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AN ANALYTICAL THEORY FOR OPTIMAL CONTROLS ON RIEMANNIAN MANIFOLDS

Abstract. The problem of interpolation in a Riemannian manifold (M, g) discussed in [9] (using variational techniques on a suitable functional fra-mework) is here reviewed, stating the main existence results for minimizers and multiplicity and regularity results for critical points of the involved functional f.

1. Introduction

In recent years, a geometric theory of the so–called Riemannian cubic polynomials has been developed [3, 5, 8, 12]. With the aim of a generalization, variational problems with Lagrangians involving higher order derivatives can be encountered in Control Theory, especially in robotics (see for instance [2], where higher order interpolation in Riemannian manifold is exploited), and has been successively developed [1, 4, 6, 10].

This paper intends to give a brief survey of the results contained in [9], where intepolating polynomials of odd order $2k + 1$ on Riemannian manifolds are studied as critical points of a functional involving covariant derivatives of k–th order of the velocity. The path space where one looks for critical points consists of curves in the Sobolev space H^{k+1}, satisfying boundary conditions on position and velocity and all its covariant derivatives up to order $k - 1$ at the initial and final point. This set is shown to be a Hilbert manifold, and the functional defined on it is shown to satisfy a compactness property (Palais–Smale condition). In this way, classical techniques from Global Analysis can be applied to recover results of multiplicity of critical points, and also existence of curves that globally minimize the functional on the path space. It is interesting to note that, in spite of the sub–Riemannian nature of the variational problem, the class of Riemannian polynomials does not contain *abnormal* minimizers.

2. The variational framework

The space of paths satisfying higher order regularity assumptions and suitable boundary conditions, which is the domain of our variational problem, is introduced here in an abstract context. The main idea is to establish some topological properties of the space to be used later (Remark 2 and Section 4) where homotopic invariants play a crucial role.

Let $\mathfrak{M}$ be a functor that associates to each compact interval $I = [a, b]$ of $\mathbb{R}$ and to each differentiable manifold M a topological space, denoted by $\mathfrak{M}(I, M)$, consisting of curves $x : I \to M$. Since the purpose is to study sets of curves satisfying boundary

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conditions involving derivatives up to the k–th order (with $k \geq 1$), it is natural to consider the case that each curve in the class $\mathfrak{M}$ be of class $\mathcal{C}^k$. For the detailed axiomatic definition of $\mathfrak{M}$, we refer the reader to [9].

REMARK 1. For the purposes of this paper, the basic examples of $\mathfrak{M}$ are the functors $\mathcal{C}^k$ and $W^{k+1,p}$, with $p \geq 1$; note however that, depending on the variational problem to be studied, many other relevant examples of regularity might be considered, like for instance a regularity of Sobolev mixed type.

Let us now fix two arbitrary points $p, q \in M$. We have an inclusion

$$\mathfrak{M}_{p,q}\big([a,b], M\big) \subset \mathcal{C}^0_{p,q}\big([a,b], M\big), \tag{1}$$

where

$$\mathfrak{M}_{p,q}\big([a,b], M\big) = \Big\{x \in \mathfrak{M}([a,b], M) : x(a) = p,\ x(b) = q\Big\}, \tag{2}$$

$$\mathcal{C}^0_{p,q}\big([a,b], M\big) = \Big\{x \in \mathcal{C}^0\big([a,b], M\big) : x(a) = p,\ x(b) = q\Big\}. \tag{3}$$

Using a classical result by Palais [13], it can be proved that (1) is actually a homotopy equivalence.

Note however that, so far, no boundary conditions, other that the initial and final position of the curve, have been imposed. To involve higher order boundary conditions, jet spaces will be used. Let us simply recall (referring the reader to [14] for more details) that, fixed $x \in M$, we can set:

$$\mathfrak{J}^k(M)_p = \big\{x \in \mathcal{C}^k\big(\,]-\varepsilon, \varepsilon[\,, M\big) : x(0) = p\big\}\Big/ \sim$$

where $\sim$ is the equivalence relation:

$$x_1 \sim x_2 \quad \Longleftrightarrow \quad \big(\varphi \circ x_1\big)^{(i)}(0) = \big(\varphi \circ x_2\big)^{(i)}(0),\ i = 1, \ldots, k$$

for some (hence for all) local chart φ of M around p; we will denote by $[x]$ the equivalence class of the curve x by the above equivalence relation. The disjoint union $\bigcup_{p \in M} \mathfrak{J}^k(M)_p$ can be canonically given a topological space structure, and will be denoted by $\mathfrak{J}^k(M)$.

