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NEW LINKS AND REDUCTIONS BETWEEN THE BROCKETT NONHOLONOMIC INTEGRATOR AND RELATED SYSTEMS

Abstract. We briefly review recently developed techniques for dealing with a class of time-dependent systems defined on Lie groups and homogeneous spaces. The techniques to treat these systems on Lie groups are a generalization of the Wei–Norman method and a reduction theorem. We apply the theory to the well-known Brockett nonholonomic integrator and to some other related systems: its generalization to second degree and another system with the same associated Lie algebra, and the generalization to third degree of the nonholonomic integrator. New relations between these systems are found by means of the geometric methods considered.

1. Introduction

The interest of differential geometric methods in control theory has been increasing in the last decades, probably due to the fact that these methods provide a deep insight into the nature and structure of problems of control theory, and they are also a very useful tool for the treatment of such problems.

This article continues a line of research in this direction, showing that some problems of geometric control theory can be related in a natural way by making use of a recently developed geometric theory for a certain class of systems defined on Lie groups and homogeneous spaces. The main novelties of this theory are two. The first is the generalization of the Wei–Norman method to systems defined on Lie groups which are not necessarily linear. The second and more important novelty consists of a reduction theorem which allows to reduce the solutions of a system on a Lie group G to that of a Lie subgroup H, provided we find a particular solution of the associated system on the homogeneous space G/H. Prior publications to this respect are [4, 9, 10], based on the theory of [5, 6, 8].

More specifically, we apply the theory to the well-known Brockett nonholonomic integrator [2], and its generalizations to second and third degree [3]. We also point out the relation of other systems in the literature with the generalization to second degree [14, 15, 16].

The paper is organized as follows. In Section 2 we briefly review the relation between certain problems formulated on Lie groups and homogeneous spaces. Then, we describe the above-mentioned generalized Wei–Norman method (see [17, 18] for the original formulation) and reduction theorem. Section 3 is devoted to the application of this theory to the mentioned examples. Firstly, we identify the Lie algebraic structure generated by the input vector fields of each system of interest. By application of the reduction theorem, we find, in particular, previously unknown relations amongst the considered systems.

2. Right-invariant systems on Lie groups and their projection on homogeneous spaces

Consider an r-dimensional connected Lie group G, and a basis $\{a_1, \ldots, a_r\}$ of the tangent space T_eG at the neutral element $e \in G$. We will denote the left and right translation maps by $L_g(g') = gg'$, $R_g(g') = g'g$, $\forall g' \in G$. The differential of a differentiable map $F : M \to N$ (resp. at $p \in M$) will be denoted by F_* (resp. F_{*p}). The Lie algebra $\mathfrak{g}$ of G consists of the subalgebra of vector fields X^L in G which are left-invariant, i.e., $L_{g*g'}(X^L)_{g'} = (X^L)_{gg'}$. They are defined by their values at the identity: if $(X^L)_e = a \in T_eG$, then $(X^L)_g = L_{g*e}(a)$. A basis of $\mathfrak{g}$ is $\{X^L_\alpha\}_{\alpha=1}^r$, where $(X^L_\alpha)_g = L_{g*e}(a_\alpha)$, $\alpha = 1, \ldots, r$. Therefore, the tangent space T_eG at the identity can be biunivocally identified with $\mathfrak{g}$, and the Lie algebra structure of the latter can be transported to the former.

Similarly, a basis of the set of right-invariant vector fields in G is $\{X^R_\alpha\}_{\alpha=1}^r$, where $(X^R_\alpha)_g = R_{g*e}(a_\alpha)$, $\alpha = 1, \ldots, r$. Left- and right-invariant vector fields generate opposite Lie algebras.

The solutions of right-invariant systems of type

$$\dot{g}(t)\, g^{-1}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ , \tag{1}$$

where the left hand side should be understood in general as $R_{g^{-1}(t)*g(t)}(\dot{g}(t))$, and $b_\alpha(t)$ are some t-dependent real functions, can be obtained by right translations of the only curve $g(t)$ solution of (1) starting from $e \in G$. In other words, $g(t)$ is the integral curve starting from $e \in G$ of the t-dependent vector field in G

$$X(g, t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, (X^R_\alpha)_g \ , \tag{2}$$

and the integral curve $\overline{g}(t)$ starting from g_0 will be $\overline{g}(t) = g(t)g_0$.

Let us consider now a transitive action $\Phi : G \times M \to M$ of G on a manifold M (which is then called a homogeneous space of G). Each choice of a point $x_0 \in M$ allows us to identify M with a space of left cosets, i.e., $M = G/H$, where H is the stability subgroup of the point x_0 with respect to Φ. Different choices for x_0 lead to conjugate subgroups. We recall that G can be regarded as the total space of the principal bundle $(G, \tau, G/H)$ over G/H, where $\tau : G \to G/H$ is the canonical projection.

The important point now [5] is that the right-invariant vector fields X^R_α are τ-projectable, the projections being the fundamental vector fields in M associated to the left action Φ. More explicitly, we have $X_\alpha(x) = X_{a_\alpha}(x) = \Phi_{x*e}(-a_\alpha)$, where $x = gH$, and $\tau_{*g} X^R_\alpha(g) = -X_\alpha(gH)$. Consequently, the vector field (2) projects onto the t-dependent vector field in M

$$X(x, t) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x) \ ,$$

giving rise in this way to a system of differential equations

$$\dot{x}^i = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha^i(x)\,, \qquad i = 1, \dots, n = \dim M\,. \tag{3}$$

This relation between both systems tell us that the solution of (3) starting from the point $x_0 \in M$, *arbitrary* but fixed, is given by $x(t) = \Phi(g(t), x_0)$, where $g(t)$ is the solution of (1) starting from the identity, compare with [13]. Note that the vector fields arising in (3) generate a Lie algebra isomorphic to $\mathfrak{g}$.

Conversely, if a system of type (3) is defined by complete vector fields generating a finite dimensional Lie algebra $\mathfrak{g}$, then it can be regarded as a system of the described type, taking a Lie group G such that the vector fields are the fundamental vector fields of its action on the manifold M given by the integration of their flows. We will call systems of type (1) on a Lie group G (resp. of type (3) on an homogeneous space M) Lie systems on G (resp. on M).

In this sense equation (1) has a universal character, since the knowledge of its solution starting from the identity is enough to write the general solution of each associated system of type (3) on each homogeneous space of G. Unfortunately, finding the desired particular solution of (1) may be a difficult problem.

