysis we can prove that $A \notin I$, when **F1**, **F2** and **F3** are satisfied. Moreover using a continuity argument on the auxiliary system (5), we prove that I is open in $[A, \infty)$ whenever **F1** and **F2** hold, see [17]. The difficult part of the proof is to show that $I \neq \emptyset$. In fact we show that, if **F1** and **F2** are satisfied there is $D \notin I$ such that $(D, \infty) \subset I$.

For this purpose we introduce a dynamical system of the form (5), using (4) with $l = q$, where q is the parameter defined in [12]. Then we show that for r small and u large we can approximate our system with an autonomous subcritical system of type (8). Through a careful analysis of the phase portrait we are able to construct a barrier set $E^\tau \subset \{(x,y) \in \mathbb{R}^2 \,|\, y \leq 0 \leq x\}$. Then, using Wazewski's principle, we show that there are $M > 0$ and $\delta > 0$ such that, for any $\tau < -M$, there is an unstable set $\tilde{W}^u(\tau) \subset E^\tau$, which intersects the y negative semi-axis in a compact connected set, say $\zeta(\tau)$. It follows that the trajectories $\mathbf{x}_q^\tau(\mathbf{Q}^u(\tau), t) \in E^\tau$ for any $t < 0$ and that $\lim_{t\to-\infty} \mathbf{x}_q^\tau(\mathbf{Q}^u(\tau), t) = \mathbf{O}$. So they correspond to regular solutions $u(d(\tau), r)$, that are positive and decreasing for $r \leq \exp(\tau)$ and they become null with nonzero slope at $r = \exp(\tau)$, so they are crossing solutions.

It follows that if **F1**, **F2** and **F3** hold there is $D > A$, such that $(D, \infty) \in I$, but $D \notin I$. Then we show that $u(D,r)$ is a monotone decreasing G.S. using a continuity argument on (5), and Theorem 20 and Corollary 5 follow.

To prove the existence of S.G.S. we have to consider system (7) and to construct a stable set $\tilde{W}^s(\tau)$ through Proposition 2. We choose $\tau < -M$, so that $\tilde{W}^u(\tau)$ crosses the y negative semi-axis, in view of Corollary 5. We denote by $B(\tau)$ the bounded set enclosed by $\tilde{W}^u(\tau)$ and the y negative semi-axis. We choose one of the uncountably many points in $\tilde{W}^s(\tau) \cap B(\tau)$, say $\mathbf{Q}^s$, and we follow backwards in t the trajectories $\mathbf{x}_q^\tau(\mathbf{Q}^s, t)$ where $\mathbf{Q}^s \in \tilde{W}^s(\tau)$. We show that $\mathbf{x}_q^\tau(\mathbf{Q}^s, t)$ is forced to stay in $B(t+\tau)$ for any $t < 0$ and we conclude that $x_q^\tau(\mathbf{Q}^s, t)$ is uniformly positive as $t \to -\infty$. Then, from Proposition 3 we deduce that the corresponding solutions $v(r)$ of (3) are monotone decreasing S.G.S.

7. Appendix: reduction of div$(g(|\mathbf{x}|)\nabla u|\nabla u|^{p-2}) + f(u, |\mathbf{x}|) = 0$ and natural dimension

In this subsection we want to show how we can pass from the analysis of radial solutions of an equation of the following class

$$\operatorname{div}(g(|\mathbf{x}|)\nabla u|\nabla u|^{p-2}) + \bar{f}(u, |\mathbf{x}|) = 0 \tag{24}$$

to the analysis of solutions of an equation of the form (3). Here again $\mathbf{x} \in \mathbb{R}^n$ and $g(|\mathbf{x}|) \geq 0$ for $|\mathbf{x}| \geq 0$.

We repeat the argument developed in Appendix B of [17]. In fact we exploit here an idea already used in [33] and [20], and we follow quite closely the latter paper, in which the concept of natural dimension is introduced. First of all observe that a radial solutions $u(r)$ of (24) satisfy the following ODE:

$$(r^{n-1}g(r)u'|u'|^{p-2})' + r^{n-1}\bar{f}(u, r) = 0. \tag{25}$$

Set $a(r) = r^{n-1}g(r)$ and assume that one of the Hypotheses below is satisfied

H1 $a^{-1/(p-1)} \in L^1[1, \infty] \setminus L^1[0, 1]$

H2 $a^{-1/(p-1)} \in L^1[0, 1] \setminus L^1[1, \infty)$

Then we make the following change of variables borrowed from [20]. Let $N > p$ be a constant and assume that Hyp. **H1** is satisfied; we define $s(r) = \left(\int_r^\infty a(\tau)^{-1/(p-1)} d\tau\right)^{\frac{-p+1}{N-p}}$. Obviously $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $s(0) = 0$, $s(\infty) = \infty$ and $s(r)$ is a diffeomorphism of $\mathbb{R}_0^+$ into itself with inverse $r = r(s)$ for $s \geq 0$. If $u(r)$ is a solution of (25), $v(s) = u(r(s))$ is a solution of the following transformed equation

$$(s^{N-1}v_s|v_s|^{p-2})_s + s^{N-1}h(s)f(v, s) = 0, \tag{26}$$

where $f(v, s) = \bar{f}(v, r(s))$ and

$$h(s) = \left(\frac{N-p}{p-1}\right)^p \left(\frac{g(r(s))^{1/p} r(s)^{n-1}}{s^{N-1}}\right)^{p/(p-1)}.$$

If we replace Hyp. **H1** by Hyp. **H2** we can define $s(r)$ as follows $s(r) = \left(\int_0^r a(\tau)^{-1/(p-1)} d\tau\right)^{\frac{p-1}{N-p}}$ and obtain again (26) from (25), with the same expression for

h. We denote by $f(v,s) = h(s)\bar{f}(v,r(s))$ and obtain (3) from (26), with r replaced by s.

REMARK 14. Note that, if for any fixed $v > 0$, $\bar{f}(v,r)$ grows like either a positive or a negative power in r for r small, we can play with the parameter N in order to have that, for any fixed $u > 0$, $f(u,0)$ is positive and bounded. E.g., if $g(r) \equiv 1$ and $\bar{f}(u,r) = r^l u|u|^{q-1}$, we can set $N = \frac{p(n+l)-n}{p+l-1}$, so that, switching from r to s as independent variable (26) takes the form

$$[s^{N-1}v_s|v_s|^{p-2}]_s + Cs^{N-1}v|v|^{q-1} = 0\,, \tag{27}$$

where $C = \left|\frac{N-p}{p-1}\right|^p \left|\frac{p-1}{N-1}\right|^{\frac{n-1}{N-p}p} > 0$. So we can directly study the spatial independent equation (27), recalling that the natural dimension is N and this changes the values of the critical exponents and the asymptotic behaviors of positive solutions as $r \to 0$ and as $r \to \infty$.

Observe that N does not need to be an integer and that in literature such an assumption is not really used to prove the results. Thus all the theorems obtained for (3) can be trivially extended to an equation of the form (25), where g satisfies either **H1** or **H2**.

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SHADOWING IN ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Shadowing deals with the existence of true orbits of dynamical systems near approximate orbits with sufficiently small local errors. Although it has roots in abstract dynamical systems, recent developments have made shadowing into a new effective tool for rigorous computer-assisted analysis of specific dynamical systems, especially chaotic ones. For instance, using shadowing it is possible to prove the existence of various unstable periodic orbits, transversal heteroclinic or homoclinic orbits of arguably the most prominent chaotic system—the Lorenz Equations. In this paper we review the current state of the theory and applications of shadowing for ordinary differential equations, with particular emphasis on our own work.

1. Introduction

In this extended introductory section we give a narrative overview of shadowing. Precise mathematical statements of relevant definitions and theorems are presented in the following sections.

Shadowing? An approximate orbit of a dynamical system with small local errors is called a *pseudo orbit*. The subject of shadowing concerns itself with the existence of true orbits near pseudo orbits; in particular, the initial data of the true orbit are near the initial data of the pseudo orbit. Shadowing is a property of hyperbolic sets of dynamical systems. It is akin to a classical result from the theory of ordinary differential equations [23]: if a non-autonomous system has a bounded solution, the variational equation of which admits an exponential dichotomy, then the perturbed system has a bounded solution nearby.

Origins? In its contemporary setting, the first significant result in shadowing, the celebrated *Shadowing Lemma*, was proved by Bowen [7] for the nonwandering sets of Axiom A diffeomorphisms. A similar result for Anosov diffeomorphisms was stated by Anosov [2], which was made more explicit by Sinai [57]. The Shadowing Lemma proved to be a useful tool in the abstract theory of uniformly hyperbolic sets of diffeomorphisms. For example, in [8] Markov partitions for basic sets of Axiom A diffeomorphisms were constructed using the Shadowing Lemma. Also shadowing can be used to give a simple proof of Smale's theorem [59] that the shift can be embedded in the neighbourhood of a transversal homoclinic point as in [47] and [39].

Discrete systems? Since Anosov and Bowen, the Shadowing Lemma has been reproved many times with a multitude of variants. A noteworthy recent development, initiated by Hammel et al. [28], [29], has been a shadowing theory for finite pseudo orbits of non-uniformly hyperbolic sets of diffeomorphisms which proved to be a powerful new paradigm for extracting rigorous results from numerical simulations of discrete chaotic systems. This development also provided new tools for establishing the existence of, for example, transversal homoclinic orbits in specific systems [21], [37]. For these recent shadowing results for discrete dynamical systems, we recommend our review article [18] followed by [21] and the references therein.

ODEs? Developing a useful notion of a pseudo orbit and establishing an appropriate Shadowing Lemma for ordinary differential equations proved to be more difficult because of the lack of hyperbolicity in the direction of the vector field. The first successful attempt in this pursuit and an accompanying Shadowing Lemma was given by Franke and Selgrade [26]. Here we will present our formulation of pseudo orbits and shadowing for ordinary differential equations as initiated in [14], [16]. Our formulation has the advantage that pseudo orbits are taken to be sequences of points and thus can be generated numerically. This permits one to garner rigorous mathematical results with the assistance of numerical simulations. Such computer-assisted shadowing techniques make an attractive complement to classical numerical analysis, especially in the investigation of specific chaotic systems.

Chaotic numerics? The key signature of chaotic systems is the sensitivity of their solutions to initial data. This poses a major challenge in numerical analysis of chaotic systems because such systems tend to amplify, often exponentially, small algorithmic or floating point errors. Here is a gloomy account of this difficulty as given by Hairer et al. [30]:

> "The solution (of the Salzman-Lorenz equations with constants and initial values $\sigma = 10$, $r = 28$, $b = 8/3$; $x(0) = -8$, $y(0) = 8$, $z(0) = 27$) is, for large values of t, *extremely* sensitive to the errors of the first integration steps. For example, at $t = 50$ the solution becomes totally wrong, even if the computations are performed in quadruple precision with $Tol = 10^{-20}$. Hence the numerical results of all methods would be equally useless and no comparison makes any sense. Therefore, we choose $t_{end} = 16$ and check the numerical solution at this point. Even here, all computations with $Tol > 10^{-7}$, say, fall into a chaotic cloud of meaningless results."

Shadowing reveals a striking silver lining of this "chaotic cloud." While it is true that this chaotic cloud has little to do with the solution having the specified initial data, it is not meaningless: the chaotic cloud is an exceedingly good approximation of another solution whose initial data is very close to the specified initial data. More generally, using the finite-time shadowing theorem in [17], it is possible to shadow numerically generated pseudo orbits of (non-uniformly hyperbolic) chaotic ordinary differential equations for long time intervals.

Chaos? There are many ways chaos can arise in a dynamical system. A common cause, as first observed by Poincaré [51] over a century ago while studying the restricted three-body problem, is the presence of transversal homoclinic points. He called such points "doubly asymptotic" because they are asymptotic both in forward and backward times to a fixed point or a periodic orbit. Birkhoff [4] proved that every homoclinic point of a two-dimensional diffeomorphism is accumulated by periodic orbits. Smale [59] confirmed Poincaré's observation by proving that a transversal homoclinic point of a diffeomorphism in dimension two and higher is contained in a hyperbolic set in which the periodic orbits are infinitely many and dense. Sil'nikov [55] showed that a similar result holds for flows. Recently, it has been conjectured by Palis and Takens [45] that generically chaotic orbits occur if and only if there is a transversal homoclinic orbit. This is indeed the case for continuous interval maps as shown in [6].

In spite of the remarkable mathematical results above, transversal homoclinic orbits are quite difficult to exhibit in specific chaotic flows. Even the periodic orbits, to which the homoclinic orbits are to be doubly asymptotic, are hard to come by. Recently in [22], we have formulated a practical notion of a pseudo homoclinic, more generally pseudo connecting, orbit and proved a shadowing theorem that guarantees the existence of transversal homoclinic, or heteroclinic, orbits to periodic orbits of differential equations. The hypotheses of this theorem can be verified for specific flows with the aid of a computer, thus enabling us to prove the existence of a multitude of periodic orbits and transversal orbits connecting them in, for example, yet again, the chaotic Lorenz Equations.

Contents? Here is a section-by-section description of the contents of the remainder of this paper:

- In Section 2, we first give definitions of an infinite pseudo orbit and its shadowing by a true orbit. Then we present two infinite-time shadowing results, one for pseudo orbits lying in hyperbolic invariant sets, and another for a single pseudo orbit in terms of a certain operator.
- In Section 3, shadowing definitions for finite pseudo orbits and a Finite-time Shadowing Theorem for non-uniformly hyperbolic systems are formulated. This theorem is significant in proving the existence of true orbits near numerically computed ones for long time intervals.
- In Section 4, the shadowing of pseudo periodic orbits is considered. The Periodic Shadowing Theorem stated here is very effective in establishing the existence of periodic orbits, including unstable ones in dimensions three and higher.
- In Section 5, the notion of a pseudo connecting orbit connecting two pseudo periodic orbits is formulated. Then a Connection Orbit Shadowing Theorem that guarantees the existence and transversality of a true connecting orbit between true periodic orbits is stated. In the particular case when the two periodic orbits coincide we have a Homoclinic Shadowing Theorem, with the aid of which existence of chaos, in the sense of Poincaré, can be rigorously established in specific systems.

- In Section 6, some of the key computational issues such as construction of good pseudo orbits, rigorous bounds on quantities that appear in the hypotheses of the shadowing theorems, and floating point computations are addressed.
- In Section 7, we present several examples to demonstrate the effectiveness of the shadowing results above as a new computer-assisted technique for establishing rigorously finite, periodic, and transversal homoclinic orbits in the quintessential chaotic system—the Lorenz Equations.
- In Section 8, we conclude our review with some parting thoughts.

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2. Infinite-time shadowing

In this section we first introduce notions of an infinite pseudo orbit and its shadowing by a true orbit of a system of autonomous differential equations. Then we state two general results that guarantee the shadowing of infinite pseudo orbits. The first result, Theorem 1, in the spirit of the classical Shadowing Lemma, is a shadowing theorem for pseudo orbits lying in a compact hyperbolic set. The second result, Lemma 1, replaces the hyperbolicity assumption with the invertibility of a certain linear operator. The second result is more general; in fact, the classical theorem follows from it. Moreover, the second result has practical applicability in numerical simulations as we shall demonstrate in later sections. Whilst the hyperbolicity assumption on a compact invariant set is not possible to verify in any realistic example (strange attractors are not usually uniformly hyperbolic), the invertibility of the operator associated with a particular pseudo orbit can frequently be established.

Consider a continuous dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1}$$

where $f : U \to \mathbb{R}^n$ is a C^2 vector field defined in an open convex subset U of $\mathbb{R}^n$. Let ϕ^t be the associated flow. Throughout this paper we use the Euclidean norm for vectors and the corresponding operator norm for matrices and linear operators, and in product spaces we use the maximum norm.

DEFINITION 1. **Definition of infinite pseudo orbit**. *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ in U, with $f(\mathbf{y}_k) \neq \mathbf{0}$ for all k, is said to be* *a δ* pseudo orbit *of Eq. (1) if there is an associated bounded sequence $\{h_k\}_{k=-\infty}^{+\infty}$ of positive times with positive* $\inf_{k\in\mathbb{Z}} h_k$ *such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \le \delta \quad for\ k \in \mathbb{Z}.$$

Next, we introduce the notion of shadowing an infinite pseudo orbit by a true orbit.

DEFINITION 2. **Definition of infinite-time shadowing**. *For a given positive number ε, a δ pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ is said to be ε*-shadowed *by a true orbit of Eq. (1) if there are points $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ on the true orbit and positive times $\{t_k\}_{k=-\infty}^{+\infty}$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \le \varepsilon \quad and \quad |t_k - h_k| \le \varepsilon \quad for\ k \in \mathbb{Z}.$$

In our first Shadowing Theorem we will assume that pseudo orbits lie in a compact hyperbolic set. For completeness, we recall the definition of a hyperbolic set as given in, for example, [47].

DEFINITION 3. **Definition of hyperbolic set**. *A set $S \subset U$ is said to be* hyperbolic *for Eq. (1) if*

(i) $f(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in S;

(ii) S is invariant under the flow, that is, $\phi^t(S) = S$ for all t;

(ii) there is a continuous splitting

$$\mathbb{R}^n = E^0(\mathbf{x}) \oplus E^s(\mathbf{x}) \oplus E^u(\mathbf{x}) \quad for\ \mathbf{x} \in S$$

such that $E^0(\mathbf{x})$ is the one-dimensional subspace spanned by $\{f(\mathbf{x})\}$, and the subspaces $E^s(\mathbf{x})$ and $E^u(\mathbf{x})$ have constant dimensions; moreover, these subspaces have the invariance property

$$D\phi^t(\mathbf{x})(E^s(\mathbf{x})) = E^s(\phi^t(\mathbf{x})), \quad D\phi^t(\mathbf{x})(E^u(\mathbf{x})) = E^u(\phi^t(\mathbf{x}))$$

under the linearized flow and the inequalities

$$\begin{aligned} \|D\phi^t(\mathbf{x})\,\xi\| &\le K_1 e^{-\alpha_1 t}\|\xi\| \quad for\ t \ge 0,\ \xi \in E^s(\mathbf{x}), \\ \|D\phi^t(\mathbf{x})\,\xi\| &\le K_2 e^{\alpha_2 t}\|\xi\| \quad for\ t \le 0,\ \xi \in E^u(\mathbf{x}) \end{aligned}$$

are satisfied for some positive constants K_1, K_2, α_1, and α_2.

Now, we can state our first shadowing theorem for infinite pseudo orbits of ordinary differential equations.

THEOREM 1. **Infinite-time Shadowing Theorem**. *Let S be a compact hyperbolic set for Eq. (1). For a given sufficiently small $\varepsilon > 0$, there is a $\delta > 0$ such that any*

δ pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) lying in S is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$. Moreover, there is only one such orbit satisfying

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad for\ k \in \mathbb{Z}.$$

In preparation for our second infinite shadowing result, we next introduce various mathematical entities. Take a fixed pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$. Let Y_k be the subspace of $\mathbb{R}^n$ consisting of the vectors orthogonal to $f(\mathbf{y}_k)$. Then let Y be the Banach space of bounded sequences $\mathbf{v} = \{\mathbf{v}_k\}_{k\in\mathbb{Z}}$ with $\mathbf{v}_k \in Y_k$, and equip Y with the norm

$$\|\mathbf{v}\| = \sup_{k\in\mathbb{Z}} \|\mathbf{v}_k\|.$$

Also, let $\tilde{Y}$ be a similar Banach space except that $\mathbf{v}_k \in Y_{k+1}$. Then let

$$L_{\mathbf{y}} : Y \to \tilde{Y}$$

be the linear operator defined by

$$(L_{\mathbf{y}}\mathbf{v})_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k,$$

where $P_k : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection defined by

$$P_k\mathbf{v} = \mathbf{v} - \frac{f(\mathbf{y}_k)^*\mathbf{v}}{\|f(\mathbf{y}_k)\|^2} f(\mathbf{y}_k).$$

So $L_{\mathbf{y}}$ is a linear operator associated with the derivative of the flow along the pseudo orbit, but restricted to the subspaces orthogonal to the vector field. This operator plays a key role in what follows. We assume that the operator is invertible with a bounded inverse:

$$\|L_{\mathbf{y}}^{-1}\| \le K.$$

Next we define various constants. We begin with

$$M_0 = \sup_{\mathbf{x}\in U} \|f(\mathbf{x})\|, \qquad M_1 = \sup_{\mathbf{x}\in U} \|Df(\mathbf{x})\|, \qquad M_2 = \sup_{\mathbf{x}\in U} \|D^2 f(\mathbf{x})\|,$$

and

$$h_{\min} = \inf_{k\in\mathbb{Z}} h_k, \quad h_{\max} = \sup_{k\in\mathbb{Z}} h_k.$$

Next, we choose a positive number $\varepsilon_0 \le h_{\min}$ such that for all k and $\|\mathbf{x} - \mathbf{y}_k\| \le \varepsilon_0$ the solution $\varphi^t(\mathbf{x})$ is defined and remains in U for $0 \le t \le h_k + \varepsilon_0$. Continuing, we define

$$\Delta = \inf_{k\in\mathbb{Z}} \|f(\mathbf{y}_k)\|, \quad \bar{M}_0 = \sup_{k\in\mathbb{Z}} \|f(\mathbf{y}_k)\|, \quad \bar{M}_1 = \sup_{k\in\mathbb{Z}} \|Df(\mathbf{y}_k)\|.$$

Now, we define the following constants in terms of the ones already given:

$$C = \max\left\{K,\ \Delta^{-1}(e^{M_1 h_{\max}} K + 1)\right\},$$
$$\overline{C} = (1 - M_1\delta C)^{-1} C,$$

and

$$
\begin{aligned}
N_1 &= 8\left[\overline{M}_0\overline{M}_1 + 2\overline{M}_1 e^{M_1(h_{\max}+\varepsilon_0)} + M_2(h_{\max}+\varepsilon_0)e^{2M_1(h_{\max}+\varepsilon_0)}\right],\\
N_2 &= 8\left[1+4C\left(M_0+e^{M_1(h_{\max}+\varepsilon_0)}\right)\right]\left[M_1\overline{M}_1 + M_2\overline{M}_0 + 2M_2 e^{M_1(h_{\max}+\varepsilon_0)}\right],\\
N_3 &= 8\left[1+4C\left(M_0+e^{M_1(h_{\max}+\varepsilon_0)}\right)\right]^2 M_1M_2.
\end{aligned}
$$

Now, we can state our second shadowing result for an infinite pseudo orbit in terms of the associated operator $L_{\mathbf{y}}$ and the constants introduced above.

LEMMA 1. **Infinite-time Shadowing Lemma**. *Let $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ be a bounded δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ such that $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \le K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1C\delta \le 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) with associated times $\{t_k\}_{k=-\infty}^{+\infty}$ and with

$$\varepsilon \le 2\bar{C}\delta.$$

Moreover this is the unique such orbit satisfying

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad for\ k \in \mathbb{Z}.$$

Notes on infinite-time shadowing: The details of the proof of the Infinite-time Shadowing Lemma are given in [22]. The idea of the proof is to set up the problem of finding a true orbit near the pseudo orbit as the solution of a nonlinear equation in a Banach space of sequences. The invertibility of the linear operator $L_{\mathbf{y}}$ implies the invertibility of another linear operator associated with the abstract problem and this enables one to apply a Newton-Kantorovich type theorem [33] to obtain the existence and uniqueness of the true orbit.

