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VECTOR BUNDLES, REFLEXIVE SHEAVES AND LOW CODIMENSIONAL VARIETIES

Dedicated to Paolo Valabrega on the occasion of his 60 th birthday

Abstract. We give a short, but hopefully up–to–date account on the theory of vector bundles and reflexive sheaves on projective spaces and hypersurfaces.

1. Introduction

The theory of vector bundles and reflexive sheaves has been developed rapidly from the 70's up to today, (see the fundamental papers [33], [35]) mainly because of its connection with the geometry of low codimensional varieties. An important motivation for the study of these objects came out from the conjecture ([34]) that any smooth 2-codimensional variety in $\mathbb{P}^r$, $r > 6$, is complete intersection. Recently, a strong impulse towards the theory followed the interest on the study of subvarieties of general hypersurfaces.

In this note, we show a general approach which, at least for rank 2, seems to us sufficient to justify the introduction and inspection of vector bundles and reflexive sheaves, in the task of understanding the geometry of projective subvarieties.

The note is also a tribute to Paolo Valabrega, in the occasion of his 60th birthday. Paolo was the advisor of my thesis and inspired all my early studies in Mathematics. He addressed me to the theory of algebraic varieties, about 25 years ago. One of his first intuition concerned the role that vector bundles play in the development of a point of view for understanding the geometrical behaviour of low codimensional varieties.

I hope that writing down some fundamental aspects of the theory of rank 2 bundles and reflexive sheaves, will be useful for propagating a point of view in the theory of projective varieties, that Paolo followed in his geometrical papers, in the last years.

2. Subvarieties of projective spaces

Many concepts that will be displayed through the note, are in fact independent from the characteristic of the ground field. Some of them (but it is not entirely clear which) do not even need the presence of a ground field itself. Nevertheless, it is more comfortable to work over an algebraically closed field of characteristic 0. So let us say that, through the note, we only consider projective spaces and projective varieties defined over the complex field $\mathbb{C}$.

In the study of varieties of codimension one in the projective space $\mathbb{P}^r$, one has little to deal with complicate relations between the local and global description.

Indeed hypersurfaces X are always described by a single equation, a homogeneous polynomial F in the ring $\mathbb{C}[x_0, \ldots, x_r]$, and properties of the hypersurface are strictly connected with properties of F. For instance, deformations of X correspond to deformations of the equation F (modulo the action of $\mathbb{C}^*$).

In principle, if one wants to describe X in terms of 0-locus of functions, it is necessary to deal with the local equations of X in the various affine pieces of $\mathbb{P}^r$. Hence, there are transition functions which display X as the 0-locus of a section of a line bundle, usually indicated as $\mathcal{O}_{\mathbb{P}^r}(d)$, d being the degree of F. However, as the global sections of $\mathcal{O}_{\mathbb{P}^r}(d)$ coincide with homogeneous polynomials of degree d, then one could also forget this description and go on with the elementary algebraic point of view.

As soon as one considers codimension 2, things become "dramatically" more complicated and new methods cannot be avoided.

First of all, given $X \subset \mathbb{P}^r$ of codimension 2, by no means one can be sure of the existence of two homogeneous polynomials F, G such that X coincides globally with the locus where $F = G = 0$. Even locally, the existence of the two defining equations is not guaranteed.

Varieties for which the two defining equations exist (on the whole projective space) are **(global) complete intersection**. In general, a variety of codimension m is a (global) complete intersection when there are $r - m$ homogeneous polynomials $F_1, \ldots, F_{r-m}$ such that X coincides with the locus $F_1 = \cdots = F_{r-m} = 0$.

When the same condition is true locally, then X is *locally complete intersection*.

Things become a little easier when we restrict to study only *smooth* subvarieties $X \subset \mathbb{P}^r$, i.e. subvarieties without singular points. In this case, one can find a Zariski open cover of $\mathbb{P}^r$ such that, on any piece of the cover, X is defined by two polynomial equations. In other words, every smooth subvariety is locally complete intersection.

Turning back to the case of smooth varieties of codimension 2, one could expect that the transition functions determined on the open cover by the equations of a smooth subvariety X, could determine a rank 2 bundle E on $\mathbb{P}^r$ such that X coincides with the 0-locus of a section of E. Unfortunately, this turns out to be false, in general. Indeed the glueing conditions are not trivial on X.

When $r = 2$ and X is a discrete set of points, then the glueing conditions are local on a finite set, so they behave friendly and one is always able to define a vector bundle E out from some local equations of X.

On the other hand, when $\dim(X) \geq 1$, the glueing condition becomes non–trivial. The resulting data define a line bundle on X, which is a twist of the canonical bundle ω_X, the line bundle defined by the differentials on X.

Serre's construction ([64]) shows that one obtains a rank 2 bundle E on $\mathbb{P}^r$ associated with X as soon as ω_X is the restriction to X of some line bundle $\mathcal{O}_{\mathbb{P}^r}(d)$. Varieties X satisfying this property are called *subcanonical*. The restriction of E to X turns out to be isomorphic to the normal bundle of X: $N_X = T_{\mathbb{P}^r|X}/T_X$.

For general varieties, the difference between ω_X and the restriction of $\mathcal{O}_{\mathbb{P}^r}(d)$ determines some *singularity* of E and E is no longer a vector bundle, but just a **re-**

flexive sheaf, i.e. a torsion free sheaf such that the double dual E^{**} is canonically isomorphic to E.

OPEN PROBLEM 1. It is still an open problem to determine a *reasonable* set of conditions which extend the previous theory to varieties of codimension greater than 2.

In codimension 3, a theorem of M. Kreuzer ([41]) shows that any set of distinct points in the 3-space can be realized as the 0-locus of a rank 3 bundle on $\mathbb{P}^3$. The bundle, however, is in general far from being unique. It is not clear (and difficult) to determine which invariants are allowed for a vector bundle of rank 3 with sections vanishing exactly on a given set Z of points. Observe that any set of distinct points is automatically subcanonical.

We do not know a precise description of curves in $\mathbb{P}^4$ which can be realized as the 0-locus of sections of a rank 3 vector bundle. A theorem of Chang proves, however, that not every smooth subcanonical curve in $\mathbb{P}^4$ shares this property ([10]).

For other varieties, observe that we have the following necessary condition:

($*$) *All the Chern classes of the tangent sheaf (or the cotangent sheaf) of X are cut on X by Chern classes of $\mathbb{P}^r$.*

In other words, the Chern classes of T_X sit in the image of the restriction map $\mathrm{Chow}(\mathbb{P}^r)\to\mathrm{Chow}(X)$.

We do not know to which extent the previous condition is also sufficient. It coincides with subcanonicity for curves, thus it is sufficient for curves in $\mathbb{P}^3$ (and for codimension 2) but it is not sufficient for curves in $\mathbb{P}^4$.

We guess that the condition could be sufficient only as soon as $2\dim(X)+1 \geq r$.

Other, more geometrical, generalizations of the notion of subcanonical varieties could be tested. For instance, it would be interesting to know if the following could be true:

(Q) *A smooth surface $X \subset \mathbb{P}^5$ is the 0-locus of a section of a rank 3 vector bundle if and only if there exists a (reflexive?) sheaf T of rank 2 on $\mathbb{P}^5$, whose restriction to X is isomorphic to the tangent sheaf T_X.*

OPEN PROBLEM 2. Useless to say that even more complicate would be to determine in general when a 2-codimensional subvariety $X \subset \mathbb{P}^r$ is the 0-locus of a rank 2 torsion free sheaf, having singularities in a given dimension ≥ 0.

The subcanonical condition can be rephrased in terms of liaison.

Roughly speaking, we say that two subvarieties X, X' of codimension 2 are (directly) *linked* by the complete intersection Y of two hypersurfaces of degrees a, b, when $Y = X \cup X'$. We will write

$$X \sim^{a,b} X'.$$

Indeed one should consider a more refined definition, which takes care of the case where X, X' have overlapping components. We refer to the paper [51] for a precise statement.

It is classical that when $X \sim^{a,b} X'$, then the canonical class ω_X of X is cut by hypersurfaces of degree $a+b-r-1$ passing through X', outside the intersection $X \cap X'$. As a consequence, when X is subcanonical, with $\omega_X = \mathcal{O}_X(e)$ $(= \mathcal{O}_{\mathbb{P}^r}(e) \otimes \mathcal{O}_X)$ then there exists a hypersurface of degree $a+b-r-1-e$ which exactly cuts $X \cap X'$ on X'. Thus X' turns out to be described by the vanishing of three homogeneous polynomials, of degrees $a, b, a+b-r-1-e$.

