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DYNAMICAL SYSTEMS: REGULARITY AND CHAOS

1. Introduction

In very general terms, we call DYNAMICAL any kind of “system” which evolves in time, starting from an initial time t_0, and whose state at any later time $t > t_0$ can be explicitly and uniquely determined from the assumed knowledge of its initial state at $t = t_0$†. One of the major goals of the theory of dynamical systems it to understand how the evolution of any such system is determined by its initial state, and, possibly, by the values of various parameters that enter its description; more specifically, to determine whether the evolution of a given system is *regular*, or *chaotic*, in a sense we shall try to describe.

We assume that the state of the system at any time $t \geq t_0$ can be described by a point x in some space $\mathcal{X}$, and that there is a functional dependence of x on t, the initial state x_0, and, possibly, some parameters $\lambda_1, \ldots, \lambda_n$. Thus, we adopt the provisional notation

$$x = x(t, x_0; \lambda)\,, \qquad \lambda := (\lambda_1, \ldots, \lambda_n)\,, \tag{1}$$

to denote this point. $\mathcal{X}$ is called the PHASE SPACE of the system, and we distinguish between *finite* and *infinite dimensional* dynamical systems, according to whether the dimension of $\mathcal{X}$ is finite or not. We also differentiate between *continuous* and *discrete* systems, according to whether the “time” variable runs over sets of the type $\mathbb{R}_{t\geq t_0}$ or $\mathbb{N}_{n\geq n_0}$.

Typical examples of dynamical systems are provided by mathematical models of physical systems that evolve in time, such as a pendulum or an electric oscillator: under certain conditions, we say that the differential equations governing the evolution of the system “generate” a corresponding (finite dimensional, continuous) dynamical system. One important example is the one-dimensional LOGISTIC equation

$$x' = \lambda x(1 - x)\,, \qquad \lambda > 0\,, \tag{2}$$

which models the growth of a population subject to mutual inhibiting interactions. In this case, $\mathcal{X} = \mathbb{R}$, the state of the corresponding system also depends on the parameter λ, and it is immediate to verify that (1) reads

$$x(t, x_0; \lambda) = \frac{x_0\,\mathrm{e}^{\lambda t}}{x_0(\mathrm{e}^{\lambda t} - 1) + 1}\,.$$

*“LEZIONE LAGRANGIANA” given on May 16th, 2006.

†More precisely, these systems are called DETERMINISTIC. One can also consider stochastic systems.

Often, one can approximate the differential equations by suitable recursive sequences, which are examples of discrete dynamical systems. For example, in Section 5 we shall examine in some detail the recursive sequence

$$x_{n+1} = \lambda x_n (1 - x_n) \, , \tag{3}$$

which is known as the "discrete logistic equation"; (3) is, evidently, the difference equation version of (2).

Examples of infinite dimensional dynamical systems are those generated by the classical heat and wave equations, or, more famously, by the Navier-Stokes equations of fluid dynamics. However, the theory is not restricted to models from physics; in fact, its rising popularity is, in large part, due to its applicability to many other applied fields, such as chemistry, biology, medicine, atmospheric sciences, and engineering. Related examples of dynamical systems could be: the changes in the density of a population in a certain environment, such as the immigration patterns in a country; the growth of bacteria in an infected organism; weather patterns in a certain region, such as the formation of clouds or the inset of vorticity in tornadoes or in the vapor trail in the wake of an airplane, the shape and propagation of flames in fires. New applications are attempted even in social and economic fields, with such examples as the spreading of a rumor in a group of people, or the variation of the prices of certain commodities or the values of a certain type of stock in the market.

This paper is organized as follows: In Sections 2 to 4 we give a brief historical outline of the origins of the theory of dynamical systems; in Sections 5 and 6 we present two very simple, and very famous, examples of one-dimensional discrete dynamical systems; in Section 7 we report some selected results on a special type of dynamical systems, called SEMIFLOWS. Finally, in Section 8 we conclude with an analysis of the dynamical system generated by the so-called LORENZ equations.

2. Poincaré and the N-body Problem.

The first ideas in the theory of dynamical systems are usually attributed to H. POINCARÉ, who, in his investigations on the three-body problem in celestial mechanics, (see [26], [27], as well as [4]) realized that what we would now call the dynamical system generated by the three-body problem, as modelled by Newton's laws of gravitation, is quite sensitive to the initial data, and could, therefore, exhibit chaotic behavior. LAGRANGE himself had already investigated this problem, together with other, related problems in astronomy. His contributions include results on the problem of the libration of the moon*, studies on the three-body problem and the perturbation of the orbits of comets caused by planets†, as well as further studies on the stability of the planetary orbits in the solar system, conducted around 1776 in Berlin (see [32]).

*This is a slightly oscillating motion of the moon, which allows more than 50% of its surface to be visible from the earth. Lagrange's original paper appeared in the *Mélanges de Philosophie et Mathématique de la Societé Royale de Turin*, t. III, 1766.

†These results earned Lagrange two prizes of the Académie des Sciences de Paris, respectively in 1772 (shared with Euler), and in 1780.

While the two body problem (e.g., motion of a single planet around the sun) was completely solved by Newton, the three-body problem (e.g., sun, earth and moon) is generally intractable. The differential system which describes the motion of N point masses, moving in accord with Newton's laws of gravitation, is conservative; one is interested in solutions that are regular, in the sense that the corresponding orbits do not collide, nor escape. In their quest for such solutions, Lagrange ([12]) and LAPLACE ([14]) realized that the mutual perturbations of the motions of the bodies described by Newton's laws are controlled by small parameters; hence, solutions could be attempted by series expansion methods. However, Poincaré found that the convergence of such series needs not be uniform, with the consequence that the stability of Lagrange and Laplace's solutions is not guaranteed. From this, Poincaré concluded that the three-body system is quite sensitive to the effects of these perturbations. In his own words ([28]; italics in the original):

> "Si nous connaissions exactement les lois de la nature et la situation de l'univers à l'instant initial, nous pourrions prédire exactement la situation de ce même univers à un instant ultérieur. Mais lors même que les lois naturelles n'auraient plus de secret pour nous, nous ne pourrions connaître la situation initiale qu'*approximativement*. Si cela nous permet de prévoir la situation ultérieure *avec la même approximation*, c'est tout ce qu'il nous faut, nous disons que le phénomène a été prévu, qu'il est régi par des lois; mais il n'en est pas toujours ainsi, il peut arriver que de petites différences dans les conditions initiales en engendrent des très grandes dans les phénomènes finaux; un petite erreur sur les premières produirait une erreur énorme sur les derniers. La prédiction devient impossible, et nous avons le phénomène fortuit."

The three-body problem[‡] already highlights three of the major features of the theory of dynamical systems: namely, that, while Newton's differential laws are completely *deterministic*, they are extremely *sensitive to their initial values*; and, yet, the motion of the planets seems to be settled into a certain kind of *asymptotic stability*[§]. In other words: Each orbit is uniquely determined by its initial value; Orbits starting arbitrarily close may differ by some large amount at any later time; Even so, there seems to be an upper bound to the distances of any two orbits starting sufficiently close.

3. Regular and Chaotic Dynamics.

Perhaps not coincidentally, Poincaré's insight came at about the same time when the new scientific and technological discoveries following the industrial revolution confirmed the unpredictability of the behavior of many complex systems. In the XIX and the first half of the XX century, scientists and engineers recognized that many systems

[‡]For extensive details, see e.g. Moser, [24].

[§]Although, in the solar system, it is *not* known whether the orbit of some planet may escape off for ever at some future time.

might behave in an apparently random way; but it was generally believed that the observed unpredictability in such systems should be produced by random external factors. Most researchers would still subscribe to the notion that an approximate knowledge of a system's initial conditions should allow them to calculate the approximate future behavior of the system. This kind of deterministic belief was famously described by Laplace ([15]):

> "Nous devons envisager l'état présent de l'univers, comme l'effet de son état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné, connaîtrait toutes les forces dont la nature est animée, et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux."

In a way, one can summarize chaos theory, by saying that Laplace's deterministic assumption very often does not hold: on the contrary, most nonlinear systems have a complex structure, and even small variations on the initial values and/or the parameters of the systems cause effects that appear completely unpredictable. In more formal

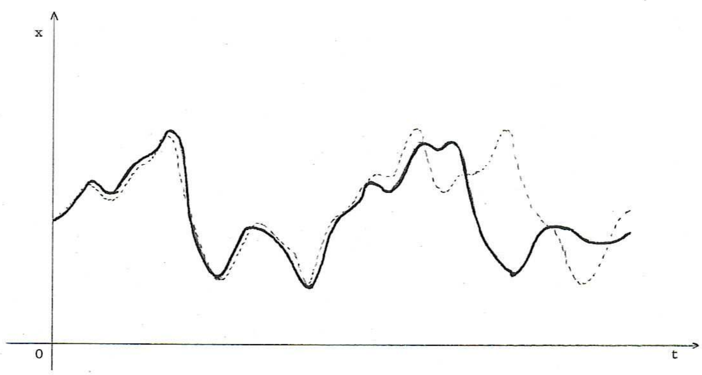


Figure 1: Propagation of a small error in the initial data.

terms, we note that Poincaré's remarks concern, in effect, the question of the asymptotic well-posedness (in the sense of the continuity with respect to the initial values) of a Cauchy problem. We can illustrate this by means of the two simple ODEs

$$x' = -x\,, \tag{4}$$

$$x' = x \tag{5}$$

The corresponding initial value problems are both well-posed on *compact* intervals; however, only the first is asymptotically well-posed. In other words, for (4), the effect of any difference in the initial values, no matter how large, becomes negligible after sufficient time; in contrast, for (5) the effect of any difference in the initial values, no matter how small, can become arbitrarily (in fact, exponentially) amplified as time increases. Thus, (5) is sensitive to its initial data, while (4) is not. We formalize these notions in the following provisional definition:

DEFINITION 1. *Let the state of a system be defined by a function* $(t, x_0) \mapsto x(t, x_0)$, *with values in a Banach space* $\mathcal{X}$, *as in* (1). *We say that the system is:*

i) REGULAR, *if it is stable in the sense of Lyapounov; that is, if for all* $x_0 \in \mathcal{X}$ *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that for all* $x_1 \in \mathcal{X}$, *and all* $t \geq 0$,

$$\|x(t, x_0) - x(t, x_1)\|_{\mathcal{X}} \leq \varepsilon \quad \text{if} \quad \|x_0 - x_1\|_{\mathcal{X}} \leq \delta \,. \tag{6}$$

ii) SENSITIVE TO ITS INITIAL VALUES, *or* CHAOTIC, *if there is* $R > 0$ *such that for all* $x_0 \in \mathcal{X}$ *and* $\varepsilon > 0$, *there are* $x_1 \in \mathcal{X}$ *and* $\bar{t} > 0$ *such that*

$$\|x(\bar{t}, x_0) - x(\bar{t}, x_1)\|_{\mathcal{X}} \geq R \quad \text{even if} \quad \|x_0 - x_1\|_{\mathcal{X}} \leq \varepsilon \,. \tag{7}$$

We remark that this definition is somewhat arbitrary, and the terms “regular” and “chaotic” are certainly not universal; indeed there are many definitions of regularity and chaos in the literature, with various degrees of mathematical rigor. On the other hand, many systems experience a complicated behavior only for a short time (called TRANSIENT), and then settle into a regular behavior. Thus, more generally, regular systems are also stable with respect to transient perturbations. An example is the set-up of a physics experiment in a laboratory, which is temporarily disturbed by the vibrations of a passing vehicle: it is evidently essential to be in a position to know whether or not the experiment will be affected by this kind of transient perturbations.

