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GENERALIZED SPLINES IN $\mathbb{R}^n$ AND OPTIMAL CONTROL*

Abstract. We give a new time-dependent definition of spline curves in $\mathbb{R}^n$, which extends a recent definition of vector-valued splines introduced by Rodrigues and Silva Leite for the time-independent case. Previous results are based on a variational approach, with lengthy arguments, which do not cover the non-autonomous situation. We show that the previous results are a consequence of the Pontryagin maximum principle, and are easily generalized using the methods of optimal control. Main result asserts that vector-valued splines are related to the Pontryagin extremals of a non-autonomous linear-quadratic optimal control problem.

1. Introduction

Polynomial splines have been extensively used in several applied areas of mathematics such as computer graphics and approximation theory. Since the early 90's, they have been used in control theory, associated to problems of aircraft control and path planning of mechanical systems. These applications originated the extension of classical spline functions to other contexts such as Riemannian manifolds, Lie groups, etc.

Another line of research started with the definition of spline functions which are not polynomial splines. One of the first generalizations in this direction are the so called *scalar generalized splines*, which were introduced in the 50's by Ahlberg, Nilson and Walsh [1]. The connection between scalar generalized splines and optimal control was established between 1995 and 1999. It turns out that splines are much more than a tool to be used in control theory. They are intrinsic to optimal control problems and appear naturally as minimizers of certain problems [9, 12].

Recently, this connection between minimality and splines was extended to a new class of spline functions in arbitrary dimensional Euclidean spaces [11]. This was accomplished by variational arguments and a more general time-invariant optimal control problem. Here, using tools from optimal control, we go a step further. We consider a class of classical linear-quadratic optimal control problems, which are not necessarily time-invariant, and recover, as corollaries, the previous results.

2. Background

In this section we give an account of scalar generalized splines, its connection to optimal control, and collect all the necessary results to be used in the sequel.

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2.1. Scalar generalized splines.

Generalized splines were first introduced in the late 50's by Ahlberg, Nilson and Walsh [1]. Consider the linear differential operator of order $p \in \mathbb{N}$

$$L = D^p \cdot + a_{p-1}(t) D^{p-1} \cdot + \cdots + a_1(t) D \cdot + a_0(t) \cdot$$

where each $a_k(t)$, $k = 0, 1, \ldots, p-1$, is a real C^p-smooth function in $[a, b]$. The operator L is acting on the space $C^m[a, b]$ of real functions defined in $[a, b]$. Its adjoint is defined by

$$L^* = (-1)^p D^p \cdot + (-1)^{p-1} D^{p-1}(a_{p-1}(t)\cdot) + \cdots - D(a_1(t)\cdot) + a_0(t) \cdot \ .$$

L^* is also acting on $C^m[a, b]$ and the scalar product for which it is computed is given by

$$\langle x_1, x_2 \rangle = \int_a^b x_1(t) x_2(t)\, dt \ .$$

Let $\Delta : a = t_0 < t_1 < \ldots < t_m = b$, $m \in \mathbb{N}$, be a partition of $[a, b]$, Ω be the family of real C^{2p-2}-smooth functions in $[a, b]$ which are C^{2p}-smooth in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$ and $f \in \Omega$.

DEFINITION 1. *The function $s : [a, b] \to \mathbb{R}$ is an interpolating generalized spline of f associated to Δ and L, if $s \in \Omega$, s is a solution of the differential equation $L^* L x = 0$ in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, and $s(t) = f(t)$ on Δ.*

DEFINITION 2. *An interpolating generalized spline of f is of type I if it is such that $s^{(k)}(t_0) = f^{(k)}(t_0)$ and $s^{(k)}(t_m) = f^{(k)}(t_m)$, for $k = 1, 2, \ldots, p-1$.*

The function f is usually omitted from the previous definitions. Instead, one has to demand that, in Definition 1, function s fulfills the interpolation condition $s(t_i) = s_i$, where s_i, $i = 0, 1, \ldots, m$, are given real numbers and, in Definition 2, that s fulfills the boundary conditions $s^{(k)}(t_0) = \eta_0^k$ and $s^{(k)}(t_m) = \eta_m^k$ where η_0^k, η_m^k, $k = 1, 2, \ldots, p-1$, are prescribed real numbers. Then, we just say that s is a *generalized spline* of type I. The next statement collects several results about generalized splines of type I which can be found in [1].