We briefly recall here two methods for dealing with such equations. The first one is based on a generalization [6, 8] of the method proposed by Wei and Norman for linear systems [17, 18]. The second one is based on a reduction theorem allowing us to reduce the problem to a similar one in a subgroup, when a solution of the associated problem in the corresponding homogeneous space is known [5].

2.1. The generalized Wei–Norman method and the reduction theorem

The main idea of the generalization of the Wei–Norman method consists on writing the curve $g(t)$ solution of (1) starting from the identity $e \in G$, in terms of the second kind canonical coordinates associated to a basis of the Lie algebra $\mathfrak{g}$. That is, we write $g(t) = \prod_{\alpha=1}^{r} \exp(-v_\alpha(t) a_\alpha)$, and then, the differential equation (1) transforms into a system of differential equations for the $v_\alpha(t)$, given by the expression [8]

$$\sum_{\alpha=1}^{r} \dot{v}_\alpha \left(\prod_{\beta<\alpha} \exp(-v_\beta(t)\, \mathrm{ad}(a_\beta)) \right) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha\,, \tag{4}$$

with initial conditions $v_\alpha(0) = 0$, $\alpha = 1, \dots, r$. The resulting system of differential equations for the functions $v_\alpha(t)$ is integrable by quadratures if the Lie algebra is solvable [17, 18], and in particular, for nilpotent Lie algebras. Note that in (4) only the abstract commutation relations of the Lie algebra $\mathfrak{g}$ are needed, and that the appearing exponentials are just matrix exponentials.

On the other hand, we can show that the problem of finding the solution of (1) starting at $e \in G$ can be reduced to the problem of solving a similar equation in a subgroup H, provided that one particular solution $\tilde{g}_1(t)$ for the system associated to the left action Φ of G on the homogeneous space $M = G/H$ is given. The result is as follows [5]:

THEOREM 1. *Every integral curve of the time-dependent vector field (2) in the group G can be written in the form $g(t) = g_1(t)\, h(t)$, where $g_1(t)$ is a curve projecting onto a solution $\tilde{g}_1(t)$ of the system of type (3) associated to the left action Φ on the homogeneous space G/H, and $h(t)$ is a solution of an equation of type (1) but for the subgroup H, given explicitly by*

$$(\dot{h}\, h^{-1})(t) = -\operatorname{Ad}(g_1^{-1}(t))\left(\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha + (\dot{g}_1\, g_1^{-1})(t)\right) \in T_e H\ . \tag{5}$$

This result allows us to understand from a group theoretical point of view, as an example, the classical integration theorems for the Riccati equation involving one, two or three of its particular solutions [7], see also [5, 8].

3. Applications for Brockett's nonholonomic integrator and related systems

3.1. Brockett nonholonomic control system

The nonholonomic integrator, introduced by Brockett (see, e.g., [2]), is well-known to be related to the three-dimensional Heisenberg–Weyl group $H(3)$, which is the lowest-dimensional non-Abelian nilpotent Lie group. Such a system is very often considered as one of the prototypical examples relating optimal control theory and extremal problems in singular or sub-Riemannian geometry, see, e.g., [1].

It consists of the control system in $\mathbb{R}^3$, with coordinates $(x,\ y,\ z)$

$$\dot{x} = b_1(t)\,, \quad \dot{y} = b_2(t)\,, \quad \dot{z} = b_2(t)x - b_1(t)y\,, \tag{6}$$

where the functions $b_1(t)$ and $b_2(t)$ are regarded as the controls. The input vector fields are

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\,, \qquad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,,$$

and taking the Lie bracket $X_3 = [X_1,\ X_2] = 2\frac{\partial}{\partial z}$ we obtain that the set $\{X_1,\ X_2,\ X_3\}$ generates the full tangent space at every point in $\mathbb{R}^3$ and the Lie algebra defined by

$$[X_1,\ X_2] = X_3\,, \qquad [X_1,\ X_3] = 0\,, \qquad [X_2,\ X_3] = 0\,, \tag{7}$$

isomorphic to the Lie algebra $\mathfrak{h}(3)$ of the Heisenberg–Weyl group $H(3)$. We will denote by $\{a_1,\ a_2,\ a_3\}$ a basis of the abstract Lie algebra $\mathfrak{h}(3)$ with commutation relations analogous to (7). Thus, the Lie system in $H(3)$ of the form (1) associated to (6) is

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = -b_1(t)a_1 - b_2(t)a_2\,. \tag{8}$$

Writing its solution, starting from the identity, as the product of exponentials

$$g(t) = \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3)\ ,$$

and applying the Wei–Norman formula (4) we find the system of differential equations $\dot{v}_1 = b_1$, $\dot{v}_2 = b_2$, $\dot{v}_3 = b_2\, v_1$, with initial conditions $v_1(0) = v_2(0) = v_3(0) = 0$. The solution can be found immediately:

(9)
$$v_1(t) = \int_0^t b_1(s)\, ds\,, \quad v_2(t) = \int_0^t b_2(s)\, ds\,, \quad v_3(t) = \int_0^t b_2(s) \int_0^s b_1(r)\, dr\, ds\,.$$

This solution can be used in order to find the general solution of the given system (6). We only need to find a suitable parametrization of the Lie group $H(3)$, and the expression of the group action with respect to which the original vector fields are the fundamental vector fields. If we take the canonical coordinates of second kind defined by $g = \exp(aa_1)\exp(ba_2)\exp(ca_3)$, when $g \in H(3)$, it can be shown that such an action reads

$$\begin{array}{rcl} \Phi : H(3) \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ ((a,\, b,\, c),\, (x,\, y,\, z)) & \longmapsto & (x - a,\; y - b,\; z + ay - bx - ab - 2c)\,, \end{array}$$

the group law being expressed as $(a,\, b,\, c)(a',\, b',\, c') = (a + a',\, b + b',\, c + c' - ba')$. Then, the general solution of (6) is $\Phi((-v_1,\, -v_2,\, -v_3),\, (x_0,\, y_0,\, z_0)) = (x_0 + v_1,\; y_0 + v_2,\; z_0 + x_0 v_2 - y_0 v_1 - v_1 v_2 + 2v_3)$, where $v_1 = v_1(t)$, $v_2 = v_2(t)$, and $v_3 = v_3(t)$ are given by (9).