Observe now that for all $t_0 \in [a,b]$ we have a well defined continuous map

$$\mathfrak{J}^k_{t_0} : \mathfrak{M}\big([a,b], M\big) \longrightarrow \mathfrak{J}^k(M)_{x(t_0)}$$

that is obtained by sending a curve x to its equivalence class in $\mathfrak{J}^k(M)_{x(t_0)}$. It can be seen that imposing boundary conditions related to retracts of the jet spaces $J^k(M)_p$ and $J^k(M)_q$ does not affect the homotopy type of the paths space. In other words, we have the following proposition (see [9] for the proof):

PROPOSITION 1. *Let $\mathcal{S}_1 \subset \mathfrak{J}^k(M)_p$ and $\mathcal{S}_2 \subset \mathfrak{J}^k(M)_q$ be retracts, and set:*

$$\mathfrak{M}_{p,q}\big([a,b], M; \mathcal{S}_1, \mathcal{S}_2\big) = \Big\{x \in \mathfrak{M}_{p,q}\big([a,b], M\big) : \mathfrak{J}^k_a(x) \in \mathcal{S}_1,\ \mathfrak{J}^k_b(x) \in \mathcal{S}_2\Big\}. \tag{4}$$

Then, the inclusion $\mathfrak{M}_{p,q}\big([a,b],M;\mathcal{S}_1,\mathcal{S}_2\big)\hookrightarrow\mathfrak{M}_{p,q}\big([a,b],M\big)$ *is an homotopy equivalence.*

Using the homotopy equivalence (1) already stated, we can immediately deduce the following corollary:

COROLLARY 1. *Under the assumptions of Proposition 1, the spaces* $\mathcal{C}^0_{p,q}\big([a,b],M\big)$ *and* $\mathfrak{M}_{p,q}\big([a,b],M;\mathcal{S}_1,\mathcal{S}_2\big)$ *have the same homotopy type.*

3. Variational formulation for Riemannian polynomials

We will now proceed with the study of the variational problem arising from the interpolation problem. Let us consider a smooth m-dimensional manifold M and a complete Riemannian metric g on M. We will denote by ∇ the covariant derivative of the Levi–Civita connection of g and by R the curvature tensor of ∇ chosen with the following sign convention: $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. We will write indifferently $R(X,Y)Z$ or $R(X,Y,Z)$, the latter notation being more appropriate when dealing with covariant derivatives of R.

For all $k\in\mathbb{N}$ and all $p\in[1,+\infty]$, we say that a curve $x:[a,b]\to M$ is of class $W^{k,p}$ if for all local chart (U,φ) of M and all interval $[c,d]\subset x^{-1}(U)$, the composition $\varphi\circ x|_{[c,d]}$ is in the Sobolev space $W^{k,p}\big([c,d],\mathbb{R}^m\big)$ of C^{k-1} curves having p-integrable weak k-th derivative. It is well known that $W^{k,p}\big([a,b],M\big)$ is an infinite dimensional Banach manifold modelled on $W^{k,p}\big([a,b],\mathbb{R}^m\big)$. We will set $H^k=W^{k,2}$, since we will be mainly interested in $H^{k+1}\big([0,1],M\big)$, which is a complete Hilbert manifold.

For all $x\in H^{k+1}\big([0,1],M\big)$, the tangent space $T_xH^{k+1}\big([0,1],M\big)$ is identified with the Hilbert space of all vector fields along x of class H^k.

It is easy to see that the map:

$$H^{k+1}\big([0,1],M\big)\ni x\mapsto\Big(\dot x(0),\frac{\mathrm{D}}{\mathrm{d}t}\dot x(0),\ldots,\tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x(0);\dot x(1),\frac{\mathrm{D}}{\mathrm{d}t}\dot x(1),\ldots,\tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x(1)\Big),$$

where $\frac{\mathrm{D}}{\mathrm{d}t}$ denotes covariant differentiation along the curve x, is a submersion taking value in the cartesian product $TM^{(k)}\times TM^{(k)}$, where $TM^{(k)}$ is the Whitney sum of vector bundles $TM\oplus\ldots\oplus TM$ (k times) with itself. Thus, if p,q are fixed points of M, and if $v_1,\ldots,v_k\in T_pM$, $w_1,\ldots,w_k\in T_qM$ are given tangent vectors, the set

$$\Gamma:=H^{k+1}\big([0,1],M;\,p,v_1,\ldots,v_k;q,w_1,\ldots,w_k\big) \tag{5}$$

consisting of those curves $x\in H^{k+1}\big([0,1],M\big)$ such that:

