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SYMMETRIC ALGEBRAS OF FINITELY GENERATED GRADED MODULES AND s-SEQUENCES

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We study properties of the symmetric algebra of finitely generated graded modules M on a Noetherian ring R, generated by s-sequences. For these modules we investigate the Eisenbud-Goto conjecture. If $R = K[X_1, \ldots, X_n]$ is a polynomial ring over a field K and M has linear syzygies, we consider the jacobian dual module of M in order to describe the Rees algebra of M.

1. Introduction

The aim of this paper is to study an interesting class of finitely generated modules M on a Noetherian ring R for which the initial ideal of the presentation ideal J of their symmetric algebra is very simple. More precisely, with respect to a special order on the variables that correspond to the generators of the module M, we have a good expression for the initial ideal of J. This area was investigated in [7], where the authors computed some algebraic invariants of $\mathrm{Sym}_R(M)$ or their bounds in terms of special ideals of the ring R. The theory gives definitive results if R is the polynomial ring in m variables on a field K of any characteristic by using the Gröbner basis theory (in the following K always denotes a field). Here we would like to study an application of previous results essentially in two directions. We have many areas of applications and this is only the starting point of investigation via s-sequences. The first is to test the Eisenbud-Goto conjecture (EGC) for the symmetric algebra of a module M generated by an s-sequence. After we have given formulations in this case, we begin to work in this direction. For regular sequences of forms in the polynomial ring (which are strong s-sequences) we prove the (EGC). If M has linear syzygies on the polynomial ring $R = K[X_1, \ldots, X_m]$, a nice construction of [12] leads to the jacobian dual module N of M. N is a finitely generated module on the ring $Q = K[Y_1, \ldots, Y_n]$ and we have the isomorphism $\mathrm{Sym}_R(M) = \mathrm{Sym}_Q(N)$. Then it is interesting to ask, when M is generated by an s-sequence, if N is generated by an s-sequence and viceversa, and this is the second area. In this context it is possible to describe the Rees algebra of the module M as a quotient of the symmetric algebra by its torsion submodule. In particular, in section 1 we give the definition of s-sequence introduced in [7], we recall some known results and we formulate the Eisenbud-Goto conjecture for the symmetric algebra of a finitely generated graded module M generated by an s-sequence on a Noetherian ring in different ways. In particular, if $R = K[X_1, \ldots, X_m]$ is the polynomial ring, we give it in terms of the annihilator ideals of the s-sequence. As an application, we verify (EGC) for a regular sequence of forms of R.
In section 2 we introduce the jacobian dual of a module M on a polynomial ring

$R = K[X_1,\dots,X_m]$. This module can be defined if the presentation matrix of M has linear entries in the variables X_i and its interest appears in many fields of commutative algebra. If this module N over the polynomial ring $K[Y_1,\dots,Y_n]$ is generated by a strong s-sequence, we obtain that the torsion submodule of $\mathrm{Sym}_R(M)$ coincides with the first annihilator ideal of the s-sequence generating N and it is an ideal of Q. The Rees algebra of M, as a quotient of $\mathrm{Sym}_R(M)$ by its torsion submodule, can be computed. As an example, we consider a monomial ideal with linear syzygies not generated by an s-sequence, whose jacobian dual is generated by an s-sequence. The idea is to address our interest to many computations in this direction.

2. Preliminaries

Let R be any Noetherian ring and M a finitely generated R-module with generators $f_1,\dots,f_n$. If we consider a presentation of M

$$R^m \xrightarrow{f} R^n \to M \to 0,$$

then f is represented by an $n\times m$ matrix (a_{ij}) with entries in R, $1\le i\le n$, $1\le j\le m$. The symmetric algebra of M on R, $\mathrm{Sym}_R(M) = \oplus_{i\ge 0}S_i(M)$, where, for each i, $S_i(M)$ is the component of degree i of $\mathrm{Sym}_R(M)$, has a presentation:

$$0 \to J \to \mathrm{Sym}_R(R^n) \to \mathrm{Sym}_R(M) \to 0$$

and $\mathrm{Sym}_R(R^n) \simeq R[Y_1,\dots,Y_n]$ is the polynomial ring on R in the variables Y_j, J is the relation ideal of $\mathrm{Sym}_R(M)$, $J = (g_1,\dots,g_m)$, with g_i form of degree 1 in the Y_j, $g_i = \sum_{j=1}^n a_{ij}Y_j$, for $i = 1,\dots,m$, then $\mathrm{Sym}_R(M) \simeq R[Y_1,\dots,Y_n]/J$.

The main problem is how to compute standard algebraic invariants of the graded algebra $\mathrm{Sym}_R(M)$ such as the dimension $\dim(\mathrm{Sym}_R(M))$, the multiplicity $\mathrm{e}(\mathrm{Sym}_R(M))$, the depth $\mathrm{depth}(\mathrm{Sym}_R(M))$ with respect to the graded maximal ideal $\mathrm{Sym}_R(M)^+ = \oplus_{i>0}S_i(M)$, the regularity $\mathrm{reg}(M)$, in terms of the corresponding invariants of special quotients of the ring R.

The first three invariants are classical. For the last invariant, we recall that $\mathrm{reg}(M)$ is the Castelnuovo-Mumford regularity of the graded module M. Its importance is briefly indicated in Eisenbud-Goto theorem which is an interesting description of regularity in terms of the graded Betti numbers of M ([2]).

They show that $\mathrm{reg}(M)$, when M is a graded finite R-module, where $R = K[X_1,\dots,X_m]$, measures the “complexity” of the minimal free resolution of M as an S-module. Therefore regularity plays an important role in many fields of commutative algebra.

More precisely, if we consider a graded minimal free resolution of M on S

$$0 \to F_\ell \to \cdots \to F_1 \to F_0 \to M \to 0,$$

if b_i is the maximum degree of the generators of the free module F_i, then

$$\mathrm{reg}(M) = \sup\{b_i - i, i \ge 0\}.$$

In other words, $\operatorname{reg}(M)$ is the smallest integer m such that for every j, the j-th syzygy module of M is generated in degree $\leq m + j$ (equivalently, $\operatorname{reg}(M) = \sup\{\beta_{i,i+j} \neq 0$, for some $i\}$, where $\beta_{i,\ell}$ are the graded Betti numbers of M).
If R is not a polynomial ring, the regularity of M can be infinite. A nice area of investigation in commutative algebra is the study of graded homogeneous algebras A generated in the same degree such that $\operatorname{reg}_A(A/m^+) = 0$ as an A-module and m^+ is the maximal graded ideal of A.