One uses the Infinite-time Shadowing Lemma to prove the Infinite-time Shadowing Theorem. The main problem is to show that hyperbolicity implies that the operator $L_{\mathbf{y}} : Y \to \tilde{Y}$ is invertible with a uniform bound on its inverse. A slightly different proof of the Infinite-time Shadowing Theorem is in an earlier publication [16].

For flows, unlike diffeomorphisms, there are various alternatives for the definition of a pseudo orbit. Should it be a sequence of points or solution segments that are functions of time? Here we have elected to use sequences of points. These choices of definitions have obvious advantages when we consider finite pseudo orbits that come from numerical computations. With such considerations in mind, the definitions of a pseudo orbit and shadowing for ordinary differential equations as given here first appeared in [14] and [16]. The problem in proving the shadowing theorem for flows is the lack of hyperbolicity in the direction of the vector field. To compensate for this, we allow a rescaling of time in our definition of shadowing.

A shadowing theorem for flows, with a somewhat different notion of shadowing, was first proved in [26]. Different versions of the shadowing theorem have been proved in [42], [34], [35], [16], and [49]. For a "continuous" shadowing theorem, see [47] and also [49]. In the latter book the author also proves a shadowing theorem for structurally stable systems, which includes that for hyperbolic systems.

3. Finite-time shadowing

Since chaotic systems exhibit sensitive dependence on initial conditions, a numerically generated orbit will diverge quickly from the true orbit with the same initial condition. However, we observe that a computed orbit is a pseudo orbit and so, according to our Infinite-time Shadowing Lemma, would be shadowed by a true orbit in the presence of hyperbolicity, albeit with slightly different initial condition. It turns out chaotic systems are seldom uniformly hyperbolic but still exhibit enough hyperbolicity that pseudo orbits can still be shadowed for long times. To this end, we formulate a finite-time shadowing theorem in this section. As in the Infinite-time Shadowing Lemma, our condition involves a linear operator associated with the pseudo orbit but now the aim is to choose a right inverse with small norm. First we recall a precise notion of shadowing of a finite pseudo orbit by an associated nearby true orbit. Then we present the Finite-time Shadowing Theorem.

DEFINITION 4. **Definition of finite pseudo orbit.** *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=0}^{N}$ is said to be a δ* pseudo orbit *of Eq. (1) if $f(\mathbf{y}_k) \neq \mathbf{0}$ and there is an associated sequence $\{h_k\}_{k=0}^{N-1}$ of positive times such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \leq \delta \quad for\ k = 0, \dots, N-1.$$

DEFINITION 5. **Definition of finite-time shadowing.** *For a given positive number ε, an orbit of Eq. (1) is said to ε-*shadow *a δ pseudo orbit $\{\mathbf{y}_k\}_{k=0}^{N}$ with associated times $\{h_k\}_{k=0}^{N-1}$ if there are points $\{\mathbf{x}_k\}_{k=0}^{N}$ on the true orbit and times $\{t_k\}_{k=0}^{N-1}$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad for\ k = 0, \dots, N \quad and \quad |t_k - h_k| \leq \varepsilon \quad for\ k = 0, \dots, N-1.$$

To state our theorem we need to develop a bit of notation and introduce certain relevant mathematical constructs. Let $\{\mathbf{y}_k\}_{k=0}^{N}$ be a δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^{N-1}$. With the subspaces Y_k and the projections P_k defined as in Section 2, we define a linear operator

$$L_{\mathbf{y}} \colon Y_0 \times \cdots \times Y_N \to Y_1 \times \cdots \times Y_N$$

in the following way: If $\mathbf{v} = \{\mathbf{v}_k\}_{k=0}^{N}$ is in $Y_0 \times \cdots \times Y_N$, then we take $L_{\mathbf{y}}\mathbf{v} = \{[L_{\mathbf{y}}\mathbf{v}]_k\}_{k=0}^{N-1}$ to be

$$[L_{\mathbf{y}}\mathbf{v}]_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k \qquad for\ k = 0, \dots, N-1.$$

The operator $L_{\mathbf{y}}$ has right inverses and we choose one such right inverse $L_{\mathbf{y}}^{-1}$ with

$$\|L_{\mathbf{y}}^{-1}\| \leq K.$$

Also we define constants as before Lemma 1 with the range of k being appropriately adjusted. Now we can state our theorem.

THEOREM 2. **Finite-time Shadowing Theorem.** *Let $\{\mathbf{y}_k\}_{k=0}^{N}$ be a δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^{N-1}$ and let $L_{\mathbf{y}}^{-1}$ be a right inverse of the operator $L_{\mathbf{y}}$ with $\|L_{\mathbf{y}}^{-1}\| \leq K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1C\delta \leq 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo orbit $\{\mathbf{y}_k\}_{k=0}^{N}$ is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=0}^{N}$ with

$$\varepsilon \leq 2\bar{C}\delta.$$

Notes on finite-time shadowing: The proof of the Finite-time Shadowing Theorem above is quite similar to that of Lemma 1 except that here we use Brouwer's fixed point theorem rather than the contraction mapping principle.

It was first observed in [28], [29] that pseudo orbits of certain chaotic maps could be shadowed for long times by true orbits, despite the lack of uniform hyperbolicity. Others, [9], [10], and [52] realized that these observations could be generalized using shadowing techniques. Here the key idea is the construction of a right inverse of small norm for a linear operator similar to the one used for infinite-time shadowing. The choice of this right inverse is guided by the infinite-time case—one takes the formula for the inverse in the infinite-time case and truncates it appropriately (see [18]). However, the ordinary differential equation case is somewhat more complicated. It is not simply a matter of looking at the time-one map and applying the theory for the map case. One must somehow "quotient-out" the direction of the vector field and allow for rescaling of time as is done in Theorem 2; this needs to be done because of lack of hyperbolicity in the direction of the vector field. That this leads to much better shadowing results is shown in [17]. For other finite-time shadowing theorems in the context of autonomous ordinary differential equations, see [11], [12], [32], [62].

4. Periodic shadowing

In simulations of differential equations apparent periodic orbits, usually asymptotically stable, are often calculated. In this section we show how shadowing can be used to verify that there do indeed exist true periodic orbits near the computed orbits. Our method can be applied even to unstable periodic orbits which are ubiquitous in chaotic systems. We first recall the notions of pseudo periodic orbit and periodic shadowing for autonomous ordinary differential equations. Then, we state a Periodic Shadowing Theorem which guarantees the existence of a true periodic orbit near a pseudo periodic orbit.

DEFINITION 6. **Definition of pseudo periodic orbit.** *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=0}^N$, with $f(\mathbf{y}_k) \neq \mathbf{0}$ for all k, is said to be a δ* pseudo periodic orbit *of Eq. (1) if there is an associated sequence $\{h_k\}_{k=0}^N$ of positive times such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \le \delta \quad for\ k = 0, \dots, N-1,$$

and

$$\|\mathbf{y}_0 - \varphi^{h_N}(\mathbf{y}_N)\| \le \delta.$$

DEFINITION 7. **Definition of periodic shadowing.** *For a given positive number ε, a δ pseudo periodic orbit $\{\mathbf{y}_k\}_{k=0}^N$ with associated times $\{h_k\}_{k=0}^N$ is said to be ε-*shadowed *by a true periodic orbit if there are points $\{\mathbf{x}_k\}_{k=0}^N$ and positive times $\{t_k\}_{k=0}^N$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ for $k = 0, \dots, N-1$, and $\mathbf{x}_0 = \varphi^{t_N}(\mathbf{x}_N)$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \le \varepsilon \quad and \quad |t_k - h_k| \le \varepsilon \quad for\ k = 0, \dots, N.$$

To decide if a pseudo periodic orbit is shadowed by a true periodic orbit we need to compute certain other quantities. Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^N$. With the subspaces Y_k and the projections P_k defined as in Section 2, we define a linear operator

$$L_{\mathbf{y}}\colon Y_0 \times Y_1 \times \cdots \times Y_N \to Y_1 \times \cdots \times Y_N \times Y_0$$

as follows: if $\mathbf{v} = \{\mathbf{v}_k\}_{k=0}^N$ then

$$\begin{aligned}(L_{\mathbf{y}}\mathbf{v})_k &= \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k, \quad for\ k = 0, \dots, N-1\\ (L_{\mathbf{y}}\mathbf{v})_N &= \mathbf{v}_0 - P_N D\phi^{h_N}(\mathbf{y}_N)\mathbf{v}_N.\end{aligned}$$

We assume the operator $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \le K$. Also we define constants as before Lemma 1 with the range of k being appropriately adjusted. Now, we can state our main theorem.

THEOREM 3. **Periodic Shadowing Theorem**. *Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of the autonomous system Eq. (1) such that the operator $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \le K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1 C\delta \le 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo periodic orbit $\{\mathbf{y}_k\}_{k=0}^N$ is ε-shadowed by a true periodic orbit $\{\mathbf{x}_k\}_{k=0}^N$ of Eq. (1) with associated times $\{t_k\}_{k=0}^N$ and with

$$\varepsilon \le 2\bar{C}\delta.$$

Moreover, this is the unique such orbit satisfying

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad for\ 0 \le k \le N.$$

Notes on periodic shadowing: The proof of the Periodic Shadowing Theorem above is in [15]. Computation of periodic orbits with long periods requires special care; this situation is addressed in [19].

Normally one expects periodic orbits to be plentiful in chaotic invariant sets. One would like to be able to prove that periodic orbits are dense in attractors like those for the Lorenz Equations. This we have not been able to do. However, in [15], [19] we were able to use shadowing techniques to prove the existence of various periodic orbits of the Lorenz Equations, including orbits with long periods. We will display some of these periodic orbits in Section 7.

The idea of using computer assistance for rigorously establishing the existence of periodic orbits occurred to other people before us. For instance, Franke and Selgrade [27] gave a computer-assisted method to rigorously prove the existence of a periodic orbit of a two-dimensional autonomous system. Other relevant studies are [1], [43], [44], [54], [58], and [61].

Using periodic shadowing techniques, one can also provide rigorous estimates for the Lyapunov exponents of periodic orbits. Details of such computations, along with the Lyapunov exponents of several periodic orbits of the Lorenz Equations, are given in [19].

5. Homoclinic shadowing

General theorems proving the existence of a transversal homoclinic orbit to a hyperbolic periodic orbit, or a transversal heteroclinic orbit connecting one periodic orbit to another periodic orbit, of a flow are few. In this section we present two such theorems. In Section 7 we will show the effective use of these theorems on specific systems.

We first introduce the definition of an infinite pseudo periodic orbit, which is just the finite pseudo periodic orbit in Definition 6 extended periodically. This is more convenient for our purposes in this section.

DEFINITION 8. **Definition of infinite pseudo periodic orbit.** *A sequence $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{\ell_k\}_{k=-\infty}^{+\infty}$ is said to be a δ* pseudo periodic orbit *of period $N \geq 1$ of Eq. (1) if $\inf_{k\in\mathbb{Z}} \ell_k > 0$, $f(\mathbf{y}_k) \neq \mathbf{0}$ for all k and*

$$\|\mathbf{y}_{k+1} - \phi^{\ell_k}(\mathbf{y}_k)\| \leq \delta \quad \text{and} \quad \mathbf{y}_{k+N} = \mathbf{y}_k,\ \ \ell_{k+N} = \ell_k \quad for\ k \in \mathbb{Z}.$$

Now we define shadowing of such a pseudo periodic orbit.

DEFINITION 9. **Definition of periodic shadowing.** *For a given positive number ε, an infinite δ pseudo periodic orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{\ell_k\}_{k=-\infty}^{+\infty}$ is said to be ε-*shadowed *by a true periodic orbit if there are points $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ and positive times $\{t_k\}_{k=-\infty}^{+\infty}$ such that $\mathbf{x}_{k+N} = \mathbf{x}_k$ and $t_{k+N} = t_k$ and $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ for all $k \in \mathbb{Z}$ and*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad and \quad |t_k - h_k| \leq \varepsilon \quad for\ k \in \mathbb{Z}.$$

We next formalize the definition of a pseudo connecting orbit, homoclinic or heteroclinic, connecting one pseudo periodic orbit to another.

DEFINITION 10. **Definition of pseudo connecting orbit.** *Consider two δ pseudo periodic orbits $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ and $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{\bar{\ell}_k\}_{k=-\infty}^{+\infty}$ and $\{\ell_k\}_{k=-\infty}^{+\infty}$. An infinite sequence $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ is said to be a δ* pseudo connecting orbit *connecting $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ to $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ if $f(\mathbf{w}_k) \neq \mathbf{0}$ for all k and*

(i) $\|\mathbf{w}_{k+1} - \phi^{h_k}(\mathbf{w}_k)\| \leq \delta$ for $k \in \mathbb{Z}$,

(ii) $\mathbf{w}_k = \bar{\mathbf{y}}_k$, $h_k = \bar{\ell}_k$ for $k \leq p$ and $\mathbf{w}_k = \mathbf{y}_k$, $h_k = \ell_k$ for $k \geq q$ for some integers $p < q$.

In particular, $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ is said to be a pseudo homoclinic orbit *if there exists τ, $0 \leq \tau < N$, such that $\bar{\mathbf{y}}_k = \mathbf{y}_{k+\tau}$ and $\bar{\ell}_k = \ell_{k+\tau}$ for all k.*

Shadowing of a pseudo connecting orbit is defined as in Definition 2 for infinite-time shadowing.

Let $L_{\mathbf{w}} : Y \to \tilde{Y}$ be the linear operator defined by

$$(L_{\mathbf{w}}\mathbf{v})_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{w}_k)\mathbf{v}_k,$$

where P_k, Y and $\tilde{Y}$ are defined as in Section 2 with $\mathbf{w}$ replacing $\mathbf{y}$. We assume the operator is invertible and that $\|L_{\mathbf{w}}^{-1}\| \leq K$. Also we define constants as before Lemma 1 with $\mathbf{y}$ replaced by $\mathbf{w}$.

With the definitions and notations above, here we state our main shadowing theorems for connecting pseudo orbits. The first theorem guarantees the existence of a true hyperbolic connecting orbit near a pseudo connecting orbit. Note that an orbit is *hyperbolic* if and only if the invariant set defined by it is hyperbolic as in Definition 3.

THEOREM 4. **Connecting Orbit Shadowing Theorem**. *Suppose that $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ and $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ are two δ pseudo periodic orbits with periods $\bar{N}$ and N, respectively, of Eq. (1). Let $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ be a δ pseudo connecting orbit of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ connecting $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ to $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$. Suppose that the operator $L_{\mathbf{w}}$ is invertible with*

$$\|L_{\mathbf{w}}^{-1}\| \leq K.$$

Then if

$$4C\delta < \varepsilon_0, \quad 4M_1C\delta \leq \min\{2,\ \Delta\}, \quad C^2\left(N_1\delta + N_2\delta^2 + N_3\delta^3\right) < 1,$$

(i) the pseudo periodic orbits $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ and $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ are ε-shadowed by true periodic orbits $\{\bar{\mathbf{x}}_k\}_{k=-\infty}^{\infty}$ of period $\bar{N}$ and $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ of period N where

$$\varepsilon \leq 2\bar{C}\delta,$$

moreover, $\phi^t(\bar{\mathbf{x}}_0)$ and $\phi^t(\mathbf{x}_0)$ are hyperbolic (non-equilibrium) periodic orbits;

(ii) the pseudo connecting orbit $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ *above is also* ε*-shadowed by a true orbit* $\{\mathbf{z}_k\}_{k=-\infty}^{\infty}$. *Moreover,* $\phi^t(\mathbf{z}_0)$ *is hyperbolic and there are real numbers* $\bar{\alpha}$ *and* α *such that* $\|\phi^t(\mathbf{z}_0) - \phi^{t+\bar{\alpha}}(\bar{\mathbf{x}}_0)\| \to 0$ *as* $k \to -\infty$ *and* $\|\phi^t(\mathbf{z}_0) - \phi^{t+\alpha}(\mathbf{x}_0)\| \to 0$ *as* $t \to \infty$.

In the special case of the theorem above when the two periodic orbits coincide, we obtain a transversal homoclinic orbit as hyperbolicity of the connecting orbit implies its transversality. Now, an additional condition is required to ensure that the connecting orbit does not coincide with the periodic orbit. The condition given in the theorem below is that there be a point on the pseudo homoclinic orbit sufficiently distant from the pseudo periodic orbit. With the setting as in the previous theorem, we state the following theorem.

THEOREM 5. **Homoclinic Orbit Shadowing Theorem**. *Suppose that* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *is a* δ *pseudo periodic orbit with period* N *of Eq. (1). Let* $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ *be a* δ *pseudo homoclinic orbit of Eq. (1) with associated times* $\{h_k\}_{k=-\infty}^{+\infty}$ *connecting* $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty} = \{\mathbf{y}_{k+\tau}\}_{k=-\infty}^{+\infty}$ *to* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$*, where* $0 \le \tau < N$*. Suppose that the operator* $L_{\mathbf{w}}$ *is invertible with*

$$\|L_{\mathbf{w}}^{-1}\| \le K.$$

Then if

$$4C\delta < \varepsilon_0, \quad 4M_1C\delta \le \min\{2,\ \Delta\}, \quad C^2\left(N_1\delta + N_2\delta^2 + N_3\delta^3\right) < 1,$$

(i) the pseudo periodic orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *above is* ε*-shadowed by a true periodic orbit* $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ *of period* N *where*

$$\varepsilon \le 2\bar{C}\delta,$$

moreover, $\phi^t(\mathbf{x}_0)$ *is a hyperbolic (non-equilibrium) periodic orbit;*

(ii) the pseudo homoclinic orbit $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ *above is also* ε*-shadowed by a true orbit* $\{\mathbf{z}_k\}_{k=-\infty}^{\infty}$. *Moreover,* $\phi^t(\mathbf{z}_0)$ *is hyperbolic and there are real numbers* $\bar{\alpha}$ *and* α *such that* $\|\phi^t(\mathbf{z}_0) - \phi^{t+\bar{\alpha}}(\mathbf{x}_0)\| \to 0$ *as* $k \to -\infty$ *and* $\|\phi^t(\mathbf{z}_0) - \phi^{t+\alpha}(\mathbf{x}_0)\| \to 0$ *as* $t \to \infty$*. Furthermore, provided there exists* r *with* $p < r < q$ *such that*

$$\|\mathbf{w}_r - \mathbf{y}_k\| > (\|f(\mathbf{y}_k)\| + 2M_1\bar{C}\delta)\frac{e^{M_1(h_{\mathbf{max}}+\varepsilon_0)-1}}{M_1} + 4\bar{C}\delta$$

for $0 \le k \le N-1$*, then* $\mathbf{z}_r$ *does not lie on the orbit of* $\mathbf{x}_0$ *and so we may conclude that* $\mathbf{z}_0$ *is a transversal homoclinic point associated with the periodic orbit* $\phi^t(\mathbf{x}_0)$.

Notes on homoclinic shadowing: We prove the theorems above partly using earlier theorems. First we prove the existence of the periodic orbits using the Periodic Shadowing Theorem. Then we use the Infinite-time Shadowing Lemma to show the existence of a unique orbit shadowing the pseudo homoclinic orbit. However, here we have to show this orbit is asymptotic to the periodic orbits. It turns out that this can be proved by a compactness argument using the uniqueness. Moreover, we need to show the transversality as well. This follows from the hyperbolicity which is proved by a rather involved argument. Detailed proofs of these theorems are available in our forthcoming paper [22].

Examples of flows with transversal connecting orbits are scarce. The papers [36], [38], and [64] have examples of connecting orbits in celestial mechanics. Note that [36] also employs shadowing methods but applies a theorem for a sequence of maps (see also [49] for such a theorem) rather than a theorem specifically for differential equations. There are also studies using shooting methods combined with interval arithmetic that attempt to establish the existence of connecting orbits; see, for example, [60] where such orbits in the Lorenz Equations are computed. In Section 7 we will exhibit the existence of a transversal homoclinic orbit in the Lorenz Equations for the classical parameter values.

There are a number of studies for effectively computing accurate approximations to finite segments of orbits of flows connecting two periodic orbits. For example, [25] and [48], inspired by [3], approximate a connecting orbit by the solution of a certain boundary value problem and they derive estimates for the error in the approximation. However, all existing work is carried out on the assumption that a true connecting orbit exists. In contrast, our Connecting and Homoclinic Shadowing Theorems provide a new computer-assisted method for rigorously establishing the existence of such orbits.

According to Sil'nikov's theorem [55], [56], and [46], the existence of a transversal homoclinic orbit implies chaos. A single transversal heteroclinic orbit does not imply chaos. However, a cycle of transversal heteroclinic orbits does imply chaos. Our Connecting Orbit Shadowing Theorem can be used to prove the existence of such cycles in, for example, the Lorenz Equations, as demonstrated in [22].

6. Implementation issues

There are two main computational issues in applying the shadowing theorems we have presented in the previous sections:

(i) Small δ: finding a suitable pseudo orbit with sufficiently small rigorous local error bound;

(ii) $||L^{-1}|| \leq K$: verifying the invertibility of the operator L (or finding a suitable right inverse) and calculating a rigorous upper bound K on the norm of the inverse.

In this section we highlight certain key ideas regarding these computational considerations, at suitable points directing the reader to references where further details can be found.

(i) Finding a suitable pseudo orbit: In the case of finite-time shadowing, one could be tempted to use a sophisticated numerical integration method with local error

tolerance control to generate a good pseudo orbit. However, in order to claim the existence of a true orbit near the computed approximate orbit, we need a *rigorous* bound on the local discretization error δ. We have found a high-order Taylor Method to be the most effective numerical integration method for this purpose. To get a rigorous δ, one must also account for the floating point errors in the calculation of $\phi^{h_k}(\mathbf{y}_k)$, which we handle using the techniques of Wilkinson [65]. Details of the implementation of the Taylor Method with floating error estimates for the Lorenz Equations are given in [17].

In the case of periodic or homoclinic shadowing, it is very difficult to find a pseudo periodic orbit or pseudo homoclinic orbit with δ small enough by a routine use of a numerical integrator using simple shooting, that is, various initial conditions are tried until one is found with a small δ. Usually the δ found in this way is not small enough to apply our theorems. To get a pseudo orbit with a smaller δ, we refine the "crude" pseudo orbit with a suitable global Newton's method. "Global" means we work with the whole pseudo orbit, not just its initial point since it turns out that working with just the initial point is not effective.