3.2. Some generalizations of Brockett's system

The nonholonomic integrator can be generalized or extended in several ways. This is the main subject of [3], in which mainly two ideas for the generalization of (6) are considered. One is to enlarge both the number of controls and the dimension of the state space in order to obtain a system of type

$$\dot{x} = b(t)\,, \quad \dot{Z} = x b^T(t) - b(t) x^T\,,$$

where x and $b(t)$ are curves in $\mathbb{R}^m$, with $m > 2$. The vectorial function $b(t)$ is the control of the system. The superscript ^{T} denotes matrix transposition, and Z is a $m \times m$ skew-symmetric matrix. This problem has been discussed also in [2].

The second general possibility considered in [3] is to enlarge the state space in order to account for higher nonlinear effects, where controllability is achieved by taking higher order Lie brackets, and eventually enlarging also the number of controls used. Depending on the number of these controls, and on the degree of the polynomial coefficients entering in the input vector fields, different hierarchies of nonholonomic control systems are constructed through a specific procedure, see [3] for details.

We will focus on two of the examples arising from the hierarchy so constructed with two control functions. These examples have been studied in [3], also in relation with the associated optimal control problems. However, our study will focus on the geometric aspects related to the theory of Section 2.

In particular, we will show that these examples are Lie systems with associated nilpotent Lie algebras of dimension five and seven, respectively. In addition, we show that the seven-dimensional Lie algebra can be regarded as a central extension of the five-dimensional one by $\mathbb{R}^2$. In turn, the five-dimensional Lie algebra can be seen as a central extension of $\mathfrak{h}(3)$ by $\mathbb{R}^2$. Therefore, we can relate these systems by means of the reduction Theorem 1. Thus, we will establish previously unknown relations of the control problem with the seven- or five-dimensional Lie algebra with the ordinary nonholonomic integrator.

Generalization to second degree of Brockett system

The first example to be considered now belongs to the hierarchy constructed in [3] for the case of two controls, and is the member with polynomial coefficient functions of the input vector fields of at most second degree. We will use a slightly different notation from that used in the cited reference.

The system of interest is the control system in $\mathbb{R}^5$, with coordinates $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)$

$$\begin{aligned} &\dot{x}_1 = b_1(t)\,, \quad \dot{x}_2 = b_2(t)\,, \quad \dot{x}_3 = b_2(t)x_1 - b_1(t)x_2\,, \\ &\dot{x}_4 = b_2(t)x_1^2\,, \quad \dot{x}_5 = b_1(t)x_2^2\,, \end{aligned} \tag{10}$$

where $b_1(t)$ and $b_2(t)$ are the control functions. This system appears as well as an approximation of the so-called plate-ball nonholonomic kinematic problem [3], which consists of a sphere rolling without slipping between two horizontal, flat and parallel plates which are separated by a distance equal to the diameter of the sphere. It is assumed that one of the plates is fixed in space and that the ball rolls because of the horizontal movement of the other plate. The geometry and the optimal control solutions of this problem have been considered in [11, 12]. In particular, it has been shown that the optimal control problem is integrable by elliptic functions, as it is the case for (10), see [3].

Now, the system (10) is of the form (3), with input vector fields $X_1 = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} + x_2^2\frac{\partial}{\partial x_5}$, $X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4}$. Taking the Lie brackets $X_3 = [X_1,\ X_2] = 2\frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} - 2x_2\frac{\partial}{\partial x_5}$, $X_4 = [X_1,\ X_3] = 2\frac{\partial}{\partial x_4}$, $X_5 = [X_2,\ X_3] = -2\frac{\partial}{\partial x_5}$, we obtain a set of vector fields which span the tangent space at each point of $\mathbb{R}^5$, therefore the system (10) is controllable. Moreover, the set $\{X_1,\ X_2,\ X_3,\ X_4,\ X_5\}$ generates the nilpotent Lie algebra defined by

$$[X_1,\ X_2] = X_3\,, \qquad [X_1,\ X_3] = X_4\,, \qquad [X_2,\ X_3] = X_5\,, \tag{11}$$

all other Lie brackets being zero. We will denote by $\{a_1,\ a_2,\ a_3,\ a_4,\ a_5\}$ a basis of the abstract Lie algebra, denoted as $\mathfrak{g}_5$, with commutation relations analogous to (11).

It is important to note that $\mathfrak{g}_5$ can be regarded as a central extension of the Lie algebra $\mathfrak{h}(3)$ by $\mathbb{R}^2$. Indeed, the center $\mathfrak{z}$ of $\mathfrak{g}_5$ is the Abelian subalgebra generated by $\{a_4,\ a_5\}$, and the factor Lie algebra $\mathfrak{g}_5/\mathfrak{z}$ is isomorphic to $\mathfrak{h}(3)$, see (7).

Analogously to what we have done in Section 3.1, we will treat briefly the Wei–Norman problem associated to the system (10), and will give the expression of the action with respect to which the vector fields X_1, X_2, X_3, X_4 and X_5 are infinitesimal generators. Afterwards, we will perform the reduction of the system (10) to another two: one of type (6), and one Lie system in $\mathbb{R}^2$.

Let us denote by G_5 the connected and simply connected nilpotent Lie group such that its Lie algebra is $\mathfrak{g}_5$. A generic Lie system of type (1) on G_5 is of the form

$$R_{g(t)^{-1}*g(t)}(\dot g(t)) = -b_1(t)a_1 - b_2(t)a_2 - b_3(t)a_3 - b_4(t)a_4 - b_5(t)a_5\,, \tag{12}$$

meanwhile that corresponding to the system (10) reads

$$R_{g(t)^{-1}*g(t)}(\dot g(t)) = -b_1(t)a_1 - b_2(t)a_2\,. \tag{13}$$

To solve (13) by the Wei–Norman method, we need to compute the adjoint representation of the Lie algebra $\mathfrak{g}_5$ with respect to the above basis. The corresponding matrices have elements

$$\begin{gathered}
[\mathrm{ad}(a_1)]_{jk} = \delta_{j3}\delta_{k2} + \delta_{j4}\delta_{k3}\,, \quad [\mathrm{ad}(a_2)]_{jk} = -\delta_{j3}\delta_{k1} + \delta_{j5}\delta_{k3}\,, \\
[\mathrm{ad}(a_3)]_{jk} = -\delta_{j4}\delta_{k1} - \delta_{j5}\delta_{k2}\,, \quad [\mathrm{ad}(a_4)]_{jk} = 0\,, \quad [\mathrm{ad}(a_5)]_{jk} = 0\,,
\end{gathered}$$