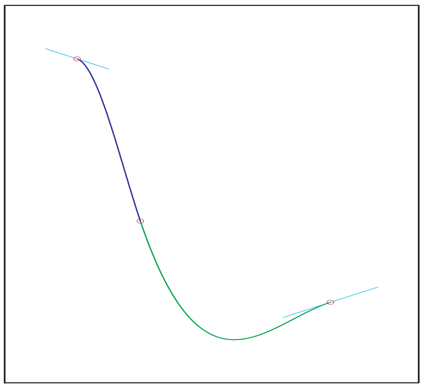
THEOREM 1 ([1]). *There exists, for each set of boundary and interpolation conditions, a unique generalized spline of type I associated with the differential operator L and the partition Δ. Moreover, this generalized spline is the unique minimizer of the functional*

$$\int_a^b (Lg)^2\, dt$$

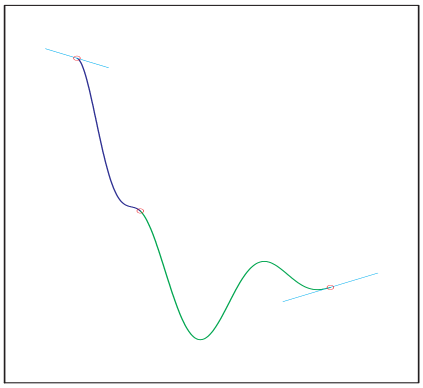
among all the functions $g \in \Omega$ that fulfill the same boundary and interpolation conditions.

REMARK 1. There are other types of boundary conditions, described in the literature, that also ensure the existence and uniqueness of the corresponding generalized spline.

We now give two examples for constant coefficient operators: an example of a cubic spline, and an example of a trigonometric spline. Let $\Delta : 0 < 1/4 < 1$ be the partition of the time interval $[0, 1]$; $s(t_0) = 3$, $s(t_1) = 1$ and $s(t_2) = 0$ be the interpolation conditions; and $\dot{s}(t_0) = -1$, $\dot{s}(t_2) = 1$ be the boundary conditions. We first consider the operator $L = D^2$. The resulting spline of type I is a C^2-smooth function in $[0, 1]$ such that $s(t) = c_{1i} + c_{2i}t + c_{3i}t^2 + c_{4i}t^3$ in each $[t_i, t_{i+1}]$ where c_{1i}, c_{2i}, c_{3i}, c_{4i} are real constants to be found. This is the classical cubic spline. Considering $L = D^2 + 144$, the resulting spline of type I is also C^2-smooth in $[0, 1]$ so that $s(t) = (c_{1i} + c_{2i}t)\cos(12t) + (c_{3i} + c_{4i}t)\sin(12t)$ in each $[t_i, t_{i+1}]$.



Cubic spline



Trigonometric spline

The most immediate generalization of scalar splines to curves in $\mathbb{R}^n$ is achieved by simply considering vector functions $g : [a, b] \to \mathbb{R}^n$, the same operator L as before, and adapted interpolation conditions, boundary conditions, and set Ω. It is obvious that each component of the resulting spline will be a scalar generalized spline, and therefore such a spline curve will always minimize the functional

$$\int_a^b \langle Lg, Lg \rangle \, dt \,,$$

where $\langle \cdot\,, \cdot \rangle$ stands for the Euclidean inner product, among all functions in Ω that fulfill the same boundary and interpolation conditions. As we shall see, from an optimal control perspective such a trivial generalization is not the natural way of extending scalar-splines to vector-valued splines.

2.2. Scalar generalized splines and optimal control.

Since the early nineties, in order to deal with applied problems from Robotics, there has been an increasing interest to combine spline curves and integral cost problems associated with linear control systems. Among theoretical developments, it was found that scalar generalized splines are minimizers of a simple optimal control problem with a linear time-invariant control system and a single control (see [9] and [12]). This discovery is of crucial importance, because it introduces a new perspective to the subject: scalar spline functions are better viewed as a consequence of the search for an optimal control, rather than a postulate imposed a priori in order to solve particular classes of problems. Given its importance, we summarize the main result here. Consider the following autonomous linear-quadratic optimal control problem:

$$
\begin{aligned}
&\min_u \quad \int_a^b u^2\,dt \\
&\text{subject to} \\
&\dot{x} = Ax + Bu \\
&x(t_0) = x_0, \quad x(t_m) = x_m \\
&x^1(t_i) = \alpha_i, \qquad i = 1, 2, \ldots, m-1\,,
\end{aligned}
\tag{1}
$$

where $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$, x^1 is the first component of the state vector, $\alpha_i \in \mathbb{R}$, u is a scalar function which is C^{n-2}-smooth in $[a, b]$ and C^n-smooth in each interval $[t_i, t_{i+1}]$. Let us assume that the state space is $\mathbb{R}^n$ and that the state vector is a C^{2n-2}-smooth function in $[a, b]$ which is also C^{2n}-smooth in each interval $[t_i, t_{i+1}]$.

THEOREM 2. *If the control system $\dot{x} = Ax + Bu$ of problem* (1) *is completely state controllable with matrices A and B in the canonical form*

$$
A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
$$

for given real numbers a_j, $j = 0, 1, \ldots, n-1$, then the optimal control problem (1) *has always a unique solution with the first component of the optimal state vector being a generalized spline of type* I *associated to the constant coefficient differential operator $L = D^n - a_{n-1}D^{n-1} - \cdots - a_1 D - a_0$.*

If the first component of the optimal state vector of problem (1) is a scalar generalized spline, the following questions come immediately to our mind: What can be said about the minimizing state trajectory of the optimal control problem? Is it some sort of a generalized spline in $\mathbb{R}^n$? The answer to these questions leads us (see Definition 4) to a new time-dependent definition of generalized spline in $\mathbb{R}^n$, which is the main contribution of the present paper.