$$\begin{aligned} x(0)&=p,\ \dot x(0)=v_1,\ldots,\tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x(0)=v_k,\\ x(1)&=q,\ \dot x(1)=w_1,\ldots,\tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x(1)=w_k \end{aligned} \tag{6}$$

is a smooth embedded closed (hence complete) submanifold of $H^{k+1}\big([0,1],M\big)$. For all $x \in \Gamma$, the tangent space $T_x\Gamma$ is identified with the closed subspace of $T_xH^{k+1}\big([0,1],M\big)$ consisting of those vector fields V such that:

$$
\begin{aligned}
V(0) &= \frac{\mathrm{D}}{\mathrm{d}t}V(0) = \ldots = \tfrac{\mathrm{D}^k}{\mathrm{d}t^k}V(0) = 0,\\
V(1) &= \frac{\mathrm{D}}{\mathrm{d}t}V(1) = \ldots = \tfrac{\mathrm{D}^k}{\mathrm{d}t^k}V(1) = 0.
\end{aligned} \tag{7}
$$

and is endowed with the Riemannian structure induced by the Hilbert space inner product $\langle V,W\rangle = \int_0^1 g\big(\tfrac{\mathrm{D}^{k+1}}{\mathrm{d}t^{k+1}}V, \tfrac{\mathrm{D}^{k+1}}{\mathrm{d}t^{k+1}}W\big)\,\mathrm{d}t$.

REMARK 2. The metric g induces a diffeomorphism Ψ between the fiber bundle $\mathfrak{J}^k(M)$ and the vector bundle $TM^{(k)}$, that for each point $\pi \in M$ is given by

$$
\mathfrak{J}^k(M)_\pi \ni [x] \stackrel{\Psi_\pi}{\longmapsto} \big(\dot x(0), \frac{\mathrm{D}}{\mathrm{d}t}\dot x(0), \ldots, \tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x(0)\big) \in T_\pi M \oplus \ldots \oplus T_\pi M. \tag{8}
$$

If $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the counterimages of $(v_1,\ldots,v_k)$ via Ψ_p and $(w_1,\ldots,w_k)$ via Ψ_q respectively, then it is easily seen that $\mathcal{S}_1$ and $\mathcal{S}_2$ are retracts of $\mathfrak{J}^k(M)_p$ and $\mathfrak{J}^k(M)_q$ respectively. It follows that Corollary 1 applies to the functional space Γ defined in (5), to conclude that Γ has the same homotopy type of the based loop space $\mathcal{C}^0_{p,q}\big([a,b],M\big)$.

In view of the above discussion, we will cast our variational problem on the space Γ. Indeed, we now give the following definition:

DEFINITION 1. *A a critical point x of the functional*

$$
f(x) = \frac{1}{2}\int_0^1 g\big(\tfrac{\mathrm{D}^k}{\mathrm{d}t^k}\dot x, \tfrac{\mathrm{D}^k}{\mathrm{d}t^k}\dot x\big)\,\mathrm{d}t. \tag{9}
$$

in Γ will be called a polynomial curve of order $2k+1$ in M *satisfying the boundary conditions* (6).

We are now going to establish the key property that the functional f needs to satisfy in order to recover results stated in Section 4. First, let us recall a couple of basic definitions. Given a smooth functional $f:\mathcal{X}\to\mathbb{R}$ on the Banach manifold $\mathcal{X}$, and given a smoothly varying Banach space structure $\mathfrak{h}$ on each tangent space $T_x\mathcal{X}$, a *Palais–Smale sequence* for f is a sequence $(x_n)_n$ in $\mathcal{X}$ such that:

1. $\lim_{n\to\infty} f(x_n) = c \in \mathbb{R}$;

2. $\lim_{n\to\infty} \|\mathrm{d}f(x_n)\|_{x_n} = 0$, where $\|\cdot\|_x$ is the norm of bounded linear functionals on $T_x\mathcal{X}$ induced by $\mathfrak{h}$.

We also recall that the functional f is said to satisfy the *Palais–Smale condition* if every Palais–Smale sequence for f has a converging subsequence in $\mathcal{X}$.

Palais–Smale condition is the compactness property required to apply classical results from Global Analysis. Of course, choosing the suitable path space is a crucial step. For instance, let us study cubic polynomials ($k = 1$) on the sphere S^2. If we take $H^2_{p,q}([0,1], S^2)$ as functional space – i.e. without fixing initial and final velocity – one can define a sequence x_n of curves making n loops around the great circle at p and q. This is a Palais–Smale sequence, but obviously does not possess any converging subsequence. Indeed, in the case we are studying, the constraints imposed by boundary conditions (6) on derivatives make things work properly, so that the following key result holds:

THEOREM 1. *The functional $f : \Gamma \to \mathbb{R}$ satisfies the Palais–Smale condition.*

REMARK 3. It may be worthwhile noticing that Palais–Smale condition holds for a whole class of functionals on Γ, also involving lower order derivatives, that contains the case introduced in (9) as a particular situation.

