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ON GENERALIZED DIFFERENTIAL QUOTIENTS OF SET-VALUED MAPS

Abstract. There are many important maps that are not differentiable in the classical way. In this paper one of the concepts of generalized differential, called Generalized Differential Quotient (abbr. GDQ), is studied. GDQ may exist for functions not differentiable in the classical sense and for set-valued maps, but it is not unique. Existence of minimal GDQs has been proved.

1. Introduction

The axiomatic definition of generalized differentiation theory (abbr. GDT) has been introduced by Hector Sussmann in [5]. He described also a few theories satisfying the axioms of GDT (see [5]). One of them is the theory of Generalized Differential Quotients (abbr. GDQs). It is known that GDQ of a single-valued or set-valued map, unlike ordinary differential, is not unique. In order to develop further Sussmann's theory we have then to answer some questions concerning GDQs: Which of GDQs are the best mathematical tools? Is any intersection of GDQs also a GDQ? The answers to these questions extend the basic issues concerning GDQ. Since GDQs are generalization of Clarke generalized gradients we will start with the definition of these gradients.

DEFINITION 1. *Let $f : X \to \mathbb{R}$ be Lipschitz near $x \in X$, where X is a finite-dimensional real linear space, and let v be a vector in X. The generalized directional derivative of f at x in the direction v, denoted $f^\circ(x; v)$, is defined as follows:*

$$f^\circ(x; v) \;=\; \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar.

DEFINITION 2. *Let X^* denote the dual space of X, i.e. the space of all bounded linear functionals $\zeta : X \to \mathbb{R}$. Let $f : X \to \mathbb{R}$ and $x \in X$. Then the generalized gradient of f at x, denoted $\partial f(x)$, is the subset of X^* given by*

$$\{\zeta \in X^* : f^\circ(x; v) \geq \langle \zeta, v \rangle \text{ for all } v \text{ in } X\}.$$

Here are some basic properties of generalized gradients (we assume that f is Lipschitz near x):

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(a) $\partial f(x)$ is a nonempty, convex, compact subset of X^*
(b) for every v in X, one has

$$f^\circ(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial f(x)\}.$$

EXAMPLE 1. Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2 \sin\frac{1}{x}$. Then f is differentiable at 0, Lipschitz near 0 and $f'(0) = 0$. We will show that $\partial f(0) = [-1, 1]$. Let us take $v \geq 0$, then we can calculate:

$$\begin{aligned} f^\circ(0;v) &= \limsup_{\substack{y\to 0\\ t\downarrow 0}} \frac{(y+tv)^2 \sin\frac{1}{y+tv} - y^2 \sin\frac{1}{y}}{t} \\ &= \limsup_{\substack{y\to 0\\ tv\downarrow 0}} \frac{v((y+tv)^2 \sin\frac{1}{y+tv} - y^2 \sin\frac{1}{y})}{tv} \\ &= \limsup_{y\to 0} (2vy \sin\frac{1}{y} - v\cos\frac{1}{y}). \end{aligned}$$

Since $\lim_{y\to 0} y \sin\frac{1}{y} = 0$, we have

$$\limsup_{y\to 0} (2vy \sin\frac{1}{y} - v\cos\frac{1}{y}) = \limsup_{y\to 0} (-v\cos\frac{1}{y}) = v.$$

By analogy for $v < 0$ we will get $f^\circ(0;v) = -v$, thus

$$f^\circ(0;v) = \begin{cases} v & \text{for } v \geq 0 \\ -v & \text{for } v < 0 \end{cases}$$

i.e. $f^\circ(0;v) = |\, v \,|$. Thus $\partial f(0)$ consists of those ζ satisfying $|\, v \,| \geq \zeta v$ for all v; that is, $\partial f(0) = [-1, 1]$.

2. Generalized differential quotients (GDQs)

If $X,\ Y$ are metric spaces and $SVM(X,Y)$ denotes the set of all set-valued maps from X to Y, then $SVM_{comp}(X,Y)$ will denote the subset of $SVM(X,Y)$ whose members are the set-valued maps from X to Y that have a compact graph. We say that a sequence $\{F_j\}_{j\in\mathbb{N}}$ of members of $SVM_{comp}(X,Y)$ *inward graph-converges* to an $F \in SVM_{comp}(X,Y)$ - and write $F_j \xrightarrow{igr} F$ - if for every open subset Ω of $X \times Y$ such that $Gr(F) \subseteq \Omega$ there exists a $j_\Omega \in \mathbb{N}$ such that $Gr(F_j) \subseteq \Omega$ whenever $j \geq j_\Omega$ ($Gr(F)$ denotes *the graph* of F, where $Gr(F) = \{(x,y) : y \in F(x)\}$).

