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# EQUIVALENCE BETWEEN THE MAXIMAL IDEALS OF THE EXTENDED WEYL ALGEBRAS $\mathbb{C}[X, Y, \xi, \frac{1}{\xi}]\langle\partial_X, \partial_Y\rangle$ AND THOSE OF THE LOCALIZED WEYL ALGEBRA $\mathbb{C}[X, Y]_X\langle\partial_X, \partial_Y\rangle$.

**Abstract.** We show first the equivalence between the maximal ideals generated by $S = \partial_x + (1 + xy)\partial_y + y$ in the extended Weyl algebra $\mathbb{C}[x, y, \xi, \frac{1}{\xi}]\langle\partial_x, \partial_y\rangle$, where $\xi^n = x$ with $n \geq 2$, and those generated by $S_n = \partial_x + \left(\frac{1}{nx^{n-1}} + \frac{xy}{n}\right)\partial_y + \frac{y}{nx^{n-1}}$, with $n \geq 2$, in the localized Weyl algebra $\mathbb{C}[x, y]_x\langle\partial_x, \partial_y\rangle$.

In the second section, by proving that $S_n$ generates a maximal ideal in the localized Weyl algebra, we conclude that $S$ generates a maximal ideal in the extended Weyl algebras. Please note that the case when $n = 2$ is already proved by the direct verification in the previous paper [2].

## 1. Some maximal and principal ideals in $\mathbb{C}[x, y, \xi, \frac{1}{\xi}]\langle\partial_x, \partial_y\rangle$ and $\mathbb{C}[x, y]_x\langle\partial_x, \partial_y\rangle$.

Let $\xi^n = x$, where $n \geq 2$, and let $A = \mathbb{C}[x, y, \xi, \frac{1}{\xi}]\langle\partial_x, \partial_y\rangle$. The case when $n = 2$ is already treated in the previous paper. Let $B = \mathbb{C}[x, y]_x\langle\partial_x, \partial_y\rangle$. Here, we prove the following proposition.

PROPOSITION 1. *The operator $S = \partial_x + (1 + xy)\partial_y + y$ generates a maximal ideal in $\mathbb{C}[x, y, \xi, \frac{1}{\xi}]\langle\partial_x, \partial_y\rangle$ if and only if the operator $S_n = \partial_x + \left(\frac{1}{nx^{n-1}} + \frac{xy}{n}\right)\partial_y + \frac{y}{nx^{n-1}}$ generates a maximal ideal in $\mathbb{C}[x, y]_x\langle\partial_x, \partial_y\rangle$.*

*Proof.* Since $n\xi^{n-1}\partial_\xi = \partial_x$, and since $\xi$ is an invertible element in $A$, therefore,

$$A = \mathbb{C}[\xi, \frac{1}{\xi}, y]\langle\partial_\xi, \partial_y\rangle = \mathbb{C}[\xi, y]_\xi\langle\partial_\xi, \partial_y\rangle.$$

The operator $S = \partial_x + (1 + xy)\partial_y + y$ is written, in terms of $\xi$, as follows:

$$S = n\xi^{n-1}\partial_\xi + (1 + \xi^n y)\partial_y + y.$$

Thus,

$$\frac{S}{n\xi^{n-1}} = \partial_\xi + \left(\frac{1}{n\xi^{n-1}} + \frac{\xi y}{n}\right)\partial_y + \frac{y}{n\xi^{n-1}}.$$

Hence, if we put $S_n = S/n\xi^{n-1}$, then $AS$ is maximal in $A$ if and only if $BS_n$ is maximal in $B$. □

Hence, in the next section, we prove that $BS_n$ is maximal in $B$, and the above proposition guarantees that $AS$ is maximal in $A$.

## 2. Proof that $BS_n$ is maximal in $B$.

First, we prove the following theorem, that we apply to prove that $BS_n$ is maximal in $B$. Here, we put $\beta = \frac{1}{nx^{n-1}} + \frac{xy}{n}$.