A computation of the previous invariants can be obtained for a finitely generated R-module that is generated by an s-sequence $f_1, \ldots, f_n$ in the sense of [7]. Consider the presentation of $\operatorname{Sym}_R(M)$

$$\operatorname{Sym}_R(M) = R[Y_1, \ldots, Y_n]/J.$$

The ideal $\operatorname{Sym}_R(M)^+$ is generated by the residue classes of the Y_i that are called f_i^*, because the variables Y_i correspond to the generators of the module $M = Rf_1 + \cdots + Rf_n$ in the presentation of $\operatorname{Sym}_R(M)$. For every $i = 1, \ldots, n$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and let $I_i = M_{i-1} :_R f_i$ be the colon ideal. We set $I_0 = (0)$ for convenience. Since $M_i/M_{i-1} \simeq R/I_i$, so I_i is the annihilator of the cyclic module R/I_i, I_i is called an annihilator ideal of the sequence $f_1, \ldots, f_i$.

Consider the polynomial ring $R[Y_1, \ldots, Y_n]$ and let $<$ be a monomial order on the monomials of $R[Y_1, \ldots, Y_n]$ in the variables Y_i such that

$$Y_1 < Y_2 < \cdots < Y_n.$$

We call $<$ an admissible order.
With respect to this term order, if $f = \sum a_\alpha \underline{Y}^\alpha$, $\underline{Y}^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$, we put $\operatorname{in}_< f = a_\alpha \underline{Y}^\alpha$, where $\underline{Y}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$.
If we assign degree 1 to each variable Y_i and degree 0 to the elements of R, we have the following facts:

1) J is a graded ideal,

2) the natural epimorphism $S \to \operatorname{Sym}_R(M)$ is a graded homomorphism of graded algebras on R, S is a graded ring and $\operatorname{Sym}_R(M)$ is a graded ring.

DEFINITION 1. *The sequence $f_1, \ldots, f_n$ is an s-sequence for M if*

$$(I_1Y_1, I_2Y_2, \ldots, I_nY_n) = in_< J.$$

If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$, the sequence is a strong s-sequence.

EXAMPLE 1. Any d-sequence of elements $a_1, \ldots, a_n$ in R is a strong s-sequence, with respect to the reverse lexicographic order on the Y_i, with $Y_1 < Y_2 < \cdots < Y_n$ ([7], Cor. 3.3).
As a consequence regular sequences, proper sequences are strong s-sequences, since they are d-sequences ([10]).
If $I = (a_1, \ldots, a_n)$, we have $\operatorname{Sym}_R(I) = \mathcal{R}_R(I)$, the Rees algebra of I.

If $R = K[X_1, \dots, X_m]$ we can use the Gröbner basis theory and Buchberger's algorithm to compute $\mathrm{in}_< J$.
If $R = K[X_1, \dots, X_m]$, then $\mathrm{Sym}_R(M) = K[X_1, \dots, X_m, Y_1, \dots, Y_n]/J$. We can introduce a term order on

$$S = K[X_1, \dots, X_m, Y_1, \dots, Y_n]$$

such that $Y_1 < Y_2 < \dots < Y_n$ and $X_i < Y_i$ for any i.
For example $X_1 < X_2 < \dots < X_m < Y_1 < Y_2 < \dots < Y_n$ is such a term order.
If G is a Gröbner basis for $J \subset K[X_1, \dots, X_m, Y_1, \dots, Y_n]$, we have $\mathrm{in}_< J = (\mathrm{in}_< G) = (\mathrm{in}_< f, f \in J)$ and if the elements of G are linear in the $Y_i s$, it follows that $f_1, \dots, f_n$ is an s-sequence for M.

REMARK 1. If $R = K[X_1, \dots, X_m]$, from the theory of Gröbner basis, if $f_1, \dots, f_n$ is an s-sequence with respect to any admissible term order $<$, then $f_1, \dots, f_n$ is an s-sequence for another admissible term order, too.

THEOREM 1 ([7]). *Suppose R is a standard graded algebra, M is a graded R-module which is generated by the homogeneous s-sequence $f_1, \dots, f_n$, where all f_i have the same degree, $I_1, \dots, I_n$ are the annihilator ideals of the sequence $f_1, \dots, f_n$. Then:*

i) $\dim \mathrm{Sym}_R(M) = \max\limits_{\substack{0 \le r \le n \\ 1 \le r_1 \le \dots \le r_n \le n}} \{\dim R/(I_{r_1} + \dots + I_{r_n}) + r\}$

ii) $e(\mathrm{Sym}_R(M)) = \sum\limits_{\substack{0 \le r \le n \\ 1 \le r_1 \le \dots \le r_n \le n}} e(R/(I_{r_1} + \dots + I_{r_n}))$,
where $\dim R/(I_{r_1} + \dots + I_{r_n}) = d - r$, $d = \dim \mathrm{Sym}_R(M)$.

For a strong s-sequence we have:

$$d = \dim \mathrm{Sym}_R(M) = \max_{0 \le r \le n} \{\dim R/I_r + r\},$$

$$e(\mathrm{Sym}_R(M)) = \sum_{\substack{0 \le r \le n \\ \dim R/I_r = d - r}} e(R/I_r).$$

THEOREM 2 ([7]). *If $R = K[X_1, \dots, X_m]$ and M is generated by elements of the same degree, which are a strong s-sequence, then*

1) $\mathrm{reg}(\mathrm{Sym}_R(M)) \le \max\{\mathrm{reg}(R/I_i), i = 1, \dots, n\} + 1$,

2) $\mathrm{depth}(\mathrm{Sym}_R(M)) \ge \min\{\mathrm{depth}(R/I_i) + i, i = 0, \dots, n\}$.

The notion of s-sequence can be useful essentially:

1) to test some conjectures for graded modules M generated by s-sequences, "via" conjectures about annihilator ideals of M, in particular we are interested to test

the Eisenbud-Goto conjecture (EGC) for $\mathrm{Sym}_R(M)$, when M has generators of the same degree and the regularity is the ordinary regularity.

The (EGC), that involves all invariants of $\mathrm{Sym}_R(M)$, can be more easily verified if M is generated by an s-sequence.

