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Control Theory and Stabil., II

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# ANALYTIC CONTROL SYSTEMS AND THEIR PROPERTIES RELATED TO OBSERVERS

**Abstract.** Some properties of analytic control systems are presented. They are connected with the indistinguishability relation and the observability property. We assume rather local observability than global observability because it is a more realistic approach to the observer problem for nonlinear systems and it is also easier to check. Then it is not possible to find an approximation of unknown state of original system, but we can find the whole class of states that are indistinguishable. This irregular observer, which gives the whole class of indistinguishable states, is called multiobserver for nonlinear system.

## 1. Analytic control systems and indistinguishability relation.

Let us consider a nonlinear analytic control system $\Sigma$ on $\mathcal{X} \subset \mathbb{R}^n$ of the following form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned}$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \Omega \subset \mathbb{R}^m$, $u_i(\cdot) \in C^\infty(\mathbb{R}_+)$, $i = 1, \ldots, m$. The maps $f$ and $h$ are analytic.

Then the following sequence

$$j^k u = \left(u, u', u'', \ldots, u^{(k)}\right)$$

will be called *$k$-jet of the control $u$*.

For a function $\varphi$ depending on $x$ and a finite jet of $u$, $j^k u$, let

$$(\mathcal{L}_f \varphi)(x, j^{k+1}u) := \frac{\partial \varphi(x, j^k u)}{\partial x} f(x,u) + \sum_{i=0}^{k} \frac{\partial \varphi(x, j^k u)}{\partial u^{(i)}} u^{(i+1)},$$

where $\frac{\partial \varphi(x,j^k u)}{\partial x} = \left(\frac{\partial \varphi(x,j^k u)}{\partial x_1}, \ldots, \frac{\partial \varphi(x,j^k u)}{\partial x_n}\right)$, $u^{(i+1)} = \left(u_1^{(i+1)}, \ldots, u_m^{(i+1)}\right)^{\mathrm{T}}$ and $\frac{\partial \varphi(x,j^k u)}{\partial u^{(i)}} = \left(\frac{\partial \varphi(x,j^k u)}{\partial u_1^{(i)}}, \ldots, \frac{\partial \varphi(x,j^k u)}{\partial u_m^{(i)}}\right)$.

If we set the infinite jet of $u$ as follows

$$j^\infty u = (u_1, \ldots, u_m, u_1', \ldots, u_m', \ldots, u_1^{(k)}, \ldots, u_m^{(k)}, \ldots).$$

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and we compute the successive derivatives of the output, we have the following family of analytic functions:

$$\begin{array}{lcl}
\psi_i^0(x, j^\infty u) & = & h_i(x), \\
\psi_i^1(x, j^\infty u) & = & \frac{\partial h_i(x)}{\partial x} f(x,u), \\
\psi_i^2(x, j^\infty u) & = & \frac{\partial \psi_i^1(x, j^\infty u)}{\partial x} f(x,u) + \frac{\partial \psi_i^1(x, j^\infty u)}{\partial u} u', \\
 & \vdots & \\
\psi_i^{k+1}(x, j^\infty u) & = & \frac{\partial \psi_i^k(x, j^\infty u)}{\partial x} f(x,u) + \sum_{i=0}^{k-1} \frac{\partial \psi_i^k(x, j^\infty u)}{\partial u^{(i)}} u^{(i+1)}, \\
 & \vdots &
\end{array}$$

For the fixed $j^\infty u$ we define

$$\mathcal{K}_{j^\infty u}(\Sigma) := \operatorname{span}_{\mathbb{R}}\{\psi_i^k(\cdot, j^\infty u), i = 1, \ldots, p, k \geq 0\}.$$

Then $\mathcal{K}_{j^\infty u}(\Sigma)$ is a space of analytic functions defined on $\mathcal{X} \subset \mathbb{R}^n$ with values in $\mathbb{R}$.