4. Reinterpreting Chaos.

Poincaré’s ideas went essentially ignored for about sixty years, even if it was generally known that simple deterministic models, such as the double pendulum or the van der Pol oscillator¶ can exhibit what appears as a totally random, or chaotic behavior. The picture changed, and the modern theory of dynamical systems was born, in the early 1960s, when E. LORENZ, an atmospheric scientist at the M.I.T., examined a system of nonlinear ODEs in $\mathbb{R}^3$, which he took as an extremely simplified approximation of the Boussinesq equations modelling the convective motion of a stratified bidimensional fluid heated by convection from below, such as air heated by the earth ([17]). The system reads

$$\begin{cases} x' = -10(x-y)\,, \\ y' = rx - y - xz\,, \\ z' = -\frac{8}{3}z + xy\,, \end{cases} \tag{8}$$

¶See e.g. the very informative simulations by Kanamaru and Thompson, [11].

and it is observed that the behavior of the orbits (that is, its dependence on the initial values) depends heavily on the values of the parameter $r > 0$, known as the Rayleigh number. In particular, for $r = 28$, Lorenz found, by numerical integration, that the system appears to be chaotic, in the sense of definition 1; and yet, while its orbits appear to not converge to any equilibria or periodic orbits, they behave as if they were almost periodic. More specifically, all orbits, in spite of their seemingly unpredictable behavior, do appear to settle in a somewhat regular, oscillating pattern, as if they were attracted to some set, more complicated than a stationary point or a periodic orbit (see Figure 2). Indeed, we can now prove that such set does exist; naturally enough, we call this set an "attractor" (see Section 8).

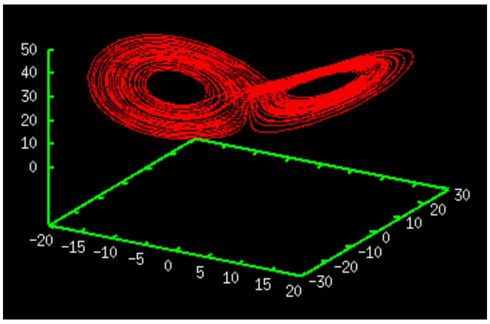


Figure 2: The Lorenz attractor.

Lorenz' paper appeared in a relatively obscure journal of atmospheric sciences, and received little attention, until it was rediscovered, in the 1970s, by a group of mathematicians who had started a rigorous study of the global behavior of dynamical systems. Among these, a precursory role was played by S. SMALE, who studied iterated maps from a topological point of view (he had been awarded the Fields medal in 1966 for his results in topology). Among his major contributions, the most famous may well be his description of the so-called "horseshoe" map ([33]). The introduction of "chaos" ‖ as a full-fledged mathematical term, and the corresponding establishment of the theory of chaotic dynamical systems as a mainstream area of research, is traditionally ascribed to J. YORKE, who in [16] gave one of the first explicit descriptions of chaotic behavior in one-dimensional systems. Almost concurrently, extensive numerical investigations were conducted on simple iterated maps, such as the logistic sequence (3) (see e.g. May, [21]**).

The realization that even quite simple discrete dynamical systems could exhibit an apparently unpredictable behavior caused a dramatic rethinking of the the relationships between chance, random causes and effects, and chaos and unpredictability (see e.g. Ruelle, [30]). For example, researchers in epidemiology began asking questions

‖ Apparently, the first recorded use of the word "chaos" in the western canon is in Hesiod's *Theogony* (VIII century BCE), in the verse I, 116: $H\tau o\iota\ \mu\varepsilon\nu\ \pi\rho\omega\tau\iota\sigma\tau\alpha\ X\alpha o\varsigma\ \gamma\varepsilon\nu\varepsilon\tau'$.

**A according to Gleick, [7], May's paper, which appeared in the wide circulation *Nature*, "made chaos popular".

such as to what extent the relatively regular evolution of a disease would be affected by a sudden perturbation, such as an inoculation campaign (would this have a transient effect, or would the changes be more dramatic?); likewise, economists would reconsider how the prices of some commodities are influenced by specific external events (such as the current price of oil). The traditional belief that changes of this kind were mostly random in nature was gradually abandoned, in favor of a reinterpretation of the observed and measured data from the point of view of chaos theory. A similar change in attitude took place in applied physics, where phenomena such as turbulence and phase transitions could be better explained in terms of fluids undergoing the transition from a smooth regime into a chaotic one, or in engineering, where, in order to prevent potentially disastrous catastrophic breakdowns of a system (such as the motion of a robot's arm), industrial designers started to routinely analyze the possibility that the dynamic response of the system may be chaotic. Even in medicine, it is now common to investigate possibilities such as whether the rhythmic vibrations of the heart are a regular or a chaotic system.

The notion that complex systems may be chaotic has, by now, entered the popular culture, with such references as the so-called "butterfly effect" [††], or the popular movies *Sliding Doors* (1998, with Gwynieth Paltrow) and *Happenstance* (2000, with Audrey Tautou). An infamous example is provided by the US presidential elections in 2000: if 600 Republican voters in Florida (out of more than five million) had not gone to the polls, or if 300 had voted differently, the world would probably not be in its present state of chaos. Unfortunately, many misconceptions abound in the popular views of chaos, due to a widespread confusion between chaos, chance and instability (chaotic systems are deterministic, and can be stable, in the sense that their irregularity persists under small perturbations, as Lorenz system illustrates). Other grave misunderstandings arise from the popular identification of chaos as the consequence of nonlinearity, specially among journalists, social scientists, and behavioral observers (see e.g. the collection of absurdities in Hayles, [9], or the examples reported in Rondoni, [29]). On the other hand, the loss of information inherent to chaotic systems can be used to more beneficial effects: one example is in cryptography, in the techniques used to encrypt a credit card number during an electronic transaction. In essence[†] , the process can be compared to the shuffling of a deck of cards, with a number of cards so large (of the order of one hundred digits) that the exact location of the "card" initially chosen by the customer is quickly lost (but can be recovered, resorting to remarkable non-trivial results of number theory).

5. Two Chaotic Sequences.

In this section we present two examples of chaotic dynamical systems, generated by iterated sequences $x_{n+1} = f(x_n)$, with f mapping the unit interval [0, 1] into itself.

[††]"A butterfly stirring the air today in Beijing can transform the storm systems next month in New York"; as reported in Gleick, [7].

[†]This example is taken from Du Sautoy, [5, ch. 10]

DEFINITION 2. *Let $(x_n)_{n\geq 0}$ be an iterated sequence defined by a function $f : \mathbb{R} \to \mathbb{R}$, that is, $x_{n+1} = f(x_n)$. For $k \in \mathbb{N}_{>0}$, denote by $f^{(k)}$ the k-th iterate of f. Let $m \in \mathbb{N}_{>0}$. The sequence is* PERIODIC OF ORDER *m, or, more briefly, m-*PERIODIC, *if for all $n \in \mathbb{N}$,*

(9) $$f^{(m)}(x_n) = x_n\,, \qquad but \quad f^{(k)}(x_n) \neq x_n \quad if \quad k < m\,.$$

5.1. Bernoulli's Sequence.

The so-called "Bernoulli's sequence" is the discrete, one-dimensional dynamical system defined by recursive sequence $x_{n+1} = f(x_n)$, with $f : [0,1] \to [0,1]$ defined by

(10) $$f(x) := 2x - \lfloor 2x \rfloor\,,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Note that f is not continuous at $x = \frac{1}{2}$.

It is easy to study the behavior of Bernoulli's sequences: $x = 0$ is the only stationary point of f, and if $x_0 = \frac{1}{2}$ or $x_0 = 1$, then $x_1 = 0$, so $x_n = 0$ for all $n \geq 1$. Consider then any two initial values x_0, y_0, in the same half interval $]0, \frac{1}{2}[$ or $]\frac{1}{2}, 1[$. As long as the corresponding successors x_n and y_n remain in the same half interval, we see that

(11) $$|x_{n+1} - y_{n+1}| = 2^{n+1}|x_0 - y_0|\,.$$

However, (11) shows that the distance between orbits grows exponentially; as a consequence, there is n_0 such that the orbits "must separate" at n_0, no matter how close they were initially (see Figure 3). In fact, let $\varepsilon := |x_0 - y_0| \ll \frac{1}{2}$: by (11), we have

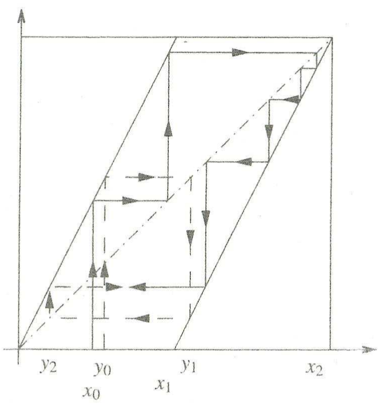


Figure 3: Bernoulli sequences.

$|y_n - x_n| > \frac{1}{2}$ as soon as $n > n_0 := \lfloor \log_2 \frac{1}{\varepsilon} \rfloor + 2$, and after this point the difference $y_{n+1} - x_{n+1}$ is no longer controllable. In fact, the evolution of Bernoulli's sequence is chaotic, in the sense of definition 1 (see e.g. [1]). One way to interpret this situation is that all information deriving from the knowledge of x_0 is eventually lost. For example, if x_0 represents the "true" initial value in an experiment, and $x_0 \pm \varepsilon$ is its actual measurement, after a number of steps equal to $\lfloor \log_2 \frac{1}{\varepsilon} \rfloor + 2$ no meaningful control of the error between the true and the approximated initial values is maintained.

This loss of control can be described explicitly. Indeed, let x_0 be represented in the binary system by the series

$$x_0 = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}, \qquad \alpha_n \in \{0, 1\}.$$

Then

$$\begin{aligned} x_1 = 2x_0 - \lfloor 2x_0 \rfloor &= \sum_{n=1}^{\infty} \frac{\alpha_n}{2^{n-1}} - \left\lfloor \sum_{n=1}^{\infty} \frac{\alpha_n}{2^{n-1}} \right\rfloor \\ &= \alpha_1 + \sum_{n=2}^{\infty} \frac{\alpha_n}{2^{n-1}} - \alpha_1 = \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{2^n}. \end{aligned}$$

This means that Bernoulli's map moves the digits of the fractional part of each number x_n one position to the left, and subtracts the unit that may so result. For example, if

$$x_0 = 0.1101001 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{128},$$

then

$$x_1 = (1 + \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{64}) - 1 = 0.101001.$$

Now, in any numerical approximation, the initial value x_0 is known only up to a finite number of digits of its fractional part. If m is this number, after m iterations of Bernoulli's map we obtain $x_m = 0$; that is, we reach the fixed point of the map. Thus, all information from x_0 is lost in a finite number of steps.