Now we describe what we do in the periodic case. Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of Eq. (1), found perhaps by simple shooting, or by concatenating segments of several orbits. In general, the δ associated with such a crude pseudo orbit will not be sufficiently small to apply our Periodic Orbit Shadowing Theorem. We want to replace this pseudo periodic orbit by a nearby one with a smaller δ. Ideally there would be a nearby sequence of points $\{\mathbf{x}_k\}_{k=0}^N$ and a sequence of times $\{t_k\}_{k=0}^N$ such that

$$\begin{aligned} \mathbf{x}_{k+1} &= \phi^{t_k}(\mathbf{x}_k) \quad \text{for } k = 0, \dots, N-1 \\ \mathbf{x}_0 &= \varphi^{t_N}(\mathbf{x}_N). \end{aligned}$$

We write $\mathbf{x}_k = \mathbf{y}_k + \mathbf{z}_k$, where $\mathbf{z}_k$ is orthogonal to $f(\mathbf{y}_k)$, and $t_k = h_k + s_k$. So we need to solve the equations

$$\begin{aligned} \mathbf{z}_{k+1} &= \phi^{h_k+s_k}(\mathbf{y}_k + \mathbf{z}_k) - \mathbf{y}_{k+1} \quad \text{for } k = 0, \dots, N-1 \\ \mathbf{z}_0 &= \varphi^{h_N+s_N}(\mathbf{y}_N + \mathbf{z}_N) - \mathbf{y}_0. \end{aligned}$$

As in Newton's method, we linearize:

$$\phi^{h_k+s_k}(\mathbf{y}_k + \mathbf{z}_k) - \mathbf{y}_{k+1} \approx f(\phi^{h_k}(\mathbf{y}_k))s_k + D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_k + \phi^{h_k}(\mathbf{y}_k) - \mathbf{y}_{k+1}.$$

Next we write

$$\mathbf{z}_k = S_k \mathbf{u}_k,$$

where $\mathbf{u}_k \in \mathbb{R}^{n-1}$ and $\{S_k\}_{k=0}^N$ is a sequence of $n \times (n-1)$ matrices chosen so that $[\,f(\mathbf{y}_k)/\|f(\mathbf{y}_k)\|\ S_k]$ is orthogonal. So now we solve the linear equations

$$\begin{aligned} S_{k+1}\mathbf{u}_{k+1} &= f(\mathbf{y}_{k+1})s_k + D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{u}_k + \mathbf{g}_k \quad \text{for } k = 0, \dots, N-1 \\ S_0\mathbf{u}_0 &= f(\mathbf{y}_N)s_N + D\phi^{h_N}(\mathbf{y}_N)S_N\mathbf{u}_N + \mathbf{g}_N \end{aligned}$$

for s_k and $\mathbf{u}_k$, where $\mathbf{g}_k = \phi^{h_k}(\mathbf{y}_k) - \mathbf{y}_{k+1}$. Multiplying each equation in the first set by S_{k+1}^* and $f(\mathbf{y}_{k+1})^*$ and multiplying the last equation by S_0^* and $f(\mathbf{y}_0)$, under the assumption of no floating point errors, we obtain

$$
\begin{aligned}
\mathbf{u}_{k+1} - A_k \mathbf{u}_k &= S_{k+1}^* \mathbf{g}_k \quad \text{for } k = 0, \dots, N-1 \\
\mathbf{u}_0 - A_N \mathbf{z}_N &= S_0^* \mathbf{g}_N
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
s_k &= -\| f(\mathbf{y}_{k+1})\|^{-2} f(\mathbf{y}_{k+1})^* \{D\phi^{h_k}(\mathbf{y}_k) S_k \mathbf{u}_k + \mathbf{g}_k\} \quad \text{for } k = 0, \dots, N-1 \\
s_N &= -\| f(\mathbf{y}_0)\|^{-2} f(\mathbf{y}_0)^* \{D\phi^{h_N}(\mathbf{y}_N) S_N \mathbf{u}_N + \mathbf{g}_N\},
\end{aligned}
$$

where $A_k = S_{k+1}^* D\phi^{h_k}(\mathbf{y}_k) S_k$ for $k = 0, \dots, N-1$ and $A_N = S_0^* D\phi^{h_N}(\mathbf{y}_N) S_N$. So the main problem is to solve Eq. (2). This is solved by exploiting the local hyperbolicity along the pseudo orbit which implies the existence of contracting and expanding directions. We use a triangularization procedure which enables us to solve forward first along the contracting directions and then backwards along the expanding directions. Note that it is numerically impossible to solve the whole system forwards because of the expanding directions.

Once the new pseudo periodic orbit $\{\mathbf{y}_k + \mathbf{z}_k\}_{k=0}^N$ with associated times $\{h_k + s_k\}_{k=0}^N$ is found, we check if its delta is small enough. If it is not, we repeat the procedure for further refinement. For complete details see [19]. A similar method to refine a crude pseudo homoclinic orbit is given in [22].

(ii) Verifying the invertibility of the operator (or finding a suitable right inverse) and calculating an upper bound on the norm of the inverse: Again we just look at the periodic case. A similar procedure is used in the finite-time and homoclinic cases but in the homoclinic case it is rather more complicated since the sequence spaces are infinite-dimensional; however, we can handle it due to the periodicity at both ends. First we outline the procedure which would be used in the case of exact computations.

To construct $L_{\mathbf{y}}^{-1}$, we need to find the unique solution $\mathbf{z}_k \in Y_k$ of

$$
\begin{aligned}
\mathbf{z}_{k+1} &= P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_k + \mathbf{g}_k, \quad \text{for } k = 0, \dots, N-1 \\
\mathbf{z}_0 &= P_N D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_N + \mathbf{g}_N.
\end{aligned}
$$

whenever $\mathbf{g}_k$ is in Y_{k+1} for $k = 0, \dots, N-1$ and in Y_0 for $k = N$. We use the $n \times (n-1)$ matrices S_k as defined in *(i)* and make the transformation

$$\mathbf{z}_k = S_k \mathbf{u}_k,$$

where $\mathbf{u}_k$ is in $\mathbb{R}^{n-1}$. Making this transformation, our equations become

$$
\begin{aligned}
\mathbf{u}_{k+1} - A_k \mathbf{u}_k &= S_{k+1}^* \mathbf{g}_k, \quad \text{for } k = 0, \dots, N-1 \\
\mathbf{u}_0 - A_N \mathbf{u}_N &= S_0^* \mathbf{g}_N,
\end{aligned}
$$

where A_k is the $(n-1) \times (n-1)$ matrix $A_k = S_{k+1}^* D\phi^{h_k}(\mathbf{y}_k) S_k$ for $k = 0, \dots, N-1$ and $A_N = S_0^* D\phi^{h_N}(\mathbf{y}_N) S_N$.

As in *(i)*, these equations are solved by exploiting the hyperbolicity which implies the existence of contracting and expanding directions.

We use a triangularization procedure which enables us to solve forward first along the contracting directions and then backwards along the expanding directions. Thus we are able to obtain a computable criterion for invertibility and a formula for the inverse from which an upper bound for the inverse can be obtained.

Of course, we have to take into account round-off and discretization error. For example, we have to work with a computed approximation to $D\phi^{h_k}(\mathbf{y}_k)$. However, this only induces a small perturbation in the operator. Another problem is that the columns of S_k, as computed, are not exactly orthonormal. Next A_k cannot be computed precisely. Finally the difference equation cannot be solved exactly but we are still able to obtain rigorous upper bounds, using standard perturbation theory of linear operators and the techniques in [65] to obtain rigorous upper bounds on the error due to floating point operations. The necessary interval arithmetic can be implemented by using IEEE-754 compliant hardware with a compiler that supports rounding mode control. For more details of these and other issues see [19] and for the finite-time and homoclinic cases see [17] and [22], respectively.

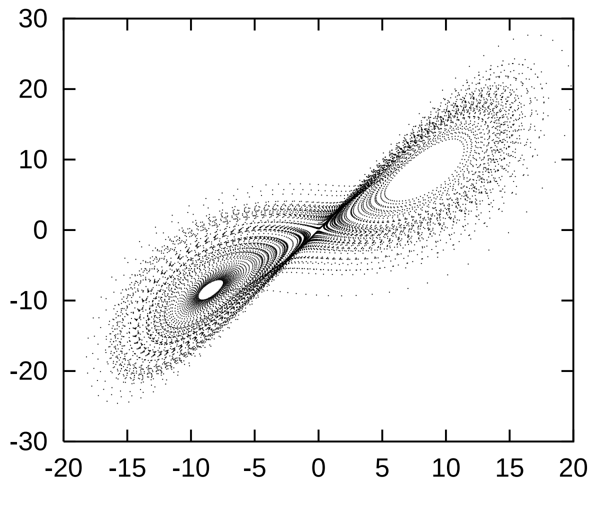
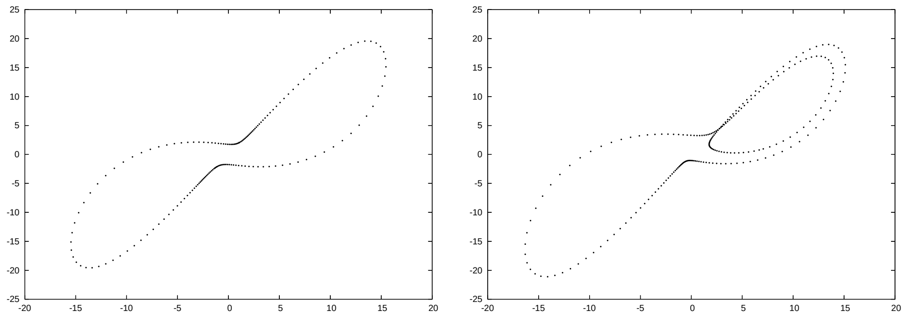


Figure 1: A pseudo orbit, with $\delta \leq 1.978 \times 10^{-12}$, of the Lorenz Equations for the classical parameter values $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ and initial data $(0,\ 1,\ 0)$ projected onto the $(x,\ y)$-plane. There exists a true orbit within $\varepsilon \leq 2.562 \times 10^{-9}$ of this pseudo orbit. For clarity, we have plotted only the first 120 time units of the pseudo orbit. Shadowing of this pseudo orbit for much longer time is possible.



(a) (b)

Figure 2: Two short pseudo periodic orbits of the Lorenz Equations for the classical parameter values projected onto the $(x,\ y)$-plane. The pseudo orbit in (a) has approximate period 1.559 time units; it is shadowed by a true periodic orbit within $\varepsilon \leq 1.800 \times 10^{-12}$. It is interesting to observe that these periodic orbits coexist with the nonperiodic orbit in Fig. 1.

7. Examples

In this section, we offer representative applications of the shadowing theorems from Sections 3, 4, and 5, using the Lorenz Equations [41]

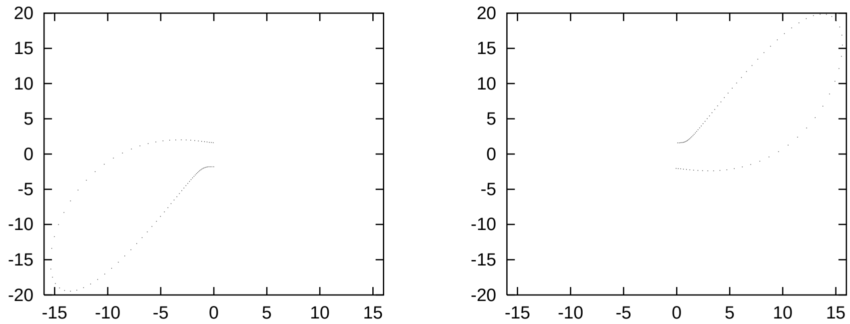
$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

with the classical parameter values $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. Using the Lyapunov function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2$, it is not difficult to establish that the set

$$U = \{(x, y, z) : \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \leq \sigma\rho^2\beta^2/(\beta - 1)\}$$

is forward invariant under the flow of the Lorenz Equations for $\sigma \geq 1$, $\rho > 0$, and $\beta > 1$. Each pseudo orbit $\{\mathbf{y}_k\}_{k=0}^N$ of the Lorenz Equations we calculate below lies inside this forward invariant ellipsoid U.

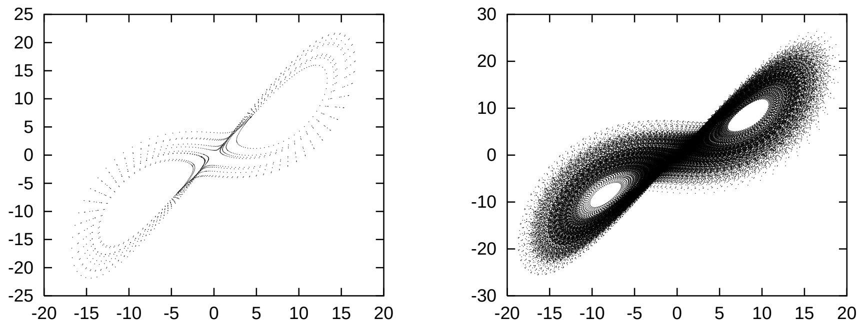
First we give an example of finite-time shadowing for the initial data $(0, 1, 0)$ used by Lorenz. The pseudo orbit $\{\mathbf{y}_k\}_{k=0}^N$ of the Lorenz Equations in Fig. 1 and the sequence of matrices approximating $D\phi^{h_k}(\mathbf{y}_k)$ are generated by applying a Taylor series method of order 31 with initial value $\mathbf{y}_0$ at $t = 0$ and with constant time step h_k. The Taylor method has the advantages that a bound for the local discretization error is easily calculated and it also allows us to use relatively large step sizes h_k. This pseudo orbit is depicted in Fig.1. It is a δ pseudo orbit with $\delta \leq 1.978 \times 10^{-13}$ and we are able to show that it is ε-shadowed by a true orbit with $\varepsilon \leq 2.562 \times 10^{-9}$ for at least 850, 000 time units. This example is taken from [17] where all the implementation details can be found. A pseudo orbit from almost all initial data of the Lorenz Equations



(X) (Y)

Figure 3: “X” and “Y” pieces of a crude pseudo periodic orbit of the Lorenz Equations for the classical parameter values projected onto the (x, y)-plane. Concatenated copies in the order XY and XYY are used as initial guesses for the Global Newton’s method to generate the refined pseudo periodic orbits in Fig. 2.

are shadowable for reasonably long time intervals. In the same reference shadowing times of many other pseudo orbits are tabulated.



(a) (b)

Figure 4: Two long pseudo periodic orbits of the Lorenz Equations for the classical parameter values projected onto the (x, y)-plane. The pseudo periodic orbit in (a) is of type XXYYXXXYYYXYXYXXXYY. The pseudo periodic orbit in (b) has approximate period 1100.787 time units; it is shadowed by a true periodic orbit within $\varepsilon \leq 5.239 \times 10^{-12}$.

Next, we present examples of periodic shadowing. Periodic orbits of the Lorenz Equations exhibit a great deal of geometric variations and they have been the subject of many publications; see, for example, [1], [5], [63]. Most of these works are either of numerical nature or pertain to a model system which is believed to capture the essential features of the actual equations. One of the first rigorous demonstrations of a specific periodic orbit of the Lorenz Equations was given in [15] using the Periodic Shadowing

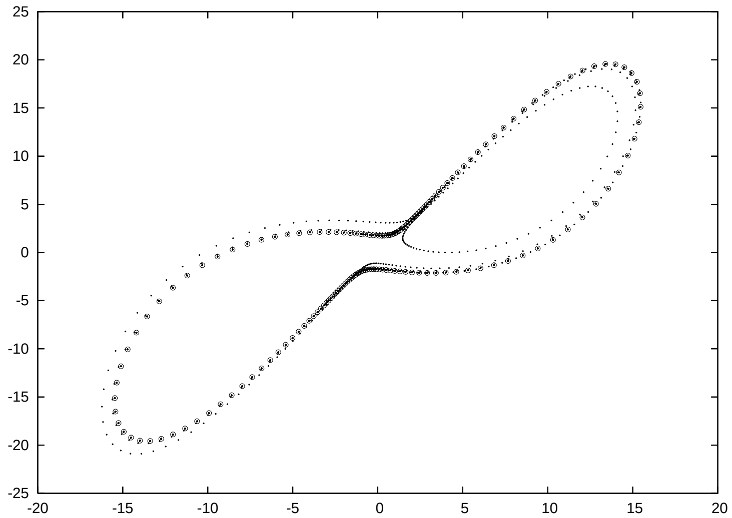


Figure 5: The pseudo periodic orbit of the Lorenz Equations from Fig. 2(a) (plotted in small circles) and a pseudo homoclinic orbit (plotted in dots) doubly asymptotic to this pseudo periodic orbit with $\delta \le 2.012 \times 10^{-12}$. There exist a true hyperbolic periodic orbit and a true transversal homoclinic orbit within $\varepsilon \le 2.266 \times 10^{-9}$ of the pseudo ones.

Theorem of Section 4.

In Fig. 2 two short pseudo periodic orbits of the Lorenz Equations are plotted. The pseudo periodic orbit in Fig. 2(a) has approximate period 1.559 and is readily visible on the computer screen by following the solution from the initial data

$$(-12.78619065852397642,\ -19.36418793711800464,\ 24.0)$$

with a decent integrator. We found the periodic orbit in Fig. 2(a) as follows: first we computed the two orbit segments X and Y in Fig. 3 whose endpoints were fairly close and concatenated them together to form a pseudo periodic orbit XY with a quite large δ. Then we used the global Newton's method as outlined in Section 6 to find a refined pseudo periodic orbit with sufficiently small δ to apply the Periodic Shadowing Theorem. With a similar concatenation method, it is possible to generate pseudo periodic orbits with various geometries. Additional pseudo periodic orbits are depicted in Figs. 2 and 4. Further computational and mathematical details of the analysis of these pseudo periodic orbits are available in [15], [19], [22].

Lastly, we give an example of homoclinic shadowing. First we took the pseudo periodic orbit in Fig. 2(a). Using our concatenation technique followed by a global Newton's method, we next found a pseudo homoclinic orbit connecting the pseudo periodic orbit to itself. This pseudo connecting orbit, which is pictured in Fig. 5, has $\delta \le 2.012 \times 10^{-12}$. The values of the significant constants associated with this pseudo connecting orbit and the requisite inequalities for the Homoclinic Orbit Shadowing

Parameters:

$$\sigma = 10.0, \quad \beta = 8.0/3.0, \quad \rho = 28.0$$

Convex set:

$$U = \{(x,\ y,\ z) : \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \le \sigma\rho^2\beta^2/(\beta - 1)\}$$

Significant quantities:

$M_0 \le 5546.1807694948548$	$K = 567.89771838622869$
$M_1 \le 87.040362193445489$	$\Delta \ge 37.640569857537358$
$M_2 \le 1.4142135623730951$	$\overline{M}_0 \le 243.91779884309116$
$h_{\max} = 0.011826110602228697$	$\overline{M}_1 \le 28.802758744306345$
$h_{\min} = 0.0019715050733614117$	$C \le 567.89771838622869$
$\varepsilon_0 = 1.0000000000000001 \times 10^{-5}$	$\overline{C} \le 567.89777438966746$
$N = 168$	$N_1 \le 57496.233185753386$
$p = 167$	$N_2 \le 2.8839060646841870 \times 10^{11}$
$q = 3940$	$N_3 \le 1.5646336856150979 \times 10^{17}$
$\tau = 76$	$\varepsilon = 2.2659685392584887 \times 10^{-9}$
$\delta = 2.0118772701071166 \times 10^{-12}$	$r = 3277$

Inequalities:

$$4C\delta \le 4.5701620454677841 \times 10^{-9} < \epsilon_0$$
$$4M_1C\delta \le 3.9778855972025359 \times 10^{-7} \le \min(2, \Delta) = 2$$
$$C^2((N_3\delta + N_2)\delta + N_1)\delta \le 3.7306585982472891 \times 10^{-2} < 1$$

Figure 6: Significant constants and the requisite inequalities for the pseudo homoclinic orbit in Fig. 5. From the Homoclinic Shadowing Theorem, there exist a true hyperbolic periodic orbit and a true transversal homoclinic orbit within $\varepsilon \le 2.266 \times 10^{-9}$ of the pseudo ones.

Theorem in Section 5 are tabulated in Fig. 6. It is evident from these numbers that the hypotheses of our Homoclinic Orbit Shadowing Theorem are fulfilled.

Thus we conclude: there exists a true transversal homoclinic orbit within $\varepsilon \le 2.266 \times 10^{-9}$ of this pseudo one. As outlined in the Introduction, the existence of such an orbit implies chaotic behavior in the vicinity.

8. Closing remarks

In the preceding pages, we have tried to demonstrate the utility and the promise of shadowing as a new development in the rigorous computer-assisted analysis of ordinary differential equations. Shadowing can be used to show that certain computer simulations of chaotic systems do indeed represent true states of the system. Moreover, shadowing is able to establish the existence of orbits with various dynamical properties such as hyperbolic periodic orbits and transversal homoclinic or heteroclinic orbits. In contrast to classical numerical analysis which tries to compute known orbits as accurately as possible, shadowing can show their existence rigorously.

One limitation of the shadowing methods presented here is that they will only work with systems which display some hyperbolicity, even if not uniform. Limits to shadowing due to fluctuating Lyapunov exponents or unstable dimension variability have been investigated in [24], [53].

In this review article we tried to give the highlights of our work in shadowing during the past decade. Due to the inevitable time and space constraints, we could not include many other developments in shadowing; we apologize to the authors of numerous excellent publications to which we could not refer. We hope we have convinced you of the utility of shadowing and that you will be interested enough to look at the details of our work and the works of the others listed in the references.

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Subalpine Rhapsody in Dynamics

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SOME RESULTS ON PERIODIC POINTS AND CHAOTIC DYNAMICS ARISING FROM THE STUDY OF THE NONLINEAR HILL EQUATIONS

Abstract. We study fixed point theorems for maps which satisfy a property of stretching a suitably oriented topological space Z along the paths connecting two disjoint subsets Z_l^- and Z_r^- of Z. Our results reconsider and extend previous theorems in [56, 59, 60] where the case of two-dimensional cells (that is topological spaces homeomorphic to a rectangle of the plane) was analyzed. Applications are given to topological horseshoes and to the study of the periodic points and the symbolic dynamics associated to discrete (semi)dynamical systems.

1. Introduction

1.1. A motivation from the theory of ODEs

In the study of boundary value problems for nonlinear ODEs, the shooting method, in spite of being sometimes considered as an old fashioned technique, is still a quite powerful and effective tool in various different situations. For instance, as a sample model, let us consider the generalized Sturm−Liouville problem for a second order equation of the form

$$u'' + f(t,u,u') = 0, \qquad (u(t_0), u'(t_0)) \in \Gamma_0\,, \quad (u(t_1), u'(t_1)) \in \Gamma_1\,,$$

where $f = f(t,x,y) : [t_0,t_1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying a locally Lipschitz condition with respect to (x,y) and Γ_0 and Γ_1 are two unbounded closed connected subsets of the plane $\mathbb{R}^2$. Using the shooting method, one can start from the Cauchy problem (for which we have the uniqueness of the solutions and their continuous dependence upon the initial values)

$$u'' + f(t,u,u') = 0, \qquad (u(t_0), u'(t_0)) = (x_0, y_0) := z_0\,,$$

with $z_0 \in \Gamma_0$ and, having denoted by $\zeta(\cdot\,; t_0, z_0)$ *the* corresponding solution of the equivalent first order system in the phase-plane

$$x' = y, \quad y' = -f(t,x,y), \tag{1}$$

with $\zeta(t_0) = z_0$, look for the intersections between Γ_1 and the set $\Gamma_0' := \{\zeta(t_1; t_0, z_0) : z_0 \in \Gamma_0\}$. Clearly, any point $z_1 \in \Gamma_0' \cap \Gamma_1$ is the value $(u(t_1), u'(t_1))$ corresponding to a solution $u(\cdot)$ of the original Sturm−Liouville problem and different points in the intersection of Γ_1 with Γ_0' are associated to different solutions as well. In the simpler

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case in which Γ_0 is the image of a continuous curve $\theta_0 : [0, 1[\to \mathbb{R}^2$, the set Γ_0' can sometimes be described as the image of a continuous curve too, by means of the map $\theta_1 : [0, 1[\ni s \mapsto \zeta(t_1; t_0, \theta_0(s))$ (this, of course, is not always guaranteed, in fact we are not assuming in this example that all the solutions of the Cauchy problems can be defined on $[t_0, t_1]$ and therefore, without further assumptions on f, it could happen that θ_1 may be not defined on the whole interval $[0, 1[$). See also [9, 10, 13, 23] for different topological approaches where the uniqueness of the solutions for the Cauchy problem is not assumed.