In general, one can see that we need $r+1$ equations to describe point by point a subvariety of codimension 2: namely, with two equations one describes a complete intersection of codimension 2, containing X. The third equation restricts the intersection to $X \cup \{$a subvariety of codimension 3$\}$ and so on, increasing the codimension of the residue by 1 at every step, up to the reach of the empty set.

Thus subcanonical varieties are exactly those which are linked to subvarieties of codimension 2 defined by 3 equations, which is the minimum allowed for non complete intersection varieties.

For curves in $\mathbb{P}^3$, this remark exhausts the possible cases, for any curve in the 3-dimensional projective space can be cut with 4 equations (see e.g. [66]).

What happens for higher dimensional object seems widely unknown, even for the initial cases.

OPEN PROBLEM 3. In $\mathbb{P}^4$, a general (smooth) surface can be described by 5 equations. Surfaces directly linked to subcanonical surface, can be cut by 3 equations.

The problem asks for a description of the geometry of surfaces which can be described with 4 equations.

Even if their canonical class ω is not cut by hypersurfaces of $\mathbb{P}^4$, they have the following property:

$(**)$ there exists a number $u \in \mathbb{Z}$ such that, in the Chow ring, the intersection $\omega_X \cdot (\omega_X \otimes \mathcal{O}_X(u))$ is 0.

The previous property should be also sufficient for a surface in $\mathbb{P}^4$ to be cut by 4 equations.

In general the geometry (existence, properties, deformations, classification) of this class of surfaces, which are, in a sense, a natural generalization of subcanonical surfaces, is unknown.

The theory could be generalized in two ways: first considering varieties of codimension 2 in $\mathbb{P}^r$, $r > 4$, cut by 4 equations, then to varieties of codimension 2 cut by any intermediate number of equations, between 4 and r.

Generalizations to subvarieties of higher codimension seem truly out of reach!

3. Subcanonical subvarieties

Let us restrict our attention to *smooth* subcanonical varieties X of codimension 2 in $\mathbb{P}^r$. Also we assume in this section $r \geq 3$, so that $\dim(X) \geq 1$.

We set e to be the number such that the canonical bundle ω_X coincides with the restriction $\mathscr{O}_X(e) = \mathscr{O}_{\mathbb{P}^r}(e) \otimes \mathscr{O}_X$. e is also the **index of speciality** of X, i.e.

$$e = \max\{i : \dim H^0(\omega_X(-i)) > 0\}.$$

Fixing e, we will also say that X is *e-subcanonical*.

In this setting, Serre's correspondence shows that there exists a vector bundle E on $\mathbb{P}^r$ and a section $s \in H^0(E)$ such that X coincides with the 0–locus of s. Identifying any homogeneous summand of the Chow ring of $\mathbb{P}^r$ with $\mathbb{Z}$, Serre's correspondence shows that the Chern classes of E are:

$$\begin{aligned} c_1 &= e + r + 1 \\ c_2 &= \deg(X). \end{aligned}$$

Locally, s is defined by two polynomials in any set of an open cover of $\mathbb{P}^r$. The Koszul complexes of the two polynomials link globally to give the fundamental exact sequence:

$$0 \to \Lambda^2 E^* = \mathscr{O}_{\mathbb{P}^r}(-c_1) \to E^* = E(-c_1) \to \mathscr{I}_X \to 0 \tag{1}$$

where $\mathscr{I}_X$ is the ideal sheaf of X.

From this description, it turns out that X is *globally* complete intersection of hypersurfaces of degree a, b if and only if sequence (1) coincides with the (global) Koszul complex of two polynomials of degree a, b, i.e. when E splits:

PROPOSITION 1. *In the previous setting, X is complete intersection of hypersurfaces of degree a, b if and only if the local Koszul complex globalizes, i.e. if and only if E decomposes in a direct sum of line bundles $E = \mathscr{O}_{\mathbb{P}^r}(a) \oplus \mathscr{O}_{\mathbb{P}^r}(b)$.*

Thus, the problem of detecting when a smooth subcanonical subvariety of codimension 2 is complete intersection is rephrased, in the language of bundles, as a particular case of the general **splitting problem**:

SP: Determine conditions under which a vector bundle E of rank n decomposes into a direct sum $\oplus \mathscr{O}_{\mathbb{P}^r}(a_i)$.

An implicit answer to the splitting problem for rank 2 in $\mathbb{P}^3$ was determined in 1942 by G. Gherardelli.

DEFINITION 1. *We say that a variety $X \subset \mathbb{P}^r$ is t-**normal** if the surjection of line bundles $\mathscr{O}_{\mathbb{P}^r}(t) \to \mathscr{O}_X(t) = \mathscr{O}_{\mathbb{P}^r}(t) \otimes \mathscr{O}_X$ yields also a surjection of the spaces of global sections $H^0(\mathscr{O}_{\mathbb{P}^r}(t)) \to H^0(\mathscr{O}_X(t))$.*

1-normal varieties are also called **linearly** *normal varieties.*

X is **arithmetically normal** *if it is t-normal for all t.*

In [28] Gherardelli proves the following result:

THEOREM 1. *A curve $C \subset \mathbb{P}^3$ is complete intersection if and only if it is* (a) *subcanonical and* (b) *arithmetically normal.*

A result of Strano ([65]) proves that a smooth 2-codimensional subvariety $X \subset \mathbb{P}^r$, $r \geq 4$, is complete intersection if and only if its general hyperplane section is.

Thus, summing up these two results, we obtain:

COROLLARY 1. *A smooth subvariety $X \subset \mathbb{P}^r$, $r \geq 3$, of codimension 2 is complete intersection if and only if it is subcanonical and its general curvilinear section is arithmetically normal.*

To understand Gherardelli's result for the splitting of rank 2 bundles, one needs to consider the cohomology of E.

REMARK 1. The natural sequence

$$0 \to \mathscr{I}_X(t) \to \mathscr{O}_{\mathbb{P}^r}(t) \to \mathscr{O}_X(t) \to 0 \tag{2}$$

shows that X is t-normal if and only if $H^1(\mathscr{I}_X(t)) = 0$. From the cohomological sequence associated to (1), one obtains

$$H^1(\mathscr{I}_X(t)) = H^1(E(-c_1 + t)).$$

Thus:

A rank 2 bundle on $\mathbb{P}^3$ decomposes in a sum of line bundles if and only if $H^1(E(i)) = 0$ for all i.

The previous splitting criterion has been extended in 1962 by A. Horrocks, to vector bundles of any rank in any projective space.

THEOREM 2 (Horrocks splitting criterion). *A vector bundle E of rank n in $\mathbb{P}^r$, $r > 1$, decomposes in a sum of line bundles if and only if:*

$$H^m(E(i)) = 0 \text{ for all } i \text{ and for all } 1 \leq m \leq \frac{r}{2} \tag{3}$$

We will refer to vector bundles satisfying condition 3 as **arithmetically Cohen-Macaulay** vector bundles (aCM for short).

Let us just observe that Gherardelli's theorem, Strano's result and Kleiman's proof of the fact that for any vector bundle E, a general section of $E(t)$ (see [39]), for $t \gg 0$, vanishes in a smooth locus, indeed imply Horrocks criterion for rank 2 vector bundles in $\mathbb{P}^r$, $r > 2$.

Horrocks criterion has been improved in several ways. A contribute of Paolo Valabrega to this theory is contained in the paper [18], where he proves that the vanishing of one single, well–determined cohomology group $H^1E(t_0)$ is enough to cause the splitting of a rank 2 vector bundle in $\mathbb{P}^3$:

THEOREM 3 (Splitting criterion in $\mathbb{P}^3$). *A vector bundle E of rank 2 in $\mathbb{P}^3$ decomposes in a sum of line bundles if and only if $H^1(E(t_0)) = 0$, where*

$$t_0 = -\frac{c_1}{2} - 1 \textit{ for even } c_1, \textit{ or}$$

$$t_0 \in \left\{ -\frac{c_1+3}{2}, -\frac{c_1+1}{2}, -\frac{c_1-1}{2} \right\} \textit{ for odd } c_1$$

Equivalently: a curve $C \subset \mathbb{P}^3$ is complete intersection if and only if it is e-subcanonical and (t_0)-normal, where

$$t_0 = \frac{e}{2} + 1 \textit{ for even } e, \textit{ or}$$

$$t_0 \in \left\{ \frac{e+1}{2}, \frac{e+3}{2}, \frac{e+5}{2} \right\} \textit{ for odd } e.$$

Examples prove that this description is sharp.