THEOREM 1. *Let $S_n = \partial_x + \beta\partial_y + \frac{y}{nx^{n-1}}$. The following statements are equivalent.*

$P_1$) *$BS_n$ is maximal in $B$.*

$P_2$) *For $R \in \mathbb{C}[x, y]_x\langle\partial_y\rangle$, $R$ is not invertible if and only if $[S_n, R] \notin \mathbb{C}[x, y]_x R$.*

$P_3$) *For $p \in \mathbb{C}[x, y]_x \setminus \{0\}$, $p$ is not invertible if and only if $p$ does not satisfy the differential equation*

$$x^{n-1}(\partial_x + \beta\partial_y)p = pr$$

*for any $r \in \mathbb{C}[x, y]_x \setminus \{0\}$.*

REMARK 1. Since $x^{n-1}(\partial_x + \beta\partial_y)(\frac{1}{x^k}) = -kx^{n-2}\frac{1}{x^k}$, and since $n \geq 2$, the condition $r \in \mathbb{C}[x, y]_x \setminus \{0\}$ in $P_3$) may be replaced by $r \in \mathbb{C}[x, y] \setminus \{0\}$ without loss of generality.

*Proof.* We, first, prove that $P_2$ implies $P_1$. Let $R = \sum_{k=0}^{N} p_k\partial_y^k \in \mathbb{C}[x, y]_x\langle\partial_y\rangle$. Then $[S_n, R]$ is of the form

$$[S_n, R] = \sum_{k=0}^{N} q_k\partial_y^k,$$

where $q_k \in \mathbb{C}[x, y]_x$. If $R$ is invertible, then $R$ is of the form $ax^s$ with $s \in \mathbb{Z}$ and with $a \in \mathbb{C} \setminus \{0\}$. In any case, $BS_n + BR = B$ and $[S_n, R] \in \mathbb{C}[x, y]_x R$ are obvious. If $R$ is not invertible, then by hypothesis we have $p_N[S_n, R] - q_N R \neq 0$, and we obtain that $\deg_{\partial_y}(p_N[S_n, R] - q_N R) \leq N - 1$. Hence $BS_n + BR$ contains a non-trivial element $p(x, y) \in \mathbb{C}[x, y]_x$. By multiplying a suitable $x^m$, we have $x^m p \in \mathbb{C}[x, y]$.

Let $x^m p = \sum_{k=0}^{L} r_k(x)y^k$, where $r_k \in \mathbb{C}[x]$, and $r_L(x) \neq 0$. If $L = 0$, then $BS_n + BR$ contains a non-trivial polynomial only in $x$, and $BS_n + BR = B$. If $L \geq 1$, then by induction, it is enough to show that $BS_n + BR$ contains a polynomial of degree less than $L$ in the variable y, with $\beta = \frac{1}{nx^{n-1}} + \frac{xy}{n}$. Now

$$[S_n, x^m p] = \left(r_L' + Lr_L\frac{x}{n}\right)y^L + \left(r_{L-1}' + \frac{1}{nx^{n-1}}Lr_L\right)y^{L-1} + \cdots$$

Since $r_L' + Lr_L\frac{x}{n} \neq 0$, we consider $\tilde{p} = \left(r_L' + Lr_L\frac{x}{n}\right)x^m p - [S_n, x^m p]$. By assumption, $\tilde{p} \neq 0$. and $\deg_y \tilde{p} \leq L - 1$.

We now prove that $P_3$ implies $P_2$. For all $R \in \mathbb{C}[x, y]_x\langle\partial_y\rangle$, we observe that $\deg_{\partial_y}[S_n, R] = \deg_{\partial_y} R$. If $\deg_{\partial_y} R = k$ and if the coefficient of $\partial_y^k$ in $R$ is $p$, then $p \neq 0$, and the coefficient of $\partial_y^k$ in $[S_n, R]$ is $p_x + \beta p_y - k\beta_y p$, where $k \geq 0$.

If $p$ is not invertible, then, directly by $P_3$, we conclude that $[S_n, R] \notin \mathbb{C}[x, y]_x R$ because $p_x + \beta p_y - k\beta_y p \neq pr$ for any $r \in \mathbb{C}[x, y]_x \setminus \{0\}$.