In [55], dealing with some boundary value problems associated to the nonlinear scalar ODE

$$u'' + q(t)g(u) = 0 \tag{2}$$

and assuming that $q : [t_0, t_1] \to \mathbb{R}$ is a continuous and piece-wise monotone function and $g : \mathbb{R} \to \mathbb{R}$, with

$$g(s)s > 0, \quad \text{for } s \neq 0,$$

a locally Lipschitz continuous mapping satisfying a condition of superlinear growth at infinity, we considered the case in which there is $\tau \in]t_0, t_1[$ such that

$$q(t) > 0 \text{ for } t \in]t_0, \tau[\quad \text{and } q(t) < 0 \text{ for } t \in]\tau, t_1[$$

and then we found in the phase-plane two conical shells $W(+) := W(r, R)$ and $W(-) := -W(r, R)$, with

$$W(r, R) := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, r^2 \leq x^2 + y^2 \leq R^2\}, \tag{3}$$

such that the following path-stretching property holds:

$(H_\pm)$ *for every path $\sigma(\pm)$ contained in $W(\pm)$ and meeting the inner and the outer circumferences at the boundary of $W(\pm)$ there are sub-paths $\gamma_i(\pm)$ $(i = 1, 2)$ contained in the domain D_φ of the map $\varphi : \mathbb{R}^2 \ni z_0 \mapsto \zeta(t_1; t_0, z_0) \in \mathbb{R}^2$ and such that $\varphi(\gamma_1(\pm))$ is contained in $W(\pm)$ and meets both the inner and the outer circumferences, as well as $\varphi(\gamma_2(\pm))$ is contained in $W(\mp)$ and meets both the inner and the outer circumferences.* *

The points $z_0 = (x_0, y_0)$ belonging to each of the sub-paths $\gamma_i(\pm)$ $(i = 2, 1)$ are initial points of system

$$x' = y, \quad y' = -q(t)g(x), \tag{4}$$

for $t = t_0$ for which the corresponding solution $\zeta(\cdot\,; t_0, z_0) = (u(\cdot), u'(\cdot))$ is such that $u(t)$ has a sufficiently large (but fixed in advance) number of zeros in the interval $]t_0, \tau[$ and then, either exactly one zero or no zeros at all (and a zero for its derivative) in the interval $]\tau, t_1[$.

*In [55] we used a definition of path which is slightly different from the one considered here in Section 1.5; however, this does not effect the validity of property $(H_\pm)$. Actually, for the moment, as long as we are in the introductory part of this work and also for the sake of simplicity in the exposition, we prefer to be a little vague about the precise concept of path we are going to use.

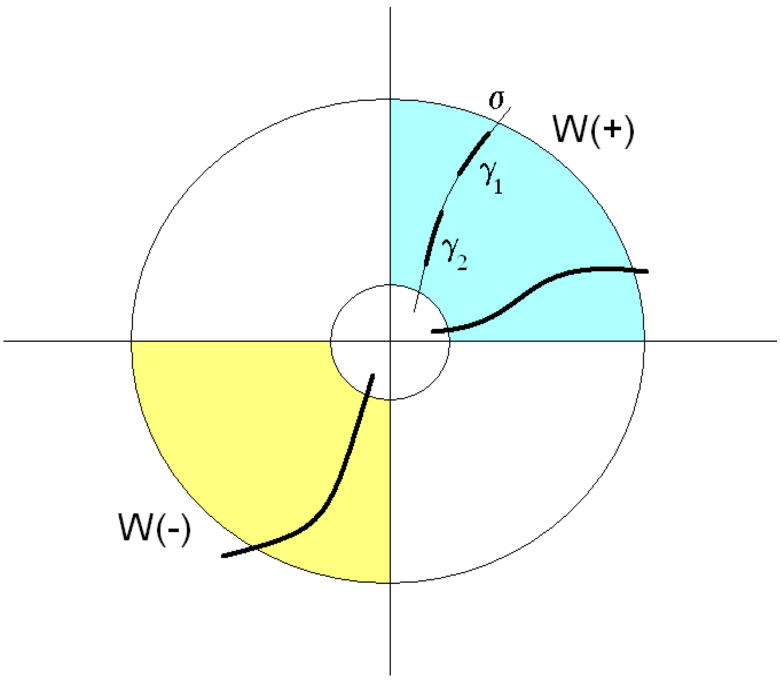

Figure 1: An illustration of the situation which occurs in [55]. Any path $\sigma = \sigma(+)$ contained in the first quadrant and meeting the inner and the outer circumferences of the conical shell $W(+)$ (like the one drawn with a thinner line) contains (at least) two sub-paths $\gamma_1 = \gamma_1(+)$ and $\gamma_2 = \gamma_2(+)$ (like those drawn with a thicker line) satisfying the following property: through the map φ, one of the two sub-paths (namely, γ_1) is transformed to a path (drawn with a thicker line) contained in the first quadrant and crossing the set $W(+)$, while the other sub-path (that is, γ_2) is transformed to a path (drawn with a thicker line) contained in the third quadrant and crossing the set $W(-)$. The same happens with respect to any path $\sigma(-)$ contained in $W(-)$ and intersecting the inner and the outer circumferences at the boundary of $W(-)$.

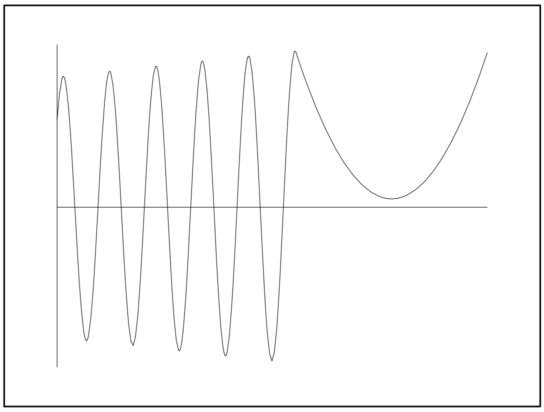

Figure 2: An illustration of the possible behavior of a solution $u(t)$ of (2) for $t \in [t_0, t_1]$, with $(u(t_0), u'(t_0))$ belonging to $\gamma_1(+)$. The solution oscillates a certain (large) number of times in $]t_0, \tau[$. At the time τ of switching from $q > 0$ to $q < 0$ the point $(u(\tau), u'(\tau))$ lies in the interior of the fourth quadrant of the phase-plane. Then, in the interval $[\tau, t_1]$ we have $u(t) > 0$ with $u'(t)$ vanishing exactly once.

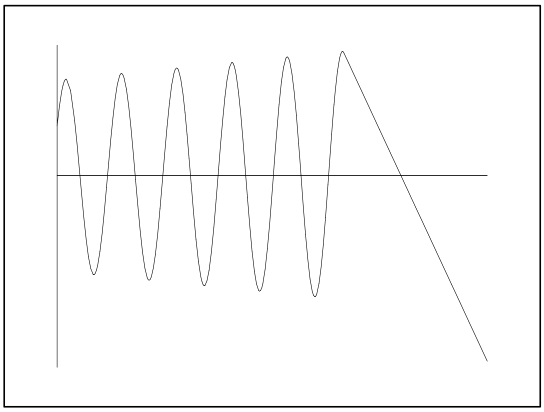

Figure 3: An illustration of the possible behavior of a solution $u(t)$ of (2) for $t \in [t_0, t_1]$, with $(u(t_0), u'(t_0))$ belonging to $\gamma_2(+)$. The solution oscillates a certain (large) number of times in $]t_0, \tau[$. At the time τ of switching from $q > 0$ to $q < 0$ the point $(u(\tau), u'(\tau))$ lies in the interior of the fourth quadrant of the phase-plane. Then, in the interval $[\tau, t_1]$ we have $u'(t) < 0$ with $u(t)$ vanishing exactly once.

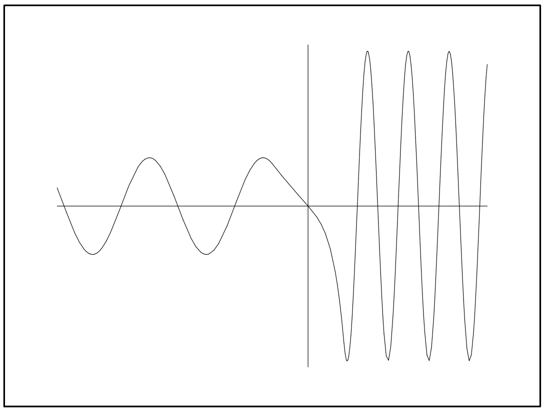

Figure 4: An illustration of the possible behavior of a solution $u(t)$ of (2) along an interval made by two subintervals where $q > 0$ which are separated by a subinterval where $q < 0$. In this latter subinterval $u' < 0$ and u vanishes exactly once.

If there are several adjacent intervals like $[t_0, t_1]$ in which the weight function $q(t)$ changes its sign, we can repeat the same argument for each of such intervals and obtain solutions which have a large number of zeros in each interval when $q(t) > 0$ and either exactly one zero, or no zeros at all (according to any finite sequence in $\{0, 1\}$ which is fixed in advance). Figure 4 and Figure 5, below, describe the situation of two positive intervals for the weight which are separated by an interval where $q < 0$. According to [55] there exist solutions with a large number of oscillations in the intervals when $q > 0$ and vanishing either once or never in the interval when the weight is negative. Both kind of solutions coexist for the same equation, the different outcome depending by small differences in the initial conditions. The graphs drawn in Figures 2-5 (using Maple software) represent an idealized situation that we use as a description of the main result in [55] and do not concern a specific equation like (2).

In [55], using a key lemma involving the path-stretching property $(H_{\pm})$, and some results adapted from Hartman [28] and Struwe [69] (which allow to find infinitely many solutions for generalized Sturm−Liouville superlinear problems when the weight function is positive), we obtained the existence of (infinitely many) solutions for various boundary value problems associated to equation (2) in the case of a nonlinear function g satisfying a condition of superlinear growth at infinity. A possible choice of g is given by $g(s) = |s|^{\alpha-1}s$ with $\alpha > 1$, which makes equation (2) a nonlinear analogue of Hill's equation (cf. [5]). Our results in [55] allow to obtain solutions having an arbitrarily large number of zeros in the intervals when $q > 0$ and either no zeros or exactly one zero (following any prescribed rule) in the intervals when $q < 0$. In some related works (see, e.g., [14, 58, 59]) it was then proved that similar results hold also with respect to other kinds of nonlinear ODEs or under different growth assumptions for $g(s)$ in (2). The interested reader can find in [4, 6, 70] as well as in [8, 14, 24, 45, 52, 53, 54, 57, 59] and the references therein further information and

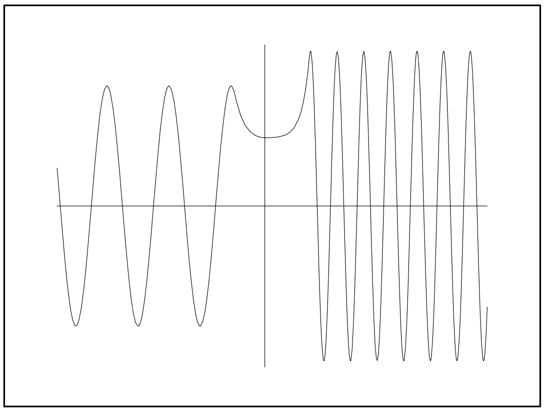

Figure 5: An illustration of the possible behavior of another solution $u(t)$ of (2) along an interval made by two subintervals where $q > 0$ which are separated by a subinterval where $q < 0$. In this latter subinterval $u > 0$ and u' vanishes exactly once.

results about the rich structure and the complex dynamics of the solutions of the nonlinear equation (2) with a sign changing weight $q(t)$.

In the case when $q(t)$ is a periodic function such that its interval of periodicity can be decomposed into a finite number of adjacent subintervals where $q(t)$ alternates its sign, a natural problem turns out to be that of the search of periodic (harmonic and subharmonic) solutions to (2). Results about the existence of infinitely many periodic solutions for the superlinear case were obtained by Butler in his pioneering work [6]. For a nonlinearity having superlinear growth at infinity, Terracini and Verzini in [70] proved the existence of periodic solutions which have an arbitrarily large (but possibly fixed in advance) number of zeros in the intervals when $q > 0$ and precisely one zero in the intervals when $q < 0$. At the best of our knowledge, this is the first result giving evidence of a very complicated behavior for the solutions of the nonlinear Hill's equations with a sign changing weight.

In view of the path-stretching property $(H_{\pm})$ and the above quoted results for the periodic problem, as a next step, one can raise the question whether it is possible to obtain fixed points (as well as periodic points) for a map like the φ considered in $(H_{\pm})$. This goal was achieved in [56] where we obtained a fixed point theorem for planar maps (subsequently reconsidered and generalized in [59, 60]) that, when applied to the case of a periodic weight function, shows that the path-stretching condition $(H_{\pm})$ implies the existence of infinitely many periodic solutions (harmonic and subharmonic) as well as the presence of a chaotic-like dynamics for the solutions of (2).

1.2. Fixed points and periodic points for planar mappings

In order to present the results in [56, 59, 60], first of all we put in a more abstract form the situation described in $(H_{\pm})$. For sake of simplicity, we give here only some of the main features of our approach, with a few comments. The interested reader is referred to [59, 60] for all the details, as well as for some remarks and comments [61] relating our results to some developments of the Conley−Ważewski theory [17, 25, 26, 47, 49, 65, 66, 67, 68, 79, 80, 81, 82]. We also notice that the terminology here is a little different than in [56, 59, 60]. The different choice in the presentation is made in order to employ some terms that can be easily adapted to the higher dimensional case, that is the scope of the second part of this article.

Let X be a Hausdorff topological space. A subset $\mathcal{R} \subseteq X$ is called a *generalized rectangle* or a *two-dimensional cell* if there is a homeomorphism η of $[0,1]^2 \subseteq \mathbb{R}^2$ onto $\mathcal{R} \subseteq X$. Given η, we put in evidence the sets

$$\mathcal{R}_l^- := \eta(\{0\} \times [0,1]), \quad \mathcal{R}_r^- := \eta(\{1\} \times [0,1]), \quad \text{and } \mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^- .$$

We define also the pair

$$\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$$

as a (generalized) *oriented rectangle* or *oriented cell*. The indexes “ l ” and “ r ” stand for “ left ” and “ right ”, respectively. Clearly, any other way of labelling two objects (like “ 0 ” and “ 1 ”) fits well. By setting

$$\mathcal{R}_b^+ := \eta([0,1] \times \{0\}) \quad \text{and } \mathcal{R}_t^+ := \eta([0,1] \times \{1\}),$$

one can also define in a dual manner the “ base ” and the “ top ” of the generalized rectangle $\mathcal{R}$. By convention, we take the orientation through the $[\,\cdot\,]^-$-set.

As an example, the conical shell $\mathcal{R} := W(+) = W(r,R)$ defined in (3) is a generalized rectangle and we can take the homeomorphism η in order to have

$$\mathcal{R}_l^- := W(+) \cap \{x^2 + y^2 = r^2\}, \quad \mathcal{R}_r^- := W(+) \cap \{x^2 + y^2 = R^2\}.$$

Next, we present a fixed point theorem for continuous maps which stretch an oriented rectangle. The main feature of our result is that we look for fixed points which belong to a given subset $\mathcal{D}$ of the domain of the map ψ under consideration. By reason of this requirement, we consider pairs $(\mathcal{D}, \psi)$.
Let $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ be two oriented rectangles (in the same Hausdorff topological space X), let $\psi : X \supseteq D_\psi \to X$ be a continuous map and let $\mathcal{D} \subseteq D_\psi \cap \mathcal{A}$.

We say that $(\mathcal{D}, \psi)$ *stretches* $\widehat{\mathcal{A}}$ *to* $\widehat{\mathcal{B}}$ *along the paths* and write

$$(\mathcal{D}, \psi) : \widehat{\mathcal{A}} \,\triangleleft\!\!\!\rightsquigarrow \widehat{\mathcal{B}},$$

if there is a compact set $\mathcal{K} \subseteq \mathcal{D}$ such that the following conditions are satisfied:

$\psi(\mathcal{K}) \subseteq \mathcal{B}$,

for every path $\sigma \subseteq \mathcal{A}$ with $\sigma \cap \mathcal{A}_l^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{K}$ with $\psi(\gamma) \cap \mathcal{B}_l^- \neq \emptyset$ and $\psi(\gamma) \cap \mathcal{B}_r^- \neq \emptyset$.

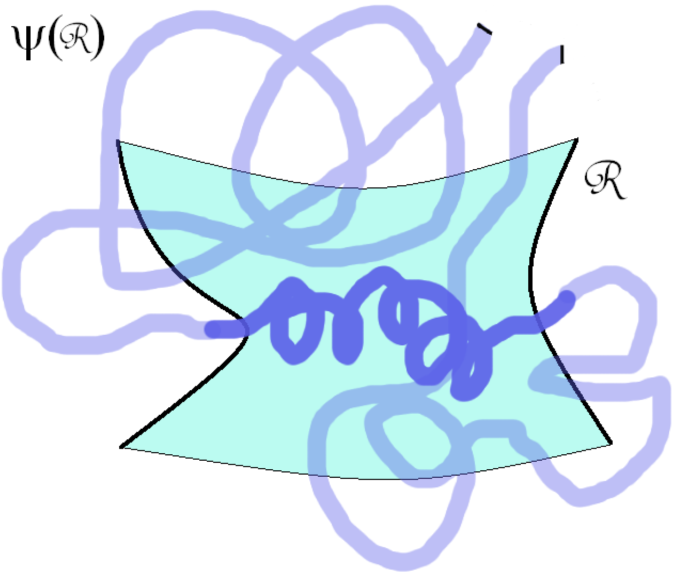

Figure 6: Suppose that a continuous mapping ψ transforms the generalized rectangle $\mathcal{R}$ (the "fat" cheese-like object) to the worm-like set $\psi(\mathcal{R})$. The two components of the $[\,\cdot\,]^{-}$-set of $\mathcal{R}$ as well as their images under ψ are represented by segments (arcs) with a darker color at the contour of the corresponding figures. The stretching property is visualized by the fact that there is a "crossing" of the "worm" through the "cheese".

To put emphasis on the role of the compact set $\mathcal{K}$ in the stretching definition, sometimes we also write

$$(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{B}}.$$

Using a result about plane continua previously applied in a different context also by Conley [11] and Butler [6] (for a proof, see [63] as well as [59, 60]), the following fixed point theorem was obtained.

THEOREM 1. *Let $\widehat{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^{-})$ be an oriented rectangle in X. If $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{R}} \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{R}}$, then there is $w \in \mathcal{D}$ (actually $w \in \mathcal{K}$) such that $\psi(w) = w$.*

An illustration of Theorem 1 is given in Figure 6.

Of course, not all the crossings fit to our purposes. For instance, Figure 7 shows an example of nonexistence of fixed points.

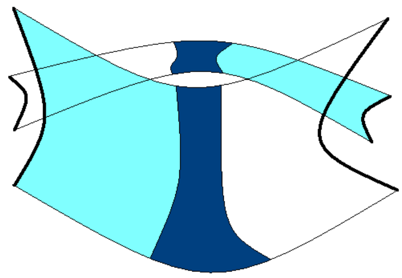

Figure 7: An example showing a case in which the stretching property is not satisfied. The white region is moved to the white one and the darker to the darker one by a homeomorphism which does not have fixed points.

In order to better understand which are the good crossings between $\mathcal{R}$ and $\psi(\mathcal{R})$ that permit to apply our fixed point theorem, we considered the following definitions of slabs of an oriented rectangle (called " slices " in [60]).

Let $\widehat{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ and $\widehat{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ be two oriented cells in X. We say that $\widehat{\mathcal{M}}$ is a *horizontal slab* of $\widehat{\mathcal{N}}$ and write

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}},$$

if $\mathcal{M} \subseteq \mathcal{N}$ and, either

$$\mathcal{M}_l^- \subseteq \mathcal{N}_l^- \quad \text{and} \quad \mathcal{M}_r^- \subseteq \mathcal{N}_r^- \,,$$

or

$$\mathcal{M}_l^- \subseteq \mathcal{N}_r^- \quad \text{and} \quad \mathcal{M}_r^- \subseteq \mathcal{N}_l^- \,.$$

Similarly, we say that $\widehat{\mathcal{M}}$ is a *vertical slab* of $\widehat{\mathcal{N}}$ and write

$$\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}},$$

if $\mathcal{M} \subseteq \mathcal{N}$ and, either

$$\mathcal{M}_b^+ \subseteq \mathcal{N}_b^+ \quad \text{and} \quad \mathcal{M}_t^+ \subseteq \mathcal{N}_t^+ \,,$$

or

$$\mathcal{M}_b^+ \subseteq \mathcal{N}_t^+ \quad \text{and} \quad \mathcal{M}_t^+ \subseteq \mathcal{N}_b^+ \,.$$

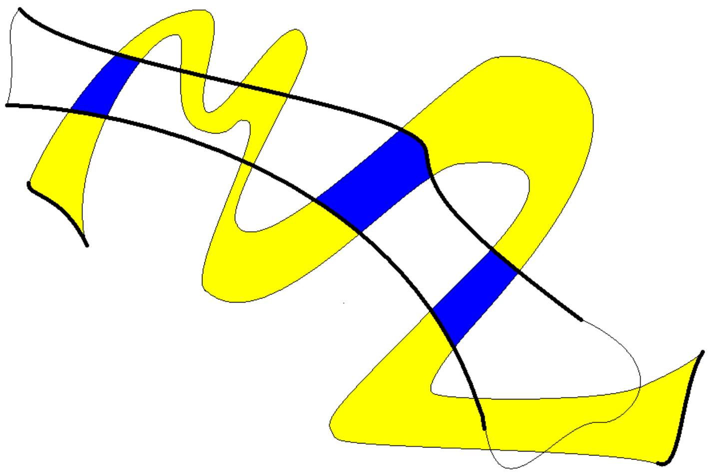

Figure 8: (taken from [60]). Example of oriented cells $\widetilde{\mathcal{R}}$ (white) and $\psi(\widetilde{\mathcal{R}})$ (light color) with crossings into three slabs (darker color). The $[\,\cdot\,]^-$-sets are indicated with a bold line. Among the five cells which are the connected components of the intersection $\psi(\mathcal{R}) \cap \mathcal{R}$, only the three painted with darker color are suitable to play the role of the $\mathcal{M}$'s for the application of Theorem 2.

Now, given three oriented rectangles (cells) in X, which are denoted by $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$, $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ and $\widehat{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$, we say that $\widehat{\mathcal{B}}$ *crosses* $\widehat{\mathcal{A}}$ *in* $\widehat{\mathcal{M}}$ and write

$$\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\},$$

if

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{A}} \quad \text{and} \quad \widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{B}}.$$

The symbol $\pitchfork$ is borrowed from the case of transversal intersections, however we point out that in our situation (although confined to sets which are two-dimensional in nature) we don't need any smoothness assumption. In fact, our setting is that of topological spaces. From the above definitions and by Theorem 1 the following result easily follows.

THEOREM 2. *Let* $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *be oriented cells in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Diamond\!\!\rightsquigarrow \widehat{\mathcal{B}}$ *and there is an oriented cell* $\widehat{\mathcal{M}}$ *such that* $\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}$, *then there exists* $w \in \mathcal{K} \cap \mathcal{M}$ *such that* $\psi(w) = w$.