REMARK 2. A brief account on the history of this result. The original proof was based on the *Gruson–Peskine speciality lemma* (see [31]), which bounds the index of speciality e.

As pointed out in [53] and [60], indeed, the use of the speciality lemma can be avoided in the proof of the result.

The same theorem has been subsequently re–obtained by a direct examination of the cohomology of vector bundles ([52]) or via the theory of the spectrum ([22]).

In the case of rank 2 bundles in $\mathbb{P}^r$, $r > 3$, by using several times the hyperplane sectional sequence:

$$0 \to E(-1) \to E \to E \otimes \mathcal{O}_H \to 0 \tag{4}$$

where H is a general hyperplane, then the splitting criterion of Theorem 3 yields:

THEOREM 4. *A vector bundle E of rank 2 in $\mathbb{P}^r$ decomposes in a sum of line bundles if and only if*

$$H^1(E(t_0)) = H^2(E(t_0-1)) = \cdots = H^i(E(t_0-i+1)) = 0, \quad i = 1, \ldots, r-2,$$

where

$$t_0 = -\frac{c_1}{2} - 1 \textit{ for even } c_1, \textit{ or}$$

$$t_0 = \textit{ either } -\frac{c_1+3}{2}, -\frac{c_1+1}{2}, -\frac{c_1-1}{2} \textit{ for odd } c_1.$$

An important improvement of the Horrocks criterion has been obtained by Evans and Griffith([29]):

THEOREM 5. *A vector bundle E of rank n in $\mathbb{P}^r$ $(r > 1)$ decomposes in a sum of line bundles if and only if:*

(5) $$H^m(E(i)) = 0 \text{ for all } i \text{ and for all } m = 1, \dots, \min\{n-1, r-1\}.$$

OPEN PROBLEM 4. For rank 2 bundles, the splitting criterion of Evans and Griffith reads:

$$E \text{ splits if and only if } H^1(E(t)) = 0 \text{ for all } t.$$

So we have two different conditions which imply the splitting of a rank 2 bundle: a "flat" condition, by theorem 5:

$$H^1(E(t)) = 0 \text{ for all } t$$

and a "tower" condition, by Theorem 4

$$H^1(E(t_0)) = H^2(E(t_0-1)) = \cdots = H^i(E(t_0-i+1)) = 0, \quad i = 1, \dots, r-2$$

where t_0 is a precise number, depending only on the first Chern class of E.

The "intersection" of these two conditions yields the following problem:

Q: Is $H^1(E(t_0)) = 0$ for some fixed t_0 enough to guarantee the splitting of a rank 2 bundle in $\mathbb{P}^r$, $r > 3$?

Indeed there are few known examples of non–splitting rank 2 bundles in $\mathbb{P}^4$, all of them essentially obtained from the **Horrocks–Mumford bundle** ([38]).

In all of these examples, $H^1E(t_0)$ does not vanish.

It is not a consequence of Theorem 3, but still true, that a rank 2 bundle E on $\mathbb{P}^2$ decomposes in a sum of line bundles if and only if a precise cohomology group $H^1(E(t_0))$ vanishes. Here the choices of numbers t_0, for which the statement holds, widens (see [4]).

Indeed, as observed in the introduction, the correspondence between subvarieties of codimension 2 in $\mathbb{P}^2$ (i.e. sets of points) and rank 2 vector bundles is weaker than in higher dimensional spaces.

In $\mathbb{P}^2$, any non-empty set of points X has some non–vanishing cohomology group $H^1(\mathscr{I}_X(t))$, thus the notion of "arithmetically normal" becomes meaningless.

Also the notion of "subcanonical" variety is useless, for the theory of lines bundles on such a finite X trivializes. One has to use instead the notion of *Cayley–Bacharach number*.

e is a Cayley–Bacharach number for the finite set $X \subset \mathbb{P}^2$ if, for any point $P \in X$, one has

$$H^0(\mathscr{I}_X(e)) = H^0(\mathscr{I}_{X-\{P\}}(e)).$$

In other words, there are no curves of degree e passing through $X - \{P\}$ and missing P.

If e is a Cayley–Bacharach number for X, then one finds a rank 2 bundle E on $\mathbb{P}^2$ and a section s of E which vanishes exactly on X. The Chern classes of E are $c_1 = e + 3$, $c_2 = \deg(X)$.

From this description, one easily understands that there are infinitely many bundles associated to X. As Cayley–Bacharach numbers associated to X may change, we find bundles associated to X with different Chern classes.

Sets of points in $\mathbb{P}^2$ which are complete intersection, are characterized by the Cayley–Bacharach theorem:

THEOREM 6 (Cayley–Bacharach). *A set of point $X \subset \mathbb{P}^2$ of degree $d = ab$ is complete intersection of curves of degree a, b if and only if $a + b - 3$ is a Cayley–Bacharach number for X.*

Among the many rank 2 bundles associated to a complete intersection $X \subset \mathbb{P}^2$, there are for sure indecomposable bundles. Nevertheless, it is possible to determine cohomological criteria for the splitting of rank 2 bundles on $\mathbb{P}^2$.

Namely, the possible sequences which give the dimension of the cohomology groups $H^1(E(t))$, as t varies, are classified in [4]. It turns out that we have a criterion for detecting, from the cohomological sequence $\dim H^1(E(t))$, the existence of a global section of some twist $E(t)$ which vanishes on a complete intersection set of points.

We refer to the paper [4] for details.

OPEN PROBLEM 5. If E is a rank 2 vector bundle on $\mathbb{P}^2$, then the possible structures of the cohomology $\oplus H^1(E(t))$, as vector spaces over $\mathbb{C}$, are known, as explained above. Nevertheless $\oplus H^1(E(t))$ is also a module over the polynomial ring. It is not known which module structures are allowed for this object.

Actually we only know a factorial characterization of the cohomology module $\oplus H^1(E(t))$, via its minimal resolution (see [21] for details).

What we have in mind is a sort of results similar to the one of Ellia and Fiorentini ([24]):

THEOREM 7. *The only indecomposable rank* 2 *bundles in $\mathbb{P}^3$ whose cohomology module $\oplus H^1(E(t))$ has trivial multiplication are associated to pairs of skew lines.*

Indeed Theorem 3 is a step in this direction. See also [37] for some bounds on the dimension of $H^1(E(t))$ for rank 2 bundles in $\mathbb{P}^3$.

OPEN PROBLEM 6. A similar description for the structure of $\oplus H^1(E(t))$, when E is a rank 2 vector bundle on $\mathbb{P}^r$, $r > 2$, has been exploited quite partially. E.g. Decker's result [21] also works in $\mathbb{P}^3$.

In fact, in $\mathbb{P}^r$, $r > 2$, we do not know a complete description of all the possible functions $\dim H^1(E(t))$ for a vector bundle E of rank 2. We just have partial results. E.g. Buraggina proved in [8] that the support of the function is connected. Also Chang gave a conjecture on the non–vanishing of cohomology groups for rank 2 bundles (see [9], [27], [20]).

When one considers codimension 2 subvarieties in $\mathbb{P}^r$, with $r \geq 6$, then a new phenomenon occurs: namely the Lefschetz principle implies that all smooth subvarieties are subcanonical (see e.g. [32]). So any smooth subvariety of codimension 2 is associated with a rank 2 vector bundle.

Classical algebraic geometers knew very well the difficulties in constructing smooth subvarieties of low codimension. For codimension 2, there are no examples of such varieties in $\mathbb{P}^r$ as soon as $r \geq 5$ (but we should mention that such a subvariety exists in $\mathbb{P}^5$, in characteristic 2, see [45]).

For $r \geq 7$, in [34] Hartshorne conjectured that:

CONJECTURE 1. All smooth subvarieties of codimension 2 in $\mathbb{P}^r$, $r \geq 7$, are complete intersection.

Equivalently:
All rank 2 bundles in $\mathbb{P}^r$, $r \geq 7$, decompose in a sum of line bundles.

It is far beyond the scopes of this note to give an account of the state of the art in the study of Hartshorne's conjecture. Enough to say that a proof of this conjecture would bound the study of rank 2 bundles over projective spaces $\mathbb{P}^r$ to the case of low r.

Turning back to the case of curves in $\mathbb{P}^3$, let us mention another suggestion which comes from the splitting Theorem 3.