5.2. The Logistic Equation.

We now discuss the behavior of the discrete logistic sequence (3), i.e.

$$x_{n+1} = \lambda x_n (1 - x_n) =: f_\lambda(x_n), \tag{12}$$

for $0 \leq \lambda \leq 4$, i.e. when the function f_λ maps the interval $[0, 1]$ into itself. When $x_0 = 0$ or $x_0 = 1$, or $\lambda = 0$, the sequence is constant for $n \geq 1$: $x_n = 0$ for all $n \geq 1$. Otherwise, its behavior depends heavily on the value of λ, being regular for $0 \leq \lambda \leq 3$, and chaotic for $3 < \lambda \leq 4$. Indeed, given any initial value $x_0 \in]0, 1[$, elementary calculus shows that, for $\lambda \in]0, 3]$, the corresponding sequence evolves as follows (see Figure 4):

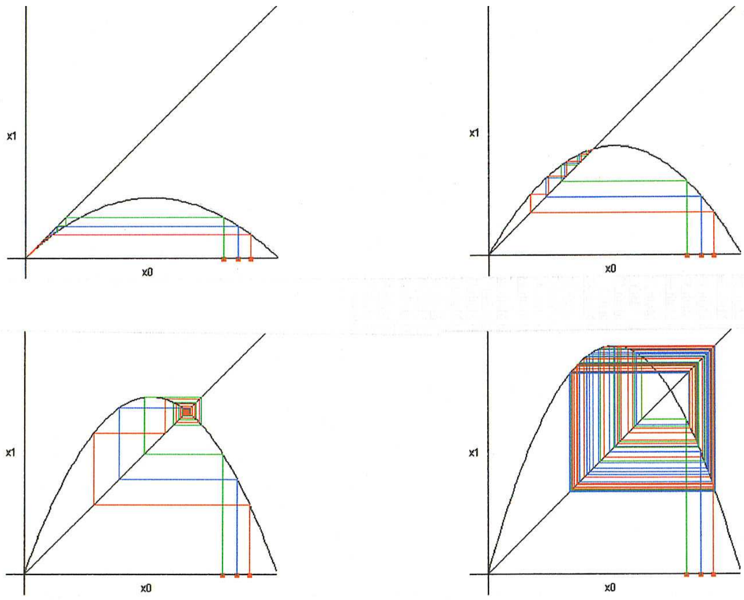


Figure 4: Logistic sequences for $\lambda = 0.95, 1.71, 2.8, 3.6$.

1) When $0 < \lambda \leq 1$, the sequence $(x_n)_{n\geq 1}$ decreases monotonically to 0, which is the unique fixed point of f_λ.

2) When $1 < \lambda \leq 2$, the sequence $(x_n)_{n\geq 1}$ increases monotonically to the limit $\ell := 1 - \frac{1}{\lambda}$, which is now a second fixed point of f_λ; note that $0 < \ell \leq \frac{1}{2}$.

3) When $2 < \lambda \leq 3$, the sequence $(x_n)_{n\geq 0}$ still converges to ℓ, but there is $n_0 \geq 1$ such that $(x_n)_{n\geq n_0}$ oscillates around ℓ; note that, now, $\frac{1}{2} < \ell \leq \frac{2}{3}$.

Since $f_\lambda{}'(0) = \lambda$, the stationary point $x = 0$ is stable if $\lambda < 1$, and unstable if $\lambda > 1$. Similarly, since $f_\lambda{}'(\ell) = 2 - \lambda$, ℓ is stable if $1 < \lambda < 3$, unstable if $\lambda > 3$. We also see directly that $\ell = 0$ and $\ell = \frac{2}{3}$ are stable when, respectively, $\lambda = 1$ and $\lambda = 3$.

Stationary points of the sequence correspond to 1-periodic orbits. To find 2-periodic orbits, we look for the stationary points of the second iterate of f_λ, i.e. for solutions of the equation

$$x = f_\lambda^{(2)}(x) = \lambda^2 x(1-x)(\lambda x^2 - \lambda x + 1)\,. \tag{13}$$

Of course, $f_\lambda^{(2)}(0) = 0$ and $f_\lambda^{(2)}(\ell) = \ell$, since a fixed point of f_λ is also a fixed point of any of its iterates. Other fixed points of $f_\lambda^{(2)}$ are found by solving (13), which we check to be equivalent to the equation

$$Q_\lambda(x) := \lambda x^2 - (1+\lambda)x + 1 + \tfrac{1}{\lambda} = 0\,.$$

The discriminant of Q_λ is $\Delta_\lambda = (\lambda+1)(\lambda-3)$; thus, for $\lambda > 3$, $f_\lambda^{(2)}$ does have two more fixed points. For $\lambda = 3$, $\Delta_3 = 0$, $Q_3(x) = 3(3x-2)^2$, and $f_3^{(2)}$ still has only the stable 1-periodic stationary orbit $\left\{\frac{2}{3}\right\}$. For $\lambda > 3$, this 1-periodic orbit becomes unstable; the two additional fixed points of $f_\lambda^{(2)}$ produce a stable 2-periodic orbit. The behavior of the sequence becomes extremely complicated as λ increases to 4 (see e.g. Moon, [23], for extensive numerical analysis, and some more details in the next section). In fact, it is not difficult to show explicitly that the logistic sequence corresponding to $\lambda = 4$ (in which case the range of f_4 is all of [0, 1]) is sensitive to its initial conditions (see Figure 5). On the other hand, for each $\lambda \in [0, 4]$, the logistic

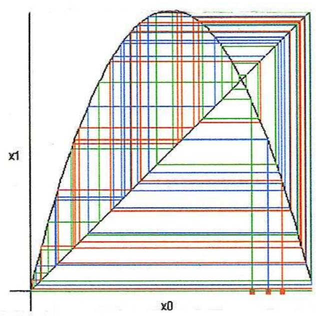


Figure 5: Chaotic behavior, $\lambda = 4$.

sequence (12) has an attractor $\mathcal{A}_\lambda \subset [0, 1]$: for $0 \le \lambda \le 1$, $\mathcal{A}_\lambda = \{0\}$; for $1 < \lambda \le 3$, $\mathcal{A}_\lambda = \{\ell\}$, while for $3 < \lambda \le 4$, $\mathcal{A}_\lambda$ is a fractal set (see Section 6.2).

6. Illustrations of Chaos.

In this section we present some example of so-called “fractal” sets, which typically provide a graphical illustration of the chaotic behavior of some discrete dynamical systems.

6.1. Bifurcation Diagrams.

Figures 6, 7 and 8 depict the graph of the map which associates to each $\lambda \in [0, 4]$ the number of stable periodic orbits of the logistic sequence (12). As we can see, for $\lambda \in [0, 3]$ this map is actually a piecewise smooth function; indeed, this is just the

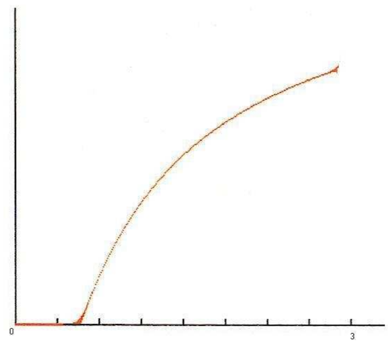


Figure 6: Graph of the function ℓ of (14), for $0 \le \lambda \le 3$.

function

$$\lambda \mapsto \ell = \max\left\{0, 1 - \tfrac{1}{\lambda}\right\}, \tag{14}$$

where ℓ is the limit of the sequence. However, as λ crosses the value $\lambda = 3$ the map becomes multi-valued, and its graph undergoes a series of more and more complicated bifurcations. For λ only slightly larger than 3, we know, from our discussion in the pre-

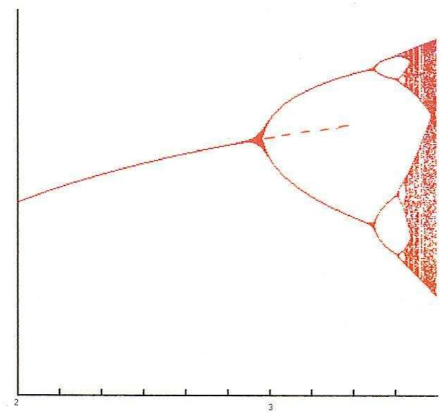


Figure 7: Bifurcation diagram for the logistic equation, $2 \le \lambda \le 3.7$.

vious section, that the logistic sequence acquires one other stable 2-periodic orbit. It is then observed that, as λ increases further, this stable 2-periodic orbit becomes unstable (at $\lambda_1 \approx 3.4495$; this corresponds to the dotted part of the diagram in Figure 7), and the sequence acquires a stable 4-periodic orbit. This pattern of period-doubling orbits persists, with the appearance of 2^n-periodic orbits for every $n \in \mathbb{N}_{>0}$ as λ increases; more precisely, each of the stable 2^n-periodic orbits becomes unstable, and, at the next stage, a stable 2^{n+1}-periodic orbit comes into existence (see Figure 7). This is followed by by a first so-called 2^∞ regime (at $\lambda_\infty \approx 3.5699$), in which there are no stable p-periodic orbit of any order p; then, 3-periodic orbits start to appear, after which the system acquires orbits of any order p. This is part of a famous result of Yorke ([16]),

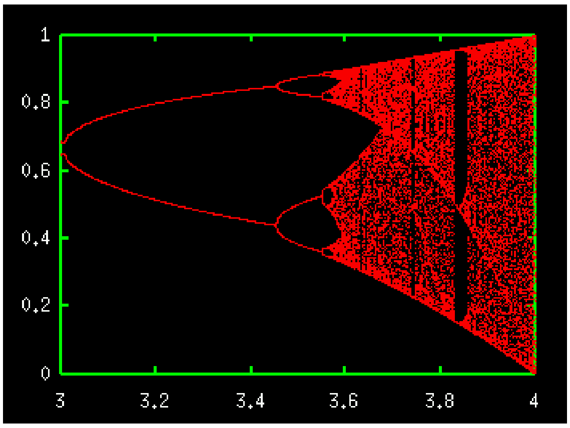


Figure 8: Bifurcation diagram for the logistic equation, $3 \le \lambda \le 4$.

and consequence of the following result, due to Sharkowski ([35]):

THEOREM 1. *Consider the following ordering in* $\mathbb{N}_{>0}$*:*

$$
\begin{aligned}
3 \succ 5 \ &\succ \ \ldots \succ 2\cdot 3 \succ 2\cdot 5 \succ \ \ldots \succ 2^2\cdot 3 \succ 2^2 \cdot 5 \succ \ \ldots \\
&\succ \ \ldots \\
&\succ \ \ldots \succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ \ \ldots \succ 2^3 \succ 2^2 \succ 2^1 \succ 1\,.
\end{aligned}
\tag{15}
$$

Let $I \subset \mathbb{R}$ *be an interval, and* f *a continuous function from* I *into itself. Suppose the dynamical system* $\mathcal{S}$ *generated by the iterated sequence* $x_{n+1} = f(x_n)$ *has a* p*-periodic orbit, and let* $q \prec p$ *in the ordering of* (15)*. Then,* $\mathcal{S}$ *has a* q*-periodic orbit. In particular, if* $\mathcal{S}$ *has a* 3*-periodic orbit, then* $\mathcal{S}$ *has a* m*-periodic orbit for all* $m \in \mathbb{N}_{>0}$.

