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APPROXIMATE NEUTRAL SURFACE OF A CONVECTION PROBLEM FOR VARIABLE GRAVITY FIELD

Abstract. The problem arises in a horizontal layer of a fluid heated from below for a gravity field variable across the layer. The linear stability problem against normal mode perturbations is an eigenvalue problem with variable coefficients. We deduce the analytical form of the secular equation. Then we present numerical approximations for the neutral curve and neutral surface. The cases of free and of rigid boundaries are considered.

1. Statement of the problem

We are concerned with the linear stability of the mechanical equilibrium of a horizontal layer of fluid heated from below when the gravity field $\mathbf{g}(z)$ is variable across the layer. Pradhan and Samal studied this problem in the framework of inviscid linearized stability theory [6]. They estimated the growth rate σ in the time dependence $e^{\sigma t}$ of the velocity and temperature fields. The effect of the nonconstancy of the gravity was pointed out. The effect is studied on a laboratory scale, but is also likely to be important in the large scale convection of planetary atmosphere [4].

The gravity field acting in the z-direction is orthogonal to the fluid layer and is assumed to depend on the vertical coordinate z [8]. For such a varying gravity field different points of the fluid experience different buoyancy forces. As a consequence, part of a fluid layer tends to become unstable while the other tends to remain stable, the mechanical equilibrium turning into a convective motion.

Experimental measurements of the Earth's upper atmosphere show that the atmospheric density decreases as the altitude increases as an approximately exponentially function of the vertical height. Taking these into account, consider a layer of heat-conducting viscous fluid contained between the planes $z = 0$ and $z = h$. The equations governing the convective motion and the conducting state are [8]

$$\text{(1)}\qquad \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \text{grad})\mathbf{v} = -\frac{1}{\rho}\text{grad}\, p + \nu\,\Delta\mathbf{v} + \mathbf{g}(z)\alpha T, \\ \text{div}\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + (\mathbf{v}\cdot \text{grad})T = k\Delta T, \end{cases} \qquad t > 0$$

where ν is the coefficient of kinematic viscosity, ρ the density, α the thermal expansion coefficient, k the thermal diffusivity, p is the pressure, T the temperature, $\mathbf{v}$ the velocity and the gravity $g(z)$ is defined by $\mathbf{g}(z) = gH(z)\mathbf{k}$, where g is a constant.

The linear stability of the conduction stationary solution characterized by $\mathbf{v} = 0$ of equations (1), written in the nondimensional form, against normal mode perturbations, is governed by a two-point problem for the ordinary differential equations [8]

$$(2) \qquad \begin{cases} (D^2 - a^2)^2 W = RH(z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RN(z)W, \end{cases}$$

where $D = \dfrac{d}{dz}$, R^2 is the Rayleigh number, a is the wave number and W and Θ are the factors in v and θ respectively, depending on z.

Consider $N(z) \equiv 1$ and $H(z) = 1 + \varepsilon h(z)$, $h(z) = -kz$, $\varepsilon \in \left[0, 1/k\right]$, $z \in (0, 1)$, $k = 1, 2, ...$, i.e. the gravity decreases linearly across the layer. Here ε represents a scale for $h(z)$. In this case, the two-point eigenvalue problem for (2) consists of the ordinary differential equations

$$(3) \qquad \begin{cases} (D^2 - a^2)^2 W = R[1 - \varepsilon kz]a^2\Theta, \\ (D^2 - a^2)\Theta = -RW \end{cases}$$

and the usual boundary conditions read

$$(4) \qquad W = DW = \Theta = 0 \quad \text{at } z = 0, 1$$

for fixed rigid walls and

$$(5) \qquad W = D^2W = \Theta = 0 \quad \text{at } z = 0, 1.$$

for free boundaries. In this paper we will treat both cases of fixed and free boundaries.

We look for the smallest eigenvalue R (the Rayleigh number) in (3)-(4), (3)-(5) defining the neutral manifold.

For $\varepsilon = 0$ the problem becomes the classical problem of convection [5].

In [2],[3] it is proved that when $H(z)N(z) \geq 0$ across the layer the principle of exchange of stabilities holds. In our case $N(z) \equiv 1$ and ε was chosen such that $H(z) \geq 0$ holds, therefore the principle of exchange of stabilities holds no matter what the boundary conditions.