If $p$ is invertible, and if $k = 0$, then $R = p = ax^s$ with $a \in \mathbb{C} \setminus \{0\}$, and with $s \in \mathbb{Z}$, hence $[S_n, R] \in \mathbb{C}[x, y]_x R$ is obvious. If $p$ is invertible and if $k \geq 1$, then we let $q$ be the coefficient of $\partial_y^{k-1}$. If we consider the element $\ell = p[S_n, R] - (p_x + \beta p_y - k\beta_y p)R$, then the coefficient of $\partial_y^k$ in $\ell$ is zero, therefore, if $[S_n, R]$ were in $\mathbb{C}[x, y]_x R$, then $\ell$ would be zero. The coefficient of $\partial_y^{k-1}$ in $\ell$ is given by

$$p\left(q_x + \beta q_y - \frac{x}{n}q - k\frac{1}{nx^{n-1}}p\right) - q\left(p_x + \beta p_y - k\frac{x}{n}p\right)$$
$$= (pq_x - qp_x) + \beta(pq_y - qp_y) - pq\frac{x}{n}(k - 1 - k) - kp^2\frac{1}{nx^{n-1}}$$

and simple calculations show that this is equal to

$$kp^2\left(\partial_x\left(\frac{q}{p}\right) + \beta\partial_y\left(\frac{q}{p}\right) + \frac{x}{n}\left(\frac{q}{p}\right) - \frac{1}{nx^{n-1}}\right).$$

This never vanishes by Corollary 1 below. Hence we conclude that $[S_n, R] \notin \mathbb{C}[x, y]_x R$.

It remains to show that $P_1$ implies $P_3$. Although the proof is essentially the same as that of Theorem 2.2 in [1], we write it again here in order to show the modifications needed.

If $BS_n$ is maximal, and if there is a non-zero solution $p$ of the differential equation, then we have $[S_n, p] = rp$ for some $r \in \mathbb{C}[x, y]_x$, the element $p$ can be considered as an element of $\mathbb{C}[x, y]_x\langle\partial_y\rangle$, and there should exist $\lambda$ and $\mu$ in $B$ such that

$$\lambda S_n + \mu p = 1.$$

If $\deg_{\partial_x} \lambda = m$, then $\deg_{\partial_x} = m + 1$, and we have

$$\sum_{k=0}^{m} C_k S_n^{k+1} + \sum_{k=0}^{m+1} D_k S_n^k p = 1$$

for some $C_k,\ D_k \in \mathbb{C}[x, y]_x\langle\partial_y\rangle$. Hence,

$$[x, \lambda S_n + \mu p] = -\sum_{k=0}^{m}(k+1)C_k S_n^k - \sum_{k=1}^{m+1} kD_k S_n^{k-1} p = 0.$$

Proceeding in this way $m$ more times, we obtain that

$$C_m + D_{m+1}p = 0.$$

Hence, we have

$$1 = \lambda S_n + \mu p = \sum_{k=0}^{m-1} C_k S_n^{k+1} + \sum_{k=0}^{m} D_k S_n^k p - D_{m+1}[p, S_n^{m+1}].$$

Since $[p, S_n^{m+1}]$ is of the form

$$\sum_{k=0}^{m} E_k S_n^k p,$$

where $E_k \in \mathbb{C}[x, y]_x \langle \partial_y \rangle$, we may write

$$1 = \sum_{k=0}^{m-1} C_k S_n^{k+1} + \sum_{k=0}^{m} (D_k + D_{m+1} E_k) S_n^k p.$$

Continuing in this way, we obtain

$$C_0 S_n + F_0 p + F_1 S_n p = 1,$$

for some $F_0$ and $F_1$ in $\mathbb{C}[x, y]_x \langle \partial_y \rangle$. Thus, $C_0 S_n + F_0 p + F_1(p S_n + r p) = 1$, and

$$(C_0 + F_1 p) S_n + (F_0 + F_1 r) p = 1$$

Here, only $S_n$ contains $\partial_x$, hence $C_0 + F_1 p = 0$. Therefore, $p$ is an invertible element.