4. Conclusions

We are now ready to state the main results, applying Global Analysis techniques. To begin, a result of existence of Riemannian polynomials can be obtained:

PROPOSITION 2. *Each homotopy class of curves joining p and q contains a polynomial curve x of order $2k + 1$ satisfying boundary conditions* (6) *and that minimizes the action functional f in that homotopy class.*

Proof. It is an easy application of the so-called *Minimax Principle* (see for instance [15]) and of Remark 2. □

We recall that if $\mathcal{X}$ is a topological space and $\mathcal{Y} \subseteq \mathcal{X}$, then the *Ljusternik-Schnirelman category* $\mathrm{cat}_{\mathcal{X}}(\mathcal{Y})$ is an homotopic invariant, given by the (possibly infinite) minimal number of closed, contractible subsets of $\mathcal{X}$ that cover $\mathcal{Y}$; we set $\mathrm{cat}(\mathcal{X}) = \mathrm{cat}_{\mathcal{X}}(\mathcal{X})$. The core of the Ljusternik–Schnirelman theory is given by the following classical result [15]:

PROPOSITION 3. *Let $\mathcal{X}$ be a complete Banach manifold and let $f : \mathcal{X} \to \mathbb{R}$ be a functional bounded from below that satisfies the Palais–Smale condition. Then, f has at least* $\mathrm{cat}(\mathcal{X})$ *critical points; moreover, if* $\mathrm{cat}(\mathcal{X}) = +\infty$*, then f has arbitrarily large critical values.*

In this case, Ljusternik–Schnirelman theory gives us a lower bound for Riemannian polynomials:

PROPOSITION 4. *There exist at least* $\mathrm{cat}\big(\mathcal{C}^0_{p,q}([0,1], M)\big)$ *polynomial curves of order* $2k+1$ *in M satisfying the boundary conditions* (6)*. If M is not contractible, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of polynomial curves of order $2k + 1$ in M satisfying*

(6), *and such that:*

$$\lim_{n\to\infty} f(x_n) = +\infty.$$

Proof. It follows immediately from Remark 2, Theorem 1 and Proposition 3. If M is not contractible, a well known result of Fadell and Husseini (see [7]) states that $\mathcal{C}^0_{p,q}\big([0,1], M\big)$ has infinite Ljusternik–Schnirelman category. □

Regularity of Riemannian polynomials is ensured by standard variational arguments instead:

PROPOSITION 5. *A Riemannian polynomial of order* $2k+1$ *satisfying boundary conditions* (6) *is smooth, and satisfies a differential equation of the form:*

$$\tag{10} \tfrac{\mathrm{D}^{2k+1}}{\mathrm{d}t^{2k+1}}\dot x + \mathbf{G}\big(\dot x, \frac{\mathrm{D}}{\mathrm{d}t}\dot x, \dots, \tfrac{\mathrm{D}^{2k}}{\mathrm{d}t^{2k}}\dot x\big) = 0,$$

where $\mathbf{G} : TM^{(2k)} \to \mathbb{R}$ *is a map given by the sum of tensor fields over* M *obtained from* R *and its covariant derivatives.*

Proof. Computing the first variation of f at x is a lengthy but straightforward process that gives

$$\tag{11} \mathrm{d}f(x)V = \int_0^1 g\Big(\tfrac{\mathrm{D}^k}{\mathrm{d}t^k}\dot x, \tfrac{\mathrm{D}^{k+1}}{\mathrm{d}t^{k+1}}V + \mathbf{F}\big(\dot x, \frac{\mathrm{D}}{\mathrm{d}t}\dot x, \dots, \tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x, V, \frac{\mathrm{D}}{\mathrm{d}t}V, \dots, \tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}V\big)\Big)\,\mathrm{d}t,$$

where $\mathbf{F} = 0$ if $k = 0$ and $F : TM^{(2k)} \to \mathbb{R}$ is defined by:

$$\tag{12} \mathbf{F}\big(\dot x, \frac{\mathrm{D}}{\mathrm{d}t}\dot x, \dots, \tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}\dot x, V, \frac{\mathrm{D}}{\mathrm{d}t}V, \dots, \tfrac{\mathrm{D}^{k-1}}{\mathrm{d}t^{k-1}}V\big) = \sum_{j=0}^{k-1} \tfrac{\mathrm{D}^j}{\mathrm{d}t^j}\big(R(V,\dot x)\tfrac{\mathrm{D}^{k-j-1}}{\mathrm{d}t^{k-j-1}}\dot x\big), \quad k \ge 1.$$