The successive period doublings exhibited by the logistic sequence is a phenomenon that was soon found to be common to many other discrete systems, such as the iterated sequences relative to the quadratic map $f_\lambda(x) = \lambda - x^2$, or the famous Hénon map[‡] , the forced damped pendulum, the van der Pol and Duffing oscillators with periodic forcing, as well as various Poincaré maps associated to Lorenz' equations (8) (see sct. 8 below). This property was first observed by Feigenbaum ([6]),

[‡]That it, the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (\lambda x(1-x)+y, \mu x)$, with $\lambda, \mu > 0$; see Hénon, [10].

who conjectured that, if λ_n denotes the value of the parameter λ at which the system undergoes the n-th period doubling, then for each such system the limit

$$\lim_{n\to+\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \tag{16}$$

exists. Sufficient conditions for the existence of such limit have been given by Lanford ([13]): its approximate value is $\delta \approx 4.6692$, a number now known at the Feigenbaum constant, and the corresponding sequence $(\lambda_n)_{n\geq 1}$ is called a FEIGENBAUM CASCADE. In particular, the limit (16) exists for all one-dimensional unimodal maps with negative Schwarz derivative; that is, smooth maps $f : I \to I$, $I \subseteq \mathbb{R}$ an interval, having only one critical point, and such that

$$D_S f := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0 \qquad \text{when} \quad f' \neq 0\,.$$

An example of a family of unimodal maps on $[0, 1]$ is $f_\lambda(x) = \lambda \sin(\pi x)$, $\lambda > 0$; for the logistic sequence (12), $\lambda_1 = 3$, and it is easily seen that for each of the maps f_λ,

$$D_S f_\lambda(x) = \frac{-6}{(1-2x)^2}\,, \qquad f_\lambda{}'(x) = 0 \quad \text{for} \;\; x = \tfrac{1}{2}\,.$$

6.2. Self-similarity, Fractals.

Another remarkable feature of the bifurcation graph in Figure 7 is its SELF-SIMILAR structure; that is, one can identify subsets of the graph that repeat themselves, with the same pattern, on smaller and smaller scales (see Figure 9). Self-similarity is a property

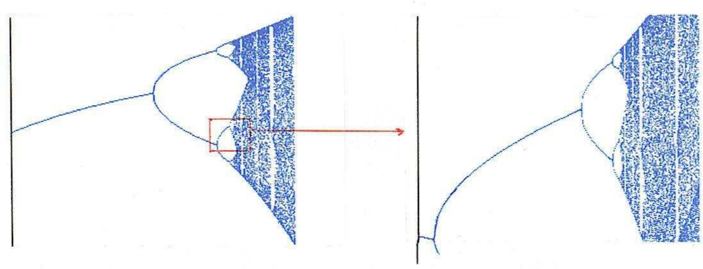


Figure 9: Self-similarity of the bifurcation diagram.

shared by many famous sets, such as the Cantor set, and the curves of Peano and Koch (see Figures 10 and 11). The notion of self-similarity was made famous by Mandelbrot, who in 1975 coined the term FRACTAL to describe sets of this kind (see e.g. [19]); namely, subsets of an Euclidean space which have a dramatically non-smooth structure, and yet possess a degree of self-similarity, associated to general properties of invariance of scale. In particular, many attractors of nonlinear dynamical systems, such

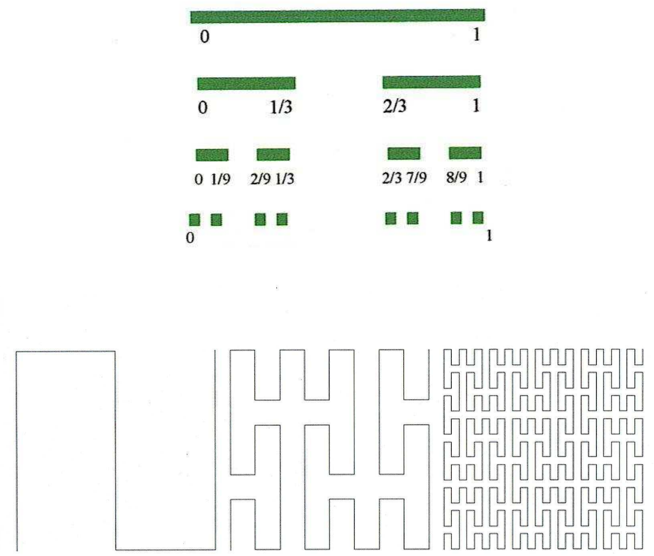


Figure 10: Cantor's Middle-third set and Peano's curve.

as those generated by the logistic equation, the Lorenz equations, the above-mentioned Hénon map, the forced or double pendulums, the van der Pol and Duffing oscillators with periodic forcing, are fractal sets. The complicated structure of these attractor is reflected in the term "strange attractor", originally introduced by Ruelle in 1971 ([31]).

The word "fractal" refers to the fact that one can associate to this kind of sets a notion of dimension, appropriately called FRACTAL DIMENSION, which, roughly speaking, gives a measure of the "non-smoothness" of the set, and generally turns out to be a fraction (see Section 6.3). For example, the fractal dimension of the attractor of the dynamical system generated by the logistic equation (12) for $\lambda = 4$ is ≈ 0.538, reflecting the fact that $\mathcal{A}_4$ is "larger" [§] than the union of a finite number of points (whose dimension is 0), but "smaller" than an interval (whose dimension is 1). Likewise, the fractal dimension of the "middle-third" Cantor set is also a number between 0 and 1.

In investigating the notion of fractal dimension, Mandelbrot famously asked "How long is the coast of Britain?" ([18]). Indeed, one can try to measure the length of a coastline with ever greater levels of accuracy, but since a typical coastline is not an Euclidean curve, these approximations will in general not converge to a finite length. Likewise, the fractal dimension of a ball of twine[¶], or of a stack of firewood, is a number between 2 and 3, reflecting the fact that these objects occupy three-dimensional space,

[§] Not in the sense of inclusions.

[¶] Gleick, [7], p. 97.

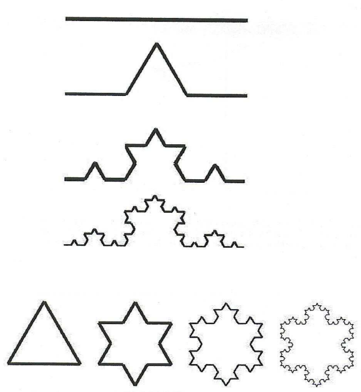


Figure 11: Koch's "snowflake"curve.

but do not completely fill a "solid" box of $\mathbb{R}^3$. In contrast, the fractal dimension of Peano's curve is 2, as a consequence of the fact that the curve completely fills a square, while the fractal dimension of Koch's curve is ≈ 1.2618, i.e. more than an Euclidean curve, but less than a surface (note that Koch's curve has infinite length, but encloses a finite area).

The availability of more and more powerful computers has allowed for increasingly more detailed investigations of the self-similarity properties of fractals; the resulting deeper understanding of the structure of these sets has found remarkable applications both in the figurative arts (see e.g. [25], and one example in Figure 12), and in various fields of applied science. For example, many structures in the human body are now considered as fractals, such as the lungs, the system of small blood vessels, the network of nervous fibers that control the heart-beat; likewise, the onset of turbulence in fluid flows is now efficiently described in terms of bifurcations from the smooth regime into a fractal.

Other famous fractals are the so-called MANDELBROT set, and the various families of JULIA sets[‖]. To describe one example of Julia set, consider the discrete dynamical system generated by the iterated sequence of Newton's method to find the roots of a complex polynomial of degree m. Given an initial guess z_0, the corresponding sequence $N(z_0) = (z_n)_{n \geq 0}$, defined by

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

will generally converge to one of the m roots; thus, each root acts as an attractor. The question is to describe the so-called "basin of attraction" of each root r_k, $k = 1, \ldots, m$;

[‖] Named after the French mathematician G. Julia (1893-1978).

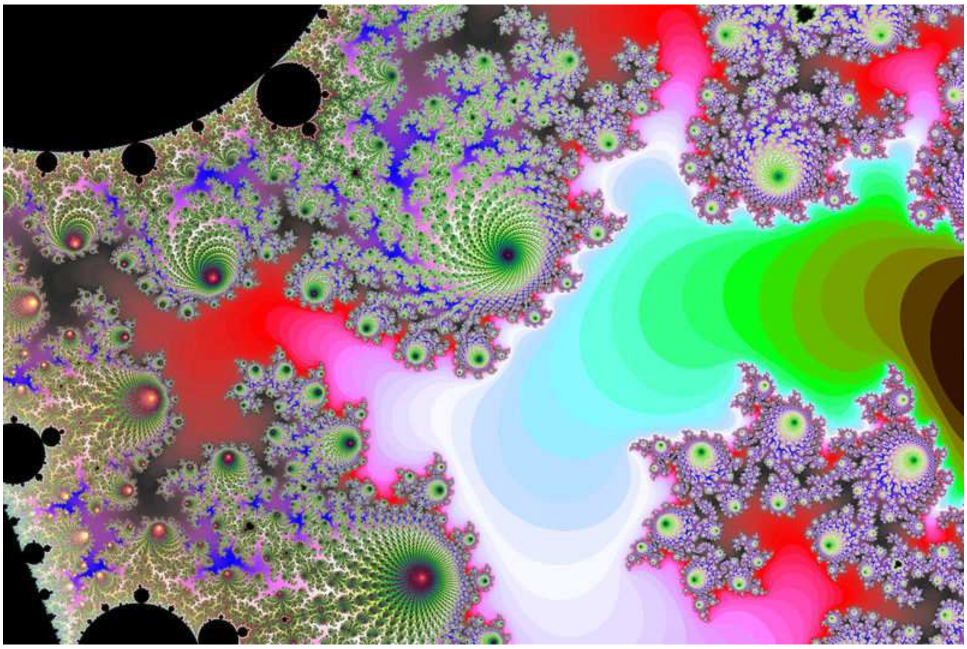


Figure 12: A fractal.

that is (see definition 8), the subsets $Z_k := \{z_0 \in \mathbb{C} \mid N(z_0) \to r_k\}$. It is found that these basins of attraction are fractal subsets of $\mathbb{C}$, which exhibit self-similar features on finer and finer scales (see Figure 13).