There are cases in which this condition is not satisfied and the principle holds. In [8], Straughan performes numerical evaluations of the Rayleigh number for $N(z) \equiv 1$, $H(z) = 1 - \varepsilon z$, $0 \leq \varepsilon \leq 1.5$. By using the energy method, he provides numerical results for other varying gravity fields too.

In applications, many varying gravity fields can be encountered. Some of these gravity fields are very important in crystal growth and many other convective flows.

2. The solution for the eigenvalue problem

Fixed boundaries

The eigenvalue problem (3)-(4) is not self-adjoint in the usual sense. However, we can apply a method frequently used in the self-adjoint case.

Since $\Theta = 0$ at $z = 0$ and $z = 1$, and $\{\sin m\pi z\}_{m\in\mathbb{N}^*}$ is a total set of $L^2(0,1)$, we have the expansion $\Theta = \sum_{m=1}^{\infty} C_m \sin m\pi z$ (see [5]).

Replacing this expansion in $(3)_1$ we obtain the fourth order non-homogeneous ordinary differential equation

$$(D^2 - a^2)^2 W = R(1 - \varepsilon kz)a^2 \sum_{m=1}^{\infty} C_m \sin m\pi z. \tag{6}$$

By using the method of variation of coefficients the general solution W of (6) reads

$$\begin{aligned} W(z) = \sum_{m=1}^{\infty} \Big\{ & A_1^m \sinh az + A_2^m \cosh az + A_3^m z \sinh az \\ & + A_4^m z\cosh az + Ra^2 \cdot \Big[\frac{C_m}{(m^2\pi^2 + a^2)^2} \sin m\pi z \\ & - \frac{C_m \varepsilon k}{(m^2\pi^2 + a^2)^2} z \sin m\pi z - \frac{4C_m \varepsilon k m\pi}{(m^2\pi^2 + a^2)^3} \cos m\pi z \Big] \Big\} \end{aligned} \tag{7}$$

Imposing the boundary conditions $W(0) = W(1) = DW(0) = DW(1) = 0$, we obtain the algebraic system in the coefficients A_1^m, A_2^m, A_3^m, A_4^m

$$\begin{cases} A_2^m = \dfrac{4C_m R\varepsilon k a^2 m\pi}{(m^2\pi^2 + a^2)^3}, \\[2ex] A_1^m \sinh a + A_2^m \cosh a + A_3^m \sinh a + A_4^m \cosh a = (-1)^m \dfrac{4C_m R\varepsilon k a^2 m\pi}{(m^2\pi^2 + a^2)^3}, \\[2ex] aA_1^m + A_4^m = -\dfrac{Ra^2 C_m m\pi}{(m^2\pi^2 + a^2)^2}, \\[2ex] aA_1^m \cosh a + aA_2^m \sinh a + A_3^m (\sinh a + a\cosh a) + A_4^m (\cosh a + a \sinh a) = \\ \quad = (-1)^m \dfrac{C_m R\varepsilon k a^2 m\pi}{(m^2\pi^2 + a^2)^2} + (-1)^{m+1} \dfrac{Ra^2 C_m m\pi}{(m^2\pi^2 + a^2)^2}. \end{cases}$$

Its root is

$$
(8)\quad \begin{cases}
A_1^m = \dfrac{4\varepsilon k R a^2 D_m m\pi\,[(-1)^m(\sinh a + a\cosh a) - a - \sinh a\cosh a]}{(m^2\pi^2 + a^2)(-a^2 + \sinh^2 a)} + \\[2ex]
\quad + \dfrac{D_m R a^2 m\pi\,[a + (-1)^{m+1}\sinh a(\varepsilon k - 1)]}{(-a^2 + \sinh^2 a)}, \\[2ex]
A_2^m = \dfrac{4R\varepsilon k D_m a^2 m\pi}{(m^2\pi^2 + a^2)}, \\[2ex]
A_3^m = \dfrac{4\varepsilon k D_m R a^3 m\pi\,\sinh a[(-1)^m a - \sinh a]}{(m^2\pi^2 + a^2)(-a^2 + \sinh^2 a)} + \\[2ex]
\quad + \dfrac{D_m R a^2 m\pi\,[-a + \sinh a\cosh a + (-1)^m(\varepsilon k - 1)(\sinh a - a\cosh a)]}{-a^2 + \sinh^2 a}, \\[2ex]
A_4^m = \dfrac{4\varepsilon k D_m a^3 R m\pi\,[a + \cosh a\sinh a + (-1)^{m+1}(\sinh a + a\cosh a)]}{(m^2\pi^2 + a^2)(-a^2 + \sinh^2 a)} + \\[2ex]
\quad + \dfrac{D_m a^2 R m\pi\,\sinh a[(-1)^m a(\varepsilon k - 1) - \sinh a]}{-a^2 + \sinh^2 a}
\end{cases}
$$

