Rend. Sem. Mat. Univ. Pol. Torino - Vol. 64, 3 (2006)

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HARMONIC MAPS BETWEEN COMPLEX SASAKIAN MANIFOLDS

Abstract. In this paper, we find two classes of harmonic maps between complex Sasakian manifolds. We give an example related to one of this classes, using the complex Heisenberg group and we study the stability of the identity map of a complex Sasakian manifold.

1. Preliminaries

Concerning the complex contact manifold, we shall recall some notions as they are presented in [1].

A complex contact manifold is a complex manifold of odd complex dimension 2n+1 together with an open covering $\{\mathcal{O}_\alpha\}$ by coordinate neighborhoods such that:

1. On each $\{\mathcal{O}_\alpha\}$ there is a holomorphic 1-form θ_α such that

$$\theta_\alpha \wedge (d\theta_\alpha)^n \neq 0;$$

2. On $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$ there is a non-vanishing holomorphic function $f_{\alpha\beta}$ such that $\theta_\alpha = f_{\alpha\beta}\theta_\beta$.

A complex contact manifold with a global complex form is called strict complex contact manifold.

On the other hand, if M is a complex manifold with almost complex structure J, Hermitian metric g and open covering $\{\mathcal{O}_\alpha\}$ by coordinate neighborhoods, M is called a complex almost contact metric manifold if it satisfies the following two conditions:

1. On each $\{\mathcal{O}_\alpha\}$ there exist 1-forms u_α and $v_\alpha = u_\alpha \circ J$ with orthogonal dual vector fields U_α and $V_\alpha = -JU_\alpha$ and (1,1) tensor fields G_α and $H_\alpha = G_\alpha J$ such that

$$G_\alpha^2 = H_\alpha^2 = -I + u_\alpha \otimes U_\alpha + v_\alpha \otimes V_\alpha,$$

$$G_\alpha J = -JG_\alpha, \;\; G_\alpha U_\alpha = 0, \;\; g(X, G_\alpha Y) = -g(G_\alpha X, Y),$$

2. On $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$,

$$u_\beta = au_\alpha - bv_\alpha, \;\; v_\beta = bu_\alpha + av_\alpha,$$

$$G_\beta = aG_\alpha - bH_\alpha, \;\; v_\beta = bG_\alpha + aH_\alpha,$$

where a and b are functions with $a^2 + b^2 = 1$.

It is proved that a complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is of the form $u - iv$ and the local tensor fields G and H are related to du and dv by

$$du(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y),$$

$$dv(X,Y) = g(X,HY) - (\sigma \wedge u)(X,Y),$$

where $\sigma(X) = g(\nabla_X U, V)$. We call a complex contact manifold with a complex almost contact metric structure satisfying these conditions, a complex contact metric manifold. Note that if the complex contact structure is strict then $\sigma = 0$.

For complex contact manifolds, there are two notions of normality (see [1]). In the present paper we adopt the definition given by B. Korkmaz (see [6]).

Thus, in order to define the normal complex contact manifolds, let us consider S and T, two tensor fields given by

$$S(X,Y) = N_G(X,Y) + 2g(X,GY)U - 2g(X,HY)V + 2(v(Y)HX - v(X)HY)$$
$$+\sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,$$
$$T(X,Y) = N_H(X,Y) - 2g(X,GY)U + 2g(X,HY)V + 2(u(Y)GX - u(X)GY)$$
$$+\sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX,$$

where N_G and N_H are the Nijenhuis tensor fields of G and H, respectively.

A complex contact metric structure is normal if

$$S(X,Y) = T(X,Y) = 0, \quad X, Y \in \mathcal{H},$$
$$S(U,X) = T(V,X) = 0, \quad X \in \chi(M),$$

where $\mathcal{H}$ is the subbundle named the horizontal subbundle and defined by the subspaces $\{X \in T_P\mathcal{O}_\alpha, P \in M; \theta_\alpha(X) = 0\}$, (see [6]).

A normal complex contact metric manifold whose complex contact structure is given by a global complex contact form, is called a complex Sasakian manifold.