and therefore

$$\begin{aligned}
\exp(-v_1\,\mathrm{ad}(a_1)) &= \mathrm{Id} - v_1\ \mathrm{ad}(a_1) + \frac{v_1^2}{2}\,\mathrm{ad}(a_1)\circ\mathrm{ad}(a_1)\,, \\
\exp(-v_2\,\mathrm{ad}(a_2)) &= \mathrm{Id} - v_2\ \mathrm{ad}(a_2) + \frac{v_2^2}{2}\,\mathrm{ad}(a_2)\circ\mathrm{ad}(a_2)\,, \\
\exp(-v_3\,\mathrm{ad}(a_3)) &= \mathrm{Id} - v_3\ \mathrm{ad}(a_3)\,, \\
\exp(-v_4\,\mathrm{ad}(a_4)) &= \mathrm{Id}\,, \qquad \exp(-v_5\,\mathrm{ad}(a_5)) = \mathrm{Id}\ .
\end{aligned}$$

Writing the solution which starts from the identity, of (13), as the product

$$g(t) = \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3)\exp(-v_4(t)a_4)\exp(-v_5(t)a_5) \tag{14}$$

and applying (4), we find the system of differential equations

$$\dot v_1 = b_1(t)\,, \quad \dot v_2 = b_2(t)\,, \quad \dot v_3 = b_2(t)v_1\,, \quad \dot v_4 = \frac{1}{2}b_2(t)v_1^2\,, \quad \dot v_5 = b_2(t)v_1v_2\,, \tag{15}$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = 0$. The solution can be found by quadratures; if we denote $B_i(t) = \int_0^t b_i(s)\,ds$, $i = 1,\ 2$, the solution reads

$$\begin{gathered}
v_1(t) = B_1(t)\,, \quad v_2(t) = B_2(t)\,, \quad v_3(t) = \int_0^t b_2(s)B_1(s)\,ds\,, \\
v_4(t) = \frac{1}{2}\int_0^t b_2(s)B_1^2(s)\,ds\,, \quad v_5(t) = \int_0^t b_2(s)B_1(s)B_2(s)\,ds\,.
\end{gathered} \tag{16}$$

We would like to remark that the system (15) is closely related to the system appearing in Examples 8.1 of [14] and 6.1 of [15], following an approach different to ours. Indeed, the system of such examples is essentially the Wei–Norman system corresponding to the equation (12) in the group G_5, when $b_5(t) = 0$, and to the factorization (14). This last system can be found as well by direct application of (4).

Following steps analogous to those of Section 3.1, we find the following expressions. Parametrizing the elements $g \in G_5$ as $g = \exp(aa_1 + ba_2 + ca_3 + da_4 + ea_5)$, the action of G_5 on $\mathbb{R}^5$ such that X_i be the infinitesimal generator associated to a_i for each $i \in \{1, \ldots, 5\}$ reads

$$\begin{aligned} \Phi : G_5 \times \mathbb{R}^5 &\longrightarrow \mathbb{R}^5 \\ ((a, b, c, d, e), (x_1, x_2, x_3, x_4, x_5)) &\longmapsto (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)\,, \end{aligned}$$

where

$$\begin{aligned} &\bar{x}_1 = x_1 - a\,, \quad \bar{x}_2 = x_2 - b\,, \quad \bar{x}_3 = x_3 + ax_2 - bx_1 - 2c\,, \\ &\bar{x}_4 = x_4 - bx_1^2 + (ab - 2c)x_1 + ac - 2d - ba^2/3\,, \\ &\bar{x}_5 = x_5 - ax_2^2 + (ab + 2c)x_2 - bc + 2e - ab^2/3\,, \end{aligned}$$

meanwhile the composition law

$$(a, b, c, d, e)(a', b', c', d', e') = (a'', b'', c'', d'', e'')$$

is given by

$$\begin{aligned} &a'' = a + a'\,, \quad b'' = b + b'\,, \quad c'' = c + c' + (ab' - ba')/2\,, \\ &d'' = d + d' + (ac' - ca')/2 + (a - a')(ab' - ba')/12\,, \\ &e'' = e + e' + (bc' - cb')/2 + (b - b')(ab' - ba')/12\,, \end{aligned} \tag{17}$$

and the neutral element is represented by $(0, 0, 0, 0, 0)$.

If, instead, we parametrize the group elements $g \in G_5$ by the second kind canonical coordinates defined by $g = \exp(aa_1)\exp(ba_2)\exp(ca_3)\exp(da_4)\exp(ea_5)$, the action reads

$$\begin{aligned} \Phi : G_5 \times \mathbb{R}^5 &\longrightarrow \mathbb{R}^5 \\ ((a, b, c, d, e), (x_1, x_2, x_3, x_4, x_5)) &\longmapsto (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)\,, \end{aligned}$$

where

$$\begin{aligned} &\bar{x}_1 = x_1 - a\,, \quad \bar{x}_2 = x_2 - b\,, \quad \bar{x}_3 = x_3 + ax_2 - bx_1 - 2c - ab\,, \\ &\bar{x}_4 = x_4 - bx_1^2 - 2cx_1 - 2d\,, \\ &\bar{x}_5 = x_5 - ax_2^2 + 2(ab + c)x_2 + 2e - ab^2\,, \end{aligned} \tag{18}$$

and the composition law $(a, b, c, d, e)(a', b', c', d', e') = (a'', b'', c'', d'', e'')$ is given by

$$\begin{aligned} &a'' = a + a'\,, \quad b'' = b + b'\,, \quad c'' = c + c' - ba'\,, \\ &d'' = d + d' - ca' + ba'^2/2\,, \\ &e'' = e + e' - cb' + ba'b' + b^2a'/2\,, \end{aligned} \tag{19}$$

the neutral element being represented as well by $(0,\ 0,\ 0,\ 0,\ 0)$ in these coordinates.