where $D_m = \dfrac{C_m}{(m^2\pi^2 + a^2)^2}$.

Replacing the functions W and Θ in equation $(3)_2$ and, imposing the condition that the obtained equation be orthogonal to the functions $sin n\pi z$, $n \in \mathbb{N}$ with respect to the inner product of $L^2(0,1)$ $(f,g) = \int_0^1 fg dz$, we obtain an infinite system of algebraic linear equations in D_m

$$
(9)\quad \begin{aligned}
\sum_{m=1}^{\infty} D_m(m^2\pi^2 + a^2)^3 \cdot \frac{\delta_{mn}}{2} &= \sum_{m=1}^{\infty} R A_1^m \frac{(-1)^{n+1} n\pi\,\sinh a}{a^2 + n^2\pi^2} \\
&+ R A_2^m \Big[\frac{n\pi}{a^2 + n^2\pi^2} + + \frac{n\pi(-1)^{n+1}\cosh a}{a^2 + n^2\pi^2}\Big] \\
&+ R A_3^m \Big[\frac{(-1)^{n+1} n\pi\,\sinh a}{a^2 + n^2\pi^2} - \frac{2an\pi\,[1 + (-1)^{n+1}\cosh a]}{(a^2 + n^2\pi^2)^2}\Big] + \\
&+ R A_4^m \Big[\frac{(-1)^{n+1} n\pi\cosh a}{a^2 + n^2\pi^2} + \frac{2an\pi(-1)^n\sinh a}{(a^2 + n^2\pi^2)^2}\Big] \\
&+ R^2 a^2 D_m \cdot \frac{\delta_{mn}}{2} - D_m R^2 a^2 \varepsilon k \cdot T_{mn} + \frac{4 D_m R^2 \varepsilon k a^2 m\pi}{(a^2 + m^2\pi^2)} \cdot U_{mn}
\end{aligned}
$$

where

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} , \qquad T_{mn} = \begin{cases} \dfrac{1}{4} & \text{if } m = n \\ \dfrac{2mn[(-1)^{m+n} - 1]}{\pi^2(n-m)^2(n+m)^2} & \text{if } m \neq n \end{cases}$$

$$U_{mn} = \begin{cases} 0 & \text{if } m = n \\ \dfrac{n[(-1)^{m+n} - 1]}{\pi(n-m)(n+m)} & \text{if } m \neq n \end{cases}$$

and the expressions (8) were taken into account.
Imposing to the determinant of this system to vanish, we are lead to the secular equation