A situation like that depicted in Figure 8 in which we have more than one good intersection between the domain and the image of a homeomorphism is typical of the horseshoe maps and thus, as a next step, we can look for the existence of a complete dynamics on m symbols, where $m \geq 2$ is the number of crossings. With this respect, we have to recall that a very general topological theory has been developed in the recent years by Kennedy, Yorke and their collaborators in a series of fundamental papers in this area (see [33, 35, 36, 37, 38]). Our goal instead is to take advantage of our simplified framework in which we consider only sets which are homeomorphic to a square and prove the existence of periodic points of any order. To this aim, we have to apply Theorem 1 to the iterates of the map ψ and select carefully some subset of the domain $\mathcal{D}$ in order to find "true" periodic points (for instance those with a long minimal period). First, however, we need a further definition, taken from [39].

We say that $\psi : X \supseteq D_\psi \to X$ has a *chaotic dynamics of coin-tossing type on k symbols* if $k \geq 2$ and there is a metrizable space $Z \subseteq X$ and k pairwise disjoint compact sets $W_1, \dots, W_k \subseteq Z \cap D_\psi$ such that, for each two-sided sequence $(s_n)_{n\in\mathbb{Z}}$ with

$$s_n \in \{1, \dots, k\}, \quad \forall\, n \in \mathbb{Z},$$

there is a sequence of points $(z_n)_{n\in\mathbb{Z}}$ with

$$z_n \in W_{s_n} \quad \text{and} \quad z_{n+1} = \psi(z_n), \quad \forall\, n \in \mathbb{Z}.$$

In other words, any possible itinerary on the sets $W_1 \dots, W_k$ is followed by some point.

As an auxiliary tool in order to obtain at the same time the existence of such kind of chaotic trajectories and also the fact that all the periodic itineraries can be followed by some periodic point, we have the following theorem (see [59]), which is also reminiscent of some results in [34] and [73].

THEOREM 3. *Assume that there is a (double) sequence of oriented rectangles $(\widehat{\mathcal{A}}_k)_{k\in\mathbb{Z}}$ and maps $((\mathcal{D}_k, \psi_k))_{k\in\mathbb{Z}}$, with $\mathcal{D}_k \subseteq \mathcal{A}_k$, such that $(\mathcal{D}_k, \psi_k) : \widehat{\mathcal{A}}_k \Subset\!\!\rightsquigarrow \widehat{\mathcal{A}}_{k+1}$ for each $k \in \mathbb{Z}$. Then the following conclusions hold:*

(a_1) *There is a sequence $(w_k)_{k\in\mathbb{Z}}$ with $w_k \in \mathcal{D}_k$ and $\psi_k(w_k) = w_{k+1}$ for all $k \in \mathbb{Z}$;*

(a_2) *For each $j \in \mathbb{Z}$ there is a compact and connected set $\mathcal{C}_j \subseteq \mathcal{D}_j$ satisfying*

$$\mathcal{C}_j \cap (\mathcal{A}_j)^+_b \neq \emptyset, \quad \mathcal{C}_j \cap (\mathcal{A}_j)^+_t \neq \emptyset$$

and such that for each $w \in \mathcal{C}_j$ there is a sequence $(y_\ell)_{\ell\geq j}$, with $y_\ell \in \mathcal{D}_\ell$ and $y_j = w$, $y_{\ell+1} = \psi_\ell(y_\ell)$ for each $\ell \geq j$;

(a_3) *If there are integers h, k with $h < k$ such that $\widehat{\mathcal{A}}_h = \widehat{\mathcal{A}}_k$, then there is a finite sequence $(z_i)_{h\leq i\leq k}$, with $z_i \in \mathcal{D}_i$ and $\psi_i(z_i) = z_{i+1}$ for each $i = h, \dots, k-1$, such that $z_h = z_k$, that is, z_h is a fixed point of $\psi_{k-1} \circ \cdots \circ \psi_h$.*

The proof of (a_1) and partially also that of (a_2) could be given by adapting to our setting the argument in [33, Lemma 3 and Proposition 5]. As to (a_3), we apply Theorem 1. The existence of the continuum (compact connected set) $\mathcal{C}_j$ in (a_2) is a byproduct of the topological lemma that we employ also in the proof of Theorem 1. We give all the main details along the proof of Theorem 11 in Section 3.2 and refer to [59] for more information.

From Theorem 3 several corollaries can be obtained. Now we just recall a few of them which are taken from [59] and [60]. Due to space limitation, we don't give here other applications to ODEs. We just mention the recent thesis by Covolan [12] which contains a detailed description of the results in [33] and those in [59, 60] and where it is shown that our theorem, when applied to the search of fixed points for the iterates of a two-dimensional map, may add some useful information (about the existence of periodic points) to the conclusions obtained in some recent articles (like, e.g., [30, 75, 76, 77]), where the theory of topological horseshoes was applied to prove the existence of a chaotic dynamics in various different models.

THEOREM 4. *Suppose that* $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *are oriented cells in* X. *If* $(\mathcal{D}, \mathcal{K}, \psi) : \widehat{\mathcal{A}} \Leftarrow\!\!\!\Rightarrow \widehat{\mathcal{B}}$ *and there are* $k \geq 2$ *oriented cells* $\widehat{\mathcal{M}}_1 \dots, \widehat{\mathcal{M}}_k$ *such that*

$$\widehat{\mathcal{M}}_i \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}, \quad \textit{for } i = 1, \dots, k,$$

with

$$\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{K} = \emptyset, \ \textit{for all } \ i \neq j, \ \textit{with } \ i, j \in \{1, \dots, k\},$$

then the following conclusion holds:

(b_1) ψ *has a chaotic dynamics of coin-tossing type on* k *symbols (with respect to the sets* $W_i = \mathcal{K}_i = \mathcal{K} \cap \mathcal{M}_i$*);*

(b_2) *For each one-sided infinite sequence* $\mathbf{s} = (s_0, s_1, \dots, s_n, \dots) \in \{1, \dots, k\}^{\mathbb{N}}$ *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{K}_{s_0}$ *with*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})^+_l \neq \emptyset, \quad \textit{and } \ \mathcal{C}^{\mathbf{s}} \cap (\mathcal{M}_{s_0})^+_r \neq \emptyset,$$

such that for each point $w \in \mathcal{C}^{\mathbf{s}}$, *the sequence*

$$z_{j+1} = \psi(z_j), \quad z_0 = w, \quad \textit{for } \ j = 0, 1, \dots, n, \dots$$

satisfies

$$z_j \in \mathcal{K}_{s_j}, \ \forall \, j = 0, 1, \dots, n, \dots \quad ;$$

(b_3) ψ *has a fixed point in each set* $\mathcal{K}_i := \mathcal{M}_i \cap \mathcal{K}$ *and, for each finite sequence* $(s_0, s_1, \dots, s_m) \in \{1, \dots, k\}^{m+1}$, *with* $m \geq 1$, *there is at least one point* $z^* \in \mathcal{K}_{s_0}$ *such that the position*

$$z_{j+1} = \psi(z_j), \quad z_0 = z^*, \quad \textit{for } \ j = 0, 1, \dots, m$$

defines a sequence of points with

$$z_j \in \mathcal{K}_{s_j}, \ \forall \, j = 0, 1, \dots, m \quad \textit{and } \ z_{m+1} = z^*.$$

As a comment to this result, we look again at Figure 8 and observe that, besides having a coin-tossing dynamics on three symbols, we have also the existence of fixed points in each of the three darker regions and, moreover, once we have labelled these

Figure 9: (compare to Figure 6). Now the worm crosses nicely the cheese for two times and we obtain a complete dynamics on two symbols (as well as periodic points of any period).

three regions with corresponding symbols, say 1, 2, 3, we have also that to any periodic sequence in $\{1, 2, 3\}$ we can find a corresponding periodic point for the map ψ which follows (along ψ) the itinerary described by the periodic sequence. Figure 8 illustrates the case in which ψ is a homeomorphism and the three good intersections are pairwise disjoint. Actually, our Theorem 3 is more flexible (cf. Corollary 2 below) and it allows to come to the same conclusion by a careful selection of disjoint subsets of the domain of the map (which is not necessarily a homeomorphism). As an illustration of this remark, let us consider Figure 9.

In this direction, two possible corollaries of Theorem 3 are the following. They correspond, respectively: (a) to the property $(H_{\pm})$ which holds with respect to the solutions of system (1) and the two conical shells $W(+)$ and $W(-)$, and (b) to the example depicted in Figure 9. We refer to [59] for the proof of both the corollaries, as well as for the proof of a more general result from which they both come.

COROLLARY 1. *Let* $\widehat{\mathcal{R}}_0 = (\mathcal{R}_0, \mathcal{R}_0^-)$ *and* $\widehat{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$, *be two oriented rectangles with* $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$ *and such that*

$$\psi : \widehat{\mathcal{R}}_i \Leftarrow\!\!\rightsquigarrow \widehat{\mathcal{R}}_j\,, \quad \forall\, i, j \in \{0, 1\}.$$

Then the following conclusions hold:

(c_1) ψ *has a dynamics of coin-tossing type with respect to the pair* $(\mathcal{R}_0, \mathcal{R}_1)$.;

(c_2) *For every sequence* $\mathbf{s} = (s_n)_n$, *with* $s_n \in \{0, 1\}$ *for each* $n \geq 0$, *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{R}_{s_0}$ *satisfying*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{R}_{s_0})^+_b \neq \emptyset, \quad \mathcal{C}^{\mathbf{s}} \cap (\mathcal{R}_{s_0})^+_t \neq \emptyset$$

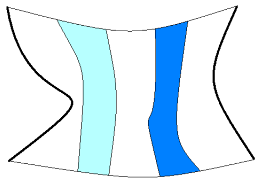

Figure 10: (compare to Figure 9). The two good crossings come from two disjoint subregions (those painted with a darker color) of the domain.

and such that for each $w \in \mathcal{C}^{\mathbf{s}}$ *there is a sequence* $(y_n)_n$, *with* $y_n \in \mathcal{R}_{s_n}$ *and* $y_0 = w$, $y_{n+1} = \psi(y_n)$ *for each* $n \geq 0$;

(c_3) *For each finite sequence* $(s_0, s_1, \ldots, s_k) \in \{0,1\}^{k+1}$ *with* $s_k = s_0$, *there is a* $w_0 \in \mathcal{R}_{s_0}$ *which generates a finite sequence* $(w_\ell)_{0\leq \ell \leq k}$ *such that*

$$\psi(w_\ell) = w_{\ell+1} \in \mathcal{R}_{s_\ell}\,, \quad \forall\, \ell = 0, \ldots, k-1$$

and $w_k = w_0$.

COROLLARY 2. *Let* $\widehat{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ *be an oriented cell and suppose that there are two disjoint compact sets* $\mathcal{D}_0$, $\mathcal{D}_1 \subseteq \mathcal{R} \cap D_\psi$ *such that*

$$(\mathcal{D}_i\,,\,\psi) : \widehat{\mathcal{R}} \Leftarrow\!\!\!\rightsquigarrow \widehat{\mathcal{R}}\,, \quad \forall\, i \in \{0,1\}.$$

Then the following conclusions hold:

(d_1) ψ *has a dynamics of coin-tossing type with respect to the pair* $(\mathcal{D}_0, \mathcal{D}_1)$;

(d_2) *For every sequence* $\mathbf{s} = (s_n)_n$, *with* $s_n \in \{0,1\}$ *for each* $n \geq 0$, *there is a continuum* $\mathcal{C}^{\mathbf{s}} \subseteq \mathcal{D}_{s_0}$ *satisfying*

$$\mathcal{C}^{\mathbf{s}} \cap (\mathcal{R})^+_b \neq \emptyset, \quad \mathcal{C}^{\mathbf{s}} \cap (\mathcal{R})^+_t \neq \emptyset$$

and such that for each $w \in \mathcal{C}^{\mathbf{s}}$ *there is a sequence* $(y_n)_n$, *with* $y_n \in \mathcal{D}_{s_n}$ *and* $y_0 = w$, $y_{n+1} = \psi(y_n)$ *for each* $n \geq 0$;

(d_3) *For each finite sequence* $(s_0, s_1, \ldots, s_k) \in \{0,1\}^{k+1}$ *with* $s_k = s_0$, *there is a* $w \in \mathcal{D}_{s_0}$ *which generates a finite sequence* $(w_\ell)_{0\leq \ell \leq k}$ *such that*

$$\psi(w_\ell) = w_{\ell+1} \in \mathcal{D}_{s_\ell}\,, \quad \forall\, \ell = 0, \ldots, k-1$$

and $w_k = w_0$.

1.3. Extensions to higher dimensions

Suppose now that we have a (non-autonomous) nonlinear differential system in $\mathbb{R}^N$ (with N possibly strictly larger than 2)

$$x' = F(t, x) \tag{5}$$

with $F : \mathbb{R}\times\Omega \to \mathbb{R}^N$ a continuous vector field (more general Carathéodory hypothesis could be considered as well) which satisfies a local Lipschitz condition with respect to $x \in \Omega$, where Ω is an open subset of $\mathbb{R}^N$. Assume also that there is $T > 0$ such that $F(t + T, x) = F(t, x)$ for every $(t, x) \in \mathbb{R} \times \Omega$. Then we can define the Poincaré's operator

$$\mathcal{Q} : \Omega \supseteq D_{\mathcal{Q}} \to \mathbb{R}^N, \qquad z_0 \mapsto \zeta(t_0 + T; t_0, z_0),$$

where $\zeta(\cdot\,; t_0, z_0)$ is the solution of (5) satisfying the initial condition $x(t_0) = z_0$. From the fundamental theory of ODEs we know that $D_{\mathcal{Q}}$ is an open subset of Ω and $\mathcal{Q}$ is a homeomorphism of $D_{\mathcal{Q}}$ onto its image $\mathcal{Q}(D_{\mathcal{Q}})$. The problems that we want to discuss concern:

(A) the search of T-periodic solutions of equation (5),

(B) the existence of "true" subharmonic solutions (that is, mT-periodic solutions of (5), for some $m \geq 2$, that are not jT-periodic for every $j = 1, \ldots, m - 1$),

(C) the evidence of a complex behavior of the solutions of (5). For instance, the existence of two sets like the sets $W(+)$ and $W(-)$ found in [55] for the systems $x_1' = x_2$, $x_2' = -q(t)g(x_1)$, where the trajectories can arbitrarily get in and out respecting any a priori fixed coin-tossing sequence of indexes (say 0 or 1 or "left" and "right") labelling the two sets.

For these goals, following a classical method [40] we rely on the study of the Poincaré's map $\mathcal{Q}$ and its iterates, or, in other words, we investigate the discrete dynamical system associated to $\mathcal{Q}$. Actually, we study general maps ψ which are not necessarily homeomorphisms (even continuity on their whole domain will be not assumed; of course, we'll need continuity on some "interesting" subsets of their domain) for which we discuss the existence of fixed points, periodic points and chaotic-like dynamics. Our main assumption is a *stretching condition along the paths* that extends to a broader setting the property $(H_{\pm})$ recalled above as well as it generalizes to higher dimension the previous treatment for the two-dimensional case considered in [56, 59, 60].

Maps which act on a topological rectangle as an expansion along some directions and a compression along the remaining ones have been widely studied in the literature. They appear, for instance, in the construction of Markov partitions (cf. [51, Appendix 2, pp.169–177]) and therefore they are crucial in the study of the multidimensional chaos. Another area in which such maps are involved concerns the search of periodic solutions for periodic non-autonomous differential systems which are partially dissipative (cf. [2, 3, 41]). In order to set a suitable list of hypotheses for a fixed point theorem concerning a continuous mapping $\psi = (\psi_u, \psi_s)$ defined on a N-dimensional rectangle $\mathcal{R} = B_u[0, 1] \times B_s[0, 1] \subseteq \mathbb{R}^N = \mathbb{R}^u \times \mathbb{R}^s$ (where we think at the u-components and the s-components as the unstable-expansive and the stable-compressive ones, respectively), a reasonable choice of assumptions to put on the map ψ along the s-component will be that of taking conditions that reduce to those of the Brouwer or of the Rothe fixed point theorems (or to analogous ones) in the special case when $u = 0$ and $s = \mathrm{N}$. On the other hand, it seems perhaps less evident which could be the best choice of assumptions to express the expansive effect along the s-components. With this respect, both conditions on the norm (like in [2, 3]) and componentwise conditions (like in [80, 81]) have been assumed. As we have already explained with some details in the first part of this Introduction, motivated by the stretching property $(H_{\pm})$ discovered in [55] for equation (2) we obtained in [56] a fixed point theorem for planar mappings where the main hypothesis requires that the map expands the paths connecting two opposite sides of a topological rectangle. Further generalizations were then given in [59, 60], but still for a setting which is basically two-dimensional in nature. We recall that an expansive condition for paths connecting the opposite faces of a N-dimensional rectangle was also considered by Kampen in [32], allowing an arbitrary number of expansive directions (see [32, Corollary 4]). However, when reduced to the special case $\mathrm{N} = 2$, Kampen's result and ours seem to differ in some relevant points. In particular, a crucial assumption of our fixed point theorem in [56] allows the map to be defined only on some subsets of the rectangle and, moreover, even when the the mapping is defined on the whole rectangle, the assumptions in [32] and those in [56] about the compressing direction are basically different. One of the main features that we ask to a fixed point theorem for expansive-compressive mappings is to depend on hypotheses that can be easily reproduced for compositions of maps. This, in turns, permits to apply the theorem to the iterates of ψ and thus obtain results about the existence of nontrivial periodic points. Since our path-stretching property well fits also with respect to this requirement (of course, it is not the only one; in fact, nice alternative approaches are available in literature), we want to address our investigations toward a suitable extension of such property to the case $\mathrm{N} > 2$.

1.4. Contents

After such a long introduction in which we surveyed some of our preceding results for the two-dimensional case, we are ready to present some new developments in the higher dimensional setting. Then the rest of this paper is organized as follows. In Section 2 we present our main result (Theorem 6) which is a fixed point for a compact map defined on a subset of a cylinder in a normed space. In order to simplify the exposition, we confine ourselves to the idealized situation in which we split our space as a product $\mathbb{R} \times X$ and indicate its elements as pairs (t, x), so that we can easily express our main assumption as an hypothesis of expansion of the paths contained in the cylinder $\mathcal{B}[a, R] = [-a, a] \times B[0, R]$ along the t-direction. The principal tool for the proof of our basic fixed point theorem is the Leray$-$Schauder continuation theorem in its strongest form asserting the existence of a continuum of solution-pairs for a nonlinear operator equation depending on a real parameter (Théorème Fondamental [42]). Such result, with its variants and extensions, is one of the main theorems of the Leray$-$Schauder topological degree theory and it has found several important applications to bifurcation (and co-bifurcation) theory [20, 21, 22, 62], to the investigation of the structure of the solution set for parameter dependent equations [19, 31, 43] and to the study of nonlinear problems in absence of a priori bounds [7, 10, 44, 46]. Thus, as a byproduct of our proof of Theorem 6 we also provide a new proof of the main fixed point result for planar maps in [57], without the need to rely on properties of plane topology.

Then, we give some variants of Theorem 6 which are analogous to the different forms in which the Schauder fixed point theorem is usually presented. In Section 3 we investigate an abstract fixed point property for topological spaces which express in a more abstract fashion the content of Theorem 6 and its variants. An analysis of such a new fixed point property allows (like in the case of the classical fixed point property) to prove that it is invariant under homeomorphisms as well as it is preserved under continuous retractions. This in turns, permits to obtain some general results in which we produce fixed points for maps defined on topological cylinders. By topological cylinders we mean sets which are obtained from a cylinder like $\mathcal{B}[a, R]$ after a deformation given by a homeomorphism. For instance, the following result (see Corollary 3 of Section 3.1) is obtained.

THEOREM 5. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[0,1] \times K$, via a homeomorphism $h : Z \to [0,1] \times K$. Define*

$$Z_l^- := h^{-1}(\{0\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Suppose that $\psi : Z \supseteq D_\psi \to Z$ is a map which is continuous on a set $\mathcal{D} \subseteq D_\psi$ and assume the following property is satisfied:

> *there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\gamma) \cap Z_l^- \neq \emptyset$, $\phi(\gamma) \cap Z_r^- \neq \emptyset$.*

Then there exists a fixed point $\tilde{z}$ of ψ with $\tilde{z} \in \mathcal{D}$ (actually, $\tilde{z} \in \mathcal{W}$).

The possibility of studying topological cylinders (instead of topological rectangles like in [59, 60]) open the way toward an extension of the results about oriented rectangles presented in Section 1.3 to higher dimensional objects possessing a privileged direction. Thus we conclude the paper with a list of possible applications to maps which stretch the paths along a direction in a $(1, \mathrm{N}-1)$-*rectangular cell* (see Section 4.1 for the corresponding definition).

1.5. Notation

Throughout the paper, the following notation is used. Let Z be a topological space and let $A \subseteq B \subseteq Z$. By $\mathrm{cl}_B A$ and $\mathrm{int}_B A$ we mean, respectively, the closure and the interior of A relatively to B (that is, as a subset of the topological space B with the topology inherited by Z). When no confusion may occur, we also set $\mathrm{cl}A$ and $\mathrm{int}A$ for $\mathrm{cl}_Z A$ and $\mathrm{int}_Z A$, respectively.

For a metric space (X, d), we denote by $B(x_0, R) := \{x \in X : d(x, x_0) < r\}$ the open ball of center $x_0 \in X$ and radius $r > 0$ and by $B[x_0, R] := \{x \in X : d(x, x_0) \leq r\}$ the corresponding closed ball. Given a map $\psi : X \supseteq D_\psi \to Y$, with X, Y metric spaces and a given subset $\mathcal{D}$ of the domain D_ψ of ψ, we say that ψ is compact on $\mathcal{D}$ if it is continuous on $\mathcal{D}$ and $\psi(\mathcal{D})$ is relatively compact in Y, that is, $\mathrm{cl}(\psi(\mathcal{D}))$ is compact.

Let Z be a topological space, let $\theta_1 : [a_1, b_1] \to Z$ and $\theta_2 : [a_2, b_2] \to Z$ be two continuous mappings (parameterized curves). We write $\theta_1 \sim \theta_2$ if there is a homeomorphism h of $[a_1, b_1]$ onto $[a_2, b_2]$ (a change of variable in the parameter) such that $\theta_2(h(t)) = \theta_1(t),\ \forall\, t \in [a_1, b_1]$. It is easy to check that $\sim$ is in fact an equivalence relation and that $\theta_1([a_1, b_1]) = \theta_2([a_2, b_2])$ whenever $\theta_1 \sim \theta_2$. By a *path* γ *in* Z we mean (formally) the equivalence class $\gamma = [\theta]$ of a continuous parameterized curve $\theta : [a, b] \to Z$. In this case, with small abuse in the notation, we write $\gamma \subseteq Z$. Since the image set $\theta([a, b])$ is the same for each $\theta : [a, b] \to Z$ with $\gamma = [\theta]$, the set

$$\bar{\gamma} := \{\theta([a, b]) : \theta \in \gamma\}$$

is well defined. Given a set $A \subseteq Z$ and a path $\gamma \subseteq Z$, we write $\gamma \cap A \neq \emptyset$ to mean that $\bar{\gamma} \cap A \neq \emptyset$, that is, for every parameterized curve θ representing γ we have that $\theta(t) \in A$ for some t in the interval-domain of θ. Given a path $\sigma \subseteq Z$, we say that $\gamma \subseteq Z$ is a *sub-path* of σ and write $\gamma \subseteq \sigma$ if there is $\theta : [a, b] \to Z$ with $[\theta] = \sigma$ such that the restriction $\theta|_{[c,d]}$, for some $[c, d] \subseteq [a, b]$, represents γ. According to these positions, given the paths $\gamma, \sigma \subseteq Z$ and a set $W \subset Z$, the condition $\gamma \subseteq \sigma \cap W$, means that γ *is a sub-path of* σ *with values in* W. If Z, Y are topological spaces and $\phi : Z \supseteq D_\phi \to Y$ is a continuous map, then for any path $\gamma \subseteq D_\phi$ and $\theta : [a, b] \to D_\phi$ such that $[\theta] = \gamma$, we have that $\phi \circ \theta : [a, b] \to Y$ is a continuous map. It is easy to check that $\phi \circ \theta_1 \sim \phi \circ \theta_2$ when $\theta_1 \sim \theta_2$ and therefore $\phi(\gamma) := [\phi \circ \theta]$ is well defined.