2) to describe the Rees algebra of the R-module M, $R = K[X_1, \dots, X_m]$,

$$\mathcal{R}_R(M) = \mathrm{Sym}_R(M)/(\mathrm{Sym}_R(M))_0$$

and to test (EGC) in the case the matrix of the relations of M is linear in the variables $X_1, \dots, X_m$. In this situation in fact we have a nice construction that collects many cases of ideals and modules (in particular those ones with linear resolution): the jacobian dual module N.
If N is the Jacobian dual of M, then natural questions arise:

i) When the jacobian dual N of M is generated by an s-sequence?

ii) If it is the case, does $\mathrm{Sym}_Q(N)$ verify (EGC)?

3. Eisenbud-Goto conjecture

There are several conjectures to connect the measures of the complexity of an algebra. One of the most important is the following:

CONJECTURE 2. (**EGC**) If A is a standard graded domain on a field K then

$$\mathrm{reg}(A) \leq e(A) - \mathrm{codim}(A),$$

where $\mathrm{codim}(A) = \mathrm{emb}\dim(A) - \dim(A)$.

If A is Cohen-Macaulay, the conjecture is true and we have equality ([2]).
We will establish the (EGC) for symmetric algebras of finitely generated graded module M generated by s-sequences.
We consider different formulations of the conjecture.

1) (**EGC1**) Eisenbud-Goto conjecture for the symmetric algebra of a module M on a standard graded algebra R, generated on R by a strong s-sequence of elements of the same degree, and such that $\mathrm{Sym}_R(M)$ is a domain

$$\mathrm{reg}(\mathrm{Sym}_R(M)) \leq e(\mathrm{Sym}_R(M)) - \mathrm{codim}(\mathrm{Sym}_R(M))$$

2) (**EGC2**) If $R = K[X_1, \dots, X_m]$, M a graded R-module generated by a strong s-sequence of elements of the same degree, Eisenbud-Goto conjecture for the symmetric algebra of M in terms of the annihilator ideals of the strong s-sequence generating M is

$$\max\{\mathrm{reg}(R/I_i) : i = 1, \dots, n\} + 1 \leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \max_{0 \leq i \leq n}\{\dim(R/I_i) + i\}$$

3) (**EGC3**) Eisenbud-Goto conjecture for any annihilator prime ideal of a strong s-sequence of the same degree > 1, generating a graded R-module M, $R = K[X_1, \dots, X_m]$, or R standard graded algebra that is a domain

(**EGC**$_i$) $\operatorname{reg}(R/I_i) \le e(R/I_i) - \operatorname{codim}(R/I_i)$, for $i = 1, \dots, n$, $\dim R/I_i = d - i$.

(**EGC'**$_i$) $\operatorname{reg}(R/I_i) \le e(R/I_i) - m + \dim(R/I_i)$, $i = 1, \dots, n$, $\dim R/I_i = d - i$.

In 2) and 3) $d = \dim \operatorname{Sym}_R(M)$.

4) The same conjecture formulated for the Rees algebra $\mathcal{R}(M)$, when $\mathcal{R}(M) = \operatorname{Sym}_R(M)$, R a standard graded domain and M generated on R by a strong s-sequence of elements of the same degree, becomes:

(**EGC1'**) $\operatorname{reg}(\mathcal{R}(M)) \le e(\mathcal{R}(M)) - \operatorname{codim}(\mathcal{R}(M))$.

If $R = K[X_1, \dots, X_m]$, M a graded finitely generated R-module generated by an s-sequence of elements of the same degree, Eisenbud-Goto conjecture of $\mathcal{R}(M) = \operatorname{Sym}_R(M)$ in terms of annihilator ideals of the strong s-sequence generating M (for example, M is an ideal of $R = K[X_1, \dots, X_m]$ generated by a d-sequence of elements of R):

(**EGC2'**) $\max\{\operatorname{reg}(R/I_i) : i = 1, \dots, n\} \le \sum_{i=1}^{n} e(R/I_i) - n$

(**EGC3'**) $\operatorname{reg}(R/I_i) \le e(R/I_i) - m + \dim(R/I_i)$

Some implications:

(**EGC2**) $\Rightarrow$ (**EGC1**)

If $R = K[X_1, \dots, X_m]$, by Theorem 1 and Theorem 2, we have:

$$\begin{aligned}
\operatorname{reg}(\operatorname{Sym}_R(M)) &\le \max\{\operatorname{reg} R/I_i : i = 1, \dots, n\} + 1 \\
&\le \sum_{i=1}^{n} e(R/I_i) - (n+m) - \max_{0 \le i \le n}\{\dim(R/I_i + i)\} \\
&\le e(\operatorname{Sym}_R(M)) - \operatorname{codim}(\operatorname{Sym}_R(M))
\end{aligned}$$

(**EGC'**$_i$) $\Rightarrow$ (**EGC**$_i$), for any ideal I_i, $i = 1, \dots, n$.

$$\operatorname{reg}(R/I_i) \le e(R/I_i) - m + \dim(R/I_i) \le e(R/I_i) - \operatorname{codim}(R/I_i).$$

(**EGC'**$_i$) for every $i = 1, \dots, n \Rightarrow$ (**EGC2**)

Since (**EGC'**$_i$) is true for every i, $i = 1, \dots, n$, we have:

$$\begin{aligned}
&\max\{\operatorname{reg} R/I_i : i = 1, \dots, n\} \\
&\quad \le \sum_{i=1}^{n} e(R/I_i) - (n+m) - (n-2)m + \dim(R/I_s) + \sum_{\substack{i=1 \\ i \ne s}}^{n} \dim(R/I_i)
\end{aligned}$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \sum_{\substack{i=1 \\ i\neq s}}^{n} (\dim(R/I_i) - m) + \dim R/I_s$$

$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \dim R/I_s + s - 1$$

since $n \leq m$.

If s is the integer such that $\dim(R/I_s) + s = \max_{1\leq i\leq n} \{\dim(R/I_i) + i\}$, then we have

$$\max\{\operatorname{reg} R/I_i : i = 1, \dots, n\}$$
$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) - (n-2)m + \dim(R/I_s) + \sum_{\substack{i=1 \\ i\neq s}}^{n} \dim(R/I_i)$$
$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \sum_{\substack{i=1 \\ i\neq s}}^{n} (\dim(R/I_i) - m) + \dim R/I_s$$
$$\leq \sum_{i=1}^{n} e(R/I_i) - (n+m) + \dim R/I_s + s - 1$$

since $\dim R/I_i - m \leq 0$, $i = 1, \dots, n$.

In order to state the (EGC) for the jacobian dual module N of M, we need some facts on N. As a consequence we will give the formulation of (EGC) for N in the next section.