DEFINITION 3. *Assume that $X,\ Y$ are metric spaces. A* regular set-valued map *from X to Y is a set-valued map F such that for every compact subset K of X, the* *restriction $F\lceil K$ of F to K belongs to $SVM_{comp}(K, Y)$ and is a limit - in the sense of inward graph-convergence - of a sequence of continuous single-valued maps from K to Y.*

We use $REG(K, M)$ to denote the set of all regular set-valued maps from K to M.

When $f : X \to Y$ is a single-valued map, then f belongs to $REG(X, Y)$ if and only if f is continuous.

An important class of examples of regular maps is provided by the following results.

THEOREM 1. *[5] Assume that K is a compact metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(K, C)$ be a set-valued map such that the graph of F is compact and the value $F(x)$ is a nonempty convex set for every $x \in K$. Then F is regular as a map from K to C.*

Thus compactness of the graph of F, in principal, guarantees regularity of F. We can relax this assumption imposing compactness of the values of F and adding upper semicontinuity of F.

THEOREM 2. *[5] Assume that X is a metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(X, C)$ be an upper semicontinuous set-valued map with nonempty compact convex values. Then $F \in REG(X, C)$.*

THEOREM 3. *[5] Assume that X, Y, Z are metric spaces. Let $F \in SVM(X, Y)$, $G \in SVM(Y, Z)$. Then the composite map $G \circ F$ belongs to $REG(X; Z)$.*

DEFINITION 4. *Let $m, n \in \mathbb{Z}_+$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ be a set-valued map, $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in \mathbb{R}^n$ ($\overline{y} \in F(\overline{x})$) and let Λ be a nonempty compact subset of $\mathbb{R}^{n\times m}$. Then an element of Λ is a $n \times m$ matrix. Let S be a subset of $\mathbb{R}^m$. We say that Λ is a generalized differential quotient (abbr. "GDQ") of F at ($\overline{x}$,$\overline{y}$) in the direction of S, and write $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ if for every positive real number δ there exist U, G such that*
1. U is a compact neighborhood of 0 in $\mathbb{R}^m$ and $U \cap S$ is compact;
2. G is a regular set-valued map from $\overline{x} + U \cap S$ to the δ-neighborhood Λ^δ of Λ in $\mathbb{R}^{n\times m}$;
3. $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ for every $x - \overline{x} \in U \cap S$

The above definition of GDQ can be extended to the one defined for manifolds (see [5]).

GDQs are generalizations of Clarke generalized gradients and, like other generalized differentials introduced by Sussmann, are not unique. Every compact overset $\tilde{\Lambda}$ of $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ also belongs to $GDQ(F; \overline{x}, \overline{y}; S)$.

Let X, Y be finite-dimensional real linear spaces. A GDQ of $F : X \twoheadrightarrow Y$ at a point $(\overline{x}, \overline{y})$ can be identified with a set of nonempty compact linear multimaps from

X to Y, where a linear multimap from X to Y is a subset of $Lin(X, Y)$.
Note that *a linear multimap* is a set of linear maps whilst *a set-valued map* is a map with values that are sets.

EXAMPLE 2. Let $f(x) = x \sin \frac{1}{x}$ and $f(0) = 0$. This function is not differentiable in the classical way. We will show that $[-1, 1] \in GDQ(f; 0, 0; \mathbb{R})$, i.e. that for every $\delta > 0$ there exist U, G such that
1. U is a compact neighborhood of 0 in $\mathbb{R}$
2. G is a regular set-valued map from U to the δ-neighborhood Λ^δ of Λ
3. $G(x) \cdot x = f(x)$ for every $x \in U$.

Let $U = [-1, 1]$ and $G(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ [-1, 1] & \text{if } x = 0 \end{cases}$

Then points 1. and 3. are satisfied. One can show using Theorem 2 that G is a regular set-valued map. Hence the point 2. is also satisfied.

EXAMPLE 3. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows $f(x) = x^2 \sin(\frac{1}{x})$ and $f(0) = 0$. Then f is Lipschitz and differentiable in the classical sense. One can show that $[-1, 1] \in GDQ(f; 0, 0; \mathbb{R})$ and $[-1, 1]$ is also the Clarke generalized gradient of f at 0. But for Lipschitz maps usually exist generalized differentials smaller then Clarke's generalized gradient, like in our case: $\{f'(0)\} \in GDQ(f; 0, 0; \mathbb{R})$

3. Minimal GDQs

In Example 3 we have seen that there are many GDQs: there exist Clarke generalized gradient and classical derivative and both of them are GDQs for the same function in the same point and in the same direction. This leads us to the following question: which GDQ one should deal with? We are interested in minimal GDQs (in the sense of inclusion of sets). They are the simplest. We are going to show that such minimal GDQs exist. In [3] we studied minimal multidifferentials of set-valued maps. Multidifferential is another concept of a generalized differential introduced by Sussmann.