$$\begin{aligned}
&- \frac{(m^2\pi^2 + a^2)^3\delta_{mn}}{2} + R\frac{(-1)^{n+1}n\pi \sinh a}{n^2\pi^2 + a^2} \cdot \Big[\frac{4\varepsilon k R a^2 m\pi [(-1)^m(\sinh a + a\cosh a)]}{(m^2\pi^2 + a^2)(a^2 - \sinh^2 a)} \\
&+ \frac{4\varepsilon k R a^2 m\pi(-a - \sinh a \cosh a)}{(m^2\pi^2 + a^2)(a^2 - \sinh^2 a)} + \frac{R a^2 m\pi [a + (-1)^{m+1}\sinh a(\varepsilon k - 1)]}{(a^2 - \sinh^2 a)}\Big] \\
&+ R \cdot \frac{n\pi(1 + (-1)^{n+1}\cosh a)}{n^2\pi^2 + a^2} \cdot \frac{4R\varepsilon k a^2 m\pi}{(m^2\pi^2 + a^2)} + R\Big[\frac{(-1)^{n+1}n\pi \sinh a}{n^2\pi^2 + a^2} \\
&\qquad - 2an\pi\Big(\frac{1}{(m^2\pi^2 + a^2)^2} + \frac{(-1)^{n+1}\cosh a)}{(m^2\pi^2 + a^2)^2}\Big)\Big] \\
&\cdot \Big[\frac{4\varepsilon k R a^3 m\pi \sinh a[(-1)^m a - \sinh a]}{(m^2\pi^2 + a^2)(a^2 - \sinh^2 a)} + \frac{R a^2 m\pi[-a + \sinh a \cosh a]}{a^2 - \sinh^2 a} + \\
&\qquad + \frac{R a^2 m\pi(-1)^m(\varepsilon k - 1)(\sinh a - a\cosh a)}{a^2 - \sinh^2 a}\Big]
\end{aligned}$$

$$\begin{aligned}
&+ R\Big[\frac{(-1)^{n+1}n\pi \cosh a}{n^2\pi^2 + a^2} + \frac{2an\pi(-1)^n \sinh a}{(n^2\pi^2 + a^2)^2}\Big] \cdot \Big[\frac{4\varepsilon k a^3 R m\pi [a + \cosh a \sinh a]}{(m^2\pi^2 + a^2)(a^2 - \sinh^2 a)} \\
&+ \frac{4\varepsilon k a^2 R m\pi [(-1)^{m+1}(\sinh a + a\cosh a)]}{(m^2\pi^2 + a^2)(a^2 - \sinh^2 a)} + a^2 R m\pi \sinh a \\
&\cdot \frac{[(-1)^m a(\varepsilon k - 1) - \sinh a]}{a^2 - \sinh^2 a}\Big] + \frac{R^2 a^2 \delta_{mn}}{2} - R^2 a^2 \varepsilon k T_{mn} + \frac{4R^2 \varepsilon k a^2 m\pi}{m^2\pi^2 + a^2} U_{mn} \\
&= 0.
\end{aligned}$$

2.1. Free boundaries

In this case the unknown function W has the the same expression as in (7), but the boundary conditions (7) are leading to the following algebraic system

$$\begin{cases} A_2^m = \dfrac{4C_m \varepsilon k m\pi Ra^2}{(m^2\pi^2+a^2)^3}, \\ A_1^m \sinh a + A_2^m \cosh a + A_3^m \sinh a + A_4^m \cosh a = (-1)^m \dfrac{4C_m R\varepsilon k a^2 m\pi}{(m^2\pi^2+a^2)^3}, \\ a^2 A_2^m + 2a A_3^m = \dfrac{2C_m \varepsilon k a^2 Rm\pi(a^2-m^2\pi^2)}{(m^2\pi^2+a^2)^3}, \\ a^2 A_1^m \sinh a + a^2 A_2^m \cosh a + A_3^m(2a\cosh a + a^2 \sinh a) \\ \qquad + A_4^m(2a\sinh a + a^2\cosh a) = \dfrac{2C_m \varepsilon k a^2 Rm\pi(-1)^m(a^2-m^2\pi^2)}{(m^2\pi^2+a^2)^3}, \end{cases}$$

the roots of which are

$$(10) \qquad \begin{cases} A_1^m = -\dfrac{a\varepsilon k Rm\pi D_m[(a^2+m^2\pi^2)(1-(-1)^m\cosh a)}{(a^2+m^2\pi^2)\sinh^2 a}, \\ \qquad\qquad + \dfrac{4a\sinh a(\cosh a + (-1)^{m+1})]}{(a^2+m^2\pi^2)\sinh^2 a} \\ A_2^m = \dfrac{4\varepsilon k a^2 Rm\pi D_m}{m^2\pi^2+a^2}, \\ A_3^m = -a\varepsilon k Rm\pi D_m, \\ A_4^m = \dfrac{a\varepsilon k Rm\pi D_m((-1)^{m+1}+\cosh a)}{\sinh a}. \end{cases}$$

The infinite system of linear equations in D_m has the same form as in (9), but where the expressions of the coefficients A_1^m, A_2^m, A_3^m, A_4^m are those in (10).
The secular equation is obtained imposing the same condition as before, i.e. that the determinant of the system to vanish. Due to the fact that it is easy to deduce this equation but its form is very complicated, we do not write it any longer.