EXAMPLE 2 (Regular sequences). Let R be a Noetherian ring and let $f_1, \dots, f_n$ be a regular sequence of elements of R. Then $f_1, \dots, f_n$ is a strong s-sequence with respect to any reverse lexicographic order on the variables $Y_1, \dots, Y_n$ such that $Y_1 < Y_2 < \dots < Y_n$ with annihilator ideals $I_1 = (0)$, $I_2 = (f_1), \dots, I_n = (f_1, \dots, f_{n-1})$ and $\mathrm{in}_< J = ((f_1)Y_2, \dots, (f_1, \dots, f_{n-1})Y_n)$.
In fact a regular sequence is a d-sequence, hence the assertion follows.

THEOREM 3. *Let R be a Noetherian ring and let $f_1, \dots, f_n$ be a regular sequence. Let $I = (f_1, \dots, f_n)$ and $\mathrm{Sym}_R(I) = R[Y_1, \dots, Y_n]/J$. Then we have:*

1) *J is minimally generated by the elements $g_{ij} = f_iY_j - f_jY_i$, $1 \leq i < j \leq n$.*
2) *If $R = K[X_1, \dots, X_m]$, the set $\{g_{ij}, 1 \leq i < j \leq n\}$ is a Gröbner basis with respect to any reverse lexicographic order on the Y_j and such that $Y_1 < \dots < Y_n$.*

Proof. 1) Put $g_{ij} = f_i Y_j - f_j Y_i$, $i < j$ and suppose that $\{g_{ij}\}_{1 \le i < j \le n}$ is not a minimal system of generators of J. Then

$$f_i Y_j - f_j Y_i = \sum_{(\rho,k) \neq (i,j)} h_{\rho k} g_{\rho k},$$

fore some i, j, $1 \le i < j \le n$.

Hence

$$f_i = \sum_{\rho > j} h_{j\rho} f_\rho + \sum_{\rho < j} h_{\rho j} f_\rho$$

and this is a contradiction.

2) We have to consider the S-couples:

i) $S(g_{ij}, g_{ik})$, $j \neq k$

ii) $S(g_{ij}, g_{kj})$, $i \neq k$

iii) $S(g_{ij}, g_{k\rho})$, $i \neq k$, $j \neq \rho$

For i), $S(g_{ij}, g_{ik}) = Y_i(-f_k Y_j + f_j Y_k)$.
For ii), $S(g_{ij}, g_{kj}) = f_j(-f_i Y_k + f_k Y_i)$.
For iii), $S(g_{ij}, g_{k\rho}) = -f_k Y_\rho g_{ij} - f_j Y_i g_{k\rho}$.
So, by Buchberger's criterion we get $\mathrm{in}_<(J) = (\mathrm{in}_< g_{ij})$. □

Now, let $R = K[X_1, \dots, X_m]$ be a polynomial ring and let I be an ideal of R generated by an R-sequence $f_1, \dots, f_n$ of homogeneous elements.

Case I: $f_1, \dots, f_n$ have the same degree a.

PROPOSITION 1. $\mathrm{reg}(\mathrm{Sym}_R(I)) \le (n-1)(a-1) + 1$.

Proof. Since $f_1, \dots, f_n$ is a strong s-sequence, then we can apply the formula

$$\mathrm{reg}(\mathrm{Sym}_R(I)) \le \max\{\mathrm{reg}(R/I_i), 1 \le i \le n\} + 1,$$

where $I_0 = I_1 = (0)$, $I_2 = (f_1), \dots, I_n = (f_1, \dots, f_{n-1})$. The result follows by the Koszul resolution for the annihilator ideals I_i, $2 \le i \le n$. □

PROPOSITION 2. *Let $I = (f_1, \dots, f_n) \subset R = K[X_1, \dots, X_m]$ be generated by a regular sequence of forms of the same degree a. Then* (EGC2') *is true.*

Proof. We have to prove that

$$\max\{\mathrm{reg}(R/I_i), 1 \le i \le n\} \le \sum_{1 \le i \le n} e(R/I_i) - n$$

that is

$$(n-1)a-(n-1) \leq \sum_{i=1}^{n} a^{i-1} - n$$

$$(n-1)a \leq a + a^2 + \cdots + a^{n-1}.$$

The assertion follows. $\square$

PROPOSITION 3. *Let $I = (f_1, \ldots, f_n) \subset R = K[X_1, \ldots, X_m]$ be generated by a regular sequence of forms of the same degree $a \geq 2$. Then* (EGC'$_i$) *is true for i, $1 \leq i \leq n$, such that the annihilator ideal I_i is a prime ideal.*

Proof. We have to prove that $\text{reg}(R/I_i) \leq e(R/I_i) - i + 1$, i.e. $(i-1)a-(i-1) \leq a^{i-1} - i + 1$ and *(EGC'$_i$)* is true. $\square$

REMARK 2.
1) For $n > 1$, in Proposition 1 we have in fact equality. The result can follow from the resolution of the algebra $\text{Sym}_R(I) = \mathcal{R}(I)$, by employing the Eagon-Northcott complex. Let $S = K[X_1, \ldots, X_m; Y_1, \ldots, Y_n]$ and let F and G be finitely generated free graded S-modules of rank 2 and m respectively. Consider a graded homomorphism of degree zero $g : G \to F$, g represented by the matrix

$$\begin{pmatrix} f_1 & \ldots & f_n \\ Y_1 & \ldots & Y_n \end{pmatrix}.$$

We can write $g : S(-a)^n \to S^2$ and we consider the Koszul complex arising from g.

$$K(g) : 0 \to \bigwedge^{n} G \otimes S(F)(-n) \to \bigwedge^{n-1} G \otimes S(F)(-n+1) \to \cdots$$
$$\cdots \to G \otimes S(F)(-1) \to S(F) \to 0,$$

where $S(F) = \text{Sym}_S(F) = S[\underline{T}] = S[T_1, T_2]$ and the differential

$$\delta : \bigwedge^{i} G \otimes S(F)(-i) \to \bigwedge^{i-1} G \otimes S(F)(-i+1)$$

is defined by

$$\delta(t_1 \wedge t_2 \wedge \ldots \wedge t_i \otimes f(\underline{T})) = \sum_{j=1}^{i} (-1)^j g(t_j) t_1 \wedge t_2 \wedge \ldots \wedge \widehat{t_j} \wedge \ldots \wedge t_i \otimes f(\underline{T}).$$