The Mandelbrot set $\mathcal{M}$ is also a subset of $\mathbb{C}$, defined as follows. Let $z \in \mathbb{C}$: then, $z \in \mathcal{M}$ if the sequence $M(z) := (z_n)_{n\geq 0}$, defined by

$$z_{n+1} = z_n^2 + z\,, \qquad z_0 = 0$$

remains in the disc $\{z \in \mathbb{C} \mid |z| \leq 2\}$. For example, 0 and $-i \in \mathcal{M}$, while $1 \notin \mathcal{M}$. Together with the Lorenz attractor, the Mandelbrot set is one of the most famous "historical" fractals (see Figure 14). In particular, for each $c \in \mathcal{M}$, one can consider the Julia set of the corresponding polynomial $P_c(z) = z^2 + c$: these sets are also fractals.

6.3. Fractal Dimension.

To introduce the definition of FRACTAL DIMENSION of a set, we start with the elementary observation that, if $C \subset \mathbb{R}^N$ is a cube of volume V, and $r > 0$, it takes exactly $M = V\,r^{-N}$ non-overlapping boxes of side r to cover C. Thus, the space dimension N is related to the numbers r and M by the formula

$$N = \frac{\ln V - \ln M}{\ln r} = \frac{\ln M}{\ln(1/r)} - \frac{\ln V}{\ln(1/r)}\,. \tag{17}$$

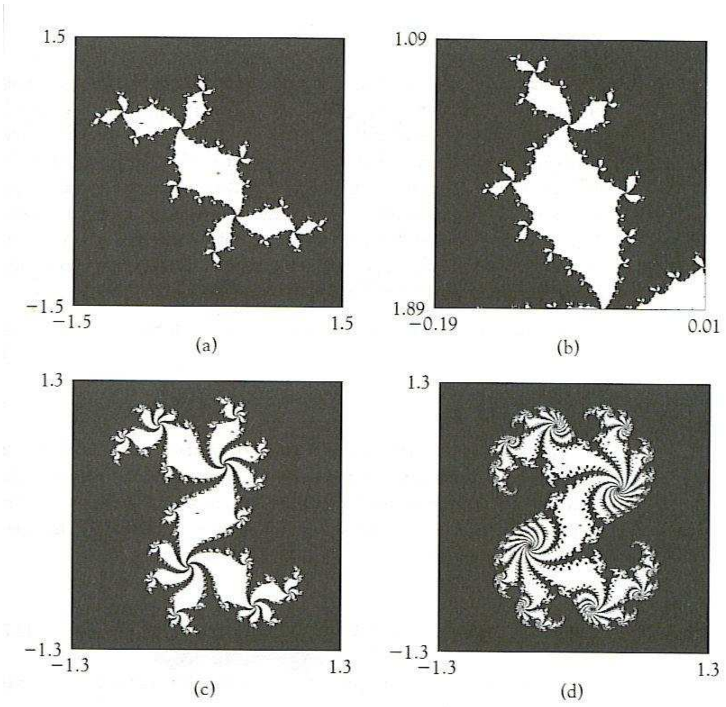


Figure 13: Julia set for the equation $z^3 - 1 = 0$.

If $r \ll 1$, the last term in (17), and thus the information on the volume of C carried by V, is negligible: this motivates the following generalization of (17).

Given a bounded set C, and $r > 0$, we denote by $M(r)$ the number of boxes of side r that are needed to cover C and, in accord with (17), we define the "box-counting" dimension of C as

$$bcd(C) := \lim_{r \to 0} \frac{\ln M(r)}{\ln(1/r)} . \tag{18}$$

For example, let C be the Cantor "middle-third" set. This set is defined as $C := \bigcap\limits_{n \geq 0} C_n$, where the sets C_n are constructed with the following inductive process. Each set C_n is the union of 2^n subintervals; starting from the set $C_0 := [0, 1]$, C_{n+1} is constructed from the previous set C_n by dividing each of the subintervals whose union is C_n into three equal subintervals, and removing all the corresponding middle intervals. For example, $C_2 := [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

Given $r \in]0, 1[$, let $n \in \mathbb{N}$ be such that

$$3^{-n-1} \leq r < 3^{-n} . \tag{19}$$

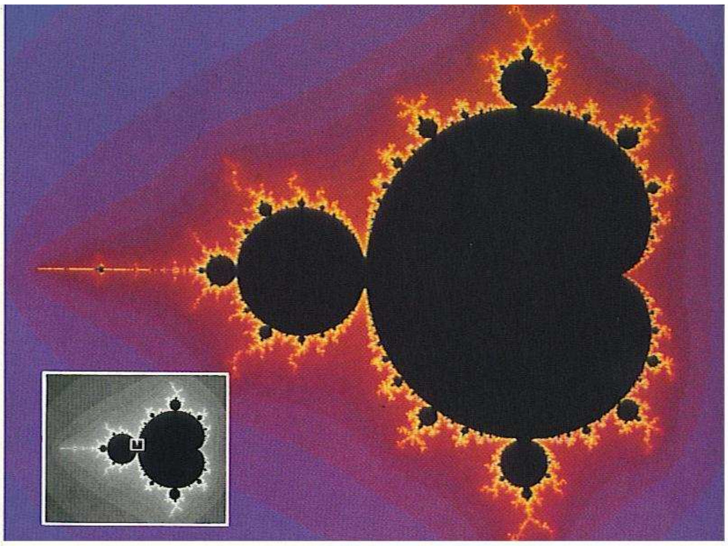


Figure 14: The Mandelbrot set.

Each of the n subintervals that make up C_n has length 3^{-n}; hence, it can be covered by exactly $(3^n r)^{-1}$ segments of length r; and since there are 2^n such subintervals in C_n, we need $M_n(r) = (\frac{2}{3})^n \frac{1}{r}$ segments of size r to cover C_n. The function under the limit in (18) is therefore

(20) $$\frac{\ln\left((\frac{2}{3})^n \frac{1}{r}\right)}{\ln\left(\frac{1}{r}\right)} = \frac{n \ \ln(\frac{2}{3}) - \ln r}{-\ln r} =: d(r,n) \, .$$

Because of (19), $r \to 0 \iff n \to +\infty$, and

$$d(r,n) \ \to \ \frac{\ln(\frac{2}{3}) + \ln 3}{\ln 3} = \frac{\ln 2}{\ln 3} \, .$$

Thus, by (18), $bcd(C) \approx 0.6309$, a non-integer value which confirms that the Cantor set C is a fractal. On the other hand, if instead of (19), we just assume that $r < 3^{-n}$, and let $r \to 0$ in (20), we obtain that for each $n \geq 0$ the fractal dimension of C_n is 1. This is of course to be expected, since each C_n is a finite union of intervals.

Likewise, for the Koch curve, at each stage $n \geq 0$ we have a polygonal K_n of length $\ell_n = (\frac{4}{3})^n$. Given again $r \in]0,1[$, and n as in (19), we see that we need $M_n(r) = (\frac{4}{3})^n \frac{1}{r}$ segments of length r to cover K_n. Hence, acting as above,

$$d(r,n) = \frac{\ln\left((\frac{4}{3})^n \frac{1}{r}\right)}{\ln(\frac{1}{r})} \to \frac{\ln 4}{\ln 3} \approx 1.2618$$

as $r \to 0$, confirming that Koch's curve is a fractal. Note that the curve's length is infinite, since $\lim \ell_n = +\infty$; of course, the fractal dimension of the surface enclosed by the curve is 2.

The definition of box-counting dimension of a set can be generalized to subsets of infinite dimensional sets, as follows.

DEFINITION 3. *Let $\mathcal{X}$ be a separable Hilbert space, and $K \subset \mathcal{X}$ be a compact subset. For $\delta > 0$, denote by $M_\delta(K)$ the smallest number of sets of diameter at most equal to δ which can cover K. The* FRACTAL DIMENSION *of K is the number*

$$\mathrm{d_F}(K) := \limsup_{\delta \to 0} \frac{\ln M_\delta(K)}{\ln(1/\delta)} . \tag{21}$$

(We include the possibility that $\mathrm{d_F}(K) = +\infty$ for some set K).

In analogy with (18), $\mathrm{d_F}(K)$ is also called the "upper box-counting" dimension of K. There are corresponding definitions of "lower box-counting" dimension and of "box-counting" dimensions of K, obtained by replacing, in (21), $\limsup$ respectively by $\liminf$ and $\lim$.

7. Dynamical Systems.

In this section, we report some of the main definitions and results in the theory of dynamical systems; in particular, we present some sufficient conditions for the existence of an attractor. For a proof of the results stated in this section, see e.g. [22].

7.1. Semiflows.

We start with the formal definition of a semiflow in a Banach space. This is a more specific notion than that of dynamical system, although the two terms are usually taken to be synonymous.

DEFINITION 4. *Let $\mathcal{X}$ be a Banach space, and $\mathcal{T}$ denote one of the set $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$. A* SEMIFLOW *on $\mathcal{X}$ is a family $\mathcal{S} = (S(t))_{t \in \mathcal{T}}$ of maps in $\mathcal{X}$ (not necessarily linear), which satisfies the* SEMIGROUP *conditions*

$$S(0) = I_{\mathcal{X}} , \tag{22}$$

$$S(t+t') = S(t)S(t') = S(t')S(t) , \tag{23}$$

for all $t, t' \in \mathcal{T}$, as well as the separate continuity conditions

$$\forall\, t \in \mathcal{T} , \qquad x \mapsto S(t)x \in C(\mathcal{X}, \mathcal{X}) , \tag{24}$$

$$\forall\, x \in \mathcal{X} , \qquad t \mapsto S(t)x \in C(\mathcal{T}, \mathcal{X}) . \tag{25}$$

If $\mathcal{T} = \mathbb{N}$, the semiflow $\mathcal{S}$ is called DISCRETE*; if $\mathcal{T} = \mathbb{R}_{\geq 0}$, $\mathcal{S}$ is called* CONTINUOUS.

A simple example of a semiflow is given by the exponential of a bounded linear operator A on a Banach space $\mathcal{X}$; that is, the family

$$\mathcal{S} := (\mathrm{e}^{tA})_{t\geq 0} \,. \tag{26}$$

More generally, semiflows are naturally generated by the solution operators associated to linear and nonlinear systems of differential equations. In fact, assume that $f : \mathbb{R}^N \to \mathbb{R}^N$ is a locally Lipschitz continuous function and, as in (1), denote by $x(t, x_0)$ the value at time t of the solution of the Cauchy problem relative to the *autonomous* system of ordinary differential equations

$$\begin{cases} x' &= f(x) \,, \\ x(0) &= x_0 \,. \end{cases} \tag{27}$$

If we define a family $\mathcal{S} = (S(t))_{t\geq 0}$ of maps in $\mathcal{X} = \mathbb{R}^N$ by

$$\mathcal{X} \ni x_0 \;\mapsto\; S(t)x_0 := x(t, x_0) \in \mathcal{X} \,, \tag{28}$$

then $\mathcal{S}$ is a semiflow on $\mathcal{X}$. Indeed: condition (22) translates the taking of the initial condition in (27); condition (23) holds because the differential equation in (27) is autonomous; condition (24) is a consequence of the Lipschitz continuity of f, and condition (25) follows from the differentiability of x in t. In this case, we say that the semiflow $\mathcal{S}$ is *generated* by the function f. For example, if $\mathcal{X} = \mathbb{R}^N$, and A is an $N \times N$ matrix, the semiflow in (26) is generated by the linear system of ODEs $x' = Ax$. In Section 8 we shall examine the continuous semiflow on $\mathbb{R}^3$, generated by Lorenz' system (8). In an analogous way, the semilinear heat equation

$$u_t - \Delta u + g(u) = f \,, \tag{29}$$

where the nonlinearity g is subject to suitable regularity and growth assumptions, the source f is independent of t (so as to insure that (29) is autonomous), and u is subject to some compatible conditions on the boundary of a bounded domain $\Omega \subset \mathbb{R}^N$, generates a semiflow in the Banach spaces $\mathcal{X} = C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, or $\mathcal{X} = L^p(\Omega)$, $p > 1$ (see e.g. Zheng, [36]).