2.3. Pontryagin's maximum principle, existence, and regularity.

The general problem of optimal control can be defined, in Lagrange form, as follows:

$$
\begin{aligned}
&\min_{(x(\cdot),u(\cdot))} \quad I[x(\cdot),u(\cdot)] = \int_{t_a}^{t_b} \mathcal{L}\left(t, x(t), u(t)\right) \, \mathrm{d}t \\
&\dot{x}(t) = \varphi\left(t, x(t), u(t)\right) \\
&(x(t_a), x(t_b)) = (\alpha, \beta) \\
&x(\cdot) \in W_{1,1}\left([t_a, t_b];\, \mathbb{R}^n\right) \\
&u(\cdot) \in \mathcal{U}\left([t_a, t_b];\, \Omega \subseteq \mathbb{R}^r\right) .
\end{aligned}
\tag{2}
$$

We assume that $\mathcal{L} : [t_a, t_b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ and $\varphi : [t_a, t_b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ are C^1-smooth functions with respect to all arguments, and that the boundary conditions, together with the class of control functions $\mathcal{U}$, are given. The standard method to solve (2) is usually based on the deductive approach: (i) a solution exists for the problem; (ii) the necessary conditions are applicable, and they identify certain candidates (so called *extremals*); (iii) subsequent elimination (if necessary) identifies the solution (or solutions). We are interested in the case where there are no restrictions on the control variables: $\Omega = \mathbb{R}^r$. The unrestricted case poses many difficulties, and the problem turns out to be a difficult one, even in special situations. As we explain next, most part of difficulties appear in the application of steps (i) and (ii).

The first general answer to (i) was given by A. F. Filippov in 1959 [7], assuming the admissible controls to be integrable ($\mathcal{U} = L_1$), and the control set Ω to be compact. As far as we assume Ω to be a noncompact set, Filippov's theorem does not apply. To solve the existence problem, we make use of the following theorem (see [4]).

THEOREM 3 ("Tonelli" Existence Theorem for (2)). *Problem* (2) *has a minimizer* $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ *with* $\tilde{u}(\cdot) \in L_1([t_a, t_b];\, \mathbb{R}^r)$*, provided there exists at least one admissible pair, and the following convexity and coercivity conditions hold:*

- (convexity) *Functions* $\mathcal{L}(t, x, \cdot)$ *and* $\varphi(t, x, \cdot)$ *are convex for all* (t, x)*;*
- (coercivity) *There exists a function* $\theta : \mathbb{R}_0^+ \to \mathbb{R}$*, bounded below, such that*

$$
\begin{gathered}
\mathcal{L}(t, x, u) \geq \theta\left(\|\varphi(t, x, u)\|\right) \quad \textit{for all } (t, x, u); \\
\lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty; \\
\lim_{\|u\| \to +\infty} \|\varphi(t, x, u)\| = +\infty \quad \textit{for all } (t, x) .
\end{gathered}
$$

REMARK 2. For the definition of convexity of $\mathcal{L}(t, x, \cdot)$ and $\varphi(t, x, \cdot)$ see [4]. In the case $\varphi = u$ one has the fundamental problem of the calculus of variations, and we get from Theorem 3 the classical Tonelli existence theorem.

Step (ii) is addressed by the Pontryagin Maximum Principle [10].

THEOREM 4 (Pontryagin Maximum Principle). *If* $(x(\cdot),\ u(\cdot))$ *is a minimizer of* (2) *and* $u(\cdot)$ *is essentially bounded,* $u(\cdot) \in L_\infty$*, then there exists* $(\psi_0,\ \psi(\cdot)) \neq 0$, $\psi_0 \leq 0$, $\psi(\cdot) \in W^n_{1,1}$*, such that the quadruple* $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ *is a Pontryagin extremal: it satisfies*

- *the* Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \psi}\,, \\ \dot{\psi} = -\dfrac{\partial H}{\partial x}\,; \end{cases} \tag{3}$$

- *the* maximality condition

$$H\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \max_{v \in \mathbb{R}^r} H\left(t, x(t), v, \psi_0, \psi(t)\right)\,; \tag{4}$$

with the Hamiltonian

$$H(t, x, u, \psi_0, \psi) = \psi_0\, \mathcal{L}\left(t, x, u\right) + \langle \psi, \varphi\left(t, x, u\right)\rangle\,. \tag{5}$$

DEFINITION 3. *A Pontryagin extremal* $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ *is said to be abnormal when* ψ_0 *is equal to zero, and normal otherwise.*

The existence is assured in the class of integrable controls ($\mathcal{U} = L_1$), while the formulation of the Pontryagin maximum principle assume the optimal controls to be essentially bounded ($\mathcal{U} = L_\infty \subset L_1$). For minimizers predicted by existence theory, Theorem 4 may fail to be valid, because the values of optimal controls can be unbounded. This is a possibility even for very simple instances of problem (2): e.g. $\mathcal{L}$ a polynomial and φ linear. One such example can be found in [2]: the problem

$$\begin{gathered} \min \int_0^1 \left(\left(x^3 - t^2\right)^2 u^{14} + \varepsilon\, u^2\right) dt \\ \dot{x}\left(t\right) = u\left(t\right) \\ x\left(0\right) = 0\,, \quad x\left(1\right) = k\,, \end{gathered} \tag{6}$$

satisfies all the hypotheses of Theorem 3; it can be proved (see [5]) that for certain choices of constants k and ε there exists a unique optimal control $u\left(t\right) = k\, t^{-1/3}$; but Theorem 4 (Pontryagin maximum principle) is not satisfied since $\dot{\psi}(t) = \mathcal{L}_x\left(t, x(t), \dot{x}(t)\right) = c\, t^{-4/3}$ is not integrable ($\psi(\cdot)$ is not an absolutely continuous function).