Since $ht(J) = n-1$, $\dim \mathcal{R}(I) = m+1 = n+m-ht(J)$ and J is perfect.
The complex

$$D_0(g) : 0 \to \Big(\bigwedge^{0} G \otimes S_{n-2}(F)\Big)^* \to \big(G \otimes S_{n-3}(F)(-1)\big)^* \to \cdots$$
$$\to \Big(\bigwedge^{n-3} G \otimes S_1(F)(-n+3)\Big)^* \to \Big(\bigwedge^{n-2} G \otimes S_0(F)(-n+2)\Big)^* \to S \to 0,$$

resolves S/J ([7], (2.16)).
Since any generator of J has degree $a+1$, the shift in the place 1 is $-a$, that is a is the shift of the generators of the module $(\wedge^{n-2}G \otimes S_0(F))^*$. Finally, the complex above is the dual of a Koszul complex. Hence:

$$\operatorname{reg}(S/J) = (n-1)a - (n-1) + 1.$$

2) If $f_1, \dots, f_n$ is a regular sequence of n forms of degree $a \geq 2$, $\mathrm{Sym}_R(I) = \mathcal{R}(I)$, $I = (f_1, \dots, f_n)$ and $\operatorname{reg}(R) \leq \operatorname{reg}\mathcal{R}(I) \leq \max\{\operatorname{reg} R + 1, \operatorname{reg} R + n(a-1)\}$ ([5], Corollary 2.6).
For $m = 1$, $\operatorname{reg} R = 0$, $\operatorname{reg}\mathcal{R}(I) = 0$. The assertion follows and (EGC) is true.
For $m > 1$, $\operatorname{reg} R = 0$ and $0 \leq \operatorname{reg}\mathcal{R}(I) \leq \max\{1, n(a-1)\} = n(a-1)$.
For $n \geq 3$, what is needed is $n(a-1) \leq \displaystyle\sum_{i=1}^{n} a^{i-1} - n + 1$, $na \leq \displaystyle\sum_{i=1}^{n} a^{i-1} + 1$. If we write $na = a + (a+a) + (n-3)a$, we have $na \leq a + a^2 + \sum_{i=4}^{n} a^{i-1}$ and (EGC) is true.
Case II: $f_1, \dots, f_n$ are forms of different degrees $d_1, \dots, d_n$, $d_1 \leq d_2 \leq \dots \leq d_n$.
Consider R as a graded ring by assigning to each variable X_i degree 0. Then $S = R[Y_1, \dots, Y_n]$ is a graded ring if we assign to each variable Y_i degree 1. Let $<$ be a monomial order on the monomials in $Y_1, \dots, Y_n$ such that $Y_1 < Y_2 < \cdots < Y_n$. Since $f_1, \dots, f_n$ is a regular sequence, it is a strong s-sequence and $\mathrm{in}_< J = (I_1 Y_1, \dots, I_n Y_n)$, $I_i = (f_1, \dots, f_{i-1})$ for $i = 1, \dots, n$.
As a consequence

$$\begin{aligned}\operatorname{reg}\mathcal{R}(I) &= \operatorname{reg} R[Y_1, \dots, Y_n]/J \leq \operatorname{reg} R[Y_1, \dots, Y_n]/\mathrm{in}_< J = \\ &= \operatorname{reg} R[Y_1, \dots, Y_n]/(I_1 Y_1, \dots, I_n Y_n) \leq \max\{\operatorname{reg}(R/I_i), 1 \leq i \leq n\} + 1.\end{aligned}$$

But the last regularity is 0, since all matrices have entries of degree 0 in the minimal graded resolution of any annihilator ideal I_i. We need $\mathcal{R}(I)$ is a standard graded algebra for the formulation of (EGC).

4. Jacobian dual

Let $R = K[X_1, \dots, X_s]$ be a polynomial ring and let E be a finitely generated R-module with presentation :

$$R^m \xrightarrow{\phi} R^n \to E \to 0$$

where the entries of the $n \times m$ matrix $A = (a_{ij})$ that represents ϕ are homogeneous linear forms.
The equations of the symmetric algebra of E, $\mathrm{Sym}_R(E) = S(E)$ are

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i \qquad j = 1, \dots, m.$$

There is a naive duality for $S(E)$, obtained from rewriting the equations f_j in the X_i's variables.

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i = \sum_{i=1}^{s} b_{ij} X_i \qquad j = 1, \dots, m$$

and $B = (b_{ij})$ is an $s \times m$ matrix of homogeneous linear forms in the Y_i's variables.

We have:

$$A^t \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = B^t \begin{pmatrix} X_1 \\ \vdots \\ X_s \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Now we put $Q = K[Y_1, \dots, Y_n]$ and consider the cokernel N of the map

$$Q^m \xrightarrow{\Psi} Q^s \to N \to 0,$$

where Ψ is the map represented by B.
N defines the Jacobian dual module of E ([12], [14]).

EXAMPLE 3. We can write the relation $f = (X_1 - 2X_2)Y_1 + (X_1 + X_2)Y_2 + X_3Y_3$ as $f = (Y_1 + Y_2)X_1 + (-2Y_1 + Y_2)X_2 + Y_3X_3$.

REMARK 3. $\mathrm{Sym}_R(E) \cong \mathrm{Sym}_Q(N)$.

EXAMPLE 4. Suppose that $A \cong B$, in the sense that the two matrices A and B have the same elements under the substitution $X_i \longrightarrow Y_i$, $n = s$. Then $R \cong Q$ and $E \cong N$.

There is a nice situation that will be interesting in the following.
Let $R = K[X_1, \dots, X_n]$, $I = m_+ = (X_1, \dots, X_n)$, $\mathrm{Sym}_R(m_+) = \mathcal{R}(m_+) = K[X_1, \dots, X_n; Y_1, \dots, Y_n]/J$, where J is generated by the binomials $X_iY_j - X_jY_i$, $1 \leq i < j \leq n$, the 2×2-minors of the $2 \times n$ matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_n \\ Y_1 & Y_2 & \dots & Y_n \end{pmatrix}.$$

The binomials in the X_i's give the dual matrix B of the relation matrix A of m_+ under the substitution $X_i \to Y_j$, $i, j = 1, \dots, n$.
Notice that the set of binomials is an universal Gröbner basis for the ideal J and this implies m_+ is generated by an s-sequence linear in the Y_i's and linear in the X_i's, too.