As in the example of the Bernoulli and logistic sequences of Section 5, a function f on $\mathbb{R}^N$ can also generate a discrete semiflow, as long as f maps a bounded set B into itself. Indeed, we can take $\mathcal{T} = \mathbb{N}$, and define $\mathcal{S}$ as the family of the successive iterates of f; that is, denoting by $f^{(0)}$ the identity in $\mathbb{R}^N$, and setting $f^{(1)} := f$, $f^{(2)} := f \circ f$, etc., $\mathcal{S} = \left(f^{(n)}\right)_{n\geq 0}$.

7.2. Absorbing and Attracting Sets.

We have mentioned many examples of chaotic systems, which experience some kind of asymptotic "order", in the sense that the orbits of the system appear to follow a rather regular pattern. This situation can often be described by the observation that the orbits remain close to some bounded set of $\mathcal{X}$, to which they seem to be attracted. In this section, we make this idea rigorous, by introducing sets that are invariant, absorbing and attracting. In the sequel, $\mathcal{S}$ denotes a semiflow on a given Banach space $\mathcal{X}$, and $\|\cdot\|$ the norm in $\mathcal{X}$.

DEFINITION 5. *Let $Y \subseteq \mathcal{X}$. Y is* POSITIVELY INVARIANT *for $\mathcal{S}$ if $S(t)Y \subseteq Y$ for all $t \geq 0$. Y is* INVARIANT *if $S(t)Y = Y$ for all $t \geq 0$.*

In other words, all orbits starting in a positively invariant set Y remain in Y forever; in addition, if Y is invariant, any of its points is on an orbit starting elsewhere in Y.

DEFINITION 6. *Let $B \subset \mathcal{X}$. B is* ABSORBING *for $\mathcal{S}$ if for all bounded set $G \subseteq \mathcal{X}$, there exists $T \geq 0$, dependent of G, such that $S(t)G \subseteq B$ for all $t \geq T$. A semiflow which admits a non-empty, bounded absorbing set is called* DISSIPATIVE.

In other words, all orbits starting in G enter B and, after possibly leaving B for a finite number of times, remain in B forever.

DEFINITION 7. *Let $A, B \subseteq \mathcal{X}$, and $x \in \mathcal{X}$. We set*

$$d(x, B) := \inf_{b \in B} \|x - b\| \,,$$

$$\partial(A, B) := \sup_{a \in A} d(a, B) \,, \tag{30}$$

$$\delta(A, B) := \max\{\partial(A, B), \partial(B, A)\} \,. \tag{31}$$

The map ∂ defined in (30) is a *semidistance* only; in fact, it is not symmetric, and the equality $\partial(A, B) = 0$ does not necessarily imply that $A = B$, as we see by taking $A \subset B$. Moreover, we can even have $\partial(A, B) = 0$ with $A \supset B$, as we see taking A to be the closure of an open set B. However, it is easy to see that if $\partial(A, B) = 0$, then $A \subseteq \overline{B}$. In particular, the map δ defined in (31) is a metric on the *closed* subsets of $\mathcal{X}$.

DEFINITION 8. *Let $A \subset \mathcal{X}$. A is an* ATTRACTOR *for $\mathcal{X}$ if it is compact, invariant, and there is a neighborhood $\mathcal{U}$ of A such that for all bounded set $B \subseteq \mathcal{U}$,*

$$\lim_{t \to +\infty} \partial(S(t)B, A) = 0 \,. \tag{32}$$

The largest neighborhood $\mathcal{U}$ of A such that (32) *holds is called the* BASIN OF ATTRACTION *of A. An attractor A is called* GLOBAL *if its basin of attraction is the whole space $\mathcal{X}$.*

It is easy to see that global attractors are unique, and, indeed, maximal with respect to set inclusion, among all compact, invariant sets of $\mathcal{X}$. Moreover, the existence of a bounded, positively invariant absorbing set is a necessary condition for the existence of an attractor. For example, let $\mathcal{S}$ be the semiflow in $\mathbb{R}$, generated by the

ODE (4) (i.e., $x' = -x$). Then, any symmetric interval $]-r, r[$, $r > 0$, is invariant and absorbing for $\mathcal{S}$, and the singleton $A = \{0\}$ is its attractor.

7.3. Finite Dimensional Attractors.

Often, global attractors have a finite fractal dimension (see definition 3). This case is of particular importance, since the corresponding dynamics is also finite dimensional. Indeed, the invariance of the attractor implies that orbits which originate in the attractor remain there for all future times; consequently, the evolution of a system on a finite dimensional attractor is essentially governed by a finite system of ODEs. In fact, it can be proven (Mañé, [20]) that, if a dynamical system possesses a finite dimensional attractor, this set coincides with the attractor of the semiflow generated by a suitable system of ODEs. This result allows us to reduce, at least in principle, the study of the asymptotic behavior of orbits which converge to a finite dimensional attractor $\mathcal{A}$ to that of the solutions of a finite dimensional system of ODEs on $\mathcal{A}$.

The possibility of actually doing so, together with the description of the corresponding system of ODEs, is one of the most challenging problems in the theory of infinite dimensional dynamical systems. Indeed, in many cases the study of the system on the attractor cannot be pursued in practice, because of various difficulties, generally related to the non-smooth structure of attractors. Other problems, of particular importance in applications, include the availability of reasonable estimates on the dimension of the attractor (and, therefore, on the corresponding system of ODEs; for example, in meteorology it is not uncommon to have estimates of the order of 10^{20}), as well as the insufficient stability of attractors under perturbations of the data. As we have seen, their numerical approximations, and the consequent propagation of errors, may then be quite difficult to control. More generally, a major goal of an effective theory would be to describe the geometric and/or topological structure of the attractor; in case the system under consideration is generated by a physical model, it would then be important to be able to translate this understanding into proper insights on relevant properties of the model.

7.4. Attractors via ω-limit sets.

In this section we present a result that constructs the global attractor of a semiflow as the ω-limit set of a bounded, positively invariant absorbing set. This construction requires some degree of regularity of the semiflow. An alternative construction is based on the method of α-contractions; see e.g. [22, §2.7].

DEFINITION 9. *The semiflow $\mathcal{S}$ is* ASYMPTOTICALLY COMPACT *if for any bounded set $B \subset \mathcal{X}$, there is $T > 0$, dependent of B, such that the set*

$$B_T := \bigcup_{t \geq T} S(t)B \tag{33}$$

has compact closure in $\mathcal{X}$.

DEFINITION 10. *Let $Y \subseteq \mathcal{X}$. The ω-LIMIT SET of Y is the set*

$$\omega(Y) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)Y} \,. \tag{34}$$

Definition 10 clearly generalizes the familiar notion of ω-limit sets in the classical theory of ODEs, such as ω-limits cycles for autonomous systems in $\mathbb{R}^2$ (see e.g. [2]).

We can then present one result on the existence of attractors:

THEOREM 2. *Let $\mathcal{S}$ be an asymptotically compact semiflow on $\mathcal{X}$, and assume that $\mathcal{S}$ admits a non-empty, bounded, absorbing set $\mathcal{B}$. Then, the ω-limit set*

$$\mathcal{A} := \omega(\mathcal{B}) \tag{35}$$

is the global attractor for $\mathcal{S}$ in $\mathcal{X}$.

For example, for the dynamical system generated by the logistic sequence (12), it can be proven that for any $\lambda \in]3, 4]$ such that the corresponding system is under the 2^∞ regime, the corresponding attractor is $\mathcal{A}_\lambda = \omega\left(\frac{1}{2}\right)$.

As another, classical example, consider the first-order system in $\mathcal{X} = \mathbb{R}^2$

$$\begin{cases} x' &= -y + x(1 - x^2 - y^2)\,, \\ y' &= x + y(1 - x^2 - y^2)\,, \end{cases} \tag{36}$$

with initial conditions

$$x(0) = x_0\,, \qquad y(0) = y_0\,.$$

Since the right side of (36) is locally Lipschitz continuous, standard results imply that (36) defines a semiflow in $\mathcal{X} = \mathbb{R}^2$. It is easy to solve (36), using polar coordinates; Figure 15 shows some of the orbits of $\mathcal{S}$. The unit disc $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ is positively invariant: orbits starting in D remain in D for all $t \geq 0$. The origin is a source for (36): orbits starting in the interior of D, except of course at O, spiral out of the origin, and converge to the unit circle ∂D. Outside D, every disc $D_\varepsilon = \{(x, y) \mid x^2 + y^2 \leq 1 + \varepsilon\}$, $\varepsilon > 0$, is positively invariant and absorbing for $\mathcal{S}$ (see Figure 16). Since ∂D is invariant, it follows that $A := \partial D$ is the attractor of $\mathcal{S}$. In fact, A is the ω-limit cycle of system (36).

For semiflows generated by dissipative evolution equations, theorem 2 can be applied as follows. The dissipativity of the equation means that is is possible to establish suitable *a priori* estimates, which imply the existence of a bounded, positively invariant absorbing set B (see e.g. Section 8 for the Lorenz system). If $\mathcal{S}$ is finite dimensional (i.e., it is generated by an autonomous system of ODEs), it follows that, for all $T > 0$, the closure of the set B_T defined in (33) is compact, since

$$B_T = \bigcup_{t \geq T} S(t)B \subseteq \bigcup_{t \geq T} B = B\,,$$

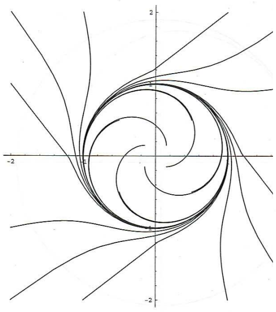


Figure 15: Orbits of (36).