3. Numerical results

In the case of rigid boundaries, a first approximation for the Rayleigh number R^2 is obtained by taking $m = n = 1$ and, correspondingly, imposing to the first-order minor in the matrix associated with the infinite system (9) to vanish [5]. This choice leads to

ε	a^2	k	R^2
0.0	9.711	1	1715.079356
0.01	9.711	1	1723.697848
0.01	9.711	4	1750.080972
0.2	9.711	1	1905.643719
0.2	9.711	3	2450.113291
0.2	12.0	1	1937.927940
0.2	12.0	3	2491.621574
0.2	14.5	1	2026.289430
0.2	16	3	2697.490529
0.3	9.711	1	2017.740395
0.3	20.7	2	3071.717938
0.5	9.711	1	2286.772413
0.5	9.711	2	3430.158727
1	10	1	3431.318766

ε	a^2	k	R^2
0.0	4.92	1	657.5133416
0.01	4.92	1	660.8174287
0.03	4.92	1	667.5262351
0.03	4.92	2	677.8488058
0.03	4.92	4	699.4822782
0.2	5.00	1	730.6101972
0.2	7.5	2	873.4107477
0.2	7.5	3	998.1837121
0.2	9.0	1	829.4751258
0.33	4.92	1	787.4411276
0.3	12	2	1245.212559
0.5	7.5	1	931.6381300
0.5	9.00	1	995.3701503
0.75	10	1	1255.126920

Table 6.1: Rayleigh number for various values of the parameters ε, k, a, when $m = n = 1$ (fixed boundaries at left, free boundaries at right).

the equation $\dfrac{(a^2+\pi^2)^3}{2} = a^2 R^2 \cdot F(a, \varepsilon k)$, where

$$
\begin{aligned}
F(a,\varepsilon,k) &= \frac{4\varepsilon k \sinh a(-\sinh a - a\cosh a)}{(\pi^2+a^2)^2(\sinh^2 a - a^2)} + \frac{4\varepsilon k \sinh a(-a - \sinh a \cosh a)}{(\pi^2+a^2)^2(\sinh^2 a - a^2)} \\
&+ \frac{\sinh a}{(\pi^2+a^2)} \cdot \frac{a + \sinh a(\varepsilon k - 1)}{\sinh^2 a - a^2} + \frac{4\varepsilon k(1+\cosh a)}{(\pi^2+a^2)^2} \\
&+ \Big(\frac{\sinh a}{\pi^2+a^2} - \frac{2a(1+\cosh a)}{(\pi^2+a^2)^2}\Big) \cdot \Big(\frac{4\varepsilon k a \sinh a}{(\pi^2+a^2)} \cdot \frac{-a-\sinh a}{\sinh^2 a - a^2} \\
&+ \frac{-a + \sinh a \cosh a - (\varepsilon k - 1)(\sinh a - a\cosh a)}{\sinh^2 a - a^2}\Big) \\
&+ \Big(\frac{\cosh a}{\pi^2+a^2} - \frac{2a \sinh a}{(\pi^2+a^2)^2}\Big) \cdot \Big(\frac{4\varepsilon k a(a + \cosh a \sinh a)}{(\pi^2+a^2)(\sinh^2 a - a^2)} \\
&+ \frac{4\varepsilon k a(\sinh a + a\cosh a)}{(\pi^2+a^2)(\sinh^2 a - a^2)} + \frac{\sinh a}{\sinh^2 - a^2} \cdot [-a(\varepsilon k - 1) - \sinh a]\Big) + \frac{2-\varepsilon k}{4},
\end{aligned}
$$

therefore $Ra = \dfrac{(a^2+\pi^2)^3}{2a^2 F(a,\varepsilon,k)}$.
Similarly, for free boundaries, the expression of the Rayleigh number can be deduced from the secular equation.