For a complex Sasakian manifold we have the following formulas

$$g((\nabla_X G)Y, Z) = -2v(X)g(HGY, Z) - u(Y)g(X,Z) - v(Y)g(JX,Z) + u(Z)g(X,Y) + v(Z)g(JX,Y), \tag{1}$$

$$g((\nabla_X H)Y, Z) = -2u(X)g(HGY, Z) + u(Y)g(JX,Z) - v(Y)g(X,Z) + u(Z)g(X,JY) + v(Z)g(X,Y), \tag{2}$$

and

$$g((\nabla_X J)Y, Z) = -2u(X)g(HY,Z) + 2v(X)g(GY,Z). \tag{3}$$

In the following let us recall some notions and some basic results concerning the harmonic maps between Riemannian manifolds, as they are presented in [7] and in [2].

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M,g) and (N,h). Define the energy density function of f, $e(f) \in C^\infty(M)$, by

$$e(f) = \frac{1}{2} tr_g(f^*h(x)) = \frac{1}{2}\sum_{i=1}^{m} (f^*h)_x(e_i, e_i), \quad x \in M,$$

where $\{e_1, ..., e_m\}$ is an orthonormal basis for the tangent space $T_x M$ at $x \in M$. If M is compact we define the energy of f by $E(f) = \int_M e(f)\nu_g$, where ν_g is the volume form of (M, g).

Denote by ∇^M, ∇^N, the Levi-Civita connections on (M, g) and (N, h) respectively. Then, for a smooth map f between (M, g) and (N, h), we define the induced connection $\widetilde{\nabla}$ on the induced bundle $f^{-1}TN$ as follows, for $X \in \chi(M)$, $V \in \Gamma(f^{-1}TN)$, define $\widetilde{\nabla}_X V \in \Gamma(f^{-1}TN)$ by $\widetilde{\nabla}_X V = \nabla^N_{dfX} V$.

The second fundamental form α of f is defined by

$$\alpha(X, Y) = \widetilde{\nabla}_X dfY - df(\nabla^M_X Y),$$

for any $X, Y \in \chi(M)$.

The tension field $\tau(f)$ of f is defined by

$$\tau(f)_x = \sum_{i=1}^{m} \alpha(e_i, e_i)(x),$$

where $\{e_1, ..., e_m\}$ is an orthonormal basis for the tangent space $T_x M$ at $x \in M$.

The map $f : M \to N$ is called a harmonic map if $\tau(f) = 0$. Note that when M is compact, $\tau(f) = 0$ is the Euler-Lagrange equation associated to the functional energy.

The author would like to thank to the referee for many valuable suggestions and comments.

2. Harmonic maps between complex Sasakian manifolds

THEOREM 1. *Let $f : M \to N$ be a smooth map between two complex Sasakian manifolds,* $(M, J, G, H, u, v, U, V, g)$, $(N, J', G', H', u', v', U', V', h)$, *such that*

$$df = G'dfG + H'dfH. \tag{4}$$

Then f is a harmonic map.

Proof. First, using the equation (4) and the definitions of the complex Sasa-kian structures on manifolds M and N, one obtains that

$$\begin{aligned} dfJ &= G'dfGJ + H'dfHJ = G'dfH - H'dfG \\ &= J'(H'dfH + G'dfG) = J'df. \end{aligned} \tag{5}$$

From (3), one obtains, for a vector field $X \in \mathcal{H}$, $\mathcal{H}$ being the horizontal subbundle on M, that $(\nabla_X J)JX = 0$ and $(\nabla_{JX} J)X = 0$, where ∇ denotes the Levi-Civita connection on M. Hence, since ∇ is a torsion free connection,

$$\nabla_X X + \nabla_{JX} JX = J[JX, X], \tag{6}$$

for any $X \in \mathcal{H}$. Note that a similar formula can be obtained on the manifold N. From (4) we have that $dfX \in \mathcal{H}'$ for any $X \in \chi(M)$, where $\mathcal{H}'$ is the horizontal subbundle on N. Using this result, the equations (5) and (6), it follows that

$$\begin{aligned} \alpha(X,X)+\alpha(JX,JX) &= \nabla'_{dfX}dfX - df(\nabla_X X) \\ &\quad + \nabla'_{dfJX}dfJX - df(\nabla_{JX}JX) \\ &= J'df[JX,X] - dfJ[JX,X] = 0, \end{aligned} \tag{7}$$

for any $X \in \mathcal{H}$, where ∇' denotes the Levi-Civita connection on N and α denotes the second fundamental form of f.