Let $\mathcal{U}_{k,B}$ be be the set of all the maps $u : \mathbb{R}_+ \to \mathbb{R}^m$ that are $k$ times differentiable in the continuous way and satisfy the following condition:

$$\left\| \frac{d^i}{dt^i} u(t) \right\| \leqslant B, \tag{1}$$

for all $t \geqslant 0$, where $0 \leqslant i \leqslant k$.
Assume that considered controls belong to the set $\mathcal{U}_{N-1,B}$. Then

$$(u(t), u'(t), \ldots, u^{(N-1)}(t)) \in \mathbb{U}_N \subset \mathbb{R}^{Nm},$$

where $\mathbb{U}_N$ is a compact subset.

Let $\chi_u(t, x_0)$ be the trajectory of the system $\Sigma$ starting at $x_0$, corresponding to the control $u$ and evaluated at time $t$. Assume that for all $x_0$ and $u$ the trajectory $\chi_u(t, x_0)$ is well defined on $[0, +\infty)$.

DEFINITION 1. *Two states $x_1, x_2 \in \mathcal{X}$ are called* indistinguishable (with respect to $\Sigma$) *if*

$$h(\chi_u(t, x_1)) = h(\chi_u(t, x_2)) \tag{2}$$

*for every $u(\cdot) \in C^\infty(\mathbb{R}_+)$ and $t \geqslant 0$. Otherwise $x_1, x_2 \in \mathcal{X}$ are called* distinguishable (with respect to $\Sigma$).

It is easy to see that distinguishable states are distinguished by some control $u$ that is applied to the system $\Sigma$.

DEFINITION 2. *A control $\upsilon : \mathbb{R}_+ \to \Omega$ is called* universal *for the system $\Sigma$, if it distinguishes all distinguishable states.*

From [2] it is known that there is an universal control for the analytic control systems considered in this article. Let $\upsilon : \mathbb{R}_+ \to \Omega$ be the universal control for the system $\Sigma$.

Define $\mathbb{K} := \mathcal{K}_{j^\infty v}(\Sigma)$.

The following equivalence holds:

THEOREM 1 ([1]). *$x_1$ and $x_2$ are indistinguishable $\Leftrightarrow$ $\psi(x_1) = \psi(x_2)$, for all function $\psi \in \mathbb{K}$.*

REMARK 1. The indistinguishability relation is an equivalence relation and it divides the state space into nonempty and disjoint classes.

The set $\mathcal{X}$ is called *global semianalytic* if there are analytic functions $f_{ij}, g_i$ such that $\mathcal{X} = \bigcup_{i=1}^{r} \left\{x \in \mathbb{R}^n \mid f_{ij}(x) > 0, g_i(x) = 0, j = 1, \ldots, s_i\right\}$. For example, the sphere and the cube are global semianalytic sets.

Assume that the set $\mathcal{X}$ is a compact global semianalytic subset of $\mathbb{R}^n$.

THEOREM 2. *There is a finite number of functions $\psi_1, \psi_2, \ldots, \psi_k \in \mathbb{K}$, such that every two states $x_1, x_2 \in \mathcal{X}$ are indistinguishable if and only if $\psi_i(x_1) = \psi_i(x_2)$, $i = 1, \ldots, k$.*

*Proof.* Let $x_1, x_2 \in \mathcal{X}$ and $\mathcal{X}$ be a compact set. From Theorem 1 we have the following equivalence:

$$\text{states } x_1 \text{ and } x_2 \text{ are indistinguishable} \\ \Updownarrow \\ \forall \psi \in \mathbb{K} \;\; \psi(x_1) - \psi(x_2) = 0.$$