Another example is given by $m_i = (X_1, \dots, X_i)$, $i < n$.

$$\mathrm{Sym}_R(m_i) = K[X_1, \dots, X_n; Y_1, \dots, Y_i]/J_i$$

where J_i is generated by the binomials $X_\ell Y_s - X_s Y_\ell$, $1 \leq \ell < s \leq i$, the 2×2 minors of the matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_i \\ Y_1 & Y_2 & \dots & Y_i \end{pmatrix}.$$

Put $S = K[Y_1, \dots, Y_i]$, $\mathrm{Sym}_R(m_i) = \mathrm{Sym}_S(N)$ and $X_1, \dots, X_i$ is an s-sequence (it is a regular sequence) for m_i and the sequence of 1-forms $x_1^*, \dots, x_i^*$ is an s-sequence for the jacobian dual N of m_i, where $x_1, \dots, x_n$ are the residue classes of $X_1, \dots, X_n$ in $S[X_1, \dots, X_n]/J_i$.

PROPOSITION 4. *Let $R = K[X_1, \dots, X_m]$ be a polynomial ring and M a graded R-module generated by forms $f_1, \dots, f_n$ of the same degree. Suppose that the relation ideal J of $\mathrm{Sym}_R(M)$ is generated by forms that are linear in both sets of variables $\underline{X}$ and $\underline{Y}$, and let N be the jacobian dual of M generated by $x_1, \dots, x_m$, where $x_1^*, \dots, x_m^*$ are the images of the elements $X_1, \dots, X_m$ in the ring $K[X_1, \dots, X_m; Y_1, \dots, Y_n]/J$.*
Suppose J has a Gröbner basis linear in the $\underline{X}$ and $\underline{Y}$ variables with respect to the reverse lexicographic order on all variables and to the two orders of variables $X_m > \dots > X_1 > Y_n > \dots > Y_1$ and $Y_n > \dots > Y_1 > X_m > \dots > X_1$.
Then M is generated by an s-sequence if and only if N is generated by an s-sequence.

Proof. It is a consequence of the previous facts. □

REMARK 4. The strong case concerns J with a universal Gröbner basis that is linear in the $\underline{X}$ and $\underline{Y}$ variables with respect to any permutation of variables.

REMARK 5. If we know the Gröbner basis of J that is linear in the variables $\underline{X}$ and $\underline{Y}$, with respect to the reverse lexicographic order and to the two orders of variables $X_m > \dots > X_1 > Y_n > \dots > Y_1$ and $Y_n > \dots > Y_1 > X_m > \dots > X_1$, then we can write the annihilator ideals of the sequences $f_1, \dots, f_n$ and $x_1, \dots, x_m$ by using lemma 3.3 of [9].
The theorem gives the annihilator ideals for the s-sequence generating the jacobian dual N of M, but the proof can be repeated to have the annihilator ideals of M.

EXAMPLE 5. Let $I = (X^2, Y^2, XY)$ that is generated by an s-sequence ([7], Examples 1.5(1)). The jacobian dual N of I is generated by an s-sequence, too, but the relation ideal J has a Gröbner basis linear in the X_i's, but not linear in the Y_i's ([7]).

Now consider $\mathrm{Sym}_R(M) = R[Y_1, \dots, Y_n]/J \cong Q[X_1, \dots, X_m]/J = \mathrm{Sym}_Q(N)$. Let $x_1^*, \dots, x_m^*$ be the images of $X_1, \dots, X_m$ mod J that we can consider as the generators of N (we denote by $x_1, \dots, x_m$ the generators of N).

We recall some propositions:

PROPOSITION 5. *Let $I \subset R$ be an ideal generated by $f_1, \dots, f_n$. Then the following conditions are equivalent:*

1) *$f_1, \dots, f_n$ is a d-sequence;*

2) *$(0 : f_1) \cap I = 0$ and $f_2, \dots, f_n$ is a d-sequence in $R/(f_1)$.*

Proof. [7], Lemma 3.1. □

PROPOSITION 6. *Let M be an R-module generated by $f_1, \dots, f_n$. Then the following conditions are equivalent:*

1) *$f_1, \dots, f_n$ is a strong s-sequence with respect to the lexicographic order induced by $Y_n > Y_{n-1} > \dots > Y_1$;*

2) *$f_1^*, \dots, f_n^*$ is a d-sequence in $\mathrm{Sym}_R(M)$.*

Proof. [7], Theorem 3.2. □

PROPOSITION 7. *Let M be a finitely generated R-module, and let R be a domain. Then $\mathrm{Sym}_R(M)$ is a domain if and only if $(\mathrm{Sym}_R(M))_0 = 0$, where $(\mathrm{Sym}_R(M))_0 \subseteq \mathrm{Sym}_R(M)$ is the torsion submodule of M ([13]).*

THEOREM 4. *Suppose N is generated by a strong s-sequence $x_1, \dots, x_m$. Then we have*

1. *$x_1^*, \dots, x_m^*$ is a d-sequence in $\mathrm{Sym}_Q(N)$;*

2. *the ideal $(0 : x_1^*)$ is generated by elements of Q;*

3. *if $(0 : x_1^*)$ is a prime ideal then*

$$\mathrm{Sym}_Q(N)/(0 : x_1^*) \cong \mathcal{R}(M).$$

Proof. 1) If N is generated by a strong s-sequence, then $x_1^*, \dots, x_m^*$ is a d-sequence in $\mathrm{Sym}_R(N)$ (by Proposition 6).
Then $(0 : x_1^*) \cap (x_1^*, \dots, x_m^*) = (0)$ and $(0 : x_1^*)$ is generated by polynomials in $Y_1, \dots, Y_n$ and we have 2).
3) Suppose $(0 : x_1^*)$ a prime ideal of $Q = K[Y_1, \dots, Y_n]$. So $\mathrm{Sym}_Q(N)/(0 : x_1^*) \cong \mathrm{Sym}_R(M)/(0 : x_1^*)$ is a domain, then $(0 : x_1^*) \subseteq (\mathrm{Sym}_R(M))_0$, where $(\mathrm{Sym}_R(M))_0$ is the torsion submodule of $\mathrm{Sym}_R(M)$. Since R is a domain, $(0 : x_1^*) = (\mathrm{Sym}_R(M))_0$ (by Proposition 7), then $\mathrm{Sym}_R(M)/(0 : x_1^*) \cong \mathcal{R}(M)$, the Rees algebra of M ([3]). □

EXAMPLE 6.

$$M = I = (X_1^2, X_2^2, X_1X_2) \subset R = K[X_1, X_2],\ \ X_2 > X_1$$

$$R^2 \xrightarrow{\varphi} R^3 \to I \to 0$$

$$A = (a_{ij}) = \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \\ -X_1 & -X_2 \end{pmatrix}$$