The relation between the first kind canonical coordinates $(a_1,\ b_1,\ c_1,\ d_1,\ e_1)$ and the second kind canonical coordinates $(a_2,\ b_2,\ c_2,\ d_2,\ e_2)$ so defined of the same group element $g \in G_5$ is

$$\begin{aligned} &a_1 = a_2\,, \quad b_1 = b_2\,, \quad c_1 = c_2 + \frac{1}{2}a_2 b_2\,, \\ &d_1 = d_2 + \frac{1}{2}a_2 c_2 + \frac{1}{12}a_2^2 b_2\,, \quad e_1 = e_2 + \frac{1}{2}b_2 c_2 - \frac{1}{12}a_2 b_2^2\,. \end{aligned}$$

The general solution of (10) can be calculated by means of the solution of the Wei–Norman system (15) as

$$\Phi((-v_1, -v_2, -v_3, -v_4, -v_5), (x_{10}, x_{20}, x_{30}, x_{40}, x_{50})) = (x_1, x_2, x_3, x_4, x_5),$$

where Φ is that of (18), $v_i = v_i(t)$, $i = 1, \ldots, 5$, are given by (16) and the vector $(x_{10},\ x_{20},\ x_{30},\ x_{40},\ x_{50}) \in \mathbb{R}^5$ denotes the initial conditions. It can be checked that the direct integration of (10) gives the same result.

There exists in the literature another control system with the same underlying Lie algebra as (10), see Example 2 of [16]. With a slightly different notation, it is the control system in $\mathbb{R}^5$, with coordinates $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)$

$$\begin{aligned} &\dot{x}_1 = b_1(t)\,, \quad \dot{x}_2 = b_2(t)\,, \quad \dot{x}_3 = b_2(t)x_1\,, \\ &\dot{x}_4 = b_2(t)x_1^2\,, \quad \dot{x}_5 = 2\,b_2(t)x_1x_2\,. \end{aligned}$$

This system is of the form (15), with the simple identification $x_1 = v_1$, $x_2 = v_2$, $x_3 = v_3$, $x_4 = 2v_4$ and $x_5 = 2v_5$. Analogous calculations to those above can be done for this case, with similar results.

We will show now the way the reduction Theorem 1 of Lie systems allows us to reduce the problem of solving (13), and hence of solving (10), to solving two other Lie systems: one of Brockett type (6), and another on the center of G_5, which can be identified with $\mathbb{R}^2$.

Using the canonical coordinates of first kind in G_5 defined by $g = \exp(aa_1 + ba_2 + ca_3 + da_4 + ea_5)$, and the composition law (17), we obtain that the adjoint representation of the group reads

$$\mathrm{Ad}(a,\ b,\ c,\ d,\ e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -b & a & 1 & 0 & 0 \\ -\frac{ab}{2} - c & \frac{a^2}{2} & a & 1 & 0 \\ -\frac{b^2}{2} & \frac{ab}{2} - c & b & 0 & 1 \end{pmatrix}. \tag{20}$$

If $g(t) = (a(t),\ b(t),\ c(t),\ d(t),\ e(t))$ is a curve in G_5 expressed in the previous coordinates, we obtain

$$
L_{g^{-1}*g}(\dot g) = \begin{pmatrix} \dot a \\ \dot b \\ \dot c + \frac{1}{2}(b\dot a - a\dot b) \\ \dot d + \frac{1}{6}(3c - ab)\dot a + \frac{1}{6}a^2\dot b - \frac{1}{2}a\dot c \\ \dot e - \frac{1}{6}b^2\dot a + \frac{1}{6}(3c + ab)\dot b - \frac{1}{2}b\dot c \end{pmatrix},
$$

(21)

$$
R_{g^{-1}*g}(\dot g) = \begin{pmatrix} \dot a \\ \dot b \\ \dot c - \frac{1}{2}(b\dot a - a\dot b) \\ \dot d - \frac{1}{6}(3c + ab)\dot a + \frac{1}{6}a^2\dot b + \frac{1}{2}a\dot c \\ \dot e - \frac{1}{6}b^2\dot a - \frac{1}{6}(3c - ab)\dot b + \frac{1}{2}b\dot c \end{pmatrix}.
$$

To perform the reduction we select the subgroup H of G_5 whose Lie algebra is the center $\mathfrak{z}$ of $\mathfrak{g}_5$, which is generated by $\{a_4, a_5\}$. Then, $\mathfrak{g}_5/\mathfrak{z} \cong \mathfrak{h}(3)$ and $G_5/H \cong H(3)$. We use the factorization

$$
(a, b, c, d, e) = (a, b, c, 0, 0)(0, 0, 0, d, e),
$$

therefore the projection reads

$$
\begin{array}{rcl} \tau : G_5 & \longrightarrow & G_5/H \\ (a, b, c, d, e) & \longmapsto & (a, b, c). \end{array}
$$

We take coordinates (y_1, y_2, y_3) in G_5/H. The left action of G_5 on G_5/H is then

$$
\begin{array}{rcl} \Phi : G_5 \times G_5/H & \longrightarrow & G_5/H \\ ((a, b, c, d, e), (y_1, y_2, y_3)) & \longmapsto & \tau((a, b, c, d, e)(y_1, y_2, y_3, d', e')) \\ & & = (y_1 + a, y_2 + b, y_3 + c + (ay_2 - by_1)/2), \end{array}
$$

where d' and e' are real numbers parametrizing a lift of (y_1, y_2, y_3) to G_5. The corresponding fundamental vector fields are

$$
\begin{aligned} X_1^H &= -\partial_{y_1} - \frac{y_2}{2}\partial_{y_3}, \quad X_2^H = -\partial_{y_2} + \frac{y_1}{2}\partial_{y_3}, \\ X_3^H &= -\partial_{y_3}, \quad X_4^H = 0, \quad X_5^H = 0, \end{aligned}
$$

which span the tangent space at each point of G_5/H, and in addition satisfy $[X_1^H, X_2^H] = X_3^H$, $[X_1^H, X_3^H] = X_4^H$ and $[X_2^H, X_3^H] = X_5^H$, or, more precisely, the commutation relations of the Heisenberg Lie algebra (7).