OPEN PROBLEM 7. Let C be a smooth curve in $\mathbb{P}^3$ of degree d and genus g. Then the index of speciality e of C ranges between 0 and $\frac{2g-2}{d}$.

e is maximal for subcanonical curves. Assume for a while that e is even, $e = 2a$. Then Theorem 3 yields that, for any degree, a subcanonical curve C is arithmetically normal if and only if $H^1(\mathscr{I}_C(a+1)) = 0$.

The other extremal case is $e = 0$. Curves for which this condition holds are called *non–special*. A theorem of Mumford ([47]) shows that, for $d > 2g - 1$, then a non special curve in $\mathbb{P}^3$ is arithmetically normal if and only if it is linearly normal, i.e. $H^1(\mathscr{I}_C(1)) = 0$. Observe that in this case also $a = e/2$ is zero, thus Mumford's condition can be written as the previous one: $H^1(\mathscr{I}_C(a+1)) = 0$.

The question is: can this pattern be extended to the intermediate cases?

Q: is there some (serious) function $\Phi(g)$ such that for smooth curves in $\mathbb{P}^3$ with even index of speciality $e = 2a$, and for $d > \Phi(g)$, arithmetically normal is equivalent to the vanishing of just the unique cohomology group $H^1(\mathscr{I}_C(a+1))$?

It could happen that the number $a + 1$ should be replaced by some more complicate expression involving e, d, g.

Of course, a similar problem can be stated for curves with odd index of speciality, or for surfaces in $\mathbb{P}^4$.

The previous problem suggests to exploit how the behaviour of subcanonical subvarieties is mirrored by general subvarieties of codimension 2. We consider here mainly the case of curves in $\mathbb{P}^3$ and surfaces in $\mathbb{P}^4$.

Fix a smooth curve C in $\mathbb{P}^3$ and call e its index of speciality. Then, glueing together the various local equations of C, one ends up with a sheaf F which is not locally free, unless C is subcanonical. F is just a reflexive sheaf of rank 2, a rank 2 "vector bundle with singularities". The *singularities* of F are the points where it is not locally free, i.e. the points at which the fiber of F jumps in dimension. These points belong to C, and their number depend on the construction of the glueing. Namely, taking any value $\epsilon \leq e$, one gets a reflexive sheaf F with Chern classes $c_1 = \epsilon + 4$, $c_2 = \deg(C)$ and $c_3 = 2g - 2 - \epsilon d$ (where g is the genus of the curve), such that there is a section $s \in H^0(F)$ which vanishes exactly along C.

As above, by taking the glueing of the local Koszul complexes, one gets an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to F \to \mathcal{I}_X(c_1) \to 0 \tag{6}$$

In this situation, F is singular exactly at $c_3 = 2g - 2 - \epsilon d$ points. Thus F is a rank 2 bundle exactly when $c_3 = 0$.

A completely analogue construction works also in $\mathbb{P}^4$ and $\mathbb{P}^5$. It could work also in higher dimensional projective spaces, but due to Barth–Larsen Theorem [32], every smooth subvariety of codimension 2 is subcanonical in such spaces, thus the reflexive sheaf theory here is useless.

One may arise, for reflexive sheaves, the same splitting problem that exists on rank 2 bundles: i.e., find cohomological conditions under which the reflexive sheaf F (is a vector bundle and) decomposes into a sum of line bundles.

It turns out easily that F decomposes if and only if, for all t, $H^1(F(t)) = H^2(F(t)) = 0$. Observe that, since F is not locally free, the H^2 sequence could be, in principle, completely different from the H^1 sequence.

The cohomological criterion has been refined by M. Roggero, for curves in $\mathbb{P}^3$ (see [54], and [60] for the extension to higher dimensional spaces):

THEOREM 8 (Roggero's criterion). *A reflexive sheaf F of rank 2 on $\mathbb{P}^3$ with first Chern classes $c_1 \in \{-1, 0\}$ is a vector bundle and decomposes into a sum of line bundles, if and only if $H^2(F(t)) = 0$ for some number t in the range $[-3+c_1, -3-c_1]$.*

Observe that the condition, via Serre duality, coincides with the one of Theorem 3 for vector bundles.

OPEN PROBLEM 8. Is there some analogue of Evans-Griffith criterion of Theorem 5 for rank 2 reflexive sheaf on higher dimensional spaces?

Can this be intersected with Roggero's criterion?

OPEN PROBLEM 9. Let us mention that Strano's criterion for the splitting of a rank 2 reflexive sheaf extends indeed to any rank. Is this true also for other splitting criteria?

The starting point of our discussion works verbatim, if we consider subvarieties of codimension 2 of some variety Y *with good geometrical properties*.

Namely, for instance, if we consider a subvariety $X \subset Y$ of codimension 2 in a smooth complete intersection variety Y, then Serre's correspondence also works: if X is subcanonical, then one can find a rank 2 bundle E *on* Y, with a section s vanishing along X. The exact sequence (1) now reads:

$$0 \to \mathscr{O}_Y(-c_1) \to E^* = E(-c_1) \to \mathscr{I}_{X,Y} \to 0. \tag{7}$$

where again we use integers to signify the tensor product by line bundles on Y belonging to the image of the restriction map:

$$\rho : \mathrm{Pic}(\mathbb{P}^r) \to \mathrm{Pic}(Y)$$

which, in our cases, will usually be a surjection.

We have:

PROPOSITION 2. *In the previous setting, E decomposes in a direct sum of line bundles $E = L \oplus M$ if and only if X is complete intersection of the* 0*–loci of a section of L and a section of M.*

It turns out that the analogy between the theory of rank 2 bundles on $\mathbb{P}^r$ and on Y, stops soon when we consider splitting criteria, even for very simple complete intersection Y.

EXAMPLE 1. Let Y be a smooth quadric in $\mathbb{P}^4$. Then there are lines X inside Y. X cannot be complete intersection of two surfaces in Y. Indeed the map $\rho : \mathrm{Pic}(\mathbb{P}^r) \to \mathrm{Pic}(Y)$ surjects, thus any surface in Y arises by intersecting Y with an hypersurface in $\mathbb{P}^4$, thus any complete intersection curve in Y has even degree.

On the other hand, X is subcanonical, so we have a rank 2 bundle E on Y, associated with X, which cannot decompose. As X is arithmetically normal in $\mathbb{P}^4$, then the exact sequence $0 \to \mathscr{O}_{\mathbb{P}^4}(-2) = \mathscr{I}_Y \to \mathscr{I}_X \to \mathscr{I}_{X,Y} \to 0$ shows that $H^1(\mathscr{I}_{X,Y}(t)) = 0$ for all t. By the exact sequence (7), this in turn implies $H^1(E(t)) = 0$ for all t.

Hence Horrocks splitting criterion does not work on a general quadric in $\mathbb{P}^4$.

One can prove (see e.g. [50]) that the rank 2 bundle associated with a line X is essentially the unique counterexample to the Horrocks splitting criterion, on a quadric hypersurface Y of $\mathbb{P}^4$.

The situation for hypersurfaces of $\mathbb{P}^4$ has been clarified by C. Madonna, who found in [42] that Horrocks splitting criterion is valid except for bundles whose first

Chern class fits in a well determined range. Indeed Madonna went over proving an extension of the splitting criterion of Theorem 3:

THEOREM 9 (Madonna's criterion). *Let $Y \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Let E be a rank 2 vector bundle on Y. Set:*

$$b := \max\{i : H^0(E(-i)) \neq 0\} \tag{8}$$

and assume

$$c_1 - 2b \leq -d + 2 \quad \text{or} \quad c_1 - 2b \geq d.$$

Then the following are equivalent:

(i) *E decomposes in a sum of line bundles $E = \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b)$;*

(ii) *$H^1(E(t)) = 0$ for all t.*

Observe that Madonna's range $-d + 2 < c_1 - 2b < d$ is empty when $d = 1$, i.e. when $Y = \mathbb{P}^3$. Also it reduces to a singleton when $d = 2$, corresponding to the case of the rank 2 bundle associated with a line.

Madonna's range for c_1 is indeed sharp. On one hand, if Y contains a line, then the associated rank 2 bundle satisfies and contradicts Horrocks splitting criterion. On the other hand, one find examples of rank 2 bundles with $c_1 = d - 1$, when Y corresponds to the pfaffian of a skew-symmetric matrix of linear forms (see [2] for details).

OPEN PROBLEM 10. Madonna's criterion has been extended to a wider class of threefolds, as complete intersection threefolds or regular subcanonical threefolds. See [42] for details.

However, an extension to more complicate classes of threefolds (for which the Serre correspondence could present some failures) is unknown.