$J = (f_1, f_2)$, where

$$f_1 = X_2Y_1 - X_1Y_3 = Y_1X_2 - Y_3X_1$$

$$f_2 = X_1Y_2 - X_2Y_3 = Y_3X_2 - Y_2X_1$$

The sequence X_1^2, X_2^2, X_1X_2 is a strong s-sequence for the ideal I ([7], Ex. 1.5(1)). Consider

$$B = (b_{ij}) = \begin{pmatrix} -Y_3 & Y_2 \\ Y_1 & -Y_3 \end{pmatrix}.$$

If $S = K[Y_1, Y_2, Y_3]$, the jacobian dual module N of I is

$$0 \longrightarrow S^2 \xrightarrow{\psi} S^2 \longrightarrow N \longrightarrow 0$$

$S(f_1, f_2) = Y_3f_1 + Y_1f_2 = (-Y_3^2 + Y_1Y_2)X_1 = f_3$.
Then a Gröbner basis w.r.t. $X_2 > X_1 > Y_3 > Y_2 > Y_1$ is $\{f_1, f_2, f_3\}$,

$$\mathrm{in}_< J = ((Y_1, Y_3)X_2, (Y_3^2 - Y_1Y_2)X_1).$$

$I_0^* = (0)$, $I_1^* = (Y_3^2 - Y_1Y_2)$, $I_2^* = (Y_1, Y_3)$, and since $I_1^* \subset I_2^*$, x_1, x_2 is a strong s-sequence for N.
From $f_3 = (-Y_3^2 + Y_1Y_2)X_1$, we have: $(0 : x_1^*) = (Y_1Y_2 - Y_3^2)$. In order to prove $(Y_3^2 - Y_1Y_2)$ is a prime ideal in $\mathrm{Sym}_Q(N)$ we remark that $J' = (f_1, f_2, Y_3^2 - Y_1Y_2)$ is a prime ideal in $\mathrm{Sym}_Q(N)$ if and only if $(Y_1Y_2 - Y_3^2)$ is a prime ideal in $\mathrm{Sym}_Q(N)$. But J' is the ideal generated by the 2×2-minors of the generic matrix

$$\begin{pmatrix} X_1 & Y_3 & Y_1 \\ X_2 & Y_2 & Y_3 \end{pmatrix}.$$

Then the assertion follows and $(Y_1Y_2 - Y_3^2)$ is a prime ideal in $\mathrm{Sym}_Q(N)$ and

$$\begin{aligned}\mathrm{Sym}_Q(N)/(0 : x_1^*) &\cong \mathrm{Sym}_R((X_1^2, X_2^2, X_1X_2))/(0 : x_1^*) \cong \mathcal{R}(I) \cong \\ &\cong R[Y_1, Y_2, Y_3]/(-Y_3X_1 + Y_1X_2, -Y_2X_1 + Y_3X_2, Y_1Y_2 - Y_3{}^2).\end{aligned}$$

EXAMPLE 7 (Monomial square-free matroidal ideals). Now we follow the notations used in [11, page 130].
Let I be a monomial ideal of $K[x_1, \dots, x_n]$ with the minimal set of generators $G(I) = \{x^{J_1}, \dots, x^{J_t}\}$, where $x^J = x_1^{j_1} \cdots x_n^{j_n}$, $J = (j_1, \dots, j_n)$ and $i = (0, 0, \dots, 1, \dots, 0)$. We set $|J| = j_1 + \cdots + j_n$.
We can associate the vector $\sum_{i=1}^t a_i \otimes x^{J_i}$ to a syzygy of I $\sum_{i=1}^t a_i x^{J_i}$, where $\otimes$ means $\otimes_K$.
For any monomial order on $K[x_1, \dots, x_n]$, we will say that

$$x_i \otimes x^J < x_k \otimes x^k \ \text{ if } \ x_i x^J < x_k x^k.$$

If $u \in G(I)$, we put $\nu_i(u) = j_i$, if $x_i^{j_i}$ appears in the monomial u.
Now, let I be a monomial ideal for which all generators have the same degree. I is matroidal if it satisfies the following exchange property ([24]):
For all $u, v \in G(I)$ and all i with $\nu_i(u) > \nu_i(v)$, there exists an integer j with $\nu_j(v) > \nu_j(u)$, such that $x_j(u/x_i) \in G(I)$.

THEOREM 5. *Let I be a matroidal square-free ideal with generators $x^{J_1}, \dots, x^{J_N}$ of the same degree. Then a minimal set of generators for the first syzygies of I has the form*

$$x_j \otimes x^{J_i} - x_t \otimes x^{J_\ell}, \qquad j + J_i = t + J_\ell, \qquad x^{J_i}, x^{J_\ell} \in G(I)$$

where $j < t$, t integer such that if $J_i = (a_1, \dots, a_n)$, $a_k = b_k$, $k = t+1, \dots, n$ and such that $b_j > a_j$, for some $x^{J_k} \in G(I)$, $J_k = (b_1, \dots, b_n)$.

Proof. In the reverse lexicographic order we can suppose that $x^{J_1} > x^{J_2} > \cdots > x^{J_N}$. Let $x^{J_i} < x^{J_k}$. Then there exists an integer t such that $a_m = b_m$ for $m = t+1, \dots, n$, $J_i = (a_1, \dots, a_n)$, $J_k = (b_1, \dots, b_n)$ and $a_t > b_t$. Hence there exists an integer j with $b_j > a_j$ such that $u' = x_j(x^{J_i}/x_t) \in G(I)$. Thus there is a syzygy of the form $x_j \otimes x^{J_i} - x_t \otimes x^{J_\ell}$, $x^{J_\ell} \in G(I)$. $\square$

EXAMPLE 8.

$$I = (x_1, x_2)(x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$$

In the reverse lexicographic order and for $x_4 > x_3 > x_2 > x_1$

$$x_4x_2 > x_3x_2 > x_4x_1 > x_3x_1.$$

We consider the mapping $Y_1 \to x^{J_4} = x_3x_1$, $Y_2 \to x^{J_3} = x_4x_1$, $Y_3 \to x^{J_2} = x_3x_2$, $Y_4 \to x^{J_1} = x_4x_2$.
The syzygies are:

$$\begin{aligned}
x_1 \otimes x^{J_2} - x_2 \otimes x^{J_4} &\longrightarrow f_1 = x_2Y_1 - x_1Y_3;\\
x_1 \otimes x^{J_1} - x_2 \otimes x^{J_3} &\longrightarrow f_2 = x_2Y_2 - x_1Y_4;\\
x_3 \otimes x^{J_3} - x_4 \otimes x^{J_4} &\longrightarrow f_3 = x_4Y_1 - x_3Y_2;\\
x_3 \otimes x^{J_1} - x_4 \otimes x^{J_2} &\longrightarrow f_4 = x_4Y_3 - x_3Y_4.
\end{aligned}$$