From the definition of the complex Sasakian manifolds and from the definition of the Levi-Civita connection, we have, on M,

$$\nabla_U U = \nabla_V V = 0.$$

From (3) it follows that $\nabla_U V = -\nabla_U JU = -J\nabla_U U = 0$ and, in the same way, $\nabla_V U = 0$. Hence $[U,V] = \nabla_U V - \nabla_V U = 0$.

Then, since $dfU, dfV \in \mathcal{H}'$, using (5) and (6), one obtains

$$\begin{aligned} \alpha(U,U)+\alpha(V,V) &= \nabla'_{dfU}dfU + \nabla'_{dfV}dfV \\ &= J'[J'dfU, dfU] = J'df[U,V] = 0. \end{aligned} \tag{8}$$

Using (7) and (8), we conclude that, for a local orthonormal basis in M, $\{e_i, Ge_i, Je_i, JGe_i, U, V\}$, $i = \overline{1,m}$, where $dim\ M = 4m+2$, adapted to the complex Sasakian structure on M, we have

$$\begin{aligned} \tau(f) &= \alpha(e_i,e_i)+\alpha(Je_i,Je_i)+\alpha(Ge_i,Ge_i) \\ &\quad + \alpha(JGe_i,JGe_i)+\alpha(U,U)+\alpha(V,V) = 0. \end{aligned}$$

Thus, f is a harmonic map. □

In order to find another class of harmonic maps, we will prove, first, the following

PROPOSITION 1. *Let $f : M \to N$ be a smooth map between two complex Sasakian manifolds,* $(M, J, G, H, u, v, U, V, g)$, $(N, J', G', H', u', v', U', V', h)$, *such that $dfG = G'df$ or $dfH = H'df$. Then $dfU = a_1U' + b_1V'$ and $dfV = a_2U' + b_2V'$, where $a_i, b_i \in \mathbb{R}$ are constants for $i = 1, 2$.*

Proof. Let us consider the case $dfG = G'df$. Then, $G'dfU = 0$ and $G'dfV = 0$. Hence $dfU, dfV \in span\{U', V'\}$. That means $dfU = a_1U' + b_1V'$ and $dfV = a_2U' + b_2V'$, where $a_i, b_i : M \to \mathbb{R}$, for $i = 1, 2$.

Let us consider the 1-forms f^*u' and f^*v', on M, defined by $f^*u'(X) = u'(dfX)$ and by $f^*v'(X) = v'(dfX)$, for $X \in \chi(M)$. It is easy to see that

$$f^*u' = a_1u + a_2v \tag{9}$$

and

$$f^*v' = b_1 u + b_2 v. \tag{10}$$

By differentiating the equation (9), one obtains

$$d(f^*u') = f^*du' = da_1 \wedge u + da_2 \wedge v + a_1 du + a_2 dv.$$

By computing this equation in (U, X), $X \in \chi(M)$, we have

$$du'(dfU, dfX) = \mathcal{L}_U a_1 u(X) - da_1(X) + \mathcal{L}_U a_2 v(X),$$

where $\mathcal{L}$ denotes the Lie derivative. Since N is a complex Sasakian manifolds we have that $du'(X, Y) = h(X, GY)$. Thus

$$da_1 = \mathcal{L}_U a_1 u + \mathcal{L}_U a_2 v \tag{11}$$

It follows that

$$da_1 \wedge u = \mathcal{L}_U a_2 v \wedge u. \tag{12}$$

By differentiating the equation (12) and by computing the obtained expression in (U, V, X), with $X \in \chi(M)$, we have $d(\mathcal{L}_U a_2) = 0$, and that means $\mathcal{L}_U a_2$ is a constant. Note that in the same way it can be proved that $\mathcal{L}_U a_1$, $\mathcal{L}_V a_i$, $\mathcal{L}_U b_i$, $\mathcal{L}_V b_i$, with $i = 1, 2$, are constants. Taking account of this results, let us differentiate again the equation (12). It follows