OPEN PROBLEM 11. Using Madonna's criterion or direct constructions, one could try to classify all rank 2 vector bundles on smooth threefolds $X \subset \mathbb{P}^4$, which does not decompose and nevertheless satisfy $H^1(E(t)) = 0$ for all t.

This has been done for $\deg(X) = 2$ ([50]), $\deg(X) = 3$ ([1]) and $\deg(X) = 4$ ([43]). For $\deg(X) = 5$, in [15] one finds a description of all the bundles with the previous property, which *could* exist on a smooth quintic. The effective existence is not known (but see [40] for some partial results).

In general, on a smooth hypersurface of degree $d > 5$, it is not known, for a fixed c_1 in Madonna's range, which values one has for the second Chern class of an indecomposable aCM rank 2 bundle.

The situation becomes only a little easier when one looks at *general* hypersurfaces X of $\mathbb{P}^4$. The situation does not change for $\deg(X) \leq 5$, while for $\deg(X) = 6$, i.e. for canonical hypersurfaces, we have:

THEOREM 10. *On a general hypersurface X of degree 6 in $\mathbb{P}^4$, the Horrocks splitting criterion holds for rank 2 bundles. Namely, a rank 2 bundle E decomposes in a sum of line bundles if and only if $H^1(E(t)) = 0$ for all t.*

As a consequence, Gherardelli's criterion holds: a smooth curve in X is complete intersection of X and two other hypersurfaces if and only if it is subcanonical and arithmetically normal.

Proof. See [16]. □

OPEN PROBLEM 12. Although one can hope to play some induction on the degree, by now we do not know whether Horrocks splitting criterion is valid for rank 2 bundles on a smooth hypersurface of degree $d > 6$ in $\mathbb{P}^4$.

REMARK 3. For rank 2 bundles on a general hypersurface of $\mathbb{P}^r$, $r > 4$, the splitting criterion works, except for quadrics in $\mathbb{P}^5$.

This has been proved for hypersurfaces of degree $d = 3, 4, 5, 6$ in $\mathbb{P}^5$ in [17] and extended to any degree in a recent paper by Mohan Kumar, Rao and Ravindra ([46]). The extension to higher dimensional projective spaces follows easily by Strano's criterion.

The case $d = 2, r > 5$ is easy. In fact, by Madonna's criterion, Horrocks splitting theorem would fail only when the quadric contains a linear space of codimension 2, which is not the case for general quadrics.

OPEN PROBLEM 13. Extend the previous theory to other classes of threefolds, as complete intersections, subcanonical threefolds, etc.

OPEN PROBLEM 14. Is there some analogue of Evans-Griffith criterion of Theorem 5 for rank 2 bundles on general hypersurfaces of $\mathbb{P}^r$, $r > 3$?

OPEN PROBLEM 15. Some particular smooth sextic threefold X in $\mathbb{P}^4$ does not satisfies Horrocks splitting criterion. E.g. if X contains a line, then this line is associated with a rank 2 bundle which is indecomposable and aCM.

The situation is similar to the Noether–Lefschetz principle for surfaces of degree $d > 3$ in $\mathbb{P}^3$: we know that, on a general surface, all line bundles are restrictions of line bundles on the projective space. However, for some particular surface, new line bundles may arise. Surfaces which do not satisfy the Noether–Lefschetz principle, fill up some subsets of the parametrizing space $\mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^3}(d)))$, which are called **Noether–Lefschetz loci**. The study of these Noether–Lefschetz loci has several applications in Algebraic Geometry.

Analogously, one may ask to describe the locus in $\mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^4}(6)))$ which parametrizes hypersurfaces of degree 6 in $\mathbb{P}^4$ bearing some indecomposable aCM rank 2 bundle.

OPEN PROBLEM 16. For the case of surfaces in $\mathbb{P}^3$, Madonna's criterion for the splitting of rank 2 vector bundles also works: Horrocks criterion is valide outside a narrow range for the invariant b of Theorem 9.

It would be interesting to know a description of all indecomposable aCM rank 2 bundles on a *general* surface of degree d of $\mathbb{P}^3$.

For $d = 2$ this is more or less well known. The case $d = 3$ has been solved by Faenzi ([25]), while the case $d = 4$ is essentially well known. For $d = 5$, a forthcoming paper ([14]) will describe the situation.

The problem is open for higher values of d. Giving a precise description of all possible shapes of the function $t \mapsto \dim(H^1(E(t)))$, for vector bundles on smooth surfaces of $\mathbb{P}^3$, seems a task. However, using a lemma on the Hilbert function of sets of points on surfaces ([13]), in a forthcoming paper [5] we are ready to give serious restriction for the possible shapes of the function.

At the end of this section, we would like to point out that, as in the case of $\mathbb{P}^3$, non–subcanonical curves C lying in smooth hypersurfaces $X \subset \mathbb{P}^4$ are associated with rank 2 reflexive sheaves F on X.

M. Valenzano (see [67]) extended to this situation a mix between Madonna's and Roggero's criteria:

THEOREM 11. *Let F be a reflexive sheaf of rank 2 on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree d. Set $b := \max\{i : H^0(E(-i)) \neq 0\}$ and assume*

$$c_1 - 2b \leq -d + 2 \quad \text{or} \quad c_1 - 2b \geq d.$$

Then F is a vector bundle and decomposes in a sum of line bundles $F = \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ if and only if

$$H^2\left(F\left(\frac{d - c_1 - 6}{2}\right)\right) = 0 \quad \textit{for even } d - c_1;$$

$$H^2\left(F\left(\frac{d - c_1 - 7}{2}\right)\right) = 0 \quad \textit{for odd } d - c_1.$$

OPEN PROBLEM 17. It is clearly not an easy task, but it could be worthy of some effort, the attempt to apply the methods used for rank 2 bundles, to obtain information on the Chern classes of reflexive sheaves on smooth hypersurfaces of $\mathbb{P}^4$ (or even more general threefolds).

This could be a way to answer some interesting question about curves contained in general threefolds of $\mathbb{P}^4$. E.g. questions like the following ones are still unanswered:

- Q1: Is it true that every smooth, connected arithmetically normal curve on a general smooth threefold of degree $d > 5$ in $\mathbb{P}^4$ has degree multiple of d?
- Q2: Is the statement true for subcanonical curves?

See [30] and [68] for a discussion on the subject.

Finally, in order to construct examples of subcanonical curves in smooth threefolds, one can try to consider double structures on curves, following the pattern introduced by Ferrand in $\mathbb{P}^3$ (see [26]). In [6] one finds some generalization to multiple structures. In [44] one can find a systematic study of such structures.

Let us point out that a generalization of Ferrand construction has been performed in Valabrega's paper [18], §3. The construction yields examples of subcanonical curves and vector bundles whose cohomological properties can be controlled rather deeply. Indeed, several pathological examples of rank 2 bundles on $\mathbb{P}^3$ can be obtained in this way (see e.g. [11]).

OPEN PROBLEM 18. Reproduce the multiple structure construction of [18] for curves and varieties in more general spaces, and exploit which kind of bundles one may obtain.

4. Families of varieties and vector bundles

The study of deformations of vector bundles and reflexive sheaves has a natural interest in projective Algebraic Geometry. Let us set the problem from the very beginning.

Any hypersurface X of the projective space $\mathbb{P}^r$ is defined by a single homogeneous equation $F \in \mathbb{C}[x_0, \dots, x_r]$. In order to deform X we need just to deform the homogeneous polynomial F. The set of all hypersurfaces that can be obtained with a deformation of X coincides thus the set of hypersurfaces defined by homogeneous equations of the same degree of F. They are parametrized by the points of a projective space

$$|O_{\mathbb{P}^r}(d)| = \mathbb{P}(H^0(O_{\mathbb{P}^r}(d)))$$

where d is the degree of F. Hence, at this level, any hypersurface is described by its degree plus a point in a projective space.

If we replace $\mathbb{P}^r$ with a (even smooth, irreducible) projective variety V, then some of the previous facts fail.

A hypersurface $X \subset V$, i.e. a subvariety of codimension 1 in V, is no longer globally defined, modulo the homogeneous ideal of V, by a unique equation. Nevertheless, as V is smooth, X is defined by a unique equation in a suitable Zariski open neighbourhood of any point of V. These equations may vary as the Zariski open set varies. They match in the overlapping of the neighbourhoods, modulo well defined transition functions. Thus X determines a line bundle L_X over V and a global section $s_X \in H^0(L_X)$, with the property that X is exactly the locus where s_X vanishes. This is true not only set-theoretically, but in a refined algebraic sense, i.e. scheme-theoretically.