Now, we factorize the solution starting from the identity of (13) as the product

$$
g_1(t)h(t) = (y_1(t), y_2(t), y_3(t), 0, 0)(0, 0, 0, d(t), e(t)),
$$

where $g_1(t)$ projects onto the solution $\tau(g_1(t)) = (y_1(t),\ y_2(t),\ y_3(t))$, with initial conditions $(y_1(0),\ y_2(0),\ y_3(0)) = (0,\ 0,\ 0)$, of the Lie system on G_5/H associated to (13),

$$\dot{y}_1 = -b_1(t)\,, \quad \dot{y}_2 = -b_2(t)\,, \quad \dot{y}_3 = \frac{1}{2}(b_2(t)y_1 - b_1(t)y_2)\,. \tag{22}$$

Thus, we reduce the problem to a Lie system in the center of G_5 for $h(t) = (0,\ 0,\ 0,\ d(t),\ e(t))$, with $h(0) = e$, i.e., $d(0) = e(0) = 0$. The expression of this last system is given by the formula (5) in Theorem 1. Using (20) and (21) we finally obtain the system

$$\dot{d} = \frac{b_1(t)}{2}\left(\frac{1}{6}y_1(t)y_2(t) - y_3(t)\right) - \frac{1}{12}b_2(t)y_1^2(t)\,,$$
$$\dot{e} = \frac{1}{12}b_1(t)y_2^2(t) - \frac{b_2(t)}{2}\left(\frac{1}{6}y_1(t)y_2(t) + y_3(t)\right)\,,$$

which is a Lie system for $H \cong \mathbb{R}^2$, solvable by two quadratures.

In this sense the original system (10) can be reduced to the system (22), which becomes the Brockett system (6) under the simple change of coordinates $x = -y_1$, $y = -y_2$ and $z = -2y_3$, and then a system in the center of G_5, identifiable with $\mathbb{R}^2$.

Generalization to third degree of Brockett system

We consider now the example from [3] which belongs to the hierarchy constructed therein with two controls, being the member with polynomial coefficient functions of the input vector fields of at most third degree.

We will show that such a system is a Lie system defined on $\mathbb{R}^8$ with an associated Lie algebra which is nilponent and seven-dimensional, to be denoted $\mathfrak{g}_7$. Moreover, this Lie algebra will be shown to be a central extension by $\mathbb{R}^2$ of $\mathfrak{g}_5$ of the preceding example, and has a four-dimensional Abelian ideal $\mathfrak{i}$ such that the factor algebra constructed with it is just $\mathfrak{h}(3)$ again.

This suggests us to apply the reduction Theorem 1 in this case by selecting the subgroups whose Lie subalgebras are the center and $\mathfrak{i}$. We will focus on this last reduction, since we feel that is the most illuminating result. Of course one could perform instead the reduction with respect to the center of the Lie group, giving rise to a Lie system with associated Lie algebra $\mathfrak{g}_5$, or by using other subgroups, yielding different realizations on lower dimensional homogeneous spaces of the system below.

The system can be solved as well by the Wei–Norman method and the computation of the corresponding action, by integration of the vector fields of a basis of the Lie algebra generated by the input vector fields, as we have done for the system (10). The treatment is similar, though longer, and will be ommited here.

The system of interest is thus the control system in $\mathbb{R}^8$, with coordinates

$(x_1, \ldots, x_8)$ (see [3])

$$\begin{aligned} &\dot{x}_1 = b_1(t)\,, \quad \dot{x}_2 = b_2(t)\,, \quad \dot{x}_3 = b_2(t)x_1 - b_1(t)x_2\,, \\ &\dot{x}_4 = b_2(t)x_1^2\,, \quad \dot{x}_5 = b_1(t)x_2^2\,, \quad \dot{x}_6 = b_2(t)x_1^3\,, \\ &\dot{x}_7 = b_1(t)x_2^3\,, \quad \dot{x}_8 = b_1(t)x_1^2x_2 + b_2(t)x_1x_2^2\,, \end{aligned} \tag{23}$$

where $b_1(t)$ and $b_2(t)$ are the control functions, that is, a system of type (3), with input vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} + x_2^2\frac{\partial}{\partial x_5} + x_2^3\frac{\partial}{\partial x_7} + x_1^2x_2\frac{\partial}{\partial x_8}\,, \\ X_2 &= \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4} + x_1^3\frac{\partial}{\partial x_6} + x_1x_2^2\frac{\partial}{\partial x_8}\,. \end{aligned}$$

Taking the Lie brackets

$$\begin{aligned} X_3 = [X_1,\ X_2] &= 2\frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} - 2x_2\frac{\partial}{\partial x_5} \\ &\quad + 3x_1^2\frac{\partial}{\partial x_6} - 3x_2^2\frac{\partial}{\partial x_7} + (x_2^2 - x_1^2)\frac{\partial}{\partial x_8}\,, \\ X_4 = [X_1,\ X_3] &= 2\frac{\partial}{\partial x_4} + 6x_1\frac{\partial}{\partial x_6} - 2x_1\frac{\partial}{\partial x_8}\,, \end{aligned}$$

$$\begin{aligned} &X_5 = [X_2,\ X_3] = -2\frac{\partial}{\partial x_5} - 6x_2\frac{\partial}{\partial x_7} + 2x_2\frac{\partial}{\partial x_8}\,, \\ &X_6 = [X_1,\ X_4] = 6\frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_8}\,, \quad X_7 = [X_2,\ X_5] = -6\frac{\partial}{\partial x_7} + 2\frac{\partial}{\partial x_8}\,, \end{aligned}$$

we obtain a set of linearly independent vector fields $\{X_1, \ldots, X_7\}$ which generates the nilpotent Lie algebra defined by

$$\begin{aligned} &[X_1,\ X_2] = X_3\,, \qquad [X_1,\ X_3] = X_4\,, \qquad [X_1,\ X_4] = X_6\,, \\ &[X_2,\ X_3] = X_5\,, \qquad [X_2,\ X_5] = X_7\,, \end{aligned} \tag{24}$$

all other Lie brackets being zero. We will denote by $\{a_1, \ldots, a_7\}$ a basis of the abstract Lie algebra $\mathfrak{g}_7$ with commutation relations analogous to (24).

Now, the center $\mathfrak{z}$ of $\mathfrak{g}_7$ is the Abelian subalgebra generated by $\{a_6, a_7\}$, and the factor Lie algebra $\mathfrak{g}_7/\mathfrak{z}$ is isomorphic to $\mathfrak{g}_5$, see (11). Moreover, $\mathfrak{g}_7$ contains an Abelian four-dimensional ideal $\mathfrak{i}$ generated by $\{a_4, a_5, a_6, a_7\}$, such that the factor Lie algebra $\mathfrak{g}_7/\mathfrak{i}$ is isomorphic to $\mathfrak{h}(3)$, see (7). Finally, note that the maximal proper ideal $\mathfrak{i}_M$ contained in $\mathfrak{g}_7$, which is Abelian, is generated by $\{a_3, \ldots, a_7\}$, the quotient being $\mathfrak{g}_7/\mathfrak{i}_M \cong \mathbb{R}^2$. We will denote by G_7 the connected and simply connected nilpotent Lie group such that its Lie algebra is $\mathfrak{g}_7$.