$$da_1 \wedge du = \mathcal{L}_U a_2(v \wedge du - dv \wedge u),$$

which computed in (U, X, Y), with $X, Y \in \mathcal{H}$, gives

$$\mathcal{L}_U a_1 du(X, Y) = -\mathcal{L}_U a_2 dv(X, Y),$$

since, from (11), $da_1(X) = 0$, for any $X \in \mathcal{H}$, and since M is a complex Sasakian manifold. Using $du(X, Y) = g(X, GY)$ and $dv(X, Y) = g(X, HY)$, one obtains

$$\mathcal{L}_U a_1 g(X, GY) = -\mathcal{L}_U a_2 g(X, HY), \tag{13}$$

for any vector fields $X, Y \in \mathcal{H}$. Taking $Y = GX$ in this last expression, it follows that

$$\mathcal{L}_U a_1 g(X, X) = \mathcal{L}_U a_2 g(X, JX) = 0,$$

for any $X \in \mathcal{H}$. But this means $\mathcal{L}_U a_1 = 0$. If we take $Y = HX$ in (13) then we have $\mathcal{L}_U a_2 = 0$. Hence, from (11), $da_1 = 0$. By a similar computation we can prove that $da_2 = db_1 = db_2 = 0$, and then a_1, a_2, b_1, b_2 are constants.

In the case $dfH = H'df$, one obtains that $H'dfU = 0$ and $H'dfV = 0$ and it is easy to see that the conclusion of the proposition can be obtained just like above. □

Now we can state the following

THEOREM 2. *Let $f: M \to N$ be a smooth map between two complex Sasakian manifolds, $(M, J, G, H, u, v, U, V, g)$, $(N, J', G', H', u', v', U', V', h)$, such that $dfG = G'df$ or $dfH = H'df$. Then, f is a harmonic map.*

Proof. As in the proof of the previous proposition, it is sufficient to consider only the case $dfG = G'df$.

First, note that if $dfG = G'df$ then for any $X \in \mathcal{H}$, $dfX \in \mathcal{H}'$ and $dfJX = -G'dfHX$.

If $X, Y \in \mathcal{H}$ then, from (1), one obtains

$$(\nabla_X G)Y = g(X, Y)U + g(JX, Y)V.$$

Taking in this equation $Y = GX$, it follows

$$(\nabla_X G)GX = 0, \quad X \in \mathcal{H}.$$

Thus

$$\nabla_X X + \nabla_{GX} GX = G[GX, X], \quad X \in \mathcal{H}. \tag{14}$$

Obviously, a similar formula holds on N.

Let us consider on M a local orthonormal frame, $\{e_i, Ge_i, Je_i, JGe_i, U, V\}$ $i = \overline{1, m}$, where $dim\ M = 4m + 2$, adapted to the complex Sasakian structure on M. Using equation (14) and $dfG = G'df$, we have

$$\alpha(e_i, e_i) + \alpha(Ge_i, Ge_i) + \alpha(Je_i, Je_i) + \alpha(JGe_i, JGe_i) = 0. \tag{15}$$

As we have seen in the proof of Theorem 1, $\nabla_U U = \nabla_V V = 0$, $\nabla_U V = \nabla_V U$ and the similar equations hold on N. Then, using Proposition 1, one obtains

$$\alpha(U, U) + \alpha(V, V) = 0 \tag{16}$$

Thus $\tau(f) = 0$, from (15) and (16). Hence f is a harmonic map.

□

3. Maps on the complex Heisenberg group

Let $H_{\mathbb{C}}$ be the complex Heisenberg group, the closed subgroup of $GL(3, \mathbb{C})$, whose elements are given by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix}$$

where $z_1, z_2, z_3 \in \mathbb{C}$. Obviously, $H_{\mathbb{C}} \simeq \mathbb{C}^3$.

If L_B denotes left translation by B, $L_B^* dz_1 = dz_1$, $L_B^* dz_2 = dz_2$, $L_B^*(dz_3 - z_2 dz_1) = dz_3 - z_2 dz_1$. The vector fields $\frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3}$, $\frac{\partial}{\partial z_2}$, $\frac{\partial}{\partial z_3}$ are dual to the 1-forms

dz_1, dz_2, $dz_3 - z_2 dz_1$ and are left invariant vector fields. Moreover, relative to the coordinates $(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ the Hermitian metric (see [1])

$$g = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 1+\mid z_2 \mid^2 & 0 & -z_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\bar{z}_2 & 0 & 1 \\ 1+\mid z_2 \mid^2 & 0 & -\bar{z}_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -z_2 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