In order to deform X in V, we may clearly take a deformation of the section $s_X \in H^0(L_X)$. So the hypersurfaces of X associated to global sections of L_X are the "first–level" deformations of X.

However, depending on the geometry of the variety V, it is possible that a deformation of the transition functions of L_X could determine also a deformation of the line bundle, which carries a deformation of the global section s. We get thus a family of line bundles and for any G in the family and any non–zero global section $t \in H^0(G)$, the locus defined by $t = 0$ is also a hypersurface obtained with a deformation of X. Line bundles G obtained from a deformation of L_X are indeed of the type $L_X + \epsilon$, where ϵ is a line bundle which lies in a variety ($\mathrm{Pic}^0(V)$) which parametrizes line bundles numerically equivalent to 0.

Thus, deformations of X are parametrized by points of a space which is a projective fibration over a subvariety of $\mathrm{Pic}^0(V)$.

$\mathrm{Pic}^0(V)$ is trivial when X is rational, or a hypersurface of dimension ≥ 2. On the other hand, when for instance V is a non–rational curve, then $\mathrm{Pic}^0(V)$ has a complicate geometrical structure, exploited in the Brill–Noether theory.

When we consider subvarieties X of codimension 2 in a projective space (or, even worse, in a subvariety Y) then the structure becomes suddenly much more difficult to describe.

First of all, we do not have a nice organization of such objects in terms of sections of some vector bundle.

An organization of this type would follow, from Serre's construction, only in the case of subcanonical varieties. For the general case, one should refer to reflexive sheaves, which however are not uniquely determined, even in their numeric features, by the subvariety X.

Restricting our attention to subcanonical varieties X, nevertheless one has a notion of "linear" deformation: fix a rank 2 bundle associated to X and a section $s \in H^0(E)$ which vanishes along X and take a deformation of s inside $\mathbb{P}(H^0(E))$. This is, roughly speaking, the concept equivalent to a deformation of local equations for subvarieties of codimension 1. Besides that, there are the deformations that one can reach by moving the pair (bundle, global section).

This sounds as a very reasonable theory, but unfortunately the space parametrizing vector bundles of rank 2, even on a projective space, may have a very intricate structure, whose understanding is far from being complete. The situation worsens if we replace $\mathbb{P}^r$ with some variety Y.

OPEN PROBLEM 19. A systematic study of the deformations of a subcanonical variety, from this point of view, has been scarcely carried on. Even a collection of fundamental results, as the space parametrizing subcanonical deformations, is sparse in a set of papers which, at least occasionally, deal with the subject.

See e.g. [63], [12], [27] for examples.

A similar problem of course could be introduced for non–subcanonical varieties, just replacing the rank 2 bundle with a rank 2 reflexive sheaf.

In any event, it seems natural to believe that the organization of deformations of 2-codimensional varieties in terms of linear deformations plus deformations of the bundle (or the reflexive sheaf) could lead to some new point of view in the theory.

One more reason to look at the vector bundle in order to understand subvarieties of codimension 2 is given by the notion of "twist".

Namely, if X is the 0–locus of a section of a rank 2 vector bundle E, then 0–loci of sections of a twist $E(n) = E \otimes \mathcal{O}_{\mathbb{P}^r}(n)$ yields subvarieties which, for many features (cohomology, deformations, etc.) are similar to X. Thus one can change X with some new subvariety X' in order to obtain geometrical information, play induction, and so on.

The general non–sense rule, in this game of passing from 0–loci of sections of E to 0–loci of sections of $E(n)$ is:

- $n \gg 0 \implies$ geometrically simpler, arithmetically more complicate
- $n \ll 0 \implies$ geometrically more complicate, arithmetically simpler.

Beware that starting with a smooth subcanonical variety X and taking sections of $E(n)$, $n \ll 0$, one may easily obtain varieties with very bad singularities (even non–reduced ones!).

However we have the following Bertini-type theorem ([39]):

THEOREM 12. *Assume that E is* generated by global sections*, i.e. there exists a surjective map $\mathcal{O}^t_{\mathbb{P}^r} \to E$ for some t. Then a general global section of E has smooth zero locus.*

Thus, by replacing E with $E(n)$, $n \gg 0$ (when possible), one may always assume that the zero locus of a general section of E is smooth.

Observe that the Castelnuovo–Mumford criterion [48], §14, provides a cohomological procedure to determine when a vector bundle (actually, any torsion free sheaf) is generated by global sections:

THEOREM 13 (Castelnuovo-Mumford criterion). *If $H^i(E(-i)) = 0$ for all $i > 0$, then E is generated by global sections.*

This also suggests the following (probably non–sense, in any case very hard):

OPEN PROBLEM 20. Determine necessary and sufficient conditions on E, n such that a general section of $E(n)$ has smooth zero locus.

OPEN PROBLEM 21. Provide conditions on E under which, for all n, $H^0(E(n)) \neq 0$ implies that a general section of $E(n)$ has smooth zero locus.

OPEN PROBLEM 22. Assume that E has a global section whose zero locus is smooth. Assume that $E(1)$ has sections which vanish in codimension 2. May one conclude that a general section of $E(1)$ has smooth zero locus?

A basic question in this theory is

Q): *Given a vector bundle E, for which n one has $H^0(E(n)) \neq 0$;*

and a similar question clearly arises for reflexive sheaves.

Consider the case $\mathrm{rank}(E) = 2$ and E has a section whose zero locus is X. Via the usual exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to E \to \mathcal{I}_X(c_1) \to 0$$

the question can be rephrased as follows:

assume we are given a subcanonical variety X of codimension 2. For which values d are there hypersurfaces of degree d which contain X?

Notice that the rephrasement, however, is not perfectly coherent. Namely

$$\min\{d : H^0(\mathcal{I}_X(d)) \neq 0\} = \min\{d : H^0(E(d - c_1)) \neq 0\}$$

unless E is *strongly normalized*, i.e. unless $\dim H^0(E) = 1$. Indeed when E is normalized, $\min\{d : H^0(E(d - c_1)) \neq 0\} = c_1$ but X lies in no hypersurfaces of degree c_1.

OPEN PROBLEM 23. Find "serious" geometrical conditions on a subcanonical variety X of codimension 2 which determine whether X is the zero locus of a section of a strongly normalized rank 2 bundle.

We know indeed that subcanonical varieties X, X' which are zero loci of sections of $E, E(n)$ respectively, for the same rank 2 bundle E, are *doubly linked*. Namely there exists a variety Y which is directly linked both to X and X'. With this respect, zero loci of sections of strongly normalized rank 2 bundles often have minimal degree in their equivalence class with respect to the relation: $X \sim Y \iff X$ is doubly linked to Y.

Going back to the main argument of this section, let us notice that the question about the minimal degree d of a hypersurface which contains a given variety X is a non trivial one. In general, X is always contained in a hypersurface of degree $\deg(X)$ ([18], §2). However this bound, which turns out to be sharp for linear subvarieties, is rather coarse.

In this setting, the main result is due to Hartshorne, who proved in [36]:

THEOREM 14. *Let F be a rank 2 reflexive sheaf in $\mathbb{P}^3$, with second Chern class $c_2 \geq 0$ and $c_1 \in \{0, -1\}$. Then $H^0F(n) \neq 0$ provided that $n > \sqrt{3c_2 + 1} - 2$ if $c_1 = 0$, or $n > \sqrt{3c_2 + (1/4)} - (3/2)$ if $c_1 = -1$.*

This result, although partial, it is indeed sharp in its range.

Improvement are possible if one knows something on the cohomology of E. For instance, Paolo Valabrega proved in [18] the following result:

THEOREM 15. *Let E be a rank 2 bundle in $\mathbb{P}^3$. Assume $H^0(E) \neq 0$ and $H^1(E(c_1)) = 0$. Then $c_2 \geq c_1\sqrt{c_1} + 2c_1 - 2\sqrt{c_1}$.*

Notice that, for the case of $\mathbb{P}^2$, almost any answer comes out from our knowledge of the shape of the function $t \mapsto \dim H^1(E(t))$ (see e.g. [4]).

Beware that Kleiman's Theorem 12 does not work for reflexive sheaves. Namely there are reflexive sheaves F such that no sections of $F(n)$ vanish in a smooth locus, for all n. Reflexive sheaves having sections with smooth zero locus are called *curvilinear*.