As in previous cases, the set of vector fields $\{X_1, \ldots, X_7\}$ can be regarded as a basis of the Lie algebra of fundamental vector fields with respect to an action of G_7

on $\mathbb{R}^8$. However, they do not span the full tangent tangent space at each point of $\mathbb{R}^8$, thus the system (23) is controllable just orbit-wise: only configurations in the same orbit with respect to the previous action can be joined, e.g., by appropriately chosen piecewise constant controls.

We concentrate now on the task of reducing the system (23) into one of type (6) and other in $\mathbb{R}^4$. The right-invariant Lie system of type (1) on G_7 corresponding to the system (23) is

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = -b_1(t)a_1 - b_2(t)a_2\,, \tag{25}$$

which is similar to (13), but now a_1, a_2, are elements of the basis of $\mathfrak{g}_7$ considered above.

If we parametrize the elements $g \in G_7$ as $g = \exp(aa_1 + ba_2 + ca_3 + da_4 + ea_5 + fa_6 + ka_7)$, it can be checked that the composition law

$$(a,\, b,\, c,\, d,\, e,\, f,\, k)(a',\, b',\, c',\, d',\, e',\, f',\, k') = (a'',\, b'',\, c'',\, d'',\, e'',\, f'',\, k'')$$

reads in these coordinates

$$\begin{aligned}
&a'' = a + a'\,, \quad b'' = b + b'\,, \quad c'' = c + c' + (ab' - ba')/2\,,\\
&d'' = d + d' + (ac' - ca')/2 + (a - a')(ab' - ba')/12\,,\\
&e'' = e + e' + (bc' - cb')/2 + (b - b')(ab' - ba')/12\,,\\
&f'' = f + f' + (ad' - da')/2 + (a - a')(ac' - ca')/12 + aa'(ba' - ab')/24\,,\\
&k'' = k + k' + (be' - eb')/2 + (b - b')(bc' - cb')/12 + bb'(ba' - ab')/24\,,
\end{aligned}$$

the neutral element being represented by $(0,\, \ldots,\, 0)$.
The adjoint representation $\mathrm{Ad}(a,\, b,\, c,\, d,\, e,\, f,\, k)$ reads

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\
-b & a & 1 & 0 & 0 & 0 & 0\\
-\frac{ab}{2} - c & \frac{a^2}{2} & a & 1 & 0 & 0 & 0\\
-\frac{b^2}{2} & \frac{ab}{2} - c & b & 0 & 1 & 0 & 0\\
-\frac{a^2b}{6} - \frac{ac}{2} - d & \frac{a^3}{6} & \frac{a^2}{2} & a & 0 & 1 & 0\\
-\frac{b^3}{6} & \frac{ab^2}{6} - \frac{bc}{2} - e & \frac{b^2}{2} & 0 & b & 0 & 1
\end{pmatrix}. \tag{26}$$

If $g(t) = (a(t),\, b(t),\, c(t),\, d(t),\, e(t),\, f(t),\, k(t))$ is a curve in G_7 expressed in the previous coordinates, we obtain

$$L_{g^{-1}*g}(\dot{g}) = \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} + \frac{1}{2}(b\dot{a} - a\dot{b}) \\ \dot{d} + \frac{1}{6}(3c - ab)\dot{a} + \frac{1}{6}a^2\dot{b} - \frac{1}{2}a\dot{c} \\ \dot{e} - \frac{1}{6}b^2\dot{a} + \frac{1}{6}(3c + ab)\dot{b} - \frac{1}{2}b\dot{c} \\ \dot{f} + \frac{1}{24}(a^2b - 4ac + 12d)\dot{a} - \frac{1}{24}a^3\dot{b} + \frac{1}{6}a^2\dot{c} - \frac{1}{2}a\dot{d} \\ \dot{k} + \frac{1}{24}b^3\dot{a} - \frac{1}{24}(ab^2 + 4bc - 12e)\dot{b} + \frac{1}{6}b^2\dot{c} - \frac{1}{2}b\dot{e} \end{pmatrix},$$

(27)

$$R_{g^{-1}*g}(\dot{g}) = \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} - \frac{1}{2}(b\dot{a} - a\dot{b}) \\ \dot{d} - \frac{1}{6}(3c + ab)\dot{a} + \frac{1}{6}a^2\dot{b} + \frac{1}{2}a\dot{c} \\ \dot{e} - \frac{1}{6}b^2\dot{a} - \frac{1}{6}(3c - ab)\dot{b} + \frac{1}{2}b\dot{c} \\ \dot{f} - \frac{1}{24}(a^2b + 4ac + 12d)\dot{a} + \frac{1}{24}a^3\dot{b} + \frac{1}{6}a^2\dot{c} + \frac{1}{2}a\dot{d} \\ \dot{k} - \frac{1}{24}b^3\dot{a} + \frac{1}{24}(ab^2 - 4bc - 12e)\dot{b} + \frac{1}{6}b^2\dot{c} + \frac{1}{2}b\dot{e} \end{pmatrix}.$$

As already indicated, to perform the reduction we select the subgroup H of G_7 whose Lie algebra is the ideal $\mathfrak{i}$ of $\mathfrak{g}_7$ generated by $\{a_4, a_5, a_6, a_7\}$. Then, $G_7/H \cong H(3)$. Taking into account the factorization

$$(a, b, c, d, e, f, k) = (a, b, c, 0, 0, 0, 0)(0, 0, 0, d, e, f, k),$$

the projection reads

$$\begin{array}{rcl} \tau : G_7 & \longrightarrow & G_7/H \\ (a, b, c, d, e, f, k) & \longmapsto & (a, b, c). \end{array}$$

We take coordinates (y_1, y_2, y_3) in G_7/H so that the left action of G_7 on G_7/H is

$$\begin{array}{rcl} \Phi : G_7 \times G_7/H & \longrightarrow & G_7/H \\ ((a, b, c, d, e, f, k), (y_1, y_2, y_3)) & \longmapsto & (y_1 + a, y_2 + b, y_3 + c + (ay_2 - by_1)/2). \end{array}$$

The corresponding fundamental vector fields are

$$X_1^H = -\partial_{y_1} - \frac{y_2}{2}\partial_{y_3}, \quad X_2^H = -\partial_{y_2} + \frac{y_1}{2}\partial_{y_3}, \quad X_3^H = -\partial_{y_3},$$
$$X_4^H = 0, \quad X_5^H = 0, \quad X_6^H = 0, \quad X_7^H = 0,$$

which span the tangent space at each point of G_7/H, and satisfy the commutation relations of the Heisenberg Lie algebra, see (7).