In order to apply the deductive method (i)–(iii) one needs to close the gap between the hypotheses of existence and necessary optimality conditions. For that, conditions beyond those of convexity and coercivity, assuring solutions $\tilde{u}(\cdot)$ to be in L_∞ and not only in L_1, must apply. To exclude the possibility of bad behavior that occurs for (6), we will focus our attention to problem (2) with Lagrangian $\mathcal{L}$ and function φ given by

$$\begin{aligned} \mathcal{L}(t, x, u) &= \langle B(t)u, B(t)u\rangle\,, \\ \varphi(t, x, u) &= A(t)x + B(t)u\,, \end{aligned} \tag{7}$$

under the hypothesis

(H1) $B(t)$ is a square matrix with full rank for all t;

(H2) the dynamical control system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is completely state controllable;

(H3) $t \to A(t)$ and $t \to B(t)$ are C^1-smooth functions.

Roughly speaking, this gives the biggest class of optimal control problems which generalize (1) in a natural way; do not admit abnormal extremals (see the next remark); and for which the gap between existence and necessary optimality conditions is automatically closed.

REMARK 3. Since there is no constraint on the control, singular trajectories are exactly projections of abnormal extremals. But due to the assumption on the linear system (it is supposed to be completely state controllable), there is no singular trajectory, and thus the optimal control problem has no abnormal extremals.

THEOREM 5 (Boundedness of optimal controls [13]). *Under the hypotheses of the existence Theorem 3, if there exist constants $c > 0$ and k such that*

$$\left|\frac{\partial \mathcal{L}}{\partial t}\right| \leq c\,|\mathcal{L}| + k\,, \quad \left\|\frac{\partial \mathcal{L}}{\partial x}\right\| \leq c\,|\mathcal{L}| + k\,,$$
$$\left\|\frac{\partial \varphi}{\partial t}\right\| \leq c\,\|\varphi\| + k\,, \quad \left\|\frac{\partial \varphi_i}{\partial x}\right\| \leq c\,|\varphi_i| + k \quad (i = 1, \ldots, n)\,;$$

then all minimizers of (2) *satisfy the Pontryagin maximum principle.*

It is a simple exercise to see that with $\mathcal{L}$ and φ defined by (7), hypotheses (H1) and (H3) imply all the conditions of Theorems 3 and 5.

3. Main results

We are interested in the following non-autonomous linear-quadratic optimal control problem:

$$(P) \qquad \begin{aligned} &\min_{u(\cdot)} \quad J\,[u(\cdot)] = \int_a^b \langle B(t)u(t), B(t)u(t)\rangle\, dt \\ &\text{subject to} \\ &\dot{x}(t) = A(t)x(t) + B(t)u(t) \\ &x(t_i) = x_i\,, \qquad i = 0, 1, \ldots, m\,, \end{aligned}$$

for a given partition $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ and fixed $x_i \in \mathbb{R}^n$. The control $u : [a, b] \to \mathbb{R}^n$ is unrestricted; the state function $x : [a, b] \to \mathbb{R}^n$ is an absolutely continuous function; $A(t)$ and $B(t)$ are $n \times n$ matrices and $B(t)$ is nonsingular. We find the minimizer of (P) by solving (P_i), $i = 0, 1, \ldots, m-1$, in the interval $[t_i, t_{i+1}]$:

$$(P_i) \qquad \begin{array}{l} \min\limits_{u(\cdot)} \quad J_i\,[u(\cdot)] = \displaystyle\int_{t_i}^{t_{i+1}} \langle B(t)u(t), B(t)u(t)\rangle\, dt \\ \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_i) = x_i, \qquad x(t_{i+1}) = x_{i+1}\,. \end{array}$$

In order to guarantee the applicability of the Pontryagin maximum principle, and the existence of a normal solution, hypotheses (H1), (H2) and (H3) of previous section are in force. Under these assumptions we can choose, without any loss of generality, $\psi_0 = -\frac{1}{2}$ in Theorem 4. The Hamiltonian (5) is then given by

$$H(t, x, u, \psi) = -\tfrac{1}{2}\, u' B(t)' B(t) u + \psi' (A(t)x + B(t)u)\,,$$

where we use the symbol prime $'$ to denote the transpose of a given vector or matrix. The Hamiltonian system (3) reduces to

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t)\,, \\ \dot{\psi}(t) = -A(t)'\psi(t)\,, \end{cases} \tag{8}$$

while from the maximality condition (4) one obtains

$$B(t)'(B(t)u(t) - \psi(t)) = 0\,.$$

This equation implies that $\psi(t) = B(t)u(t)$ and hence

$$u(t) = B(t)^{-1}\psi(t) \tag{9}$$

is the unique Pontryagin extremal control. Thus, due to Theorem 5, u given by (9) must be optimal.