The relations matrix of I is

$$\begin{pmatrix} x_2 & 0 & x_4 & 0 \\ 0 & x_2 & -x_3 & 0 \\ -x_1 & 0 & 0 & x_4 \\ 0 & -x_1 & 0 & -x_3 \end{pmatrix}$$

and the dual matrix is

$$\begin{pmatrix} -Y_3 & -Y_4 & 0 & 0 \\ Y_1 & Y_2 & 0 & 0 \\ 0 & 0 & -Y_2 & -Y_4 \\ 0 & 0 & Y_1 & Y_3 \end{pmatrix}$$

Consider the order $x_4 > \cdots > x_1 > Y_4 > \cdots > Y_1$, J has a Gröbner basis linear in the x_i variables. In fact $J = (f_1, f_2, f_3, f_4)$ and a Gröbner basis of J is

$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ with $f_5 = x_4(Y_2Y_3 - Y_1Y_4)$, $f_6 = x_2(Y_2Y_3 - Y_1Y_4)$, $\mathrm{in}_< J = ((Y_3, Y_4)x_1, (Y_3Y_2)x_2, (Y_2, Y_4)x_3, (Y_3Y_2)x_4)$, $I_1^* = (Y_3, Y_4)$, $I_2^* = (Y_3Y_2)$, $I_3^* = (Y_2, Y_4)$, $I_4^* = (Y_3Y_2)$.
The jacobian dual N of I is generated by an s-sequence $x_1^*, \dots, x_4^*$ that is not a strong s-sequence.
The torsion submodule of $\mathrm{Sym}_R(I)$ can be read in the dual matrix:

$$0 : x_1^* = \left(\begin{vmatrix} -Y_1 & Y_2 \\ Y_3 & Y_4 \end{vmatrix} \right) = (Y_1Y_4 - Y_3Y_2).$$

REMARK 6. The ideal I is not generated by an s-sequence. The Gröbner basis of J is not linear in the variables $Y_1, \dots, Y_4$ for any admissible order such that $Y_4 > Y_3 > Y_2 > Y_1$. Then we are forced in this case to study the invariants of $\mathrm{Sym}_Q(N) \cong \mathrm{Sym}_R(M)$ "via" the jacobian dual.
The computation of the annihilator ideals of the s-sequence generating N can be done by the lemma 3.2 of [9]. Moreover this lemma can be used to compute the annhilator ideals of a generating s-sequence of M, changing the variables x_i's with the Y_j's and for a term order on all variables x_i's and Y_j's, that is admissible for x_i and for Y_j, for example $x_n > x_{n-1} > \dots > x_1 > Y_n > Y_{n-1} > \dots > Y_1$ or $Y_n > Y_{n-1} > \dots > Y_1 > x_n > x_{n-1} > \dots > x_1$.

In general, it is possible that M is not generated by an s-sequence and N is generated by an s-sequence. If this is the case (Ex. 8), we can obtain the Rees algebra of M by the quotient $\mathrm{Sym}_R(M)/I_1{}^*$, $I_1{}^* = 0 : x_1{}^* = (\mathrm{Sym}_R(M))_0$, where $I_1{}^*$ is a prime ideal. Then (EGC) can be true for $\mathcal{R}(M)$ that is a domain,

$$(\mathrm{EGC1}^*) \quad \mathrm{reg}(\mathcal{R}(M)) \le e(\mathcal{R}(M)) - \mathrm{codim}(\mathcal{R}(M)).$$

Moreover we have, if $R = K[X_1, \dots, X_m]$, M a graded finitely generated R-module, N the jacobian dual on $Q = K[Y_1, \dots, Y_n]$, generated by an s-sequence of elements of Q of the same degree, Eisenbud-Goto conjecture for the symmetric algebra $\mathrm{Sym}_Q(N)$, in terms of the annihilator ideals $I_1^*, \dots, I_m^*$ of Q.
(EGC2*) $\max\{\mathrm{reg}(Q/I_i^*) : i = 1, \dots, m\} + 1 \le \sum_{i=1}^m e(Q/I_i^*) - (n+m) +$
$+ \max_{0 \le i \le m}\{\dim(Q/I_i^*) + i\}$
(EGC$_i^*$) $\mathrm{reg}(Q/I_i^*) \le e(Q/I_i^*) - \mathrm{codim}(Q/I_i^*)$, for $i = 1, \dots, m$.
(EGC$_i^{*\prime}$) $\mathrm{reg}(Q/I_i^*) \le e(Q/I_i^*) - n + \dim(Q/I_i^*)$, for $i = 1, \dots, m$.

EXAMPLE 9. $I = (X^2, Y^2, XY)$, $\mathrm{Sym}_R(I)$ verifies the inequality of (EGC2), $I_0 = I_1 = (0)$, $I_2 = (X^2)$, $I_3 = (X, Y)$.

$$\mathrm{reg}(R/I_2) = 1,\ \mathrm{reg}(R/I_3) = 0,\ e(R/I_2) = 2,\ e(R/I_3) = 1$$

then:

$$\max\{\mathrm{reg}(R/I_i), i = 1, 2, 3\} < \sum_{i=1}^{3} e(R/I_i) - 2$$

is true. For the jacobian dual: $I_1^* = (Y_3^2 - Y_1Y_2)$, $I_2^* = (Y_1, Y_3)$

$$\operatorname{reg}(Q/I_1^*) = 1,\ \operatorname{reg}(Q/I_2^*) = 0,\ e(Q/I_1^*) = 2,\ e(Q/I_2^*) = 1$$

then the inequality:

$$\max\{\operatorname{reg}(Q/I_i^*), i = 1, 2\} < \sum_{i=0}^{2} e(Q/I_i^*) - 2 + 1$$

is verified.

REMARK 7. Let M be a graded module on $R = K[X_1, \dots, X_m]$, let N be the jacobian dual of M. Suppose that $x_1, \dots, x_m$ is a strong s-sequence for N and $(0 : x_1^*)$ is a prime ideal of Q. Then $x_1^*, \dots, x_m^*$ is a d-sequence in $\mathcal{R}(M) = \operatorname{Sym}_Q(N)/(0 : x_1^*)$.

Our attention actually applies to prove the conjecture via the annihilator ideals of the jacobian dual of large classes of monomial ideals with linear resolution.

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