For general reflexive sheaves, Paolo Valabrega and M. Roggero proved the following bound ([59]), which works for any Chern classes:

THEOREM 16. *Let F be a reflexive sheaf in $\mathbb{P}^3$ having a non zero global section. Then $H^0(F(t)) \neq 0$ for some $t \leq c_1 + 2\sqrt{3c_2 + 1 + 3c_1/4}$.*

The same authors extended these methods to produce similar bounds for subcanonical surfaces and rank 2 bundles in $\mathbb{P}^4$.

OPEN PROBLEM 24. Repeat this analysis for rank 2 vector bundles and reflexive sheaves on threefolds different from $\mathbb{P}^3$, as general hypersurfaces, complete intersections, etc.

When the vector bundle or reflexive sheaf E on $\mathbb{P}^3$ is strongly normalized, then the minimal degree of a surface containing the zero locus of a section of E evidently does no longer represent the minimal level of twist such that E has sections. Instead it is connected with the level of a *second section* of E:

DEFINITION 2. *Let F be a reflexive sheaf of rank 2 on $\mathbb{P}^3$. Call α the minimal integer such that $H^0(F(\alpha)) \neq 0$. Then for all $n \geq \alpha$ the space $H^0(F(n))$ contains all the products of a section of $H^0(F(\alpha))$ by forms of degree $n - \alpha$. These products vanishes along a surface.*

Define β as the minimal integer such that

$$\dim H^0(F(\beta)) > \dim H^0(\mathcal{O}_{\mathbb{P}^3}(n-\alpha)).$$

In other words, β is the first level of twist such that $F(\beta)$ has a section which does not depend algebraically from a minimal section.

REMARK 4. If $\dim H^0(F) > \dim H^0(F(-1)) = 0$ and C is the zero locus of a (minimal) section of E, then $c_1 + \beta$ is the minimal degree of a surface containing C.

If Y is the smooth irreducible zero locus of a section of $F(n)$, for $n \gg 0$, then Y is contained in a complete intersection curve of minimal type $(n + \alpha + c_1, n + \beta + c_1)$.

Roggero proved in [53] that the general section of $F(n)$ vanishes along a curve exactly when either $n = \alpha$ or $n \geq \beta$.

Thus the knowledge of this second section level has relevant applications in the theory of vector bundles and curves.

In particular, it allows to start with the procedure of liaison, which produces exactly the chance of reading properties of a given curve in some other, hopefully simpler, variety.

In this setting, the best known results has been found by Paolo Valabrega, Roggero and Nollet. In a series of papers ([55], [57], [58], [59]) culminating with [49], they prove the following:

THEOREM 17. *Let F be a rank 2 reflexive sheaf, with $c_1 = 0, -1$. Consider the numbers α, β defined in 2. Let T_0 be the largest real root of the polynomial:*

$$T^3 - (6c_2 + 6\alpha c_1 + 6\alpha^2 + 1)T + 3(2\alpha + c_1)(c_2 + c_1\alpha + \alpha^2).$$

Then $\beta \leq T_0 - \alpha - c_1 - 1$.

The same paper contains applications of this result to determine the minimal degree of a surface containing a given curve, where the word *curve* in this setting means any generically complete intersection subscheme of pure dimension 1 (possibly reducible and non reduced). This generalization of the question about the geometry of a curve to such non–trivial varieties produced recently a huge amount of literature, which we do not record here.

OPEN PROBLEM 25. A similar analysis for rank two bundles and reflexive sheaves on surfaces and threefolds which are not projective spaces is only at a very initial stage.

OPEN PROBLEM 26. Only few ideas are known about the extensions of these theories to codimension 3 or higher, e.g. to rank 3 bundles.

Finally, let us point out quickly another huge field of researches in this theory.

As we said at the beginning of the section, besides the study of curves or varieties arising from the section of one fixed vector bundle, one may *deform* the bundle itself, just as one does in the study of linear systems.

Classification spaces (Moduli spaces) for vector bundles are extensively studied in the literature, and we will not even try to present here an overview of the theory, neither a short patch of it.

Let us simply recall a long–standing open problem, which is strictly related to the above discussion on the cohomological properties of vector bundles.

It is well known that, in the border of most Moduli spaces of vector bundles, one finds also non–locally free sheaves, which are, in general, just torsion free.

For instance, starting with the decomposable bundle $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$, whose general sections correspond to an elliptic quartic, complete intersection of two quadrics Q_1, Q_2, and moving the quadrics to acquire a common plane, there is a limit curve C which splits in the union of an elliptic plane curve and a line. Accordingly, the bundle $\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$ is the general element of a family whose special member E_0, which corresponds to C, is not locally free. Notice that, in any event, $H^1(E_0(n)) = 0$ for all n.

The cohomology groups are semi–continous in flat families, and there are examples of flat deformations in which the general elements have some vanishing cohomology groups which do not vanish for the special element (see e.g. [12]). Nevertheless there is no known example of splitting bundles of rank 2 which degenerate to non–splitting ones. After Horrocks criterion, this means that we do not know whether $H^1(E(n)) = 0$ for all n and for all general elements in the family, implies that $H^1(E_0(n)) = 0$ for any special element which is locally free.

As sections of $E(n)$ vanish on smooth varieties, for $n \gg 0$, the problem can be exactly rephrased as follows:

- Are there *flat* families of *smooth* curves in $\mathbb{P}^3$ whose general member is complete intersection, while some special member is not?

The problem, although apparently quite basic, seems very hard. Only partial answers are known.

E.g. one gets a negative answer by imposing that all the members of the family belong to surfaces of small degree ([19]).

On the other hand, Mohan Kumar proved in [45] that there are examples of this type in positive characteristic.

For an overview of the setting of the theory, one can read the paper [19].

The interesting feature of this problem resides in the (quite unusual) fact that no precise feeling seems available, through the mathematical community, on what the answer should be.

This justifies the absence of conjectures.

References

[1] ARRONDO E. AND COSTA L., *Vector bundles on Fano 3-folds without intermediate cohomology*, Comm. Algebra **28** (2000), 3899–3911.

[2] BEAUVILLE A., *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.

[3] BEORCHIA V. AND FRANCO D., *On the moduli space of t'Hooft bundles*, Ann. Univ. Ferrara Sez. VII **47** (2001), 253–268.

[4] BOCCI C. AND CHIANTINI L., *The cohomology of rank 2 bundles in* $\mathbb{P}^2$, in preparation.

[5] BOCCI C. AND CHIANTINI L., *On the cohomology of rank 2 bundles in* $\mathbb{P}^3$, in preparation.

[6] BORATYNSKI M., *Multiple structures on projective curves in* $\mathbb{P}^3$, preprint.

[7] BUCHWEITZ R.-O., GREUEL G.-M. AND SCHREYER F.-O., *Cohen–Macaulay modules on hypersurface singularities. II*, Invent. Math. **88** (1987), 165–182.

[8] BURAGGINA A., *The intermediate cohomology modules of a rank two vector bundle on* $\mathbb{P}^3$ *are not split*, Comm. Algebra **25** (1997), 999–1008.

[9] CHANG M.C., *A bound on the order of jumping lines*, Math. Ann. **262** (1983), 511–516.

[10] CHANG M.C., *On the zero sets of vector bundles*, Chinese J. Math. **21** (1993), 109–114.

[11] CHIANTINI L., *On* 4*-generated bundles and directly linked subcanonical curves*, J. Algebra **99** (1986), 239–262.

[12] CHIANTINI L., *Obstructions of t'Hooft curves*, Rend. Sem. Mat. Univ. Pol. Torino **46** (1988), 217–231.

[13] CHIANTINI L. AND CILIBERTO C., *Towards a Halphen theory of linear series on curves*, Trans. Amer. Math. Soc. **351** (1999), 2197–2212.

[14] CHIANTINI L. AND FAENZI D., *ACM bundles on the quintic surface*, in preparation.

[15] CHIANTINI L. AND MADONNA C., *ACM bundles on a general quintic threefold*, Le Matematiche **55** (2000), 239–258.

[16] CHIANTINI L. AND MADONNA C., *A splitting criterion for rank* 2 *vector bundles on a general sextic threefold*, Internat. J. Math. **15** (2004), 341–359.

[17] CHIANTINI L. AND MADONNA C., *ACM bundles on general hypersurfaces of* $\mathbb{P}^5$ *of low degree*, Collect. Math. **56** (2005), 85–96.

[18] CHIANTINI L. AND VALABREGA P., *Subcanonical curves and complete intersections in projective* 3*–space*, Ann. Mat. Pura Appl. **138** (1984), 309–330.