Such a pair $(x_1, x_2) \in \mathcal{X} \times \mathcal{X} =: \mathcal{Z}$ is in the indistinguishability relation. Define $\tilde{\psi}(x_1, x_2) := \psi(x_1) - \psi(x_2)$, for each function $\psi \in \mathbb{K}$. Let

$$\mathcal{P} = \{(x_1, x_2) \mid \; \tilde{\psi}(x_1, x_2) \; = 0, \; \psi \in \mathbb{K}\} \subset \mathcal{Z},$$

and let $\mathcal{O}_\mathcal{Z}$ be the ring of germs of analytic functions on a compact set $\mathcal{Z}$. From the Frisch theorem [3] the ring $\mathcal{O}_\mathcal{Z}$ is a Noetherian ring, so each ideal is finitely generated in this ring. Let $\mathcal{I}$ be the ideal of the ring $\mathcal{O}_\mathcal{Z}$ generated by the germs of the functions $\tilde{\psi}$ on $\mathcal{Z}$. Then

$$\mathcal{I} = \left\langle \tilde{\psi}_{1_\mathcal{Z}}, \ldots, \tilde{\psi}_{k_\mathcal{Z}} \right\rangle,$$

where $\tilde{\psi}_{i_\mathcal{Z}}$ are germs of functions $\tilde{\psi}_i$ in $\mathcal{Z}$, $i = 1, \ldots, k$. All germs $\tilde{\psi}_{i_\mathcal{Z}} \in \mathcal{I}$ satisfy $\tilde{\psi}_{i_\mathcal{Z}}(x_1, x_2) = 0$, where $x_1$ and $x_2$ are indistinguishable. But $\tilde{\psi}_{i_\mathcal{Z}}$ is a germ of $\tilde{\psi}_i$ on $\mathcal{Z}$ and we can assume that $\tilde{\psi}_i(x_1, x_2) = \psi_i(x_1) - \psi_i(x_2)$, for $i = 1, \ldots, k$ and $\psi_i \in \mathbb{K}$. Thus functions $\psi_1, \ldots, \psi_k$ determine the indistinguishable relation on a compact set $\mathcal{X}$. □

The indistinguishability relation is connected with the observability property.

DEFINITION 3. *System $\Sigma$ is* observable *if any two distinct points from the state space are distinguishable.*

DEFINITION 4. *System* $\Sigma$ *is* locally observable at $x_0$ *if there is a neighborhood* $V$ *of* $x_0$ *such that for every* $x \in V$, $x$ *and* $x_0$ *are distinguishable.* $\Sigma$ *is* locally observable *if it is locally observable at every point.*

REMARK 2. If $\Sigma$ is locally observable and we assume additionally that $\mathcal{X}$ is a compact set then any indistinguishability class in $\mathcal{X}$ is finite.

Local observability is a weaker property than observability which means that any two distinct points are distinguishable. Observability is a better property than local observability from a theoretic point of view, but it is much harder to check it. No efficient algorithm for checking it is known (for $n \geq 1$).

From the theorem 2 we know that a finite number of functions from $\mathbb{K}$ distinguish states that are distinguishable on the compact set $\mathcal{X}$. Then there is $N \in \mathbb{N}$ such that the below mentioned map $\mathrm{S}\Phi_N$ is related to the indistinguishability relation.

Let $\mathrm{S}\Phi_N : \mathcal{X} \times \mathbb{R}^{(N-1)\cdot m} \to \mathbb{R}^{N\cdot p} \times \mathbb{R}^{(N-1)\cdot m}$ be the map defined as follows:

$$\mathrm{S}\Phi_N(x, j^{N-2}u) \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x, u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \\ j^{N-2}u \end{pmatrix}.$$

For considered analytic systems we have the following property:

THEOREM 3. *If the system* $\Sigma$ *is analytic and it is defined on a compact global semianalytic set* $\mathcal{X}$, *then there is a continuous map* $\varphi : \mathbb{R}^{Np} \times \mathbb{R}^{Nm} \to \mathbb{R}^p$ *such that*

$$\mathcal{L}_f^N h = \varphi(h, \ldots, \mathcal{L}_f^{N-1} h, u, u', \ldots, u^{(N-1)}), \tag{3}$$

*for some* $N \in \mathbb{N}$.