At last we recall a known definition. Let Z be a topological space. We say that Z is *arcwise connected* if, given any two points $P, Q \in Z$ with $P \neq Q$, there is a continuous map $\theta : [a, b] \to Z$ such that $\theta(a) = P$ and $\theta(b) = Q$. In such a situation, we'll also write $P, Q \in \gamma$, where $\gamma = [\theta]$. In the case of a Hausdorff topological space Z, the image set $\theta([a, b])$ turns out to be a locally connected metric continuum (a Peano space according to [29]). Then, the above definition of arcwise connectedness is equivalent to the fact that, given any two points $P, Q \in Z$ with $P \neq Q$, there exists an *arc* (that is the homeomorphic image of a compact interval) contained in Z and having P and Q as extreme points (see, e.g., [18, p.29], [29, pp.115–131] or [71]).

2. A fixed point theorem in normed spaces and its variants

2.1. Main results

Let $(X, \|\cdot\|)$ be a normed space and suppose that

$$\phi = (\phi_1, \phi_2) : \mathbb{R} \times X \supseteq D_\phi \to \mathbb{R} \times X$$

is a map (not necessarily continuous on its whole domain D_ϕ even if, in the sequel, we assume the continuity of ϕ on some relevant subset $\mathcal{D}$ of D_ϕ).

Let $\mathcal{D} \subseteq D_\phi$ be a given set (in our applications we'll usually take $\mathcal{D}$ closed, for instance, $\mathcal{D} = \mathcal{W}$ of Theorem 6 below, but such an assumption for the moment is not required). We are looking for fixed points of ϕ belonging to $\mathcal{D}$, i.e., we want to prove the existence of a pair $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{D}$ which solves the equation

$$\begin{cases} t = \phi_1(t, x) \\ x = \phi_2(t, x). \end{cases}$$

Our first result is the following.

THEOREM 6. *Let* $\mathcal{B}[a, R] := [-a, a] \times B[0, R]$ *and define*

$$\mathcal{B}_l := \{(-a, x) : \|x\| \le R\}, \quad \mathcal{B}_r := \{(a, x) : \|x\| \le R\}$$

the left and the right bases of the cylinder $\mathcal{B}[a, R]$. *Assume that*

$$\phi \textit{ is compact on } \mathcal{D} \cap \mathcal{B}[a, R]$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{B}[a, R]$ *such that the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{B}[a, R]$ *with* $\sigma \cap \mathcal{B}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{B}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{B}[a, R]$ *and* $\phi(\gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{B}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. First of all we observe that, as a consequence of Dugundji Extension Theorem and Mazur's Lemma (see, e.g., [64, p.22] in the case of Banach spaces or [16, Th.2.5, p.56] for a general situation), there exists a *compact* operator $\tilde{\phi}$ defined on $\mathbb{R} \times X$ which extends ϕ restricted to $\mathcal{W}$, i.e.

$$\tilde{\phi} : \mathbb{R} \times X \to \mathbb{R} \times X, \quad \tilde{\phi}|_{\mathcal{W}} = \phi|_{\mathcal{W}}.$$

Consider also the projection

$$P_R : X \to B[0, R], \quad P_R(x) := x \, \min\{1, R\,\|x\|^{-1}\}$$

and define the compact operator

$$\psi = (\psi_1, \psi_2), \quad \psi_1(t, x) := \tilde{\phi}_1(t, x), \quad \psi_2(t, x) := P_R(\tilde{\phi}_2(t, x))$$

Note that if $\bar{z} = (\bar{t}, \bar{x})$ is a fixed point of ψ with

$$\bar{z} \in \mathcal{W} \quad \text{and} \quad \phi_2(\bar{t}, \bar{x}) \in B[0, R], \tag{6}$$

then $\bar{t} = \psi_1(\bar{z}) = \tilde{\phi}_1(\bar{z}) = \phi_1(\bar{z})$ and $\bar{x} = \psi_2(\bar{z}) = P_R(\tilde{\phi}_2(\bar{z})) = P_R(\phi_2(\bar{z})) = \phi_2(\bar{t}, \bar{x})$, so that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ.

We study now the auxiliary fixed point problem

(7) $$x = \psi_2(t, x), \quad x \in X$$

where we take, for a moment, $t \in [-a, a]$ as a parameter.

Observe that, by definition, $\psi_2(t, x) \in B[0, R]$ for every (t, x) and therefore, for any $r > R$, it follows that

$$x - \psi_2(t, x) \neq 0, \quad \forall\, t \in [-a, a],\ \forall\, x \in \partial B(0, r).$$

Thus the Leray−Schauder topological degree

$$d_0 := \deg(I - \psi_2(t, \cdot), B(0, r), 0)$$

is well defined and is constant with respect to $t \in [-a, a]$. Using the compact homotopy $h_\lambda(x)$ defined by

$$(\lambda, x) \mapsto x - \lambda\psi_2(t, x), \quad \text{with } \lambda \in [0, 1] \text{ and } x \in B[0, r]$$

we find that $h_\lambda(x) \neq 0$ for every $\lambda \in [0, 1]$ and $x \in \partial B(0, r)$ and therefore $d_0 = \deg(I, B(0, r), 0) = 1$. Hence, the Leray−Schauder Théorème Fondamental [42] implies that the solution set

$$\Sigma := \{(t, x) \in [-a, a] \times B(0, r) : x = \psi_2(t, x)\}$$

is nonempty and contains a continuum (compact and connected set) $\mathcal{S}$ such that

$$p_1(\mathcal{S}) = [-a, a],$$

where we have denoted by $p_1 : \mathbb{R} \times X \to \mathbb{R}$, $p_1(t, x) = t$, the projection of the product space onto its first factor (see also [44] for more information about this fundamental result). Since the projection of $\mathcal{S}$ onto the t-axis covers the interval $[-a, a]$, we obtain

$$\mathcal{S} \cap \mathcal{B}_l \neq \emptyset, \quad \mathcal{S} \cap \mathcal{B}_r \neq \emptyset.$$

By the definition of P_R it is clear also that

(8) $$p_2(\mathcal{S}) \subseteq B[0, R] \subseteq B(0, r),$$

where we have denoted by $p_2 : \mathbb{R} \times X \to X$, $p_2(t, x) = x$, the projection of the product space onto its second factor.

Let now $\varepsilon \in]0, r - R[$ be a fixed number and consider a covering of $\mathcal{S}$ by a finite number of open balls of the form $]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \varepsilon)$, with $(t_i, x_i) \in \mathcal{S}$. Without loss of generality, we can suppose that $-a \le t_1 < t_2 \ldots t_{i-1} < t_i \ldots t_N \le a$, where N is the number of the balls required for the covering. The set

$$\mathcal{U}_\varepsilon := \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, \varepsilon) \subseteq] - a - \varepsilon, a + \varepsilon[\times B(0, r),$$

is open and connected. Hence, it is arcwise connected as well. Therefore, there is a continuous map $\theta : [0,1] \to \mathcal{U}_\varepsilon$ with $\theta(0) \in \mathcal{B}_l$ and $\theta(1) \in \mathcal{B}_r$ and, without loss of generality (i.e., possibly cutting off some points of the interval and changing the parameter for the curve) we can also assume that

$$\theta_1(s) := p_1(\theta(s)) \in [-a, a], \quad \forall s \in [0,1].$$

Next, we define the new curve $\zeta(s) = (\zeta_1(s), \zeta_2(s))$, with

$$\zeta_1(s) := p_1(\theta(s)) = \theta_1(s), \quad \zeta_2(s) := P_R(p_2(\theta(s))) = P_R(\theta_2(s))$$

and observe that $\zeta(\cdot)$ satisfies the following properties:

(I_1) $\zeta(s) \in \mathcal{V}_\varepsilon \cap \mathcal{B}[a, R], \quad \forall s \in [0,1]$;

(I_2) $\zeta(0) \in \mathcal{B}_l$ and $\zeta(1) \in \mathcal{B}_r$;

where we have set

$$\mathcal{V}_\varepsilon := \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i + \varepsilon[\times B(x_i, 2\varepsilon).$$

To check (I_1), let us set $x := \zeta_2(s)$ and assume that $\|x\| > R$ as well as $x \in B(x_i, \varepsilon)$, for some i. Then,

$$\begin{aligned}\left\| \frac{Rx}{\|x\|} - x_i \right\| &= \|Rx - \|x\|\, x_i\| / \|x\| \\ &\leq \frac{R}{\|x\|} \|x - x_i\| + (\|x\| - R) \frac{\|x_i\|}{\|x\|} < 2\varepsilon.\end{aligned}$$

The proofs of all the remaining cases for the verification of (I_1) are obvious.

From (I_1) and (I_2), it follows that the path $\sigma := [\zeta]$ is contained in the cylinder $\mathcal{B}[a, R]$ and it has a nonempty intersection with the left and the right bases of $\mathcal{B}[a, R]$. Then, by hypothesis (H), we know that there exists a sub-path γ of σ, such that $\gamma \subseteq \mathcal{W}$ with $\phi(\gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{B}_r \neq \emptyset$. Let $\xi = (\xi_1, \xi_2) : [0,1] \to \mathbb{R} \times X$ be a continuous map such that $[\xi] = \gamma$. By the above assumptions, we have that

(J_1) $\xi(s) \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \forall s \in [0,1]$;

(J_2) $\phi(\xi(0)) \in \mathcal{B}_l$ and $\phi(\xi(1)) \in \mathcal{B}_r$;

(J_3) $\phi(\xi(s)) \in \mathcal{B}[a, R], \quad \forall s \in [0,1]$;

are satisfied.

We consider now the continuous map $g : [0,1] \ni s \mapsto \xi_1(s) - \phi_1(\xi(s))$, where $\xi_1(s) = p_1(\xi(s))$. Since $\xi(s) \in \bar{\gamma} \subseteq \bar{\sigma}$, we have that $\xi_1(0) \geq -a$ and therefore $g(0) \geq -a - (-a) = 0$. Similarly, one can check that $g(1) \leq a - a = 0$. By Bolzano's Theorem, we conclude that there exists some $\hat{s} = \hat{s}_\varepsilon \in [0,1]$ such that, setting

$$\hat{t} = \hat{t}_\varepsilon := \xi_1(\hat{s}), \quad \hat{x} = \hat{x}_\varepsilon := \xi_2(\hat{s}), \quad \hat{z} = \hat{z}_\varepsilon := (\hat{t}, \hat{x}),$$

we find that

$$\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \hat{t} = \phi_1(\hat{z}), \quad \phi_2(\hat{z}) \in B[0, R].$$

By the definition of $\tilde{\phi}$ and ψ, it is clear that

$$\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}, \quad \hat{t} = \psi_1(\hat{z}), \quad \phi_2(\hat{z}) \in B[0, R].$$

Moreover, for each $\hat{z} \in \mathcal{V}_\varepsilon \cap \mathcal{W}$, there is $\hat{z}_i \in \mathcal{S}$ such that $||\hat{z} - \hat{z}_i|| < 2\varepsilon$.
Then, letting $\varepsilon = \varepsilon_n \searrow 0$ and passing to a subsequence on the corresponding $\hat{z}_n$'s (thanks to the compactness of $\mathcal{S}$), we can find a point

$$\bar{z} = (\bar{t}, \bar{x}) \in \mathcal{S}$$

such that (by the continuity of ϕ and ψ and the closure of $\mathcal{W}$)

$$\bar{z} \in \mathcal{S} \cap \mathcal{W}, \quad \bar{t} = \psi_1(\bar{z}), \quad \phi_2(\bar{z}) \in B[0, R],$$

follows. The fact that $\mathcal{S}$ is contained in the solution set Σ of (7) implies that

$$\bar{x} = \psi_2(\bar{t}, \bar{x})$$

and therefore $\bar{z} \in \mathcal{W}$ is a fixed point of ψ (since we have already proved that $\bar{t} = \psi_1(\bar{t}, \bar{x})$). The fact that $\phi_2(\bar{t}, \bar{x}) \in B[0, R]$, implies (in view of (6) and the remarks at the beginning of the proof) that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ. $\square$

REMARK 1. The first part of the proof of Theorem 6 can be used to obtain the following result where we use a slightly different expansive condition which is inspired from [33, 36, 37]. We recall that here by a *continuum* we mean a compact and connected set.

THEOREM 7. *With the notation of Theorem 6, assume that ϕ is compact on $\mathcal{D} \cap \mathcal{B}[a, R]$ and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{B}[a, R]$ such that the assumption*

(H') *for every continuum $\sigma \subseteq \mathcal{B}[a, R]$ with $\sigma \cap \mathcal{B}_l \neq \emptyset$ and $\sigma \cap \mathcal{B}_r \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\Gamma) \cap \mathcal{B}_l \neq \emptyset$, $\phi(\Gamma) \cap \mathcal{B}_r \neq \emptyset$,*

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D}$, with $\phi(\tilde{z}) = \tilde{z}$.

Proof. We follow the proof of Theorem 6 till to (8). Now, assumption (H') guarantees the existence of a continuum $\Gamma \subseteq \mathcal{S} \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{B}[a, R]$ and $\phi(\Gamma) \cap \mathcal{B}_l \neq \emptyset$,

$\phi(\Gamma) \cap \mathcal{B}_r \neq \emptyset$. This means that there are points $Q_1 = (q_1^1, q_2^1)$, $Q_2 = (q_1^2, q_2^2) \in \Gamma$ such that $\phi_1(Q_1) = -a$ and $\phi_1(Q_2) = a$. This implies that $p_1(Q_1) - \phi_1(Q_1) = q_1^1 + a \geq -a + a = 0$ and $p_1(Q_2) - \phi_1(Q_2) = q_1^2 - a \leq a - a = 0$. By the Bolzano's Theorem we can conclude that there exists a point $\bar{z} = (\bar{t}, \bar{x}) \in \Gamma \subseteq \mathcal{S} \cap \mathcal{W}$ such that $\bar{t} = \psi_1(\bar{z})$ and, by (8), $\phi_2(\bar{z}) \in B[0, R]$. At this point, we can complete our argument as in the proof of Theorem 6. Indeed, the fact that $\mathcal{S}$ is contained in the solution set Σ of (7) implies that

$$\bar{x} = \psi_2(\bar{t}, \bar{x})$$

and therefore $\bar{z} \in \mathcal{W}$ is a fixed point of ψ. The fact that $\phi_2(\bar{t}, \bar{x}) \in B[0, R]$, implies (in view of (6) and the remarks at the beginning of the proof of Theorem 6) that $\bar{z} \in \mathcal{W}$ is a fixed point of ϕ. $\square$

REMARK 2. In our theorems we have confined ourselves to the case of compact maps. Extensions can be given to more general operators like, locally compact, k-contractive, etc., provided that a decent degree theory is available (see [15, 27, 50] for the corresponding definitions).

2.2. Results related to Theorem 6

Like in the case of the Schauder fixed point theorem, we give now some variants of Theorem 6 (see Theorem 8 and Theorem 9 below). As in Theorem 6 we assume that X is a normed space and

$$\phi : \mathbb{R} \times X \supseteq D_\phi \to \mathbb{R} \times X$$

is a map which is continuous on a set $\mathcal{D} \subseteq D_\phi$. For the subsequent proofs, we systematically check condition (H) by taking as a representation of the path σ a continuous curve $\theta(s)$ which is parameterized on the interval $[0, 1]$ and look for a suitable restriction of $\theta(s)$ with $s \in [s_0, s_1] \subseteq [0, 1]$, as a representation of a sub-path $\gamma \subseteq \sigma$.
We start with a preliminary lemma.

LEMMA 1. *Let $\rho = (\rho_1, \rho_2) : \mathcal{B}[a, R] := [-a, a] \times B[0, R] \to \mathcal{B}[a, R]$ be a continuous map such that, for each $t \in [-a, a]$, $\rho_1(t, x) = t$ and $\rho_2(t, \cdot)$ is a retraction of $B[0, R]$ onto its image. Define*

$$\mathcal{R} := \cup_{t \in [-a,a]} \{\rho(t, x) : x \in B[0, R]\} = \rho(\mathcal{B}[a, R])$$

and

$$\mathcal{R}_l := \rho(\mathcal{B}_l),\ \mathcal{R}_r := \rho(\mathcal{B}_r).$$

Assume that

$$\phi \textit{ is compact on } \mathcal{D} \cap \mathcal{R}$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$ such that the assumption

(H) *for every path $\sigma \subseteq \mathcal{R}$ with $\sigma \cap \mathcal{R}_l \neq \emptyset$ and $\sigma \cap \mathcal{R}_r \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\gamma) \subseteq \mathcal{R}$ and $\phi(\gamma) \cap \mathcal{R}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{R}_r \neq \emptyset$,* *holds. Then there exists* $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. Given the compact operator $\phi : \mathcal{D} \cap \mathcal{R} \to \mathbb{R} \times X$ we define

$$\psi(t,x) := \phi(\rho(t,x)) = \phi(t, \rho_2(t,x)), \quad \text{for } (t,x) \in \mathcal{D}' := \rho^{-1}(\mathcal{D} \cap \mathcal{R}).$$

Clearly, ψ is compact on $\mathcal{D}' = \mathcal{D}' \cap \mathcal{B}[a, R]$. We also define

$$\mathcal{W}' := \rho^{-1}(\mathcal{W}) \subseteq \mathcal{B}[a, R] =: \mathcal{B}.$$

Consider now a continuous parameterized curve $\theta = (\theta_1, \theta_2) : [0,1] \to \mathcal{B}[a, R]$ such that $\theta(0) \in \mathcal{B}_l$ (that is, $\theta_1(0) = -a$) and $\theta(1) \in \mathcal{B}_r$ (that is, $\theta_1(1) = a$). Then for the curve $\vartheta : [0,1] \ni s \mapsto \rho(\theta(s))$, it holds that

$$\vartheta(s) \in \mathcal{R},\ \forall s \in [0,1] \quad \text{and} \quad \vartheta(0) \in \mathcal{R}_l\,,\ \vartheta(1) \in \mathcal{R}_r\,.$$

By assumption (H) referred to $\mathcal{R}$, there exists a restriction of ϑ to an interval $[s_0, s_1] \subseteq [0,1]$ such that $\vartheta(s) \in \mathcal{W}$ for every $s \in [s_0, s_1]$ and, moreover,

$$\phi(\vartheta(s)) \in \mathcal{R},\ \forall s \in [s_0, s_1],$$

as well as

$$\phi(\vartheta(s_0)) \in \mathcal{R}_l \ \text{ and } \ \phi(\vartheta(s_1)) \in \mathcal{R}_r\,, \quad \text{or} \quad \phi(\vartheta(s_1)) \in \mathcal{R}_l \ \text{ and } \ \phi(\vartheta(s_0)) \in \mathcal{R}_r\,.$$

Just to fix one of the two possible cases for the rest of the proof, suppose that the first possibility occurs (the treatment of the other case is exactly the same, modulo minor changes in the role of s_0 and s_1). Then, by the definition of ϑ, $\mathcal{W}'$ and ψ, we can also write that

$$\theta(s) \in \mathcal{W}', \quad \forall s \in [s_0, s_1],$$

and

$$\psi(\theta(s)) \in \mathcal{R} \subseteq \mathcal{B},\ \forall s \in [s_0, s_1]$$

$$\psi(\theta(s_0)) \in \mathcal{R}_l \subseteq \mathcal{B}_l\,,\ \psi(\theta(s_1)) \in \mathcal{R}_l \subseteq \mathcal{B}_r\,.$$

We have thus proved that assumption (H) of Theorem 6 is satisfied with respect to the operator ψ and the cylinder $\mathcal{B}[a, R]$ and therefore Theorem 6 guarantees that there exists for ψ a fixed point $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W}' \subseteq \mathcal{D}'$, with $\psi(\tilde{z}) = \tilde{z}$. As a last step, we just recall that the range of ψ coincides with the range of ϕ and that ρ (as a retraction) is the identity on $\mathcal{R}$. This implies that $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{R}$ with $\phi(\tilde{z}) = \tilde{z}$. □

THEOREM 8. *Let* $C \neq \emptyset$ *be a closed convex subset of the normed space* X. *Let* $\mathcal{C} := [-a, a] \times C$ *and define*

$$\mathcal{C}_l := \{(-a, x) : x \in C\}, \quad \mathcal{C}_r := \{(a, x) : x \in C\}.$$

Assume that

$$\phi \text{ is compact on } \mathcal{D} \cap \mathcal{C}$$

and there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{C}$ *such that the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{C}$ *with* $\sigma \cap \mathcal{C}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{C}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{C}$ *and* $\phi(\gamma) \cap \mathcal{C}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{C}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{C}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. By assumption, the operator $\phi = (\phi_1, \phi_2)$ is compact on $\mathcal{D} \cap \mathcal{C}$, hence, $\phi(\mathcal{D} \cap \mathcal{C})$ is bounded in $\mathbb{R} \times X$. In particular, there is $R > 0$ such that,

$$||\phi_2(t, x)|| < R, \quad \forall z = (t, x) \in \mathcal{D} \cap \mathcal{C}.$$

As a consequence of the Dugundji Extension Theorem, the closed convex set $C' := C \cap B[0, R]$ is a retract of $B[0, R]$ (actually, it is a retract of the whole space X, but for us it is more convenient to restrict the retraction to $B[0, R]$). We denote by $\varrho : B[0, R] \to C'$ such a continuous retraction.

If we define now $\mathcal{C}' := [-a, a] \times C'$ and

$$\mathcal{C}'_l := \{(-a, x) : x \in C'\}, \quad \mathcal{C}'_r := \{(a, x) : x \in C'\},$$

as well as $\mathcal{D}' := \mathcal{D} \cap \mathcal{C}'$ and $\mathcal{W}' := \mathcal{W} \cap \mathcal{C}'$, we find that

(H) for every path $\sigma \subseteq \mathcal{C}'$ with $\sigma \cap \mathcal{C}'_l \neq \emptyset$ and $\sigma \cap \mathcal{C}'_r \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}'$ with $\phi(\gamma) \subseteq \mathcal{C}'$ and $\phi(\gamma) \cap \mathcal{C}'_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{C}'_r \neq \emptyset$,

holds. At last, we define the continuous retraction

$$\rho = (\rho_1, \rho_2) : \mathcal{B}[a, R] := [-a, a] \times B[0, R] \to \mathcal{B}[a, R],$$

$$\rho_1(t, x) = t, \ \ \rho_2(t, x) = \varrho(x)$$

and easily check that now the set $\mathcal{C}'$ plays here the same role as the set $\mathcal{R}$ in Lemma 1. Then, according to Lemma 1, there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W}' \subseteq \mathcal{W}$, with $\phi(\tilde{z}) = \tilde{z}$. The proof is complete. □

THEOREM 9. *Let* $K \neq \emptyset$ *be a compact convex subset of the normed space* X. *Let* $\mathcal{K} := [-a, a] \times K$ *and define*

$$\mathcal{K}_l := \{(-a, x) : x \in K\}, \quad \mathcal{K}_r := \{(a, x) : x \in K\}.$$

Suppose that there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$ *such that* ϕ *is continuous on* $\mathcal{W}$ *and the assumption*

(H) *for every path* $\sigma \subseteq \mathcal{K}$ *with* $\sigma \cap \mathcal{K}_l \neq \emptyset$ *and* $\sigma \cap \mathcal{K}_r \neq \emptyset$, *there is a sub-path* $\gamma \subseteq \sigma \cap \mathcal{W}$ *with* $\phi(\gamma) \subseteq \mathcal{K}$ *and* $\phi(\gamma) \cap \mathcal{K}_l \neq \emptyset$, $\phi(\gamma) \cap \mathcal{K}_r \neq \emptyset$,

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$, *with* $\phi(\tilde{z}) = \tilde{z}$.