[19] CHIANTINI L. AND VALABREGA P., *Some problems on the degeneration of projective varieties*, Sem. Dip. Mat. Bologna (1985), 13–22.

[20] CHIANTINI L. AND VALABREGA P., *On some properties of subcanonical curves and unstable bundles*, Comm. Algebra **15** (1987), 1877–1887.

[21] DECKER M., *Monads and cohomology modules of rank* 2 *vector bundles*, Compositio Math. **76** (1990), 7–17.

[22] ELLIA PH., *Sur la cohomologie de certains fibrés de rang deux sur* $\mathbb{P}^3$, Ann. Univ. Ferrara Sez. VII **38** (1992), 217–227.

[23] ELLIA PH., *Some vanishings for the cohomology of stable rank two vector bundles on* $\mathbb{P}^3$, J. Reine Angew. Math. **451** (1994), 1–14.

[24] ELLIA PH. AND FIORENTINI M., *Quelques remarques sur les courbes arithmétiquement Buchsbaum de l'espace projectif*, Ann. Univ. Ferrara Sez. VII **33** (1987), 89–111.

[25] FAENZI D., *Rank* 2 *arithmetically Cohen–Macaulay bundles on a non–singular cubic surface*, preprint math.AG/0504492 (2005).

[26] FERRAND D., *Courbes gauches et fibrés de rang* 2, C. R. Acad. Sci. Paris Sér. A **281** (1975), 345–347.

[27] FRANCO D., *Vanishing for the cohomology of certain families of stable rank two vector bundles on* $\mathbb{P}^3$, Ann. Mat. Pura Appl. **170** (1996), 133–145.

[28] GHERARDELLI G., *Sulle curve sghembe algebriche intersezioni complete di due superficie*, Rend. Reale Accad. d'Italia **IV** (1943), 128–132.

[29] EVANS E.G. AND GRIFFITH P., *The syzygy problem*, Ann. of Math. **114** (1981), 323–333.

[30] GRIFFITHS P. AND HARRIS J., *Algebraic geometry and local differential geometry*, Ann. Sci. École Norm. Sup. **12** 4 (1979), 355–452.

[31] GRUSON L. AND PESKINE C., *Genre des courbes de l'espace projectif*, Lecture Notes in Math. **687**, Springer, 1978, pp. 31–59.

[32] HARTSHORNE R., *Varieties of small codimension in projective space*, Bull. A.M.S. **80** (1974), 1017–1032.

[33] HARTSHORNE R., *Stable vector bundles of rank* 2 *on* $\mathbb{P}^3$, Math. Ann. **238** (1978), 229–280.

[34] HARTSHORNE R., *Algebraic vector bundles on projective spaces: a problem list*, Topology **18** (1979), 117–128.

[35] HARTSHORNE R., *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[36] HARTSHORNE R., *Stable reflexive sheaves. II*, Invent. Math. **66** (1982), 165–190.

[37] HARTSHORNE R. AND SOLS I., *Stable rank* 2 *vector bundles on* $\mathbb{P}^3$ *with* $c_1 = -1$, $c_2 = 2$, J. Reine Angew. Math. **325** (1981), 145–152.

[38] HORROCKS G. AND MUMFORD D., *A rank* 2 *vector bundle on* $\mathbb{P}^4$ *with 15,000 symmetries*, Topology **12** (1973), 63–81.

[39] KLEIMAN S., *Geometry on Grassmannians and applications to splitting bundles and smoothing cycles*, Publ. Math. I.H.E.S. **36** (1969), 281–297.

[40] KLEY H.P., *Rigid curves in complete intersection Calabi–Yau threefolds*, Compos. Math. **123** (2000), 185–208.

[41] Kreuzer M., *On 0–dimensional complete intersections*, Math. Ann. **292** (1992), 43–58.

[42] Madonna C., *A splitting criterion for rank 2 vector bundles on hypersurfaces in* $\mathbb{P}^4$, Rend. Sem. Mat. Univ. Pol. Torino **56** (1998), 43–54.

[43] Madonna C., *Rank-two vector bundles on general quartic hypersurfaces in* $\mathbb{P}^4$, Rev. Mat. Complut. **13** (2000), 287–301.

[44] Manolache N., *Multiple structures on smooth support*, Math. Nach. **167** (1994), 157–202.

[45] Mohan Kumar N., *Smooth degeneration of complete intersection curves in positive characteristic*, Invent. Math. **104** (1991), 313–319.

[46] Mohan Kumar N., Rao A.P. and Ravindra V., *Arithmetically Cohen–Macaulay bundles on hypersurfaces*, preprint math.AG/0507161 (2005).

[47] Mumford D., *Curves and their Jacobians*, The University of Michigan Press, Ann Arbor, 1975.

[48] Mumford D., *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies **59**, Princeton University Press, 1966.

[49] Nollet S., Roggero M. and Valabrega P., *On curve sections of rank two reflexive sheaves*, Comm. Algebra **28** (2000), 5531–5540.

[50] Ottaviani G., *Some extensions of Horrocks criterion to vector bundles on Grassmannians and quadrics*, Ann. Mat. Pura Appl. **155** (1989), 317–341.

[51] Peskine C. and Szpiro L., *Liaison des variétés algébriques. I*, Invent. Math. **26** (1974), 271–302.

[52] Popescu S., *On the splitting criterion of Chiantini and Valabrega*, Rev. Roumaine Mat. Pures Appl. **33** (1988), 883–887.

[53] Roggero M., *Some remarks about the speciality theorem of Gruson and Peskine*, Atti Accad. Sci. Torino **119** (1985), 253–256.

[54] Roggero M., *On the rank 2 reflexive sheaves and the subcanonical curves in* $\mathbb{P}^3$, Comm. Algebra **16** (9) (1988), 1779–1790.

[55] Roggero M. and Valabrega P., *Chern classes and cohomology for rank 2 reflexive sheaves on* $\mathbb{P}^3$, Pacific J. Math. **150** (1991), 383–395.

[56] Roggero M. and Valabrega P., *Some vanishing properties of the intermediate cohomology of a reflexive sheaf on* $\mathbb{P}^n$, J. Algebra **170** (1994), 307–321.

[57] Roggero M. and Valabrega P., *Sulle sezioni di un fascio riflessivo di rango 2 su* $\mathbb{P}^3$*: casi estremi per la prima sezione*, Atti Accad. Pelor. Pericolanti **LXXIII** (1995), 103–111.

[58] Roggero M. and Valabrega P., *On the second section of a rank 2 reflexive sheaf on* $\mathbb{P}^3$, J. Algebra **180** (1996), 67–86.

[59] Roggero M. and Valabrega P., *On the smallest degree of a surface containing a space curve*, Boll. U.M.I. Sez. B **1** 8 (1998), 123–138.

[60] Roggero M. and Valabrega P., *The speciality lemma, rank 2 bundles and Gherardelli–type theorems for surfaces in* $\mathbb{P}^4$, Compositio Math. **139** (2003), 101–111.

[61] Roggero M. and Valabrega P., *A few splitting theorems for rank 2 vector bundles on* $\mathbb{P}^4$, preprint (2004).

[62] Roggero M., Valabrega P. and Valenzano M., *Rank two bundles and reflexive sheaves on* $\mathbb{P}^3$ *and corresponding curves: an overview*, in: "Geometric and Combinatorial Aspects of Commutative Algebra (Messina, 1999)", Lecture Notes in Pure and Appl. Math. **217**, Dekker, New York 2001, 327–343.

[63] Sernesi E., *L'unirazionalità della varietà dei moduli delle curve di genere dodici*, Ann. Scuola Norm. Pisa **8** (1981), 405–439.

[64] Serre J.-P., *Sur les modules projectif*, Secr. Math. Paris (1961).

[65] Strano R., *A characterization of complete intersection curves in* $\mathbb{P}^3$, Proc. Amer. Math. Soc. **104** (1988), 711–715.

[66] SZPIRO L., *Lectures on equations defining space curves*, Tata Institute of Fundamental Research, Bombay, 1979.

[67] VALENZANO M., *Fasci riflessivi di rango due su una varietà algebrica proiettiva non singolare tridimensionale*, Tesi di dottorato, Università di Genova, Genova 2001.

[68] VOISIN C., *Sur une conjecture de Griffiths et Harris*, in: "Algebraic Curves and Projective Geometry (Trento, 1988)", Lecture Notes in Math. **1389**, Springer, Berlin 1989, 270–275.

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