Then, Euler–Lagrange equations (10) are obtained by successive integration by parts in (11), keeping in mind that the conditions (7) imply that all the boundary terms arising from these integration by parts vanish. □

EXAMPLE 1. Euler–Lagrange equation (10) for the cubic case ($k = 1$) is

$$\tfrac{\mathrm{D}^3}{\mathrm{d}t^3}\dot x - R\big(\dot x, \frac{\mathrm{D}}{\mathrm{d}t}\dot x\big)\,\dot x = 0,$$

whereas fifth order polynomials ($k = 2$) satisfy

$$\tfrac{\mathrm{D}^5}{\mathrm{d}t^5}\dot x + R\big(\frac{\mathrm{D}}{\mathrm{d}t}\dot x, \tfrac{\mathrm{D}^2}{\mathrm{d}t^2}\dot x\big)\dot x - R\big(\dot x, \tfrac{\mathrm{D}^3}{\mathrm{d}t^3}\dot x\big)\dot x = 0.$$

Finally, Morse theory can be applied, at least whenever $(p, v_1, \dots, v_k)$ and $(q, w_1, \dots, w_k)$ are not *conjugate* by Riemannian polynomials, that is it does not exist a critical point $x \in \Gamma$ of f such that the second variation $\mathrm{d}^2 f(x)$ of f at x is degenerate. Let us recall that, if $f : \mathcal{X} \to \mathbb{R}$ is a $\mathcal{C}^2$ map on a Hilbert manifold, f is a

Morse function if all its critical points are *nondegenerate* (i.e. $d^2 f(x_0)$ is represented by an invertible self–adjoint operator on $T_{x_0}\mathcal{X}$, whenever x_0 is a critical point), and the *Morse index* $m(x_0)$ is defined to be the index of the bilinear form $d^2 f(x_0)$.

In general, the central result of Morse theory can be stated as follows [15, 11]:

PROPOSITION 6. *Let $\mathcal{X}$ be a complete Hilbert manifold, $f : \mathcal{X} \to \mathbb{R}$ a Morse function which is bounded from below and satisfies the Palais–Smale condition. Then, given any coefficient field $\mathbb{F}$ and denoted by $\beta_n(\mathcal{X};\mathbb{F})$ the n-th* Betti's number *of $\mathcal{X}$ (i.e., the dimension of the n-th singular homology vector space with coefficient in $\mathbb{F}$), and by $\mathfrak{P}(\mathcal{X};\mathbb{F})(z) = \sum_{n=0}^{\infty} \beta_n(\mathcal{X};\mathbb{F})z^n$ the* Poincaré polynomial *of $\mathcal{X}$ with coefficients in $\mathbb{F}$, then there exists a formal power series $Q(z) = \sum_{n=0}^{\infty} q_n z^n$ with coefficients $q_n \in \mathbb{N}\bigcup\{+\infty\}$ such that the following identity between formal power series holds:*

$$\sum_{x \text{ critical point of } f} z^{m(x)} = \mathfrak{P}(\mathcal{X};\mathbb{F})(z) + (1+z)Q(z). \tag{13}$$

In our case, it can be showed [9] that $f : \Gamma \to \mathbb{R}$ is a Morse function if and only if $(p, v_1, \ldots, v_k)$ and $(q, w_1, \ldots, w_k)$ are not conjugate by Riemannian polynomials. Then, using Remark 2, Theorem 1 and the above proposition, one gets the following form for (13):

$$\sum_{n=0}^{\infty} \kappa_n z^n = \mathfrak{P}\big(\mathcal{C}^0_{p,q}\big([0,1],M\big);\mathbb{F}\big) + (1+z)Q(z), \tag{14}$$

where κ_n is the the number of Riemannian polynomials of order $2k+1$ in M satisfying the boundary conditions (6) and having Morse index n. In particular, if β_n^0 denotes the n–th Betti number of $\mathcal{C}^0_{p,q}\big([0,1],M\big)$, we can state the following result:

PROPOSITION 7. *Assuming that $(p, v_1, \ldots, v_k)$ and $(q, w_1, \ldots, w_k)$ are not conjugate by Riemannian polynomials of order $2k+1$, then there are at least $\sum_{n=0}^{\infty} \beta_n^0$ Riemannian polynomials of order $2k+1$ in M satisfying (6). Moreover, the number of such polynomials is either infinite or odd.*

Proof. It follows from the above discussion, whereas last statement comes from (14) setting $z = 1$. □

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