*Proof.* Let $\upsilon$ be the universal control for the analytic control $\Sigma$. From Theorem 2 there exists $N \in \mathbb{N}$ such that the following functions

$$\psi_i(x) := h_i(x),\ \psi_{i,\upsilon}(x) := \mathcal{L}_f h_i(x, \upsilon), \ldots, \psi_{i,j^{N-2}\upsilon}(x) := \mathcal{L}_f^{N-1} h_i(x, j^{N-2}\upsilon),$$

distinguish states that are distinguishable. It means that states $x_1$ and $x_2$ are indistinguishable if and only if we have

$$\begin{aligned} \psi_i(x_1) &= \psi_i(x_2), \\ \psi_{i,\upsilon}(x_1) &= \psi_{i,\upsilon}(x_2), \\ &\vdots \\ \psi_{i,j^{N-2}\upsilon}(x_1) &= \psi_{i,j^{N-2}\upsilon}(x_2), \end{aligned}$$

when $i = 1, \ldots, p$.

Then for every jet $j^{N-1}u \in \mathbb{U}_N \subset \mathbb{R}^{Nm}$ the function $x \mapsto \mathcal{L}_f^N h_i(x, j^{N-1}u)$, $i = 1, \ldots, p$ does not distinguish states from the set

$$[x] = \{\tilde{x} \in \mathcal{X} : \ \Phi_{N, j^{N-2}v}(x) = \Phi_{N, j^{N-2}v}(\widetilde{x})\},$$

where $\Phi_{N, j^{N-2}u} : \mathcal{X} \to \mathbb{R}^{Np}$ is defined by

$$\Phi_{N, j^{N-2}u}(x) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x, u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \end{pmatrix}. \tag{4}$$

Denote by $\widetilde{\mathcal{X}}$ the quotient space of $\mathcal{X}$ with respect to the indistinguishability relation ($\widetilde{\mathcal{X}} := \mathcal{X}/_\sim$). Let $\mathrm{S}\widetilde{\Phi}_N : \widetilde{\mathcal{X}} \times \mathbb{U}_N \to \operatorname{im} \Phi_{N, j^{N-2}u} \times \mathbb{U}_N \subset \mathbb{R}^{Np} \times \mathbb{R}^{Nm}$ be the map defined as follows:

$$\mathrm{S}\widetilde{\Phi}_N([x], u, u', \ldots, u^{(N-1)}) := \left(\Phi_{N, j^{N-2}u}(x), u, u', \ldots, u^{(N-1)}\right). \tag{5}$$

We impose the quotient topology on $\widetilde{\mathcal{X}}$. Then $\widetilde{\mathcal{X}}$ is compact and the map $\mathrm{S}\widetilde{\Phi}_N$ is continuous and injective, so $\mathrm{S}\widetilde{\Phi}_N$ is homeomorphism onto its image, which is closed (from lemma 4.1 of [4]). Then $\Psi := \mathrm{S}\widetilde{\Phi}_N^{-1}$ is a continuous map from $\mathrm{S}\widetilde{\Phi}_N(\widetilde{\mathcal{X}} \times \mathbb{U}_N)$ to $\widetilde{\mathcal{X}} \times \mathbb{U}_N$. The map $\Psi$ is such that

$$\Psi\left(h(x), \mathcal{L}_f h(x, u), \ldots, \mathcal{L}_f^{N-1} h(x, j^{N-2}u), j^{N-1}u\right) = \left([x], j^{N-1}u\right). \tag{6}$$