From equation $\psi(t) = B(t)u(t)$ and from equation $\dot{\psi}(t) = -A(t)'\psi(t)$ of system (8) we get the matrix differential equation

$$\frac{d}{dt}(B(t)u(t)) + A(t)'B(t)u(t) = 0\,. \tag{10}$$

Introducing the matrix differential operator $L = D - A(t)$, the control system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ can be written as

$$Lx(t) = B(t)u(t) \tag{11}$$

and equation (10) as

$$L^* B(t)u(t) = 0\,, \tag{12}$$

where $L^* = -D - A(t)'$ is the adjoint operator of L. From (11) and (12) we conclude that the minimizing state trajectory is a solution of the differential equation

$$L^* L x(t) = 0\,,$$

which can be written as

$$\ddot{x}(t) + (A(t)' - A(t))\,\dot{x}(t) - (A(t)'A(t) + \dot{A}(t))\,x(t) = 0\,.$$

We have just proved Lemma 1.

LEMMA 1. *Under hypotheses (H1)-(H3) the optimal control u is, in each interval* $[t_i, t_{i+1}]$*,* $i = 0, 1, \ldots, m-1$*, a solution of the matrix differential equation* $L^* B(t)u = 0$ *with* $L^* = -D - A(t)'$ *the adjoint operator associated to the operator* $L = D - A(t)$*. The corresponding optimal state trajectory x is such that* $L^* L x = 0$ *in each interval* $[t_i, t_{i+1}]$*.*

An explicit expression for the optimal state trajectory and for the optimal control can be obtained in terms of the *state transition matrix*. These results are stated in the following Theorem. We refer the reader to [3] for the definition, and properties, of the state transition matrix.

THEOREM 6. *The optimal state trajectory of problem* (P) *has, in each interval* $[t_i, t_{i+1}]$*,* $i = 0, 1, \ldots, m-1$*, the explicit expression*

$$x(t) = \Phi(t, t_i)x_i + \left(\int_{t_i}^{t} \Phi(t, s)\Phi(t_i, s)'\, ds\right) S^{-1}(\Phi(t_i, t_{i+1})x_{i+1} - x_i) \tag{13}$$

where Φ *is the state transition matrix associated to* $\dot{x} = A(t)x$*, and S is the symmetric matrix given by*

$$\int_{t_i}^{t_{i+1}} \Phi(t_i, s)\Phi(t_i, s)'\, ds\,.$$

Furthermore, the optimal control of problem (P) *has, in each interval* $[t_i, t_{i+1}]$*,* $i = 0, 1, \ldots, m-1$*, the explicit expression*

$$u(t) = B(t)^{-1}\,\Phi(t_i, t)'\,S^{-1}(\Phi(t_i, t_{i+1})\,x_{i+1} - x_i)\,. \tag{14}$$

Proof. (Theorem 6) Since $\psi = B(t)u$, the Hamiltonian system takes the form

$$\begin{cases} \dot{x} = A(t)x + \psi\,, \\ \dot{\psi} = -A(t)'\psi\,. \end{cases} \tag{15}$$

From equation $\dot{\psi} = -A(t)'\psi$ we get $\psi(t) = \Phi(t_i, t)'\psi(t_i)$. The substitution of ψ in equation $\dot{x} = A(t)x + \psi$ of system (15) generates $\dot{x} = A(t)x + \Phi(t_i, t)'\psi(t_i)$. The solution of this complete differential equation, with initial condition $x(t_i) = x_i$, is given by

$$x(t) = \Phi(t, t_i)x_i + \int_{t_i}^{t} \Phi(t, s)\Phi(t_i, s)'\psi(t_i)\, ds. \tag{16}$$

Now, we just have to find $\psi(t_i)$. Using the other initial condition $x(t_{i+1}) = x_{i+1}$ we get

$$
\begin{aligned}
x_{i+1} &= \Phi(t_{i+1}, t_i)x_i + \left(\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, s)\Phi(t_i, s)'\, ds\right) \psi(t_i) \\
&= \Phi(t_{i+1}, t_i)x_i + \Phi(t_{i+1}, t_i) \left(\int_{t_i}^{t_{i+1}} \Phi(t_i, s)\Phi(t_i, s)'\, ds\right) \psi(t_i)\,.
\end{aligned}
$$

If we denote the symmetric matrix

$$
\int_{t_i}^{t_{i+1}} \Phi(t_i, s)\Phi(t_i, s)'\, ds
$$

by $S(t_i, t_{i+1})$, or simply by S, we can write

$$
\Phi(t_{i+1}, t_i)^{-1} x_{i+1} - x_i = S\,\psi(t_i) \Leftrightarrow \Phi(t_i, t_{i+1})x_{i+1} - x_i = S\,\psi(t_i).
$$