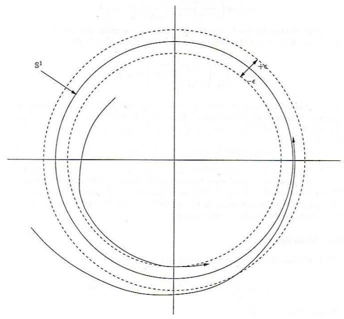


Figure 16: The absorbing sets and attractor for (36).

and $\overline{B}$ is compact. Hence, $\mathcal{S}$ is asymptotically compact, and theorem 2 guarantees that $\omega(B)$ is its attractor. In the case of infinite dimensional systems, such as those generated by autonomous PDEs, the asymptotic compactness of the semiflow is no longer an automatic consequence of its dissipativity, and has to be established separately. In practice, this requires a regularity result for solutions of the PDE, to hold for sufficiently large t. In the case of parabolic equations, this may be expected, as a consequence of the smoothing effect of the corresponding solution operator. In contrast, dissipative hyperbolic equations do not enjoy this smoothing property, and we must resort to a modification of theorem 2, whereby the semiflow $\mathcal{S}$ is required to be asymptotically compact only up to a uniformly decaying perturbation; that is, to satisfy a decomposition $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, where $\mathcal{S}_1$ is asymptotically compact, and for all bounded sets $G \subset \mathcal{X}$,

$$\lim_{t\to+\infty} \sup_{g\in G} \|S_2(t)g\| = 0\,.$$

As an example of the results that it is possible to obtain in this way, we consider, for $\varepsilon \geq 0$, the semilinear evolution equation

$$\varepsilon\, u_{tt} + u_t - \Delta u + u^3 - u = f \tag{37}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$, with u subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$, assumed smooth. For $\varepsilon = 0$, equation (37) is parabolic, while for $\varepsilon > 0$ it is dissipative hyperbolic. Assuming that the source term f is independent of t, so that (37) is autonomous, we can prove the following results:

THEOREM 3. *Let $f \in L^2(\Omega)$. Then:*

i) If $\varepsilon = 0$, (37) generates a semiflow $\mathcal{S}$ in $\mathcal{X} := L^2(\Omega)$, which admits a global attractor $\mathcal{A} \subset \mathcal{X}$. In fact, $\mathcal{A}$ is compact in $H^2(\Omega) \cap H^1_0(\Omega)$.

ii) If $\varepsilon > 0$, (37) generates a semiflow $\mathcal{S}$ in $\mathcal{X} := H^1_0(\Omega) \times L^2(\Omega)$, which admits a global attractor $\mathcal{A} \subset \mathcal{X}$. In fact, $\mathcal{A}$ is compact in $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$.

iii) In either case, $\mathcal{A}$ *contains all stationary solutions of* (37)*, together with their unstable manifolds.*

For a proof of theorem 3, see e.g. Zheng, [36], or [22], as well as Grasselli-Pata, [8], for the regularity of $\mathcal{A}$ when $\varepsilon > 0$, and Babin-Vishik, [3], for the structure of $\mathcal{A}$. Note that, in the hyperbolic case, the phase space $\mathcal{X}$ is a product space: in fact, $\mathcal{S}$ is defined by

$$\mathcal{S}(t)(u_0, u_1) = (u(t, \cdot), u_t(t, \cdot)) ,$$

where u is the solution of (37) corresponding to the initial values (u_0, u_1).

In definitions 6, 8 and 28, the insistence in considering *bounded* sets can be interpreted as an illustration of the attempt to control the possible errors in the determination of the initial state of a system, as mentioned in the introduction. Indeed, an initial state u_0 of a system is in general known only within a certain approximation; however, any such approximation is in an explicitly identifiable bounded set, which should contain u_0. For example, if we must approximate $u_0 = \sqrt{2}$ up to three decimal digits, we can work in the bounded interval [1.4139, 1.4141].

7.5. Poincaré Sections.

Given a continuous semiflow $\mathcal{S} = (S(t))_{t\geq 0}$, it is often possible to construct a discrete semiflow $\tilde{\mathcal{S}} = (S^n)_{n\geq 0}$, whose asymptotic behavior allows us to deduce information on that of $\mathcal{S}$. One way to do so is to choose an arithmetic sequence $(t_n)_{n\in\mathbb{N}}$, with $t_0 = 0$ and $t_{n+1} = t_n + \tau$ for some $\tau > 0$, and to define the maps $S^n : \mathcal{X} \to \mathcal{X}$ by

$$S^n u_0 := S(t_n)u_0 , \qquad u_0 \in \mathcal{X} .$$

Clearly, $\tilde{\mathcal{S}}$ is a semiflow (since the values t_n are equidistant), and each of the points $u_n := S^n u_0$ lies on the orbit starting at u_0. This choice defines a map $\Phi : \mathcal{X} \to \mathcal{X}$, by $u_{n+1} = \Phi(u_n)$. Maps constructed in this way are called STROBOSCOPIC MAPS. In the case of finite dimensional systems, a similar kind of map, called POINCARÉ MAP, can be constructed in the following way. We fix a hyperplane $\Sigma \subset \mathbb{R}^N$, called a POINCARÉ SECTION; given $u_0 \in \mathbb{R}^N$, and the corresponding orbit $\gamma := (S(t)u_0)_{t\geq 0}$, we consider the sequence of the "first returns" points of γ on Σ; that is, the points u_n defined by the successive intersections of γ with Σ ** (see Figure 17). The sequence $(u_n)_{n\in\mathbb{N}}$ can then be considered as a recursive sequence on Σ, implicitly defined by a map $u_{n+1} = \Phi_\Sigma(u_n)$. The map Φ_Σ (which is not necessarily stroboscopic) is called a POINCARÉ MAP associated to the semiflow $\mathcal{S}$; evidently, different sections Σ define different maps Φ_Σ.

Poincaré maps can thus be used to study the asymptotic behavior of a continuous semiflow, by reducing it to a discrete one. For example, if the initial value problem (27) has a periodic solution with period T, the Poincaré map with sampling synchronized with the period, i.e. with $t_n = nT$, will have a fixed point (see Figure 18). Of course, for a given system of ODEs it may not be clear how to find suitable sampling sequences

**Poincaré maps are sometimes also known as "first return" maps.

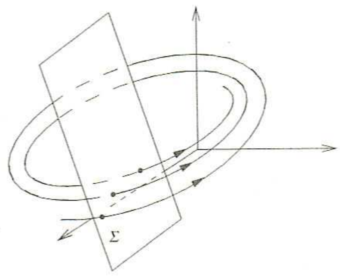


Figure 17: A Poincaré section.

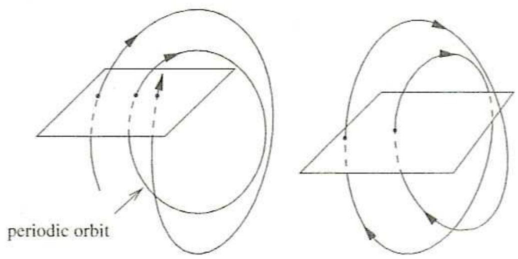


Figure 18: Periodic and 2-periodic orbits produce fixed points in a Poincaré section.

$(t_n)_{n\in\mathbb{N}}$, and extensive numerical experimentation may well be required. In Section 8 below, we show how to construct a Poincaré section for the Lorenz system (8), with $r = 28$, which allows us to deduce that, for this value of r, Lorenz' equations are indeed chaotic.

7.6. Exponential Attractors and Inertial Manifolds.

1. As we have mentioned in Section 7.3, the non-smooth structure of attractors causes non trivial difficulties in the practical study of the long-time behavior of a semiflow. On the other hand, there are systems whose attractors do not exhibit this kind of difficulties, because they are imbedded into a finite dimensional Lipschitz manifold $\mathcal{M} \subset \mathcal{X}$, and the orbits converge to this manifold with a uniform exponential rate (as opposed to requirement (32) in definition 8 of the attractor, which carries no information on the rate of convergence of the orbits to the attractor, other than this rate is uniform for all orbits starting in the same bounded set B). Such a set $\mathcal{M}$ is called an INERTIAL MANIFOLD of the semiflow. As in the case of finite dimensional attractors, when a semiflow admits an inertial manifold, its evolution on this manifold is governed by a finite system of ODEs, called the INERTIAL FORM of the semiflow. Since orbits converge to the inertial manifold with a uniform exponential rate, the dynamics on the manifold will be a much better approximation of the long time behavior of the semiflow. In fact, the uniform rate of convergence of the orbits to the manifold makes these systems extremely stable under perturbations and numerical approximations.

For semiflows generated by a semilinear evolution equation like (37), or, more generally, by an abstract ODE of the type

$$u_t + Au = f(u)\,, \tag{38}$$

where A is an unbounded linear operator on a Banach space $\mathcal{X}$, it is of particular interest to construct inertial manifolds having the structure of a smooth graph; that is, of the form

$$\mathcal{M} = \{x + m(x) \mid x \in \mathcal{X}_1\}\,, \tag{39}$$

where $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ is decomposed into a closed linear subspace $\mathcal{X}_1$ of finite dimension, and its algebraic complement $\mathcal{X}_2$, and $m\colon \mathcal{X}_1 \to \mathcal{X}_2$ is a Lipschitz continuous map. Inertial manifolds of the type (39) allow us to embed the global attractor $\mathcal{A}$ in $\mathbb{R}^N$, with $N = \dim\mathcal{X}_1 = \dim\mathcal{M}$; moreover, if A commutes with the continuous projector π_1 from $\mathcal{X}$ onto $\mathcal{X}_1$, the asymptotic behavior of the solution of (38) is governed by the N-dimensional inertial form system

$$x' = -Ax + \pi_1 f(x + m(x))\,, \tag{40}$$

which is a *finite* system of ODEs in $\mathcal{X}_1$. Note that (40) has a Lipschitz continuous right-hand side if f is Lipschitz continuous.

Unfortunately, there are not may examples of systems which are known to admit an inertial manifold; among these, we mention the semiflows generated by a number of reaction-diffusion equations of "parabolic" type, and by the corresponding hyperbolic (small) perturbations of these equations. A typical model is that of the so-called Chafee-Infante equations in $\mathbb{R}^1$, which can be put in the form (37). In general, inertial sets of type (39) for the semiflow generated by (38) can be found if the operator A satisfies a relatively restrictive condition on its spectrum, called the SPECTRAL GAP CONDITION. In essence, this is a requirement that the eigenvalues of A should be spaced with sufficiently large gaps, so as to allow the linear term of equation (38), i.e. the term Au, to "dominate" (in a suitable sense) the nonlinear term, i.e. $f(u)$[††]. We refer to [22, ch. 5] for more details.

2. As we have discussed, attractors are somewhat "unfriendly" sets for the study of the long-time behavior of a system, because of their generally non-smooth structure, and the slow rate of convergence of the orbits to them. In contrast, inertial manifolds are much more "friendly", given their smooth structure and the exponential rate of convergence of the orbits. A sort of intermediate situation may occur, whereby a semiflow admits a so-called EXPONENTIAL ATTRACTOR[‡‡]. More precisely, exponential attractors do not necessarily have a smooth structure, but retain at least three of the features

[††]Spectral gap conditions of various type were originally introduced in the context of Navier-Stokes equations in two dimensions of space.

[‡‡]These sets are also sometimes called "inertial sets" in the literature.

of inertial manifolds that attractors do not necessarily have: the finite dimensionality, the exponential convergence of the orbits, and a high degree of stability with respect to approximations. This means that when an exponential attractor exists, after an "exponentially short" transient the dynamics of the system are essentially governed by a finite system of ODEs (the classical image is that of an airplane, landing at a "fast" speed and then "slowly" taxiing to the arrival gate).