For every state $x \in \mathcal{X}$ and every jet $j^{N-1}u \in \mathbb{U}_N \subset \mathbb{R}^{Nm}$ we have

$$\begin{aligned} &\mathcal{L}_f^N h\left(x, u, u', \ldots, u^{(N-2)}, u^{(N-1)}\right) = \mathcal{L}_f^N h\left([x], j^{(N-1)u}\right) = \mathcal{L}_f^N h\left(\Psi\left(Y, j^{N-1}u\right)\right) \\ &= \left(\mathcal{L}_f^N h \circ \Psi\right)\left(Y, j^{N-1}u\right) \end{aligned}$$

where $Y := \Phi_{N, j^{N-2}u}(x)$. The function $\mathcal{L}_f^N h \circ \Psi$ is continuous and it is defined on $\Phi_{N, j^{N-2}u}(\mathcal{X}) \times \mathbb{U}_N$, where $\Phi_{N, j^{N-2}u}(\mathcal{X})$ is a compact set. Then there is a continuous extension of $\mathcal{L}_f^N h \circ \Psi$ on the entire $\mathbb{R}^{Np} \times \mathbb{U}_N$ (it is known from Tietze's theorem). Let $\varphi : \mathbb{R}^{Np} \times \mathbb{U}_N \to \mathbb{R}^p$ be the continuous extension of the function $\mathcal{L}_f^N h \circ \Psi$, so for all $(Y, j^{N-1}u) \in \Phi_{N, j^{N-2}u}(\mathcal{X}) \times \mathbb{U}_N$ we have

$\varphi\left(Y, j^{N-1}u\right) = \left(\mathcal{L}_f^N h \circ \Psi\right)\left(Y, j^{N-1}u\right).$ □

REMARK 3. Theorem 3 is useful to construct a system that estimates output and its successive derivatives up to order $N - 1$.

EXAMPLE 1. Let $\Sigma$ be the system defined on $\mathbb{R}^2$ as follows:

$$\begin{aligned} \dot{x}_1 &= x_1^2 u \\ \dot{x}_2 &= x_1 x_2 u \\ y &= x_1^2 + x_2^2, \end{aligned}$$

where $u \in \Omega = (0, +\infty)$ i $y \in \mathbb{R}$. Functions $h(x) = x_1^2 + x_2^2$ and $\mathcal{L}_f h(x, u) = 2x_1 \cdot (x_1^2 + x_2^2)u$ belong to the family $\mathbb{K}$ and determine the indistinguishability relation on the compact set $\mathcal{X} \subset \mathbb{R}^2$. Then

$$\mathcal{L}_f^2 h(x, u, \dot{u}) = \varphi_2(h, \mathcal{L}_f h, u, \dot{u}) \begin{cases} 3\frac{(\mathcal{L}_f h)^2}{2h} + \mathcal{L}_f h \cdot \frac{\dot{u}}{u}, & \text{if } h \neq 0, \\ 0, & \text{if } h = 0. \end{cases}$$

The function $\varphi_2$ is not smooth, but it is continuous on the following set:

$$\text{im } \mathrm{S}\Phi_2 = \{(z_1, z_2, u) \mid \ z_2^2 \leqslant 4z_1^3 u^2\},$$

where $\mathrm{S}\Phi_2(x, u) = \begin{pmatrix} x_1^2 + x_2^2 \\ 2x_1(x_1^2 + x_2^2)u \\ u \end{pmatrix}$.

From Tietze's theorem there exists a continuous extension $\varphi : \mathbb{R}^2 \times \Omega \times \mathbb{R} \to \mathbb{R}$ of $\varphi_2$. It may look as follows:

$$\varphi(z_1, z_2, u, \dot{u}) \begin{cases} \frac{3z_2^2}{2z_1} + z_2 \frac{\dot{u}}{u}, & \text{if } z_2^2 < 4z_1^3 u^2 \\ \frac{3}{2}\sqrt[3]{4u^2} z_2^{\frac{4}{3}} + z_2 \frac{\dot{u}}{u}, & \text{if } z_2^2 \geqslant 4z_1^3 u^2 \end{cases}.$$

## 2. Multiobservers for analytic control systems

DEFINITION 5. *The system $\mathcal{M}$ of the following form*

$$\dot{z} = F(z, u, y) \tag{7}$$
$$\hat{x} = g(z, u, y) \tag{8}$$

is called *a multiobserver* of the control system $\Sigma$, if

(i) the input $y$ is the output of $\Sigma$,

(ii) $g$ is multifunction whose values are finite subsets of $\mathbb{R}^n$,

(iii) $\lim\limits_{t \to +\infty} d(x(t); \hat{x}(t)) = 0$, for every trajectory $z(\cdot)$ of the system (7),

(iv) $F$ and $g$ are continuous.