is a left invariant metric on $H_{\mathbb{C}}$, but is not a Kähler metric. The form $\theta = \frac{1}{2}(dz_3 - z_2 dz_1)$ is a complex contact structure on $H_{\mathbb{C}}$. So, the structure tensors may be taken globally. Let J be the standard almost complex structure on $\mathbb{C}^3$, $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$. Since θ is holomorphic, let $\theta = u - iv$, $v = u \circ J$. Also set $4\frac{\partial}{\partial z_3} = U + iV$; then, $u(X) = g(U, X)$ and $v(X) = g(V, X)$. In complex coordinates G and H are given by (see [1])

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{z}_2 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -iz_2 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i\bar{z}_2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $H_{\mathbb{C}}$ endowed with this structure, is a complex Sasakian manifold.

In the following, let us consider a smooth map $f : H_{\mathbb{C}} \to H_{\mathbb{C}}$. After a straightforward computation one obtains that $dfG = Gdf$ if and only if the complex components of f, denoted by w_1, w_2, w_3, verify the following equations.

$$\text{(17)} \qquad \begin{cases} \dfrac{\partial w_1}{\partial z_1} = \dfrac{\partial \overline{w}_2}{\partial \bar{z}_2}, \qquad \dfrac{\partial w_1}{\partial z_2} = -\dfrac{\partial \overline{w}_2}{\partial \bar{z}_1} \\[2ex] \dfrac{\partial w_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{w}_2}{\partial z_2}, \qquad \dfrac{\partial w_1}{\partial \bar{z}_2} = -\dfrac{\partial \overline{w}_2}{\partial z_1} \\[2ex] z_2\dfrac{\partial \overline{w}_2}{\partial z_1} = -\dfrac{\partial w_3}{\partial \bar{z}_2}, \quad z_2\dfrac{\partial \overline{w}_2}{\partial z_2} = \dfrac{\partial w_3}{\partial \bar{z}_1} + \bar{z}_2\dfrac{\partial w_3}{\partial \bar{z}_3} \\[2ex] z_2\frac{\partial \overline{w}_2}{\partial \bar{z}_1} = -\dfrac{\partial w_3}{\partial z_2}, \quad z_2\dfrac{\partial \overline{w}_2}{\partial \bar{z}_2} = \dfrac{\partial w_3}{\partial z_1} + z_2\dfrac{\partial w_3}{\partial z_3} \\[2ex] \dfrac{\partial w_i}{\partial z_3} = \dfrac{\partial w_i}{\partial \bar{z}_3} = 0, \quad i = 1, 2. \end{cases}$$

Such a map is harmonic by Theorem 2.

If, in addition, we assume that f is a holomorphic map, that is $\frac{\partial w_i}{\partial \bar{z}_\alpha} = 0$, for any

$i, \alpha = \overline{1,3}$, then, solving the equations 17, we have that $dfG = Gdf$ if and only if

$$\left\{ \begin{array}{l} w_1 = az_2 + bz_1 + c_1 \\ w_2 = -\overline{a}z_1 + \overline{b}z_2 + c_2 \\ w_3 = \frac{1}{2}az_2^2 + bz_3 + c_3 \end{array} \right. , \tag{18}$$

where a, b, c_i are some complex constants.

Next, one obtains that $dfH = Hdf$ if and only if w_1, w_2, w_3 verify the following equations.

$$\left\{ \begin{array}{ll} \dfrac{\partial w_1}{\partial z_1} = \dfrac{\partial \overline{w}_2}{\partial \overline{z}_2}, & \dfrac{\partial w_1}{\partial z_2} = -\dfrac{\partial \overline{w}_2}{\partial \overline{z}_1} \\ \dfrac{\partial w_1}{\partial \overline{z}_1} = -\dfrac{\partial \overline{w}_2}{\partial z_2}, & \dfrac{\partial w_1}{\partial \overline{z}_2} = \dfrac{\partial \overline{w}_2}{\partial z_1} \\ z_2 \dfrac{\partial \overline{w}_2}{\partial z_1} = \dfrac{\partial w_3}{\partial \overline{z}_2}, & -z_2 \dfrac{\partial \overline{w}_2}{\partial z_2} = \dfrac{\partial w_3}{\partial \overline{z}_1} + \overline{z}_2 \dfrac{\partial w_3}{\partial \overline{z}_3} \\ z_2 \frac{\partial \overline{w}_2}{\partial \overline{z}_1} = -\dfrac{\partial w_3}{\partial z_2}, & z_2 \dfrac{\partial \overline{w}_2}{\partial \overline{z}_2} = \dfrac{\partial w_3}{\partial z_1} + z_2 \dfrac{\partial w_3}{\partial z_3} \\ \dfrac{\partial w_i}{\partial z_3} = \dfrac{\partial w_i}{\partial \overline{z}_3} = 0, \quad i = 1, 2. & \end{array} \right. \tag{19}$$