LEMMA 1. *[2] Let U be an open subset of a topological space X. If the family $\{K_s\}_{s \in S}$ of closed subsets of the space X includes at least one compact set and $\bigcap_{s \in S} K_s \subset U$ then there exists such a finite set $\{s_1, ..., s_n\} \subset S$ such that $K_{s_1} \cap K_{s_2} \cap ... \cap K_{s_n} \subset U$.*

The following lemma is easy to prove.

LEMMA 2. *Let Λ_k, $k \in \mathbb{N}$ be a sequence of descending compact sets*

$$\Lambda_1 \supset \Lambda_2 \supset ...$$

and let $\bigcap_k \Lambda_k = \Lambda$ (then Λ is compact and $\Lambda \neq \emptyset$). Then for every $\delta > 0$ there exist δ'

and $k \in \mathbb{N}$ such that

$$\Lambda_k \subset \Lambda_k^{\delta'} \subset \Lambda^{\delta}$$

THEOREM 4. *Let $F \in SVM(\mathbb{R}^n, \mathbb{R}^m)$ and $\Lambda_k \in GDQ(F; \overline{x}, \overline{y}; \mathbb{R}^n)$ for $k = 1, 2, \ldots$. Let us assume that $\Lambda_1 \supset \Lambda_2 \supset \ldots$ and $\bigcap_k \Lambda_k = \Lambda$. ($\Lambda \neq \emptyset$ and Λ is compact as implied by definition of Λ_k). Then*

$$\Lambda \in GDQ(F; \overline{x}, \overline{y}; \mathbb{R}^n)$$

Proof. Without loss of generality we can assume that $\overline{x} = 0$ and $\overline{y} = 0$. We have to show that for every $\delta > 0$ there exist U_δ and G_δ such that conditions of definition of GDQ are satisfied for Λ.

From the assumption we have that for every $\delta > 0$ and every $k \in \mathbb{N}$ there exist $U_{\delta,k}$ and $G_{\delta,k}$ such that $U_{\delta,k}$ is a compact neighborhood of 0 in $\mathbb{R}^n$, $G_{\delta,k} \in REG(U_{\delta,k}, \Lambda_k^{\delta})$ where Λ_k^{δ} is a δ-neighborhood of Λ_k in $\mathbb{R}^{n \times m}$ and $G_{\delta,k}(x) \cdot x \subseteq F(x)$ for every $x \in U_{\delta,k}$ and $k \in \mathbb{N}$.

It is sufficient to show that for every $\delta > 0$ there exist $\delta' < 0$ and Λ_k with its neighborhood $\Lambda_k^{\delta'}$ contained in δ-neighborhood of Λ. But this fact follows from Lemma 2. So if we put $U_\delta = U_{\delta',k}$ and $G_\delta = G_{\delta',k}$ then from the arbitrary of choice of δ we get what we wanted to show. □

COROLLARY 1. *If the set of GDQs of a set-valued map F at $(\overline{x}, \overline{y})$ is not empty then there exists in this set at least one minimal GDQ at this point in the sense of inclusion of sets.*

Proof. From Kuratowski-Zorn Lemma, a family of sets with the property that descending sequences have a lower bound, possesses a minimal element. In our case such a sequence (D_k) of GDQs has a lower bound – their intersection D. Thus there exists for the family of GDQ a minimal element D, which we call a minimal GDQ .

□

COROLLARY 2. *An intersection of every descending sequence of $GDQs$ of F at $(\overline{x}, \overline{y})$ in the direction of S is again a GDQ of F at the same point in the same direction.*

Proof. Immediately from the definition of GDQ and from Theorem 4. □

COROLLARY 3. *Every element Λ of $GDQ(F; \overline{x}, \overline{y}; S)$ contains a minimal element of $GDQ(F; \overline{x}, \overline{y}; S)$ in the sense of inclusion of sets.*

Proof. Assume that Λ does not contain such a minimal element. Let us construct a descending sequence of element of $GDQ(F; \overline{x}, \overline{y}; S)$. Then from Corollary 2 we have that intersection of this sequence is a minimal GDQ and we have a contradiction.

□

EXAMPLE 4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Then one can show that $[-1, 1] \in GDQ(f; 0, 0; \mathbb{R})$ and that this is the minimal GDQ. This interval is also Clarke generalized gradient of f at 0. However for $f(x) = x^2 \sin \frac{1}{x}$ and $f(0) = 0$ the same interval is again Clarke generalized gradient of f at 0, while the minimal GDQ is just the ordinary derivative at 0 equal 0.

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