Let $u(t)$ be the smooth control applied to the system $\Sigma$. Assume we know the output $y(t)$. Then we consider the system $\mathcal{M}_\theta^1$ defined on $\mathbb{R}^{Np}$ with inputs $u(t)$, $u'(t)$, $\dots$, $u^{(N-1)}(t)$ and $y(t)$ as follows:

$$\dot{Z} = (\mathrm{A} - \mathrm{K}_\theta \mathrm{C})Z + \mathrm{K}_\theta y + \mathrm{b}\varphi(\mathrm{Z}, u, u', \dots, u^{(N-1)}),$$

where $Z \in \mathbb{R}^{Np}$, $\mathrm{A} = \begin{pmatrix} 0 & \mathrm{I}_p & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathrm{I}_p & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \mathrm{I}_p \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{Np\times Np}$, $\mathrm{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{I}_p \end{pmatrix}_{Np\times p}$,

$\mathrm{C} = \begin{pmatrix} \mathrm{I}_p & 0 & \dots & 0 \end{pmatrix}_{p\times Np}$, $\Delta_\theta = \begin{pmatrix} \theta\mathrm{I}_p & 0 & 0 & \dots & 0 \\ 0 & \theta^2\mathrm{I}_p & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \theta^N\mathrm{I}_p \end{pmatrix}_{Np\times Np}$, $\theta \in \mathbb{R}_+$

and $\theta \gg 1$, $\mathrm{K}_\theta \Delta_\theta \mathrm{K}$. $\mathrm{K} \in \mathbb{R}^{Np\times p}$ is an matrix such that $\mathrm{A} - \mathrm{KC}$ is a Hurwitz matrix, and the function $\varphi$ comes from Theorem 3. $b \gg a$ denotes that $\frac{a}{b}$ is small enough.

THEOREM 4. *Assume that $\varphi$ is a Lipschitz function with Lipschitz constant $L_{\mathcal{X},B}$, inputs $u$ belong to $\mathcal{U}_{N-1,B}$. For every $Z(0) \in \mathbb{R}^{Np}$, $x(0) \in \mathcal{X}$ and $u(\cdot) \in \mathcal{U}_{N-1,B}$ the solution of the system (7) satisfies the following inequality*

$$\left\| Z(t) - \Phi_N \left( x(t), u(t), \dots, u^{(N-2)}(t) \right) \right\| \le \theta^{N-1}\beta \exp\left[ -\frac{1}{2}\left( \theta\alpha - 2\omega_{\mathcal{X},B} \right) t \right] \cdot$$
$$\cdot \left\| Z(0) - \Phi_N(x(0), u(0), \dots, u^{(N-2)}(0)) \right\|,$$

*where $\alpha, \beta$ are some positive constants, $\omega_{\mathcal{X},B}$ is a constant which depends on $\mathcal{X}$ and $B$. Above mentioned inequality holds for all $t$ such that $\chi_u(t, x_0) \subset \mathcal{X}$.*

*Proof.* Similar to the main theorem in [5] (see also [4], [6] for earlier results in the same fashion). □

REMARK 4. The system $\mathcal{M}_\theta^1$ gives an exponential estimation of the successive derivatives of the outputs and their successive derivatives up to order $N-1$, based on the knowledge of: the output $y(t)$, which is observed, and the $u(t), u'(t), \dots, u^{N-1}(t)$, that are applied to the system.