Since matrix S is always non-singular, we get $\psi(t_i) = S^{-1}(\Phi(t_i, t_{i+1})x_{i+1} - x_i)$. Finally, from (16), we obtain the equality (13):

$$
x(t)_{|t\in[t_i,t_{i+1}]} = \Phi(t, t_i)x_i + \left(\int_{t_i}^{t} \Phi(t, s)\Phi(t_i, s)'\, ds\right) S^{-1}(\Phi(t_i, t_{i+1})x_{i+1} - x_i).
$$

The second part of the theorem is a direct consequence of equation (9). From previous calculations we have

$$
\psi(t) = \Phi(t_i, t)'\, S^{-1}(\Phi(t_i, t_{i+1})x_{i+1} - x_i)
$$

and thus, equality (14) follows immediately. □

REMARK 4. From the proof of Theorem 6 it follows, by direct calculations, that the optimal value for the integral functional $J_i[\cdot]$ of problem (P_i) is given by

$$
(\Phi(t_i, t_{i+1})x_{i+1} - x_i)'\, S^{-1}\, (\Phi(t_i, t_{i+1})x_{i+1} - x_i)\,.
$$

REMARK 5. We have seen that in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, the optimal state trajectory of problem (P) is a solution of the matrix differential equation

$$
\ddot{x}(t) + (A(t)' - A(t))\,\dot{x}(t) - (A(t)'A(t) + \dot{A}(t))\,x(t) = 0
$$

which does not depend on the matrix $B(t)$. This is natural since we can make the substitution $u \mapsto v = B(t)u$ in the problem (P) and thus eliminate the presence of matrix $B(t)$ in all further calculations.

REMARK 6. When problem (P) is autonomous, the first part of Theorem 6 reduces to Theorem 2.12 in [11].

Lemma 1 and Theorem 6 give the main motivation for our definition of generalized time-dependent spline in $\mathbb{R}^n$. Let L be the linear matrix differential operator of order p

$$L = D^p \cdot - A_{p-1}(t) D^{p-1} \cdot - \cdots - A_1(t) D \cdot - A_0(t) \cdot \,, \tag{17}$$

where each $A_j(t)$, $j = 0, 1, \ldots, p-1$, is a real square $n \times n$ C^p-smooth matrix function in $[a, b]$. The operator L is acting on the space $C^m[a, b]$ of real vector functions defined in $[a, b]$. The adjoint of L, denoted by L^*, is defined as

$$\begin{aligned} L^* = (-1)^p D^p \cdot + (-1)^p D^{p-1}(A'_{p-1}(t)\cdot) + \ (-1)^{p-1} D^{p-2}(A'_{p-2}(t)\cdot) + \\ + \cdots + D(A'_1(t)\cdot) - A'_0(t) \cdot \ . \end{aligned}$$

L^* is also acting on $C^m[a, b]$ and the scalar product for which it is computed is given by

$$\langle x_1, x_2 \rangle = \int_a^b x_1(t)' x_2(t) \, dt \, .$$

Consider

$$\Delta : a = t_0 < t_1 < \ldots < t_m = b \tag{18}$$

to be a partition of $[a, b]$, and let Ω represent the set of all $\mathbb{R}^n$-valued functions defined in $[a, b]$ which are of class C^{2p-2} in $[a, b]$ and of class C^{2p} in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$.

DEFINITION 4 (Generalized time-dependent spline in $\mathbb{R}^n$). *A function $s : [a, b] \to \mathbb{R}^n$ is an interpolating generalized spline of $f \in \Omega$, associated to Δ* (18) *and L* (17)*, if $s \in \Omega$, s is a solution of the matrix differential equation $L^* L x = 0$ in each interval $[t_i, t_{i+1}]$, $s(t) = f(t)$ on Δ (interpolation conditions), and $s^{(k)}(t_0) = f^{(k)}(t_0)$, $s^{(k)}(t_m) = f^{(k)}(t_m)$, for $k = 1, 2, \ldots, p-1$ (boundary conditions).*

REMARK 7. Definition 4 includes, as particular cases, the scalar Definition 2 and the definition introduced in [11].

REMARK 8. As done in the scalar case, the interpolating function $f \in \Omega$ can be omitted in Definition 4.

REMARK 9. The function $x(t)$, $t \in [a, b]$, given in each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, by (13), is a generalized time-dependent spline in $\mathbb{R}^n$ in the sense of Definition 4.

REMARK 10. For $L = D^p$ the solutions of $L^* L x = 0$ give polynomial splines in $\mathbb{R}^n$ with all the components being scalar polynomial splines of degree $2p-1$. This is, as mentioned at the end of §2.1, the immediate generalization of scalar polynomial splines to vector-valued splines, and the one found in the literature.

We have seen that generalized splines associated to an operator L of order $p = 1$ are related to the optimal control problem (P). For $p > 1$, there corresponds an optimal control problem with higher-order dynamic $x^{(p)} = \sum_{j=0}^{p-1} A_j(t)x^{(j)} + B(t)u$. This higher-order optimal control problem can be easily written in form (P). For that we introduce new state variables, reducing the control system of order p to a first-order control system. This is the same to say that when L is an operator of order $p > 1$, the homogeneous differential equation $L^*Lx = 0$ of order $2p$ can be reduced to a first order differential equation, just by increasing the dimension of the matrices $A_j(t)$, $j = 0, 1, \ldots, p-1$.