Exponential attractors for semiflows generated by an ODE such as (38) can be constructed if the operator A satisfies a less restrictive condition on its spectrum, called the DISCRETE SQUEEZING PROPERTY. This property essentially translates a dichotomy principle, whereby either the system is exponentially contracting on a fixed compact set $\mathcal{B} \subset \mathcal{X}$, or the evolution of the difference of two solutions originating in $\mathcal{B}$, when expressed as a Fourier series with respect to the eigenvectors of the operator A, can be controlled by a finite number of terms of the series. In other words, in this series the tail can be dominated by its complementary finite sum.

Not surprisingly, it turns out that there are many more systems that admit an exponential attractor than systems that are known to admit an inertial manifold. The main reason for this difference is that all known inertial manifolds are closed (as are those of type (39)), and therefore the existence of a compact absorbing set (which is a necessary condition for the existence of a global attractor) also yields directly the existence of an exponential attractor. Moreover, inertial manifolds are much more regular than exponential attractors. Finally, at least for evolution equations like (38), the existence of inertial manifolds in general requires the validity of the strong squeezing property, while for the existence of an exponential attractor it is sufficient to assume the discrete squeezing property, which is a much weaker condition. Indeed, roughly speaking, the strong squeezing property also translates a sort of dichotomy principle, whereby either the difference of two motions can never leave a certain cone, or, if it does, the distance between the motions decays exponentially. In contrast, the discrete squeezing property only requires that either the difference of two motions is in a cone *at a specific time* (as opposed to for all times), or, if not, the distance between the motions decays exponentially. Again, we refer to [22, ch. 4] for more details.

Finally, we remark that when a semiflow admits a global attractor $\mathcal{A}$ and a closed inertial manifold $\mathcal{M}$ (or an exponential attractor $\mathcal{E}$) then $\mathcal{A} \subseteq \mathcal{M}$ (respectively, $\mathcal{A} \subseteq \mathcal{E}$). In particular, in this case the dimension of $\mathcal{A}$ is finite. Neither implication $\mathcal{E} \subseteq \mathcal{M}$ nor $\mathcal{M} \subseteq \mathcal{E}$ needs to hold; on the other hand, if the semiflow admits both a compact, positively invariant absorbing set G, and a closed inertial manifold $\mathcal{M}$, then the set $\mathcal{E} := \mathcal{M} \cap G$ is also a compact set, which is positively invariant (being the intersection of two positively invariant sets) and exponentially attracting*. That is, $\mathcal{E}$ is an exponential attractor, and $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{M}$.

*At least if $\mathcal{M}$ possesses a more specific type of attractivity property, called "exponential tracking property".

8. The Attractor of Lorenz' Equations.

In this section, we study the behavior of solutions to Lorenz' system (8), in relation to the parameter $r > 0$. Letting $u := (x, y, z) \in \mathbb{R}^3$, we rewrite (8) in the compact form

$$u' = f(u)\,, \tag{41}$$

with obvious definition of f. By standard results in ODEs, it is easy to check that, for all values of r, and for all initial values $u_0 = (x_0, y_0, z_0)$, (41) has a unique global solution, and generates a semiflow $\mathcal{S}$ on $\mathcal{X} = \mathbb{R}^3$. In proposition 1 below, we show that $\mathcal{S}$ is dissipative: then, as a consequence of theorem 2, Lorenz equations (41) do have a compact attractor $\mathcal{A}$ in $\mathbb{R}^3$. For certain values of r, the structure of this attractor is relatively well understood. Although most detailed information can be obtained by means of extensive numerical experimentation, we present here some results that can be established by simple analytical techniques. For most details, we refer to [22, sct. 1.5]; for a more extensive study of Lorenz' equations, see e.g. Sparrow, [34].

We first show that Lorenz' equations are dissipative.

PROPOSITION 1. *The semiflow $\mathcal{S}$ defined by Lorenz' equations (41) admits a family of bounded, positively invariant absorbing balls in $\mathbb{R}^3$.*

Sketch of Proof. Let $\kappa := (0, 0, r + \sigma) \in \mathbb{R}^3$, $u_0 := (x_0, y_0, z_0)$, and set $\varphi(t) := |S(t)u_0 - \kappa|^2$. Multiplying (41) by $2u$, we can deduce that φ satisfies the exponential inequality

$$\varphi'(t) + 2\varphi(t) \le M := \frac{8}{3}\,(r + 10)^2\,.$$

After integration, we obtain that, for all $t \ge 0$,

$$0 \le \varphi(t) \le \mathrm{e}^{-2t}\varphi(0) + \tfrac{1}{2}\,M(1 - \mathrm{e}^{-2t})\,. \tag{42}$$

From this, we easily deduce that for all $\varepsilon \ge 0$, the balls $B_\varepsilon := B\left(\kappa, \tfrac{1}{2}\,M + \varepsilon\right)$ are positively invariant. If $\varepsilon > 0$, these balls are also absorbing. To see the latter, let G be a bounded set of $\mathbb{R}^3$. Given $\varepsilon > 0$, there is $R > 0$ such that $G \subseteq B(\kappa, R)$. Let $u_0 \in G$. Then, $\varphi(0) \le R^2$, so that (42) implies that

$$\varphi(t) \le R^2\mathrm{e}^{-2t} + \tfrac{1}{2}\,M(1 - \mathrm{e}^{-2t})\,,$$

from which it follows that $\varphi(t) \le \frac{1}{2}\,M + \varepsilon$ for all $t \ge T_\varepsilon$, with

$$T_\varepsilon := \max\left\{0, \frac{1}{2}\,\ln\left(\frac{2R^2 - M}{2\varepsilon}\right)\right\}\,.$$

Note that T_ε depends on G, via R. □

Next, we study the equilibrium points of (41). It is immediate to see that, if $r \leq 1$, the origin O is the only equilibrium point of (41), while if $r > 1$ there are the two other equilibrium points

$$C_\pm := (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)\,, \quad b := \tfrac{8}{3}\,.$$

Considering the Jacobian matrix $f'(u)$, we find that $f'(0)$ has three real eigenvalues (of which one is $-b$). If $r < 1$, these eigenvalues are all negative; thus, the unique equilibrium point O is a stable node, and is in fact the attractor of the system (i.e., $\mathcal{A} = \{O\}$). If $r > 1$, one of the eigenvalues of $J(O)$ is positive, so the origin is an unstable saddle, with a 2-dimensional stable manifold $\mathcal{M}^{\mathrm{s}}(O)$ attracted by O, and a one-dimensional unstable manifold $\mathcal{M}^{\mathrm{u}}(O)$ repelled by it. At the points $C_\pm$, we find again that at least one eigenvalue is real negative, and the others are either real negative too, or have negative real part, if and only if $r < r_* = \frac{470}{19} \approx 24.737$. It follows that, if $1 < r < r_*$ the stationary points $C_\pm$ are stable nodes, and every orbit converges to one of these points. In this case, the attractor $\mathcal{A}$ of Lorenz' system consists of the points C_-, C_+, and the unstable manifold $\mathcal{M}^{\mathrm{u}}(O)$ connecting C_- to C_+. If instead $r > r_*$, the stationary points O, C_+ and C_- are all unstable. In this case, the corresponding attractor $\mathcal{A}$ is more difficult to describe; as we have mentioned in Section 4, it can be shown (see e.g. Sparrow, [34]) that $\mathcal{A}$ is a fractal set, of dimension ≈ 2.06. Near $C_\pm$, orbits arrive along the stable manifolds $\mathcal{M}^{\mathrm{s}}(C_\pm)$ (corresponding to the real negative eigenvalue of $J(C_\pm)$), and spiral out along the two-dimensional surface $\mathcal{M}^{\mathrm{u}}(C_\pm)$. This behavior was first discovered by Lorenz, who observed the so-called "butterfly" attractor for $r = 28$ (see Figure 19).

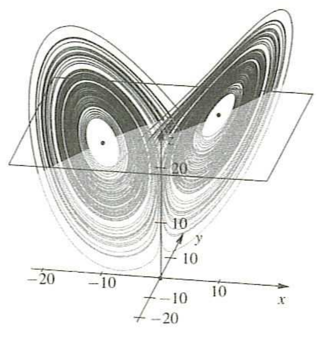
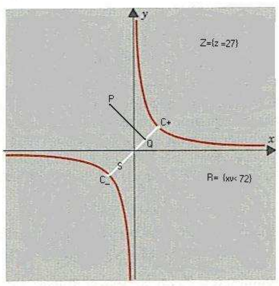


Figure 19: The attractor of Lorenz equations.

Finally, we construct a particular Poincaré section for (41). When $r = 28$, the equilibrium points of (8) are the origin O, and $C_\pm = (\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$. Let f be as in (41), and consider the hyperplane $Z := \{z = 27\}$. Since $f(x, y, 27) \cdot (0, 0, 1) = xy - 72$, it follows that orbits cross Z downwards in the region $R = \{(x, y, 27) \in Z \mid xy < 72\}$. A little more work shows that, in fact, orbits can cross R only in the subregions $R_\pm = \{(x, y, 27) \in R \mid \pm x > 0, \pm y > 0\}$ (see Figure 20). Moreover,



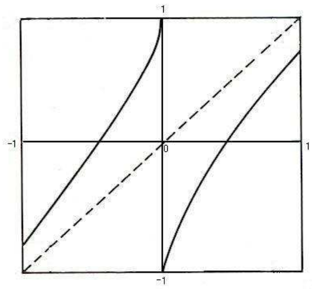


Figure 20: The Poincaré section on Z [Left].

Figure 21: Qualitative graph of φ, the first return map on S [Right].

orbits that enter R in R_+ must turn back around C_+, and, symmetrically, orbits that enter R in R_- must turn back around C_-. Let $S \subset R$ be the segment joining C_- to C_+, parametrized by a function $[-1, 1] \ni r \mapsto Q = q(r) \in S$. Given a point $P \in R_- \cup R_+$, let $Q \in S$ be the projection of P onto S, and let r be the coordinate of Q under q. If P_n and P_{n+1} are two successive return points on Z of an orbit of (41), and r_n, r_{n+1} are the coordinates of the corresponding projections Q_n and Q_{n+1} on S, the orbit defines a "projected first return" map $\varphi : [-1, 1] \to [-1, 1]$, by

$$r_{n+1} = \varphi(r_n) . \tag{43}$$

In this way, we have constructed a one-dimensional Poincaré section S, and a corresponding Poincaré map φ. A little analysis shows that the graph of φ has the general form shown in Figure 21; note that φ is not defined at $r = 0$ (because orbits intersecting the z-axis converge to the origin, and therefore never cross R again). Since the graph of φ is qualitatively similar to that of the Bernoulli map f of (10), and the Bernoulli sequence is chaotic, a topological argument can be constructed, to deduce that the Poincaré map defined by (43) is also chaotic. It follows that, for $r = 28$, system (8) is chaotic, as conjectured by Lorenz.

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I dedicate this Lecture to the memory of my uncle Riccardo Milani, late Professor Emeritus of Zoology at the University of Pavia.

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