Now, we factorize the solution of (25) starting from the identity as the product

$$g_1(t)h(t) = (y_1(t), y_2(t), y_3(t), 0, 0, 0, 0)(0, 0, 0, d(t), e(t), f(t), k(t)),$$

where $g_1(t)$ projects onto the solution $\tau(g_1(t)) = (y_1(t),\ y_2(t),\ y_3(t))$, with initial conditions $(y_1(0),\ y_2(0),\ y_3(0)) = (0,0,0)$, of the Lie system on G_7/H associated to (25), which coincides with (22). In this way we reduce to a Lie system in $H \cong \mathbb{R}^4$ for $h(t) = (0,\ 0,\ 0,\ d(t),\ e(t),\ f(t),\ k(t))$, with $h(0) = e$, i.e., $d(0) = e(0) = f(0) = k(0) = 0$. We calculate it according to the formulas (5), (26) and (27), i.e.,

$$\begin{aligned}
\dot{d} &= \frac{b_1(t)}{2}\left(\frac{1}{6}y_1(t)y_2(t) - y_3(t)\right) - \frac{1}{12}b_2(t)y_1^2(t)\,,\\
\dot{e} &= \frac{1}{12}b_1(t)y_2^2(t) - \frac{b_2(t)}{2}\left(\frac{1}{6}y_1(t)y_2(t) + y_3(t)\right)\,,\\
\dot{f} &= -\frac{1}{24}y_1(t)\left(b_1(t)(y_1(t)y_2(t) - 8y_3(t)) - b_2(t)y_1^2(t)\right)\,,\\
\dot{k} &= -\frac{1}{24}y_2(t)\left(b_1(t)y_2^2(t) - b_2(t)(y_1(t)y_2(t) + 8y_3(t))\right)\,,
\end{aligned} \tag{28}$$

which is solvable by quadratures. Thus, we have reduced the solution of the system (25), and hence of (23), to solve first (22) (i.e., essentially the nonholonomic integrator) and then (28), in order to reconstruct the complete solution of (25).

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References

[1] BELLAÏCHE A., *The tangent space in sub-Riemannian geometry*, in “Sub-Riemannian Geometry” (Eds. Bellaïche A. and Risler J.-J.), Birkhäuser, Basel 1996.

[2] BROCKETT R.W., *Control theory and singular Riemannian geometry*, in: “New Directions in Applied Mathematics” (Eds. Hilton P.J. and Young G.S.), Springer–Verlag, New York 1982.

[3] BROCKETT R.W. AND DAI L., *Non-holonomic kinematics and the role of elliptic functions in constructive controllability*, in: “Nonholonomic Motion Planning” (Eds. Li Z.X. and Canny J.F.), Kluwer, Norwell, MA 1993.

[4] CARIÑENA J.F., CLEMENTE-GALLARDO J. AND RAMOS A., *Motion on Lie groups and its applications in control theory*, Rep. Math. Phys. **51** (2003), 159–170.

[5] CARIÑENA J.F., GRABOWSKI J. AND RAMOS A., *Reduction of time-dependent systems admitting a superposition principle*, Acta Appl. Math. **66** (2001), 67–87.

[6] CARIÑENA J.F., MARMO G. AND NASARRE J., *The nonlinear superposition principle and the Wei–Norman method*, Int. J. Mod. Phys. A **13** (1998), 3601–3627.

[7] CARIÑENA J.F. AND RAMOS A., *Integrability of the Riccati equation from a group theoretical viewpoint*, Int. J. Mod. Phys. A **14** (1999), 1935–1951.

[8] CARIÑENA J.F. AND RAMOS A., *A new geometric approach to Lie systems and physical applications*, Acta Appl. Math. **70** (2002), 43–69.

[9] CARIÑENA J.F. AND RAMOS A., *Lie systems in control theory*, in: “Contemporary trends in nonlinear geometric control theory and its applications” (Eds. Anzaldo-Meneses A., Bonnard B., Gauthier J.-P. and Monroy-Pérez F.), World Scientific Publishers, Singapore 2002.

[10] CARIÑENA J.F. AND RAMOS A., *Applications of Lie systems in quantum mechanics and control theory*, in: “Classical and quantum integrability” (Eds. Grabowski J., Marmo G. and Urbański P.), Banach Center Publications **59**, Institute of Mathematics, Polish Academy of Sciences, Warszawa 2003.

[11] JURDJEVIC V., *The geometry of the plate-ball problem*, Arch. Rational Mech. Anal. **124** (1993), 305–328.

[12] JURDJEVIC V., *Optimal control problems on Lie groups: crossroads between geometry and mechanics*, in: "Geometry of feedback and optimal control" (Eds. Jakubczyk B. and Respondek W.), Marcel Dekker, New York 1993.

[13] JURDJEVIC V., *Geometric control theory*, Cambridge University Press, Cambridge 1997.

[14] LAFFERRIERE G. AND SUSSMANN H.J., *Motion planning for controllable systems without drift*, in: "Proc. 1991 IEEE International Conference on Robotics and Automation", IEEE, New York 1991.

[15] LAFFERRIERE G. AND SUSSMANN H.J., *A differential geometric approach to motion planning*, in: "Nonholonomic Motion Planning" (Eds. Li Z.X. and Canny J.F.), Kluwer, Norwell, MA 1993.

[16] NIKITIN S., *Control synthesis for Čaplygin polynomial systems*, Acta Appl. Math. **60** (2000), 199–212.

[17] WEI J. AND NORMAN E., *Lie algebraic solution of linear differential equations*, J. Math. Phys. **4** (1963), 575–581.

[18] WEI J. AND NORMAN E., *On global representations of the solutions of linear differential equations as a product of exponentials*, Proc. Amer. Math. Soc. **15** (1964), 327–334.

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