Under our hypotheses, it is possible to write the optimal control problem (P) as a problem of the calculus of variations with higher-order derivatives. This is done by showing that an arbitrary admissible pair $(x(\cdot), u(\cdot))$ of (P) can be always expressed in terms of higher order derivatives of a single vector valued function (see [6]). From Theorem 6 we obtain:

THEOREM 7. *Given the operator L (17) and the partition Δ (18), there exists a unique generalized spline in $\mathbb{R}^n$ for each set of boundary and interpolation conditions. This generalized spline is the unique solution of the following higher-order problem of the calculus of variations:*

$$\int_a^b \langle Lg, Lg\rangle \, dt \quad \to \quad \min ,$$

among all the functions $g \in \Omega$ that satisfy the same boundary and interpolation conditions.

4. Examples

We give two examples for which the state and control spaces are $\mathbb{R}^2$. We denote the components of the state vector x by x_1 and x_2; the components of the control vector u by u_1 and u_2. The first example is

$$\min_{u=(u_1,u_2)'} \quad \int_0^2 {u_1}^2 + {u_2}^2 \, dt$$

subject to the control system

$$\begin{cases} \dot{x}_1 = t^2 x_2 + u_2 \, , \\ \dot{x}_2 = -t^2 x_1 + u_1 \, , \end{cases} \tag{19}$$

and the interpolating conditions

$$x(t_0 = 0) = (0, 0)', \quad x(t_1 = 1) = (1, 0.5)', \quad x(t_2 = 2) = (-0.25, 1)' .$$

As far as the control system (19) is non-autonomous, this example is not covered by the results in [11]. We have the time interval $[0, 2]$ and its partition $\Delta : a = 0 < 1 <$

$2 = b$. The associated state transition matrix is given by

$$\Phi(t, t_i) = \begin{pmatrix} \cos\left(\frac{t^3 - t_i{}^3}{3}\right) & \sin\left(\frac{t^3 - t_i{}^3}{3}\right) \\ -\sin\left(\frac{t^3 - t_i{}^3}{3}\right) & \cos\left(\frac{t^3 - t_i{}^3}{3}\right) \end{pmatrix}.$$

The linear dynamic is completely state controllable. Such a conclusion follows immediately from the fact that the symmetric matrix

$$W = \int_{\tau_0}^{\tau_1} \Phi(\tau_0, s) B(s) B(s)' \Phi(\tau_0, s)' \, ds$$

is positive definite for some $\tau_1 > \tau_0$ with $\tau_0, \tau_1 \in [0, 2]$. This is a classical test for complete controllability which is due to Kalman [8]. Since B and Φ are orthogonal matrices, the matrix W is simply

$$\begin{pmatrix} \tau_1 - \tau_0 & 0 \\ 0 & \tau_1 - \tau_0 \end{pmatrix}.$$

The optimal control is, in each interval $[t_i, t_{i+1}]$, solution of the equation

$$L^* B u = 0 \Leftrightarrow \begin{cases} \dot{u}_1 + t^2 u_2 = 0 \\ \dot{u}_2 - t^2 u_1 = 0. \end{cases}$$

We get

$$u(t)_{|t \in [t_i, t_{i+1}]} = \begin{pmatrix} -\sin\left(\frac{t^3 - t_i{}^3}{3}\right) c_{1i} + \cos\left(\frac{t^3 - t_i{}^3}{3}\right) c_{2i} \\ \cos\left(\frac{t^3 - t_i{}^3}{3}\right) c_{1i} + \sin\left(\frac{t^3 - t_i{}^3}{3}\right) c_{2i} \end{pmatrix} \tag{20}$$

where c_{1i} and c_{2i} are real constants to be found. The corresponding generalized spline in $\mathbb{R}^2$, solution of equation

$$L^* L x = 0 \Leftrightarrow \begin{cases} \ddot{x}_1 - 2t^2 \dot{x}_2 - t^4 x_1 - 2t x_2 = 0 \\ \ddot{x}_2 + 2t^2 \dot{x}_1 + 2t x_1 - t^4 x_2 = 0 \end{cases}$$

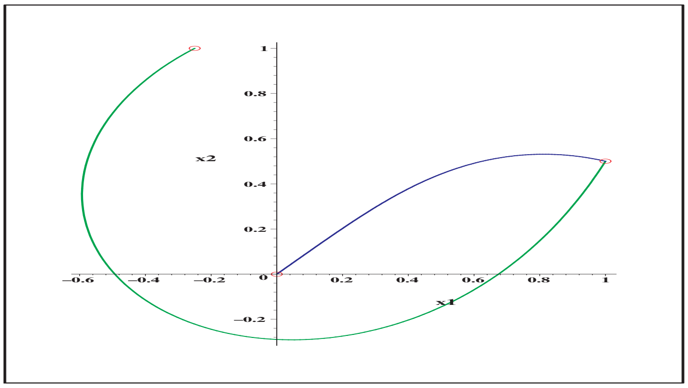
in each interval $[t_i, t_{i+1}]$, is given by $x(t)_{|t \in [t_i, t_{i+1}]} = (x_1(t), x_2(t))'$ with

$$x_1(t) = \cos\left(\frac{t^3 - t_i{}^3}{3}\right)(x_1(t_i) + (t - t_i)\, c_{1i}) + \sin\left(\frac{t^3 - t_i{}^3}{3}\right)(x_2(t_i) + (t - t_i)\, c_{2i})$$