We performed numerical evaluations for the Rayleigh number also for $m = n = 2$ and $m = n = 3$. In these cases, the value of the Rayleigh number tends to decrease but the variation are so small, that we can consider that the first approximation is a good one. The approximate values of the Rayleigh number for various values of ε, a, k in the case when the two boundaries are fixed or free are presented in Table 3 ($m = n = 1$),

ε	a^2	k	R^2
0.0	9.711	1	1715.079356
0.01	9.711	1	1723.675581
0.01	9.711	4	1749.751329
0.2	9.711	1	1899.344809
0.2	9.711	3	2401.252634
0.2	12.0	1	1928.750633
0.2	12.0	3	2434.718875
0.2	14.5	1	2014.541043
0.2	16	3	2628.936104
0.3	9.711	1	2004.771332
0.3	20.7	2	2987.518777
0.5	9.711	1	2253.100613
0.5	9.711	2	3254.966193
1	10	1	3254.119930

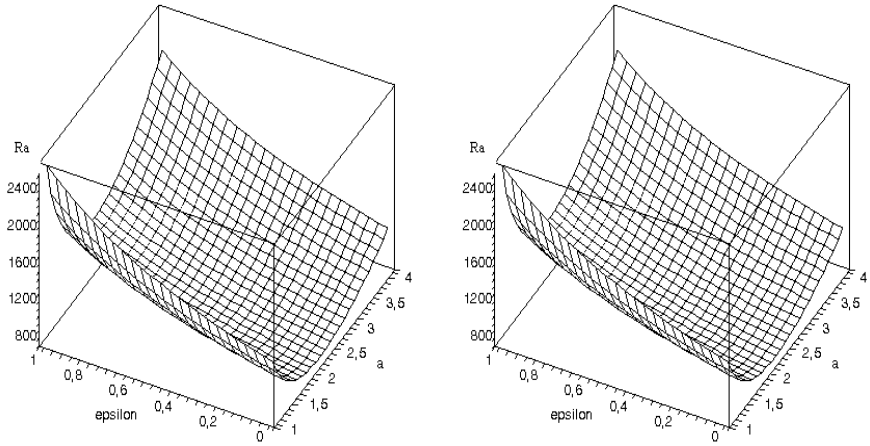
ε	a^2	k	R^2
0.0	4.92	1	657.5133415
0.01	4.92	1	660.8173455
0.03	4.92	1	667.5254645
0.03	4.92	2	677.8455763
0.03	4.92	4	699.4680828
0.2	5.00	1	730.5647663
0.2	7.5	2	873.0339301
0.2	7.5	3	996.9203050
0.2	9.0	1	829.3918569
0.33	4.92	1	787.2880261
0.3	12	2	1242.757781
0.5	7.5	1	930.9240665
0.5	9.00	2	994.4723300
0.75	10	1	1251.093151

Table 6.2: Rayleigh number for various values of the parameters ε, k, a, when $m = n = 2$ (fixed boundaries at left, free boundaries at right).

ε	a^2	k	R^2
0.0	9.711	1	1707.937389
0.01	9.711	1	1716.519675
0.01	9.711	4	1742.787785
0.2	9.711	1	1897.531721
0.2	9.711	3	2436.543844
0.2	12.0	1	1929.268869
0.2	12.0	3	2476.807746
0.2	14.5	1	2016.847663
0.2	16	3	2679.673115
0.3	9.711	1	2008.866882
0.3	20.7	2	3049.235234
0.5	9.711	1	2275.346836
0.5	9.711	2	3390.531858
1	10	1	3391.116464

ε	a^2	k	R^2
0.0	4.92	1	657.5133415
0.01	4.92	1	660.8173465
0.03	4.92	1	667.5254639
0.03	4.92	2	677.8455763
0.03	4.92	4	699.4680828
0.2	5.00	1	730.5647668
0.2	7.5	2	873.0339289
0.2	7.5	3	996.9202924
0.2	9.0	1	829.3918563
0.33	4.92	1	787.2880261
0.3	12	2	1242.757709
0.5	7.5	1	930.9240629
0.5	9.00	1	994.4723231
0.75	10	1	1251.092996

Table 6.3: Rayleigh number for various values of the parameters ε, k, a, when $m = n = 3$ (fixed boundaries at left, free boundaries at right).