It is well known that a holomorphic map between two Kähler manifolds is harmonic (see [7], p. 146). It is easy to see, from (17) and (19), that $dfG = Gdf$ and $dfH = Hdf$ implies $\frac{\partial w_i}{\partial \overline{z}_\alpha} = 0$, for any $i, \alpha = \overline{1,3}$. This means that f is a holomorphic map. Thus we have found a class of harmonic maps on the complex Heisenberg group, which are necessarily holomorphic maps.

4. Stability on complex Sasakian manifolds with constant GH-sectional curvature

In [4] the authors proved a theorem related to the stability of the identity map on a compact Sasakian manifold with constant φ-sectional curvature. We will prove a similar result related to the identity map on a complex Sasakian manifold with constant GH-sectional curvature.

Let (M, g) be compact Riemannian manifold and let $f : M \to N$ be a harmonic map. Consider a smooth variation $f_{s,t}$, with $s, t \in (-\epsilon, \epsilon)$ and $f_{0,0} = f$. The corresponding variation fields are denoted by Y and Z. Then the Hessian of a harmonic map f, H_f, is defined as follows

$$H_f(Y, Z) = \frac{\partial^2}{\partial s \partial t}(E(f_{s,t})) \mid_{(s,t)=(0,0)} .$$

The second variation formula is (see [7])

$$H_f(Y,Z) = \int_M h(J_f(Y), Z)\nu_g$$

where J_f is called the Jacobi operator and is a second order selfadjoint elliptic differential operator acting on the space $\Gamma(f^{-1}(TN))$, of the form

$$J_f(Y) = -\sum_{i=1}^{n}(\widetilde{\nabla}_{X_i}\widetilde{\nabla}_{X_i} - \widetilde{\nabla}_{\nabla_{X_i}X_i})Y - \sum_{i=1}^{n} R^N(Y, dfX_i)dfX_i,$$

for $Y \in \Gamma(f^{-1}(TN))$, and $\{X_1, ..., X_n\}$ a local orthonormal frame on TM. The operator $\overline{\Delta}_f$ defined by

$$\overline{\Delta}_f(Y) = -\sum_{i=1}^{n}(\widetilde{\nabla}_{X_i}\widetilde{\nabla}_{X_i} - \widetilde{\nabla}_{\nabla_{X_i}X_i})Y,$$

is called the rough Laplacian.

The index of a harmonic map $f : M \to N$ is the dimension of the largest subspace of $\Gamma(f^{-1}(TN))$ on which the Hessian H_f is negative definite. A harmonic map f is said to be stable if the index of f is zero and, otherwise, is said to be unstable.

Let $(M, J, G, H, u, v, U, V, g)$ be a compact $(4m+2)$-dimensional complex Sasakian manifold. For a unit vector $X \in \mathcal{H}_m$, the plane in T_mM spanned by X and $Y = aGX + bHX$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$ is called a GH-plan section and its sectional curvature the GH-sectional curvature of the plane section, (see [6]). In [6] the expression of the Ricci tensor for a complex contact metric manifold with constant GH-sectional curvature, c, was obtained. For a complex Sasakian manifold with constant GH-sectional curvature, c, this expression is

(20)

$$\begin{aligned} Ric(X,Y) = &\sum_{i=1}^{m}[g(R(e_i,X)Y,e_i) + g(R(Ge_i,X)Y,Ge_i) + g(R(Je_i,X)Y,Je_i) \\ &+ g(JGe_i,X)Y,JGe_i)] + g(R(U,X)Y,U) + g(R(V,X)Y,V) \\ = &[(m+2)c + 3m + 2]g(X,Y) \\ &+ [-(m+2)c + m - 2](u(X)u(Y) + v(X)v(Y)), \end{aligned}$$

where $\{e_i, Ge_i, Je_i, JGe_i, U, V\}$, $i = \overline{1,m}$, is a local orthonormal basis.