Proof. This result is an immediate consequence of Theorem 8, with the position $C := K$. □

REMARK 3. Variants of Theorem 8 and Theorem 9 can be obtained, as a consequence of Theorem 7, using condition (H') instead of condition (H). For instance, Theorem 9 could be accompanied by the following.

THEOREM 10. *Under the same positions of Theorem 9, suppose that there is a closed subset $\mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$ such that ϕ is continuous on $\mathcal{W}$ and the assumption*

(H') *for every continuum $\sigma \subseteq \mathcal{K}$ with $\sigma \cap \mathcal{K}_l \neq \emptyset$ and $\sigma \cap \mathcal{K}_r \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\phi(\Gamma) \subseteq \mathcal{K}$ and $\phi(\Gamma) \cap \mathcal{K}_l \neq \emptyset$, $\phi(\Gamma) \cap \mathcal{K}_r \neq \emptyset$,*

holds. Then there exists $\tilde{z} = (\tilde{t}, \tilde{x}) \in \mathcal{W} \subseteq \mathcal{D} \cap \mathcal{K}$, with $\phi(\tilde{z}) = \tilde{z}$.

3. Extensions, remarks and consequences

3.1. The "stretching along the paths" fixed point property

Let Z be a topological space. According to a well known definition, *Z has the fixed point property* (FPP) if every continuous map of Z into itself has at least a fixed point. The FPP is invariant by homeomorphisms and it is preserved under continuous retractions. Thus, by the Brouwer fixed point theorem, we know that any topological space which is homeomorphic to (a retract of) a closed ball of a finite dimensional normed space has the FPP. It is the aim of this section to show that something similar (even if not exactly the same) holds with respect to the assumption of "stretching along the paths" (H) and the corresponding fixed point result in Theorem 6.

We consider now the following situation.

DEFINITION 1. *Assume that Z is a topological space and Z_l^-, Z_r^- are two nonempty disjoint subsets of Z. We set*

$$Z^- := Z_l^- \cup Z_r^-$$

and define

$$\widetilde{Z} := (Z, Z^-).$$

We call $\widetilde{Z}$ a two-sided oriented space or simply an oriented space. In view of our applications below which concern the case of arcwise connected spaces and where the family of paths connecting Z_l^- to Z_r^- is involved, we call $\widetilde{Z}$ a path-oriented space when Z is arcwise connected.

We say that $\widetilde{Z}$ has the fixed point property for maps stretching along the paths (in the sequel referred as FPP-γ) if Z is arcwise connected and, for every pair $(\mathcal{D}, \psi)$, satisfying the following conditions:

(i_1) $\mathcal{D} \subseteq Z$;

(i_2) *$\psi : \mathcal{D} \to Z$ is continuous;*

(i_3) *there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \cap Z_l^- \neq \emptyset$ and $\phi(\gamma) \cap Z_r^- \neq \emptyset$;*

there exists at least a fixed point of ψ in $\mathcal{D}$.

DEFINITION 2. *Suppose we have two arcwise connected topological spaces Z, Y and assume that Z_l^-, Z_r^- are two nonempty disjoint subsets of Z with $Z^- = Z_l^- \cup Z_r^-$, as well as Y_l^-, Y_r^- are two nonempty disjoint subsets of Y, with $Y^- = Y_l^- \cup Y_r^-$. Defining, as above $\widetilde{Z} = (Z, Z^-)$ and $\widetilde{Y} = (Y, Y^-)$ the corresponding path-oriented spaces, we say that the pair $(\mathcal{D}, \psi)$ stretches $\widetilde{Z}$ to $\widetilde{Y}$ along the paths and write*

$$(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\rightsquigarrow \widetilde{Y},$$

if the conditions

- *(j_1)* $\mathcal{D} \subseteq Z$;
- *(j_2)* $\psi : \mathcal{D} \to Y$ *is continuous;*
- *(j_3) there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \cap Y_l^- \neq \emptyset$ and $\phi(\gamma) \cap Y_r^- \neq \emptyset$;*

hold.
Accordingly, we have that $\widetilde{Z}$ has the FPP-γ if and only if for every pair $(\mathcal{D}, \psi)$ with $(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\rightsquigarrow \widetilde{Z}$, there is at least a fixed point of ψ in $\mathcal{D}$.

REMARK 4. The definition of a map stretching along the paths was introduced in [56, 57] and refined in [59, 60] in the case of two-dimensional oriented cells. Our Definition 2 above is a generalization of the previous cited one as it reduces to [60] in the situation considered therein. We note that in [60], as well as in the other preceding papers, the map ψ was allowed to be defined possibly on some larger domains. However, up to a restriction, we can always enter in the case of Definition 2 when we consider the situation described in [60].
In the definition of path-oriented space, as well as in the subsequent stretching condition, the order in which we label the two sets Z_l^- and Z_r^- (or Y_l^- and Y_r^-) has no effect at all.
We also point out that given two nonempty disjoint sets W_l and W_r of a topological space W, the condition that there exists a path $\sigma \subseteq W$ with $\sigma \cap W_l \neq \emptyset$ and $\sigma \cap W_r \neq \emptyset$ is equivalent to the existence of a continuous map $\theta : [0, 1] \to W$ with $\theta(0) \in W_l$ and $\theta(1) \in W_r$.
The choice of the notation $\widetilde{Z}$ instead of $\widehat{Z}$ (previously considered in Section 1.2 and next again in Section 4.1) comes from the fact that, even if all the applications we present here are for the $\widehat{[\cdot]}$-sets, nonetheless, the oriented spaces $\widetilde{[\cdot]}$ are, in principle, more general. Thus we prefer to think to the $\widehat{[\cdot]}$-sets as some particular cases of the $\widetilde{[\cdot]}$-sets.
Finally, we mention that some analogous definitions, previously introduced in the literature (see, for instance, the concept of quadrilateral set given in [36]) also fit with our definition of oriented space.

The next two lemmas extend to the case of the FPP-γ two corresponding classical results about the usual fixed point property. Their proof is quite standard and therefore it is omitted.

LEMMA 2. *Let Z, Y be two arcwise connected topological spaces and let $h : Z \to Y$ be a homeomorphism. Suppose Z_l^- and Z_r^- are nonempty disjoint subsets of Z and set $Y_l^- = h(Z_l^-)$, $Y_r^- = h(Z_r^-)$. Then $\widetilde{Z}$ has the FPP-γ if and only if $\widetilde{Y}$ has the FPP-γ.*

Proof. We leave the proof as an exercise. □

LEMMA 3. *Let Z, Y be two arcwise connected topological spaces with $Y \subseteq Z$ and let $r : Z \to Y$ be a continuous retraction. Suppose Y_l^- and Y_r^- are nonempty disjoint subsets of Y and set $Z_l^- = r^{-1}(Y_l^-)$, $Z_r^- = r^{-1}(Y_r^-)$. Then $\widetilde{Y}$ has the FPP-γ if $\widetilde{Z}$ has the FPP-γ.*

Proof. The proof follows the same argument (mutatis mutandis) of that of Lemma 1 and therefore it is omitted. □

COROLLARY 3. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[-1, 1] \times K$, via a homeomorphism $h : Z \to [-1, 1] \times K$. Define*

$$Z_l^- := h^{-1}(\{-1\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Then $\widetilde{Z}$ has the FPP-γ.

Lemma 2 and Lemma 3 together with Theorem 6 (or its variants) permit to give some straightforward examples with some geometrical meaning. For simplicity, we confine ourselves to subsets of a finite dimensional space E.

EXAMPLE 1. Let $Z \subseteq E$ be a compact set which is homeomorphic to the closed unit ball $B[0, 1] \subseteq \mathbb{R}^{\mathrm{N}}$, with $\mathrm{N} \geq 1$. Let $P, Q \in Z$ with $P \neq Q$ be two given points. Let $\psi : E \supseteq D_\psi \to E$ be a continuous map and suppose that $\mathcal{E} \subseteq D_\psi$ is a closed set such that *for every path $\sigma \subseteq Z$, with $P, Q \in \sigma$, there is a sub-path $\gamma \subseteq \sigma \cap \mathcal{E}$ with $\psi(\sigma) \subseteq Z$ and $P, Q \in \psi(\gamma)$*. Then ψ has at least a fixed point in $\mathcal{E} \cap Z$.

Proof. We discuss only the case $\mathrm{N} \geq 2$, since for $\mathrm{N} = 1$, the result is obvious. Let us consider the cylinder

$$\mathcal{C} := \{(x_1, \dots, x_{\mathrm{N}-1}, x_{\mathrm{N}}) : ||(x_1, \dots, x_{\mathrm{N}-1})|| \leq 1, \ |x_{\mathrm{N}}| \leq 1\}$$

on which we select as a right and left sides the south and the north bases respectively:

$$\mathcal{C}_l := \{x = (x_1, \dots, , x_{\mathrm{N}}) \in \mathcal{C} : x_{\mathrm{N}} = -1\},$$

$$\mathcal{C}_r := \{x = (x_1, \dots, , x_{\mathrm{N}}) \in \mathcal{C} : x_{\mathrm{N}} = 1\}$$

and we also set $\mathcal{C}^- := \mathcal{C}_l \cup \mathcal{C}_r$. Theorem 6 implies that the path-oriented space $\widetilde{\mathcal{C}} = (\mathcal{C}, \mathcal{C}^-)$ has the FPP-γ and therefore Lemma 3 ensures that the same fixed point property holds also with respect to retracts of $\widetilde{\mathcal{C}}$. The closed unit ball $B[0,1]$ is a retract of the cylinder $\mathcal{C}$ through the continuous map ϱ defined by

$$\varrho(x) = (x_1 \min\{1, \delta(x)\}, \dots, x_{N-1} \min\{1, \delta(x)\}, x_N),$$

where

$$\delta(x) = \delta(x_1, \dots, x_{N-1}, x_N) := \frac{\sqrt{1 - x_N^2}}{\sqrt{x_1^2 + \cdots + x_{N-1}^2}}.$$

Hence, the path oriented space $\widetilde{B} = (B, B^-)$, with $B = B[0,1]$, $B^- = B_l^- \cup B_r^-$, $B_l^- = \{\text{South pole}\}$ and $B_r^- = \{\text{North pole}\}$ has the FPP-γ. Finally, Lemma 2 implies the the FFP-γ holds for every oriented space in which the base space Z is homeomorphic to a closed ball and we select as Z_l^- and Z_r^- two different points of Z. This concludes the proof. $\square$

EXAMPLE 2. Consider the cone

$$K = \{(x_1, \dots, x_k, x_{k+1}) : ||(x_1, \dots, x_k)|| \le x_{k+1} \le 1\} \subseteq \mathbb{R}^{k+1}$$

and select the point $0 = (0, \dots, 0, 0) \in K$ and the base $K_l = \{(x_1, \dots, x_k, 1) : ||(x_1, \dots, x_k)|| \le 1\} \subseteq K$. Let $Z \subseteq E$ be a compact set which is homeomorphic to K, by a homeomorphism $h : Z \to K$. Define $Z_r^- = h^{-1}(\{0\})$ and $Z_l^- = h^{-1}(K_l)$. Then $\widetilde{Z}$ has the FPP-γ.

Proof. It is possible to obtain our claim by suitably adapting the argument employed in the proof of Example 1. We omit the details. $\square$

REMARK 5. We observe that one could define a fixed point property (say FPP-Γ) for maps satisfying a condition which extends property (H') to general (oriented) topological spaces. Then, after having obtained from Theorem 7 a result analogous to Lemma 2, the following corollary can be proved.

COROLLARY 4. *Let $K \neq \emptyset$ be a compact convex subset of a normed space. Let Z be a compact topological space which is homeomorphic to $[-1,1] \times K$, via a homeomorphism $h : Z \to [-1,1] \times K$. Define*

$$Z_l^- := h^{-1}(\{-1\} \times K), \quad Z_r^- := h^{-1}(\{1\} \times K).$$

Then, for every pair $(\mathcal{D}, \psi)$, satisfying the following conditions:

(i$_1$) $\mathcal{D} \subseteq Z$;

(i$_2$) $\psi : \mathcal{D} \to Z$ *is continuous;*

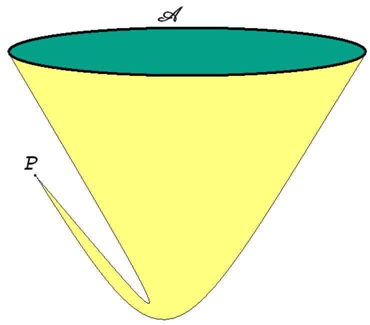

Figure 11: A possible illustration of Example 2 in $\mathbb{R}^3$, where we have denoted by P the point $h^{-1}(\{0\})$ and by $\mathcal{A}$ the surface $h^{-1}(K_l)$ of the deformed cone Z. According to our result there exists at least a fixed point $z = \psi(z) \in \mathcal{W}$, for any continuous map ψ defined on a closed subset $\mathcal{W}$ of Z and with values in Z having the property that any path σ in Z and joining P to $\mathcal{A}$ contains a sub-path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \subseteq Z$ and $\psi(\gamma)$ joining P to $\mathcal{A}$.

(i_3') there is a closed set $\mathcal{W} \subseteq \mathcal{D}$ such that, for every continuum $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$, there is a continuum $\Gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\Gamma) \cap Z_l^- \neq \emptyset$ and $\phi(\Gamma) \cap Z_r^- \neq \emptyset$;

there exists at least a fixed point of ψ in $\mathcal{D}$.

In the present paper we do not further pursue the research in this direction and confine ourselves to the study of the stretching condition along the paths. Investigations toward the fixed point properties for maps satisfying an expansive conditions with respect to other kind of connected sets will be considered elsewhere.

3.2. Further definitions and consequences

As a next step, we give now some simple (but nevertheless useful) properties about the stretching along the paths condition. Unless otherwise specified, all the spaces involved are arcwise connected topological spaces. When we consider a triple (Z, Z_l^-, Z_r^-), we always assume that Z_l^- and Z_r^- are nonempty disjoint subsets of Z. First of all, we consider two further definitions.

DEFINITION 3. *Let $\widetilde{Z} = (Z, Z^-)$ and $\widetilde{Y} = (Y, Y^-)$ be two path-oriented spaces with Y a subspace of Z. We say that $\widetilde{Y}$ is a horizontal slab of $\widetilde{Z}$ and write*

$$\widetilde{Y} \subseteq_h \widetilde{Z},$$

if every path $\gamma \subseteq Y$ with $\gamma \cap Y_l^- \neq \emptyset$ and $\gamma \cap Y_r^- \neq \emptyset$ is such that $\gamma \cap Z_l^- \neq \emptyset$ and

$\gamma \cap Z_r^- \neq \emptyset$.
Similarly, we say that $\widetilde{Y}$ is a vertical slab of $\widetilde{Z}$ and write

$$\widetilde{Y} \subseteq_v \widetilde{Z},$$

if every path $\sigma \subseteq Z$ with $\sigma \cap Z_l^- \neq \emptyset$ and $\sigma \cap Z_r^- \neq \emptyset$ contains a sub-path $\gamma \subseteq Y$ such that $\gamma \cap Y_l^- \neq \emptyset$ and $\gamma \cap Y_r^- \neq \emptyset$.

REMARK 6. The definition of slabs generalizes the case of rectangles with horizontal and vertical sides parallel to the contracting and expanding directions in the Smale horseshoe (see, for instance, [72, Section 2.3]). In our general setting of a topological space Z oriented by the paths connecting two disjoint subsets Z_l^- and Z_r^- and in view of Theorem 6, we consider as horizontal-expanding the "direction" along (Z_l^-, Z_r^-) (of course, in a very vague sense and taking also into account the fact that in our setting "horizontal" and "vertical" are merely conventional terms). Definition 3 generalizes the analogous concepts of "slices" considered in [60] in the setting of oriented two-dimensional cells and recalled in Section 1.2 as well as some possibilities considered in [61] for N-dimensional cells (namely, the case in which there is a one-dimensional expansive direction). Note that our definitions are purely topological in nature and therefore we do not need (like in [72, Section 2.3]) the slabs to be described by means of graphs of Lipschitz functions. We refer to Figure 14 as a possible picture of horizontal and vertical slabs in a simple situation.

Having available in the general setting the definition of slabs, we can now borrow from [60] and [61] the next definition (compare also to the corresponding definition in Section 1.2).

DEFINITION 4. *Let $\widetilde{Z} = (Z, Z^-)$, $\widetilde{Y} = (Y, Y^-)$ and $\widetilde{X} = (X, X^-)$ be three path-oriented spaces with X, Y, Z subspaces of the same topological space W and $X \subseteq Y \cap Z$.*
We say that $\widetilde{Y}$ crosses $\widetilde{Z}$ in $\widetilde{X}$ and write

$$\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\},$$

if

$$\widetilde{X} \subseteq_h \widetilde{Z} \quad \text{and} \quad \widetilde{X} \subseteq_v \widetilde{Y}.$$

REMARK 7. As already remarked in [60] and [61], in our setting, the definition of $\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\}$, covers very general situations, in particular also when there is no way to define any kind of transversal intersection. A possible illustration is given in Figure 15 of Section 4.

LEMMA 4. *The following properties hold:*

(e_1) *if $(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\!\rightsquigarrow \widetilde{Y}$ and $(\mathcal{E}, \phi) : \widetilde{Y} \Leftarrow\!\!\!\rightsquigarrow \widetilde{X}$, then $(\mathcal{F}, \phi \circ \psi) : \widetilde{Z} \Leftarrow\!\!\!\rightsquigarrow \widetilde{X}$ for $\mathcal{F} = \mathcal{D} \cap \psi^{-1}(\mathcal{E})$;*

(e_2) *if* $\phi : Z \to Y$ *is a homeomorphism such that* $\phi(Z_l^-) = Y_l^-$ *and* $\phi(Z_r^-) = Y_r^-$ *(or* $\phi(Z_l^-) = Y_r^-$ *and* $\phi(Z_r^-) = Y_l^-$*), then* $(Z, \phi) : \widetilde{Z} \Leftarrow\!\!\!\!\rightsquigarrow \widetilde{Y}$ *;*

(e_3) *if* $(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\!\!\rightsquigarrow \widetilde{Y}$, *then* $(\mathcal{D} \cap X \cap \psi^{-1}(W), \psi) : \widetilde{X} \Leftarrow\!\!\!\!\rightsquigarrow \widetilde{W}$, *for every* $\widetilde{X} \subseteq_h \widetilde{Z}$ *and every* $\widetilde{W} \subseteq_v \widetilde{Y}$ *;*

(e_4) *if* $(\mathcal{D}, \psi) : \widetilde{Z} \Leftarrow\!\!\!\!\rightsquigarrow \widetilde{Y}$, *then* ψ *has a fixed point in* $\mathcal{D} \cap X$, *for every* $\widetilde{X} \in \{\widetilde{Z} \pitchfork \widetilde{Y}\}$, *having the FPP-*$\gamma$ *.*

Proof. The above properties follow immediately by the corresponding definitions. □

Now we are in position to consider a sequence of spaces and maps and obtain a result which is in line with [34] and [73] and extend to a general setting some results [59, Theorem 2.2], [60, Theorem 4.2] previously obtained in the two-dimensional setting. For simplicity, we confine ourselves to the framework of compact metric spaces. This simplifies somehow our proofs. We point out, however, that some of the properties exposed in the next Theorem 11 would be still true in some more general situations.

THEOREM 11. *Suppose that there is a (double) sequence of path-oriented spaces*

$$(\widetilde{X}_k)_{k \in \mathbb{Z}} = ((X_k, X_k^-))_{k \in \mathbb{Z}} ,$$

where, for each $k \in \mathbb{Z}$, X_k *is a compact and arcwise connected metric space. Denote by* $(X_k^-)_l$ *and* $(X_k^-)_r$ *the two sides of* X_k^- *. Assume that there is a sequence of maps* $((\mathcal{D}_k, \psi_k))_{k \in \mathbb{Z}}$ *, such that*

$$(\mathcal{D}_k, \psi_k) : \widetilde{X}_k \Leftarrow\!\!\!\!\rightsquigarrow \widetilde{X}_{k+1} , \quad \forall\, k \in \mathbb{Z}.$$

Then the following conclusions hold:

(a_1) *There is a sequence* $(w_k)_{k \in \mathbb{Z}}$ *with* $w_k \in \mathcal{D}_k$ *and* $\psi_k(w_k) = w_{k+1}$ *for all* $k \in \mathbb{Z}$*;*

(a_2) *For each* $j \in \mathbb{Z}$ *there is a compact set* $\mathcal{C}_j \subseteq \mathcal{D}_j$ *such that for each* $w \in \mathcal{C}_j$ *there exists a sequence* $(y_\ell)_{\ell \geq j}$ *, with* $y_\ell \in \mathcal{D}_\ell$ *and* $y_j = w$, $y_{\ell+1} = \psi_\ell(y_\ell)$ *for each* $\ell \geq j$*.*
The compact set $\mathcal{C}_j$ *satisfies the following separation property:*

$$\mathcal{C}_j \cap \sigma \neq \emptyset, \text{ for each path } \sigma \subseteq X_j \text{ with } \sigma \cap (X_k^-)_l \neq \emptyset \text{ and } \sigma \cap (X_k^-)_r \neq \emptyset;$$

(a_3) *If there are integers* h, k *with* $h < k$ *such that* $\widetilde{X}_h = \widetilde{X}_k$ *and* $\widetilde{X}_h$ *possesses the FPP-*γ*, then there is a finite sequence* $(z_i)_{h \leq i \leq k}$ *, with* $z_i \in \mathcal{D}_i$ *and* $\psi_i(z_i) = z_{i+1}$ *for each* $i = h, \ldots, k-1$*, such that* $z_h = z_k$ *.*

Proof. As in [59, Theorem 2.2] we prove the three properties in the reverse order. First of all we observe that, for each $i \in \mathbb{Z}$, there is a closed (and hence compact) set $W_i \subseteq \mathcal{D}_i \subseteq X_i$ such that, for every path $\sigma \subseteq X_i$ with $\sigma \cap (X_i)_l^- \neq \emptyset$ and $\sigma \cap (X_i)_r^- \neq \emptyset$, there is a sub-path $\gamma \subseteq \sigma \cap W_i$ with $\psi_i(\gamma) \cap (X_{i+1})_l^- \neq \emptyset$ and $\psi_i(\gamma) \cap (X_{i+1})_r^- \neq \emptyset$.