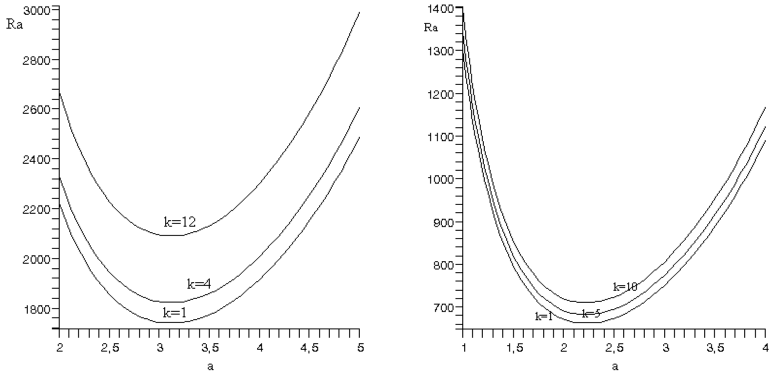


a) fixed boundaries b) free boundaries

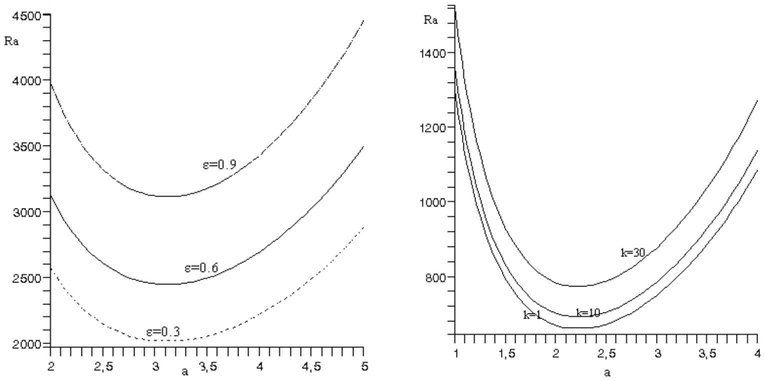
Figure 1: The surface $Ra(\varepsilon, a, k = 1)$.

Table 3 ($m = n = 2$), Table 3 ($m = n = 3$).
For $\varepsilon = 0$, we regain the known [5] critical values, $Ra = 1715.07935$, $Ra = 1715.079356$ and $Ra = 1707.937389$ for $a = \sqrt{9.711}$, and $m = n = 1$, $m = n = 2$, $m = n = 3$ respectively (for rigid boundaries) and $Ra = 657.5133416$, $Ra = 657.5133415$ and $Ra = 657.5133415$ for $a = \sqrt{4.92}$ and $m = n = 1$, $m = n = 2$ and $m = n = 3$, respectively (for free boundaries).
The neutral surfaces from Figure 1 are drawn based on numerical evaluations performed for the wave number $a \in [2, 5]$ (for fixed boundaries), $a \in [1, 4]$(for free boundaries) and the parameter $\varepsilon \in [0, 1]$. For particular values for the parameters ε, k we obtain Ra_c as a function of parameter a only, i.e. the neutral curve. These figures show that the increase in ε (i.e. the decrease of the $h(z)$) enlarges the domain of stability.
Figure 2 presents, for comparison, three neutral curves on each graph when ε is kept constant and k is varying. Then, Figure 3 presents neutral curves for ε varying and k a constant.

For rigid boundaries, the increase of the parameter k, and, so, a rapidly linearly decreasing gravity field leads to an enlargement of the stability domain (Figure 2 a)) and an increasing of the Rayleigh number. In the case of free boundaries, the situation is quite the same: as the parameter k is increasing, the Rayleigh number also is increasing (Figure 2 b)). A similar situation holds when we keep the parameter k constant and we vary the parameter ε (Figure 3).
We studied separately the variation of the neutral curve with ε and k for the sake of comparison. In fact, in our particular case we could consider a single parameter εk.



a) fixed boundaries $\varepsilon = 0.03$ b) free boundaries $\varepsilon = 0.015$



a) fixed boundaries $\varepsilon = 0.3$ b) free boundaries $\varepsilon = 0.01$

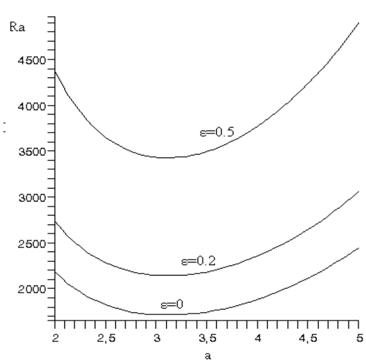
Figure 2: The function $Ra(a)$ for various values of k.