Let $1_M : M \to M$ be the identity map. Obviously $d1_MG = Gd1_M$. Using the second variational formula one obtains

$$H_{1_M}(Y,Y) = \int_M g(\overline{\Delta}Y, Y)\nu_g - \sum_{i=1}^{4m+2}\int_M g(R(Y,X_i)X_i, Y)\nu_g,$$

where $\{X_1, ..., X_{4m+2}\}$ is an orthonormal frame on TM.

Now we can continue just like in [4]. Let us recall the Weitzenböck formula for 1-forms, (see [2]). Let E be a vector bundle over a n-dimensional Riemannian manifold M. For any 1-form $\omega \in A^1(E)$ we have

$$\Delta_1\omega = \overline{\Delta}\omega - \rho(\omega),$$

where $\rho(\omega)(X) = \sum_{i=1}^{n} R(X, e_i)(\omega(e_i)) - \sum_{i=1}^{n} \omega(R(X, e_i)e_i)$, for any $X \in \chi(M)$. Here Δ_1 is the Laplacian of E-valued 1-forms and $\overline{\Delta}$ is the rough Laplacian of $A^1(E)$.

Using the Weitzenböck formula for $E = M \times \mathbb{R}$, we have

$$\Delta_1 Y = \overline{\Delta}Y + \sum_{i=1}^{4m+2} R(Y, X_i)X_i$$

and it follows that

$$H_{1_M}(Y, Y) = \int_M g(\Delta_1 Y, Y)\nu_g - 2\sum_{i=1}^{4m+2} \int_M g(R(Y, X_i)X_i, Y)\nu_g.$$

Let λ_1 be the first eigenvalue of the Laplacian Δ_g acting on $C^\infty(M)$, and ψ a non-constant eigenfunction such that $\Delta_g\psi = \lambda_1\psi$. Let $Y = grad\ \psi \neq 0$. Since $\Delta_1 d\psi = d\Delta_g\psi = \lambda_1 d\psi$, using (20), we have

$$H_{1_M}(Y, Y) = [\lambda_1 - 2(mc + 3m + 2c + 2)]\int_M g(Y, Y)\nu_g +$$

$$+2[(m + 2)c - m + 2]\int_M [(u(Y))^2 + (v(Y))^2]\nu_g.$$

We can state the following

THEOREM 3. *Let M be a compact $(4m + 2)$-dimensional complex Sasakian manifold with constant GH-sectional curvature, c, such that $c \leqslant \frac{m-2}{m+2}$. If the first eigenvalue of the Laplacian Δ_g acting on $C^\infty(M)$, λ_1, satisfies $\lambda_1 < 2(mc + 3m + 2c + 2)$, then the identity map 1_M is unstable.*

References

[1] BLAIR D.E, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser Boston 2002.

[2] EELLS J. AND LEMAIRE L., *Selected topics in harmonic maps*, Am. Math. Soc, Conf. Board Math. Sci. **50** (1983).

[3] FOREMAN B., *Complex contact manifolds and hyperkähler geometry*, Kōdai Math. J. **23** (2000), 12–26.

[4] GHERGHE C., IANUS S. AND PASTORE A.M, *CR-manifolds, harmonic maps and stability*, J.Geom., **71** (2001), 42–53.

[5] IANUS S. AND PASTORE A.M., *Harmonic maps on contact metric manifold*, Ann. Math. Blaise Pascal **2** 2 (1995), 43–53.

[6] KORKMAZ B., *Normality of complex contact manifolds*, Rocky Mountain J. Math. **30** (2000), 1343–1380.

[7] URAKAWA H., *Calculus of variation and harmonic maps*, Am. Math. Soc., Translations of Mathematical Monographs, Providence, Rhode Island 1993.

AMS Subject Classification: 58E20, 53C25.

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Lavoro pervenuto in redazione il 23.03.2005 e, in forma definitiva, il 04.11.2005.