If our original system is not observable, then it is not possible to find an approximation of unknown state $x(t)$, but we can find the whole class of states that are indistinguishable. When the system is locally observable then indistinguishable states form a discrete set, i.e. each point is isolated from the others. If the set $\mathcal{X}$ is compact (closed and bounded) then any indistinguishable class in $\mathcal{X}$ is finite. Denote the class of states indistinguishable from $x$ by

$$[x] = \{\tilde{x} \in \mathcal{X} : \Phi_{N,j^{N-2}u}(x) = \Phi_{N,j^{N-2}u}(\tilde{x})\},$$

where for every jet $j^{N-1}u \in \mathbb{U}_N \subset \mathbb{R}^{Nm}$ we have $\Phi_{N,j^{N-2}u} : \mathcal{X} \to \mathbb{R}^{Np}$ and

$$\Phi_{N,j^{N-2}u}(x) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x,u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, j^{N-2}u) \end{pmatrix}. \tag{9}$$

From the proof of Theorem 3 we have the continuous map $\Psi : S\widetilde{\Phi}_N(\widetilde{\mathcal{X}} \times \mathbb{U}_N) \to \widetilde{\mathcal{X}} \times \mathbb{U}_N$, such that (6) holds.

Define a mulifunction $g : \mathbb{R}^{Np} \times \mathbb{U}_N \to \widetilde{\mathcal{X}}$ as follows

$$g(z_1, z_2, \ldots, z_N, u, u^{(1)}, \ldots, u^{(N-1)}) := \pi_1 \circ \Psi,$$

where $\pi_1 : \widetilde{\mathcal{X}} \times \mathbb{R}^{Nm} \to \widetilde{\mathcal{X}}$ is a projection on the first component, i.e. $\pi_1([x], U) = [x]$.

To reconstruct the class of indistinguishable states we use the multifunction $g$, so the second equation of our multiobserver has the following form:

$$\hat{x} = g(z_1, z_2, \ldots, z_N, u, u^{(1)}, \ldots, u^{(N-1)}).$$

This irregular observer (multiobserver) for nonlinear system $\Sigma$ consists of dynamics approximating the successive derivatives of the output of the original system and of output giving the whole class of indistinguishable states.

In Example 1 the dynamical part of the multiobserver of $\Sigma$ is described by the following equations:

$$\begin{aligned} \dot{z}_1 &= -k_1\theta z_1 + z_2 + k_1\theta y \\ \dot{z}_2 &= -k_2\theta^2 z_1 + k_2\theta^2 y + \varphi(z_1, z_2, u, \dot{u}), \end{aligned}$$

where $u(\cdot) \in \mathcal{U}_{1,B}$, $k_1, k_2 \in \mathbb{R}$ and $\begin{pmatrix} -k_1 & 1 \\ -k_2 & 0 \end{pmatrix}$ is Hurwitz matrix, $\theta \in \mathbb{R}_+$ ($\theta \gg 1$) and

$$\varphi(z_1, z_2, u, \dot{u}) \begin{cases} \frac{3z_2^2}{2z_1} + z_2\frac{\dot{u}}{u}, & \text{if } z_2^2 < 4z_1^3u^2 \\ \frac{3}{2}\sqrt[3]{4u^2}z_2^{\frac{4}{3}} + z_2\frac{\dot{u}}{u}, & \text{if } z_2^2 \geqslant 4z_1^3u^2 \end{cases}.$$

To reconstruct the class of states which are indistinguishable we use the following multifunction:

$$\hat{x}(t) = \begin{cases} \left\{\left(\frac{z_2}{2uz_1}, \frac{\sqrt{4z_1^3u^2 - z_2^2}}{2z_1u}\right), \left(\frac{z_2}{2uz_1}, -\frac{\sqrt{4z_1^3u^2 - z_2^2}}{2z_1u}\right)\right\}, & \text{if } z_2^2 < 4z_1^3u^2 \\ \left\{\left(\sqrt[3]{\frac{z_2}{2u}}, 0\right)\right\}, & \text{if } z_2^2 \geqslant 4z_1^3u^2 \end{cases}.$$

This multifunction will be the output of the multiobserver.

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