Conversely, if $p$ is an invertible element, then $p$ is of the form

$$p = ax^s, \text{ where } s \in \mathbb{Z} \text{ and } a \in \mathbb{C} \setminus \{0\}.$$

Then, $x^{n-1}(\partial_x + \beta \partial_y) a x^s = s a x^{n-1+s-1} = r a x^s$ follows with $r = s x^{n-2}$. □

Hence, in order to show that $BS_n$ is maximal in $B$, we prove the following.

THEOREM 2. *Suppose that $p \in \mathbb{C}[x, y]_x \setminus \{0\}$. Then $p$ is not an invertible element in $\mathbb{C}[x, y]_x \setminus \{0\}$ if and only if $p$ does not satisfy the equation*

$$x^{n-1}(\partial_x + \beta \partial_y) p = pr$$

*for any $r \in \mathbb{C}[x, y] \setminus \{0\}$.*

*Proof.* Here, we adopt the notation $L_n(p) = (\partial_x + \beta \partial_y) p$. If $p$ is invertible, then $p = ax^s$, and $x^{n-1} L_n(p) = pr$ is obvious with $r = s x^{n-2}$. If $p$ is not invertible, we utilize the following equalities with $i \geq 1$ and $j \geq 1$:

$$L_n(x^i y^j) = i x^{i-1} y^j + \frac{j}{n} x^{i+1-n} y^{j-1} + j x^{i+1} y^j$$

$$L_n(y^j) = \frac{j}{n} x^{1-n} y^{j-1} + j x y^j,$$

$$L_n(x^i) = i x^{i-1}$$

and

$$L_n(x^{-i} y^j) = -i x^{-i-1} y^j + \frac{j}{n} x^{-i+1-n} y^{j-1} + j x^{-i+1} y^j.$$

If there were $p$ such that $L_n(p) = p\frac{r}{x^{n-1}}$, then let $M = \deg_y p$. By the third equality, we see that $M > 0$. Moreover, the first equality tells us that $\frac{r}{x^{n-1}} = Mx + b$ for some constant $b$.

The second equality shows that $p$ also depends on $x$ or $x^{-1}$. Since $L_n(p)$ depends on $\frac{1}{x}$, so does $p$. Let $N = \deg_{\frac{1}{x}} p$. Then $\deg_{\frac{1}{x}} p\frac{r}{x^{n-1}} = N - 1$, but $\deg_{\frac{1}{x}} L_n(p) = N + 1$ by the forth equality. Hence, we obtain a contradiction. □

The following corollary of Theorem 2 is used in the proof of Theorem 1.

COROLLARY 1.

$$\left(\partial_x + \beta\partial_y + \frac{x}{n}\right)\frac{q}{p} \neq \frac{1}{nx^{n-1}}$$

*for all* $p \in \mathbb{C}[x,y]_x \setminus \{0\}$ *and* $q \in \mathbb{C}[x,y]_x$.

*Proof.* Note that, without loss of generality, we may suppose that $p$ and $q$ are in $\mathbb{C}[x,y]$, and that $p$ and $q$ are mutually prime.

Let $\beta = \frac{1}{nx^{n-1}} + \frac{xy}{n}$. If there were such $p$ and $q$, then we would have

$$\frac{p(q_x + \beta q_y) - q(p_x + \beta p_y)}{p^2} + \frac{xpq}{np^2} = \frac{1}{nx^{n-1}}.$$

Hence,

$$nx^{n-1}pq_x + p(1 + x^n y)q_y - nx^{n-1}qp_x - q(1 + x^n y)p_y + x^n pq = p^2.$$

Thus,

$$p(nx^{n-1}q_x + (1 + x^n y)q_y - p) - q(nx^{n-1}p_x + (1 + x^n y)p_y + x^n p) = 0.$$

Since $p$ and $q$ are mutually prime, there exists some $r \in \mathbb{C}[x,y]$ such that

$$nx^{n-1}p_x + (1 + x^n y)p_y + x^n p = pr.$$

Hence,

$$x^{n-1}(p_x + \beta p_y) = p\frac{r - x^n}{n},$$

which contradicts Theorem2. □

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