Proof of (a_3). Let $k = h + s$ for some $s \geq 1$. Let us also set $\widetilde{\mathcal{Z}} = \widetilde{X}_h = \widetilde{X}_k$, and $\psi = \psi_{k-1} \circ \cdots \circ \psi_h$. By property (e_1) of Lemma 4 we have that $(\mathcal{D}, \psi) : \widetilde{\mathcal{Z}} \Leftarrow\!\!\!\leadsto \widetilde{\mathcal{Z}}$, with

$$\mathcal{D} := \{z \in W_h \, : \, \psi_j \circ \cdots \circ \psi_{h+1} \circ \psi_h(z) \in W_{j+1}\, , \text{for } j = h, \ldots, k-1\}.$$

Then the assumption about the FPP-γ for X_h implies the existence of a fixed point $z^* \in \mathcal{D}$ for ψ. Moreover, $z_h := z^* \in \mathcal{D}_h$ and, setting $z_i = \psi_{i-1} \circ \cdots \circ \psi_h(z_h)$, for $i = h+1, \ldots, k$, the verification of the properties in (a_3) is straightforward.

Proof of (a_2). The situation described here is similar to that considered in [33, The Expander Lemma, p.417]. Without loss of generality, assume $j = 0$. Define the closed set

$$\mathcal{S} = \{x \in W_0 \, : \, \psi_\ell \circ \cdots \circ \psi_0(x) \in W_{\ell+1}\, , \ \forall\, \ell = 0, 1, 2, \ldots\}.$$

Let $\gamma_0 \subseteq X_0$ be a path intersecting both the components of X_0^-. By the stretching assumption between $\widetilde{X}_0$ and $\widetilde{X}_1$, there is a sub-path $\gamma_1 \subseteq W_0 \subseteq \mathcal{D}_0$ such that $\psi_0(\gamma_1) \subseteq X_1$ and with $\psi_0(\gamma_1)$ meeting both the components of X_1^-. On the other hand, the path $\psi_0(\gamma_1) = \sigma_1$ contains a sub-path $\sigma_2 \subseteq W_1 \subseteq \mathcal{D}_1$ such that $\psi_1(\sigma_2) \subseteq X_2$ and with $\psi_1(\sigma_2)$ intersecting both the components of X_2^-. We also define $\gamma_2 = \{x \in \gamma_1 : \psi_0(x) \in \sigma_2\}$. Then, by induction, we can find a sequence of nonempty compact sets contained in X_0

$$W_0 \supseteq \gamma_0 \supseteq \gamma_1 \supseteq \gamma_2 \supseteq \ldots \gamma_n \supseteq \gamma_{n+1} \supseteq \ldots$$

with $\psi_i(\gamma_i) \subseteq W_{i+1} \subseteq \mathcal{D}_{i+1}$ and such that $\psi_i \circ \cdots \circ \psi_0(\gamma_{i+1})$ is a path in X_{i+1} meeting both the components of X_{i+1}^-. Taking a point $w \in \cap_{n=0}^{\infty} \gamma_n$ we have that $\psi_\ell \circ \cdots \circ \psi_0(w) \in W_{\ell+1} \subseteq \mathcal{D}_{\ell+1}$ for each $\ell \geq 0$. Thus, any path $\gamma_0 \in X_0$ intersecting both the components of X_0^- contains a point of $\mathcal{S}$ which generates a sequence as in (a_2).

Proof of (a_1). A diagonal argument (see, e.g., [33, Proposition 5] or [56, Theorem 2, (w_4)]) allows to prove (a_1) as a consequence of (a_2). We give a sketch of it for the reader's convenience. By (a_2) we have that for each $n = 1, 2, \ldots$ there is a compact set $\mathcal{C}_{-n} \subseteq W_{-n} \subseteq \mathcal{D}_{-n}$ such that

$$K_{j+1,n} := \psi_j \circ \cdots \circ \psi_{-n}(\mathcal{C}_{-n}) \subseteq W_{j+1} \subseteq \mathcal{D}_{j+1}\, .$$

We take a point $y_{j,n} \in K_{j,n}$, for each $j \geq -n+1$, in order to form the infinite matrix

$$\begin{array}{ccccccccc}
 & & & & y_{0,1} & y_{1,1} & \cdots & y_{j,1} & \cdots \\
 & & & y_{-1,2} & y_{0,2} & y_{1,2} & \cdots & y_{j,2} & \cdots \\
 & & y_{-2,3} & y_{-1,3} & y_{0,3} & y_{1,3} & \cdots & y_{j,3} & \cdots \\
 & \cdots & & & & & & & \\
y_{-n+1,n} & \cdots & y_{-2,n} & y_{-1,n} & y_{0,n} & y_{1,n} & \cdots & y_{j,n} & \cdots \\
 & \cdots & & & & & & &
\end{array}$$

where, for each n and j, we have that $\psi_j(y_{j,n}) = y_{j+1,n}$. Now, a standard compactness and diagonal argument (or, from another point of view, the fact that the product of countably many sequentially compact spaces is sequentially compact) allows to pass to the limit on each “column” along a common subsequence of indexes in order to find, for each $j \in \mathbb{Z}$, a point $w_j \in W_j \subseteq \mathcal{D}_j$ and the continuity of ψ_j implies also that $\psi(w_j) = w_{j+1}$, $\forall\, j \in \mathbb{Z}$.

□

3.3. A final remark about sub-paths

We conclude this section with a remark about the stretching condition that we have chosen for property (H) and its variants and consequences. As pointed out in the Introduction, our definition is mainly motivated by our previous applications to ODEs and, more precisely, to our paper [55] where we obtained the stretching property $(H_{\pm})$ in a concrete example of the planar system (1). In that specific example, the proof was carried on by considering a continuous parameterized curve defined in the interval $[0, 1[$ and with an unbounded image in $\mathbb{R}^2$. Subsequently, in our search of fixed points for general continuous mappings defined on two-dimensional cells we used a definition of path as the continuous image of an interval. As shown in [60] as long as we are concerned with fixed points of a single mapping defined on a topological rectangle, there is no effect on the possible different choices in the definitions. With this respect, consider also Theorem 6 and Theorem 7 which show how, as long as we are looking for the existence of a fixed point, the stretching condition for paths and that for continua are both sufficient to obtain the desired result. Things, however, seem to be somehow more complicated when we focus our attention on the search of fixed points for the composition of maps (and thus, in particular, for the iterations of a given map). In such a case, the possibility of considering parameterized curves (modulo some equivalent relation like in Section 1.5) instead of images of curves as sets embedded in a space, looks simpler from the point of view of stating some hypotheses which are easily verifiable through the composition of maps. In the next example we try now to express better our point of view which lead to the choice of definition for a path considered in Section 1.5.

EXAMPLE 3. We define the function

$$g : [0, 3\pi] \to \mathbb{R}, \quad g(t) := \max\{t - 4\pi, \min\{t, \pi - t\}\}$$

and the continuous curve

$$\theta = (\theta_1, \theta_2) : [0, 3\pi] \to \mathbb{R}^2, \quad \theta_1(t) := \cos(g(t)), \quad \theta_2(t) := \sin(g(t)) = \sin(t).$$

Observe that for the path $\gamma = [\theta]$, the image set is

$$\bar{\gamma} = \{\theta(t) : t \in [0, 3\pi]\} = S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

The motion of the point $\theta(t)$ along the circumference S^1 can be described as follows: we start for $t = 0$ at the point $P = (1, 0)$, we move on S^1 in the counterclockwise sense till to the point $P' = (0, 1)$ for $t = \pi/2$. At this moment, the point $\theta(t)$ starts moving to the reverse direction (clockwise sense) till it reaches again the point P' at

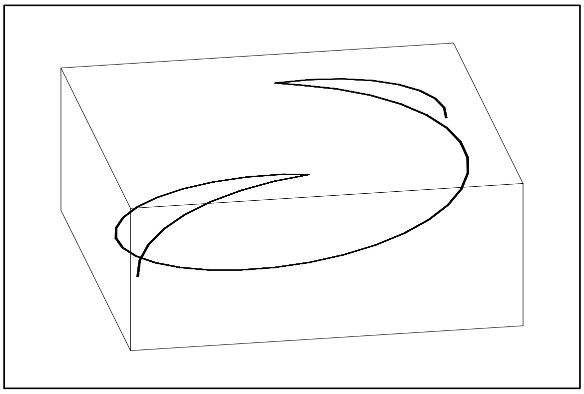

Figure 12: The bold line viewed from a suitably chosen point of perspective represents the arc in $\mathbb{R}^3$ parameterized by $t \mapsto (t, \theta_1(t), \theta_2(t))$, for $t \in [0, 3\pi]$.
The plot has been performed by Maple software.

the time $t = 5\pi/2$. Finally, the motion switches again to the counterclockwise sense and the point $Q = (-1, 0)$ is reached at the time $t = 3\pi$.
Let us set now $Z = S^1$ and $Z_l^- = \{P\}$, $Z_r^- = \{Q\}$ (we intentionally take this choice to show that the terms "left" and "right" are merely conventional and their order is not important). According to our definitions, γ is a path in Z with $\gamma \cap Z_l^- \neq \emptyset$ and $\gamma \cap Z_r^- \neq \emptyset$. If we consider now the arcs $\Gamma^{\text{upper}} := \{(x, y) \in S^1 : y \geq 0\}$ and $\Gamma^{\text{lower}} := \{(x, y) \in S^1 : y \leq 0\}$ which are contained in $\bar{\gamma}$, we see that while Γ^{lower} is the image set of a sub-path of γ, Γ^{upper} is not the image of any sub-path of γ.

4. Applications to topological cells in finite dimensional spaces, periodic points and topological dynamics

In this section we propose an application of the results in Section 3 to the setting of [26, 56, 57, 59, 60, 61, 82].

4.1. Definitions

Let X be a Hausdorff topological space. We define a $(1, \mathrm{N} - 1)$-*rectangular cell of* X as a pair

$$\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}),$$

where $\mathcal{N} \subseteq X$ is a compact set and $c_{\mathcal{N}} : \mathcal{N} \to [-1, 1]^{\mathrm{N}} \subseteq \mathbb{R}^{\mathrm{N}}$ is a homeomorphism of $\mathcal{N}$ onto its image $[-1, 1]^{\mathrm{N}}$. Sometimes, it will be convenient to put in evidence the Hausdorff topological space X containing a given cell $\mathcal{N}$ and the dimension N of the codomain of the homeomorphism. In such a situation, we'll write

$$\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}; X, \mathrm{N}).$$

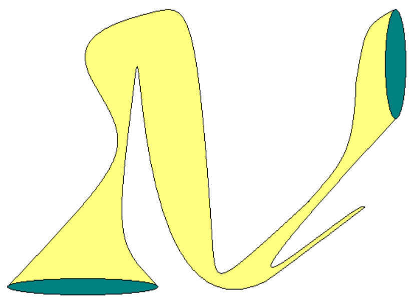

Figure 13: A possible picture of a $(1,2)$-rectangular cell $\widehat{\mathcal{N}}$, where we have put in evidence the two components of the $\mathcal{N}^-$ set which are painted with a darker color.

Our definition of $(1, \mathrm{N}-1)$-rectangular cell is borrowed from that of *h*-set given by Zgliczyński and Gidea [82, Definition 1] and considered also by Pireddu and Zanolin in [61]. However, we point out that, differently than in [82] and [61], we don't assume here $\mathcal{N}$ to be a subset of $\mathbb{R}^{\mathrm{N}}$ and moreover in the present case the homeomorphism $c_{\mathcal{N}}$ is defined only on $\mathcal{N}$ whence in the above cited articles $c_{\mathcal{N}}$ was defined on the whole space X. We also define the sets

$$\mathcal{N}_l^- := {c_{\mathcal{N}}}^{-1}(\{-1\} \times [-1,1]^{\mathrm{N}-1}), \quad \mathcal{N}_r^- := {c_{\mathcal{N}}}^{-1}(\{1\} \times [-1,1]^{\mathrm{N}-1}),$$

conventionally called *the left and the right faces of* $\widehat{\mathcal{N}}$, as well as the set

$$\mathcal{N}^- := \mathcal{N}_l^- \cup \mathcal{N}_r^- \,.$$

If we define now

$$\widetilde{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-),$$

we have that $\widetilde{\mathcal{N}}$ is a path-oriented spaces which possesses the FPP-γ.

Our definition of oriented cell $\widehat{\mathcal{N}}$ fits with that of $(1, \mathrm{N}-1)$-*window* considered by Gidea and Robinson in [25] and, in the special case $\mathrm{N} = 2$, is equivalent to that of two-dimensional *oriented cell* by Papini and Zanolin in [60].

For completeness we also recall the form that the stretching condition takes with respect to the path-oriented spaces determined by the rectangular cells that we have just defined.

Let $\widehat{\mathcal{A}} = (\mathcal{A}, c_{\mathcal{A}}\,; X, \mathrm{N}_1)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, c_{\mathcal{B}}\,; Y, \mathrm{N}_2)$ be two rectangular cells contained in the Hausdorff topological spaces X and Y, respectively. Let $\phi : X \supseteq D_\phi \to Y$ be a map (not necessarily continuous on its whole domain D_ϕ) and let us consider a set $\mathcal{D} \subseteq D_\phi$.

DEFINITION 5. *We say that the pair $(\mathcal{D}, \phi)$ stretches $\widehat{\mathcal{A}}$ to $\widehat{\mathcal{B}}$ along the paths and write*

$$(\mathcal{D}, \phi) : \widehat{\mathcal{A}} \Leftarrow\!\!\!\rightsquigarrow \widehat{\mathcal{B}},$$

if ϕ is continuous on $\mathcal{D}\cap\mathcal{A}$ and, moreover, there is a compact set $\mathcal{W} \subseteq \mathcal{D}\cap\mathcal{A}$ such that, for every path $\sigma \subseteq \mathcal{A}$ with $\sigma \cap \mathcal{A}_l^- \neq \emptyset$ and $\sigma \cap \mathcal{A}_r^- \neq \emptyset$, there is a path $\gamma \subseteq \sigma \cap \mathcal{W}$ with $\psi(\gamma) \cap \mathcal{B}_l^- \neq \emptyset$ and $\phi(\gamma) \cap \mathcal{B}_r^- \neq \emptyset$.

Observe that this definition coincides with

$$(\mathcal{D}', \phi) : \widetilde{\mathcal{A}} \Leftarrow\!\!\!\rightsquigarrow \widetilde{\mathcal{B}},$$

according to Definition 2, for

$$\mathcal{D}' = \mathcal{D} \cap \mathcal{A} \cap \phi^{-1}(\mathcal{B}).$$

As in [60] we introduce now some special subsets of a cell which are crucial for our applications.

DEFINITION 6. *Let $\widehat{\mathcal{M}} = (\mathcal{M}, c_{\mathcal{M}}; X, d_1)$ and $\widehat{\mathcal{N}} = (\mathcal{N}, c_{\mathcal{N}}; X, d_2)$ be two rectangular cells of the same topological space X and let $\widetilde{\mathcal{M}} = (\mathcal{M}, \mathcal{M}^-)$ and $\widetilde{\mathcal{N}} = (\mathcal{N}, \mathcal{N}^-)$ be the corresponding path-oriented spaces. We say that $\widehat{\mathcal{M}}$ is a horizontal slab of $\widehat{\mathcal{N}}$ and write*

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}},$$

if $\widetilde{\mathcal{M}} \subseteq_h \widetilde{\mathcal{N}}$, that is, if $\mathcal{M} \subseteq \mathcal{N}$ and if every path $\gamma \subseteq \mathcal{M}$ with $\gamma \cap \mathcal{M}_l^- \neq \emptyset$ and $\gamma \cap \mathcal{M}_r^- \neq \emptyset$ is such that $\gamma \cap \mathcal{N}_l^- \neq \emptyset$ and $\gamma \cap \mathcal{N}_r^- \neq \emptyset$.
Similarly, we say that $\widehat{\mathcal{M}}$ is a vertical slab of $\widehat{\mathcal{N}}$ and write

$$\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}},$$

if $\widetilde{\mathcal{M}} \subseteq_v \widetilde{\mathcal{N}}$, that is, if $\mathcal{M} \subseteq \mathcal{N}$ and if every path $\sigma \subseteq \mathcal{N}$ with $\sigma \cap \mathcal{N}_l^- \neq \emptyset$ and $\sigma \cap \mathcal{N}_r^- \neq \emptyset$ contains a sub-path $\gamma \subseteq \mathcal{M}$ such that $\gamma \cap \mathcal{M}_l^- \neq \emptyset$ and $\gamma \cap \mathcal{M}_r^- \neq \emptyset$.

REMARK 8. Note that in order to have $\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}}$ it is equivalent to require that $\mathcal{M} \subseteq \mathcal{N}$ and either

$$\widehat{\mathcal{M}}_l^- \subseteq \widehat{\mathcal{N}}_l^-, \quad \widehat{\mathcal{M}}_r^- \subseteq \widehat{\mathcal{N}}_r^-$$

or

$$\widehat{\mathcal{M}}_l^- \subseteq \widehat{\mathcal{N}}_r^-, \quad \widehat{\mathcal{M}}_r^- \subseteq \widehat{\mathcal{N}}_l^- .$$

In this manner, our definition of horizontal slab reduces to the one of horizontal slice in [60, Def.1.2]

As in Section 3, we can now borrow from [60] and [61] the next definition.

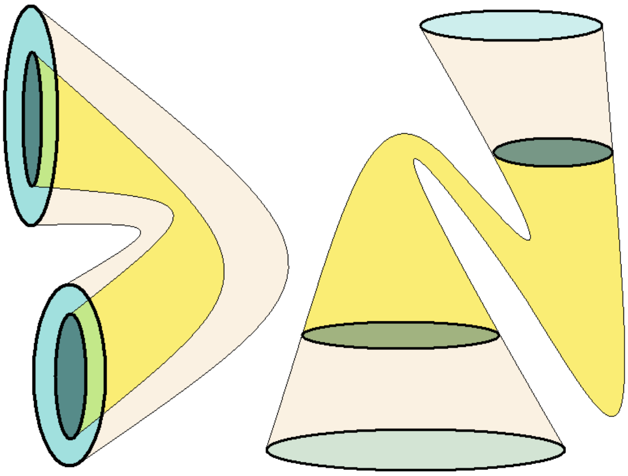

Figure 14: Examples of $\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{N}}$ and of $\widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{N}}$ (the left and the right figures, respectively). The painted areas represent $\widehat{\mathcal{M}}$ as embedded in $\widehat{\mathcal{N}}$. The contours of $[\,\cdot\,]^-$-sets for the oriented cells $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are indicated with a bold line.

DEFINITION 7. *Let $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{M}}$ be three rectangular cells with $\mathcal{A}, \mathcal{B}, \mathcal{M}$ subspaces of the same topological space X and suppose that $\mathcal{M} \subseteq \mathcal{A} \cap \mathcal{B}$.*
We say that $\widehat{\mathcal{B}}$ crosses $\widehat{\mathcal{A}}$ in $\widehat{\mathcal{M}}$ and write

$$\widehat{\mathcal{M}} \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\},$$

if

$$\widehat{\mathcal{M}} \subseteq_h \widehat{\mathcal{A}} \quad \text{and} \quad \widehat{\mathcal{M}} \subseteq_v \widehat{\mathcal{B}}.$$

4.2. Applications

At this step, we can just reconsider the same main results from [59, 60] already proved for the stretching property in the case of generalized two-dimensional cells and extend them to $(1, \mathrm{N}-1)$-rectangular cells. For instance, we have the following (compare to Theorem 4).

THEOREM 12. *Suppose that $\widehat{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widehat{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ are oriented cells in X. If $(\mathcal{D}, \psi) : \widehat{\mathcal{A}} \Leftrightarrow\!\!\!\rightsquigarrow \widehat{\mathcal{B}}$ and there are $k \geq 2$ oriented cells $\widehat{\mathcal{M}}_1 \ldots, \widehat{\mathcal{M}}_k$ such that*

$$\widehat{\mathcal{M}}_i \in \{\widehat{\mathcal{A}} \pitchfork \widehat{\mathcal{B}}\}, \quad \text{for } i = 1, \ldots, k,$$

with

$$\mathcal{M}_i \cap \mathcal{M}_j \cap \mathcal{D} = \emptyset, \text{ for all } i \neq j, \text{ with } i, j \in \{1, \ldots, k\},$$

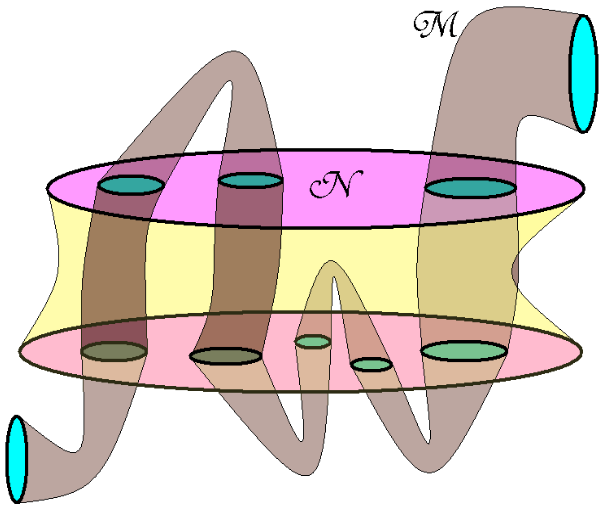

Figure 15: In $\mathbb{R}^3$, the $(1, 2)$-rectangular cell $\widehat{\mathcal{N}}$ (the cheese shaped set) is crossed by the $(1, 2)$-rectangular cell $\widehat{\mathcal{M}}$ (the snake-like set). Among the four intersections of $\mathcal{M}$ with $\mathcal{N}$, the first two (counting from the left and painted by a darker color) belong to $\{\widehat{\mathcal{N}} \pitchfork \widehat{\mathcal{M}}\}$.

then the following conclusion holds:

- ψ *has a chaotic dynamics of coin-tossing type on k symbols (with respect to the sets* $\mathcal{K}_i := \mathcal{D} \cap \mathcal{M}_i$*).*
- ψ *has a fixed point in each set* $\mathcal{K}_i := \mathcal{D} \cap \mathcal{M}_i$ *and, for each finite sequence* $(s_0, s_1, \ldots, s_m) \in \{1, \ldots, k\}^{m+1}$, *with* $m \geq 1$, *there is at least one point* $z^* \in \mathcal{K}_{s_0}$ *such that the position*

$$z_{j+1} = \psi(z_j), \quad z_0 = z^*, \quad \textit{for } j = 0, 1, \ldots, m$$

defines a sequence of points with

$$z_j \in \mathcal{K}_{s_j}, \ \forall\, j = 0, 1, \ldots, m \quad \textit{and } z_{m+1} = z^*.$$

REMARK 9. The two conclusions in Theorem 12 corresponds to (a_1) and (a_3) of Theorem 11. We could derive from (a_2) also a conclusion about the existence of a continuum of initial points which generate any (fixed) forward itinerary and thus obtain an extension of the conclusion (b_2) of Theorem 4. This one as well as some related topics, which require a more careful treatment, will be discussed elsewhere.

As shown by this example, from Theorem 11 and the definitions of stretching, slabs and crossings adapted to the case of $(1, \mathrm{N} - 1)$-rectangular cells, we have now available all the tools which are needed in order to achieve a full extension of the topological results contained in [59, 60] and partially recalled in Section 1.2, to maps which expand the arcs along one direction. A more complete investigation on this subject will appear in a future work.

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