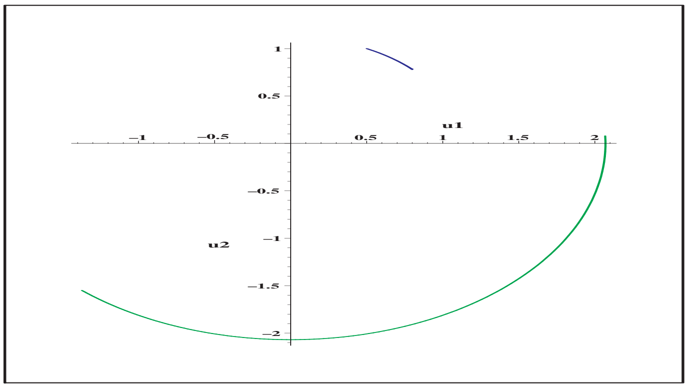
and

$$x_2(t) = -\sin\left(\frac{t^3 - t_i{}^3}{3}\right)(x_1(t_i) + (t - t_i)\, c_{1i}) + \cos\left(\frac{t^3 - t_i{}^3}{3}\right)(x_2(t_i) + (t - t_i)\, c_{2i})$$

where c_{1i} and c_{2i} are the same constants that appear in formula (20). As expected, the resulting spline is a continuous vector function and the optimal control function is discontinuous at $t = t_1$.



First example – generalized spline in $\mathbb{R}^2$



First example – optimal control

We now apply our results to the autonomous situation treated in [11]. Consider the optimal control problem

$$\min_{u=(u_1,u_2)'} \quad \int_0^4 {u_1}^2 + 2u_1u_2 + 2{u_2}^2\, dt$$

subject to

$$\begin{cases} \dot{x}_1 = -x_2 + u_2 \\ \dot{x}_2 = 2x_1 + u_1 + u_2, \end{cases}$$
$$x(t_0 = 0) = (0,0)', \quad x(t_1 = 1) = (1, 0.5)',$$
$$x(t_2 = 2) = (-0.25, 1)' \quad \text{and} \quad x(t_3 = 4) = (1, -1)'.$$

The optimal state trajectory is the generalized spline which, in each interval $[t_i, t_{i+1}]$, is solution of equation

$$L^*Lx = 0 \Leftrightarrow \ddot{x} + (A' - A)\,\dot{x} - (A'A)\,x = 0 \Leftrightarrow \begin{cases} \ddot{x}_1 + 3\dot{x}_2 - 4x_1 = 0 \\ \ddot{x}_2 - 3\dot{x}_1 - x_2 = 0. \end{cases}$$

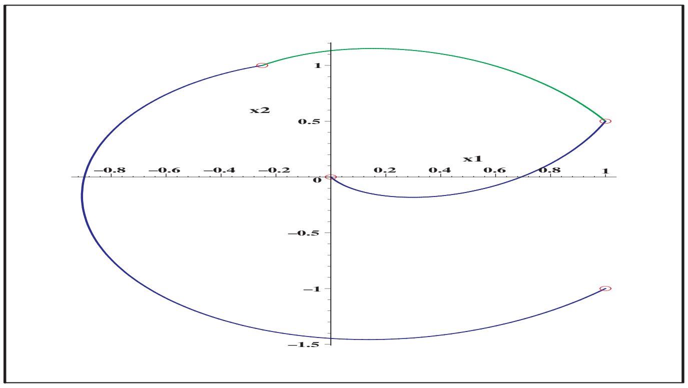
We get, in each interval $[t_i, t_{i+1}]$,

$$x_1(t) = \sin(\sqrt{2}\,t)(\tfrac{3t}{4}c_{1i} + c_{4i}) + \cos(\sqrt{2}\,t)(\tfrac{3\sqrt{2}}{8}c_{1i} + \tfrac{3t}{4}c_{2i} + c_{3i})$$

and

$$x_2(t) = \sin(\sqrt{2}\,t)(c_{1i} + \tfrac{3\sqrt{2}\,t}{4}c_{2i} + \sqrt{2}\,c_{3i}) + \cos(\sqrt{2}\,t)(-\tfrac{3\sqrt{2}\,t}{4}c_{1i} + \tfrac{1}{4}c_{2i} + \sqrt{2}\,c_{4i})$$

where c_{1i}, c_{2i}, c_{3i} and c_{4i} are real constants to be found in each interval.



Second example – generalized spline in $\mathbb{R}^2$

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