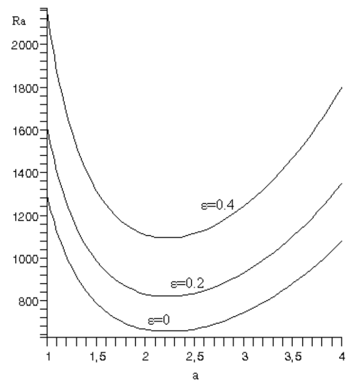
4. Conclusion

In this paper, for a horizontal layer in the case of varying gravity field (linearly decreasing across the layer), we deduced the analytical expression of the secular equation in the form of an infinite determinant equal to zero and perform numerical computations for some approximations of it.

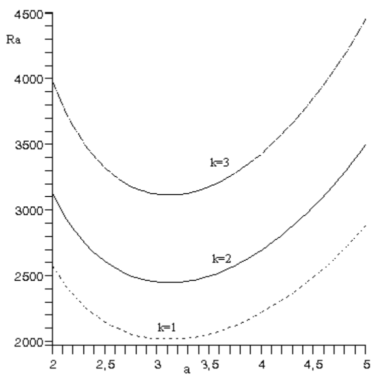
The direct Chandrasekar-Galerkin technique [5] is used to investigate the associated eigenvalue problem in both cases of fixed and free boundaries.

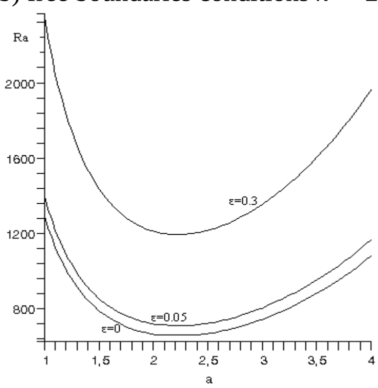
The numerical approximations of the Rayleigh number in some specific gravity field were obtained for some values of the parameters. The approximate neutral curves and surfaces show the influences of each parameter on the value of the Rayleigh number: the domain of stability decreases as the gravity is increased. Our numerical results proved to agree quite well to those from [5], [8].





a) fixed boundary conditions $k = 2$ b) free boundaries conditions $k = 2$.





a) fixed boundary conditions $k = 3$ b) free boundaries conditions $k = 3$.

Figure 3: The function $Ra(a)$ for various values of ε and $k = 2$.

References

[1] HARRIS D.L. AND REID W.H., *On orthogonl functions which satisfy four boundary conditions*, The Astrophysical J., **33** 3 (1958), 429–453.

[2] HERRON I., *On the principle of exchange of stabilities in Rayleigh-Bénard convection*, SIAM J. Appl. Math. **61** 4 (2000), 1362–1368.

[3] HERRON I., *Onset of convection in a porous medium with internal heat source and variable gravity*, IJES **39** (2001), 201–208.

[4] PRADHAN G. AND SAMAL P., *Thermal instability of a fluid layer under variable body forces*, J. Math. Anal. Appl. **122** (1987), 1167–1181.

[5] STRAUGHAN B., *Convection in a variable gravity field*, J. of Math. Anal. Appl. **140** (1989), 467–475.

[6] CHANDRASEKAR S., *Hydrodynamic and hydromagnetic stability*, Oxford University Press, 1961.

[7] GEORGESCU A., *Hydrodynamic stability theory*, Kluwer, Dordrecht 1985.

[8] STRAUGHAN B., *The energy method, stability, and nonlinear convection*, 2nd ed., Springer, Berlin 2003.

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