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# REMARKS ON STABILITY OF INVERTED PENDULA

**Abstract.** Using linearization principle and tools from chronological calculus one establishes a criteria for stabilization of, usually unstable, equilibrium position of Double Inverted Pendula when subject an arbitrary fast oscillation. Both, planar and spherical cases are considered.

## 1. Introduction

Problem of stability and stabilization of, usually unstable, upper equilibrium position of inverted pendulum has been intensively studied. Almost no bibliography is provided here. Some interesting references to the earlier work (starting from the beginning of 1900) can be found in [8]. Two publications introducing the readers into the field of vibrational control and vibrational mechanics are [5] and [11]. Here we just quote a "textbook result" from [10, Chap 5] concerning stability of an inverted pendulum of length $l$ with vertically (harmonically) oscillating point of suspension. Stability of equilibrium position is assured whenever number of oscillations in one unit of time is greater or equal then $\frac{1}{a}\sqrt{\frac{3}{64}}\omega$, where $\omega^2 = g/l$ and $a$ is the amplitude of oscillation of the suspension point.

In [6, 7], one of us considered problem of stability for time-periodic systems, in particular the problem of stabilizing equilibrium position of an inverted pendulum when its pivot is subject to an arbitrary fast oscillation. Conditions of stability were established there by application of technics from chronological calculus and averaging theory.

Following the techniques used in [4, 7, 6] one intends to derive similar conditions for double pendulum in planar and spherical cases.

The paper is organized as follows. Section 2 is devoted to chronological calculus and to some classical results on stability. In Section 3 equations for mechanical system (planar case) are set, the corresponding monodromy matrix is computed and the proof of main result for planar inverted pendulum is presented. The spherical case is treated in Section 4.

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## 2. Preliminaries

### 2.1. Variational formula of chronological calculus

Chronological calculus (see [1] and [9]) has been introduced by Agrachev and Gamkrelidze in [1] for nonlinear time-varying systems on smooth manifolds. Since the system to be considered in this work is linear, one presents some reformulations for (time-varying) linear case.

Consider the time-varying system of linear ordinary differential equations

$$\dot{z}(t) = A(t)\, z(t), \qquad z(0) = z_0 \tag{1}$$

where $z(t) \in \mathbb{R}^n$ and $A(t)$ is a matrix-valued function depending continuously on $t$. Considering its solutions $z(t; z_0)$ one can introduce a flow of linear maps $P^t$ : $z_0 \mapsto z(t, z_0)$. Obviously, $P^t$ is the unique solution to the matrix ordinary differential equation with initial condition

$$\dot{P}^t = A(t)\, P^t, \qquad P^0 = I_n.$$

Following [1], one calls the flow $P^t$ right chronological exponential of $A(t)$ and denotes it

$$P^t = \overrightarrow{\exp} \int_0^t A(\tau)\, d\tau.$$

One can also define left chronological exponential $\overleftarrow{\exp} \int_0^t A(\tau)\, d\tau$ as solution of $\dot{P}^t = P^t\, A(t),\ P^0 = I_n$.

Using this notation, the solution of problem (1) is represented as

$$z(t) = \overrightarrow{\exp} \int_0^t A(\tau)\, d\tau\, z_0 = P^t\, z_0.$$

The flow $\overrightarrow{\exp} \int_0^t A(\tau)\, d\tau$ admits a series expansion in the form

$$\overrightarrow{\exp} \int_0^t A(\tau)\, d\tau = \mathrm{Id} + \sum_{n=1}^{\infty} \underset{0 \le \tau_n \le \ldots \le \tau_1 \le t}{\int \cdots \int} A(\tau_n) \ldots A(\tau_1)\, d\tau_n \ldots d\tau_1.$$

In (1), take $A(t) = B(t) + C(t)$, considering $B(t)$ as reference matrix of coefficients and $C(t)$ as its perturbation. There holds so called chronological calculus variational formula

$$\overrightarrow{\exp} \int_0^t (B(\tau) + C(\tau))\, d\tau = \overrightarrow{\exp} \int_0^t \left( \overrightarrow{\exp} \int_0^\tau \mathrm{ad}\, B(\theta)\, d\theta \right) C(\tau)\, d\tau \,\circ \\ \circ\, \overrightarrow{\exp} \int_0^t B(\tau)\, d\tau \tag{2}$$

where “ad” is related to the matrix commutator as

$$\operatorname{ad} A(t)\, B(t) = -[A(t), B(t)] = B(t)\, A(t) - A(t)\, B(t)$$

and $Q^t = \overrightarrow{\exp} \int_0^t \operatorname{ad} B(\theta)\, d\theta\, d\tau$ is the solution of the operator differential equation

$$\frac{d}{dt} Q^t = Q^t \circ \operatorname{ad} B(t), \qquad Q^0 = Id.$$

### 2.2. Formal expansion for $\ln \overrightarrow{\exp} \int_0^t A_\tau d\tau$

Let $A(t) \stackrel{def}{=} A_t$ be a matrix-valued function with time-varying entries. One is interested in defining formal expansion

$$\Lambda_{0,t}(A_\tau) = \ln \overrightarrow{\exp} \int_0^t A_\tau d_\tau = \sum_{m=1}^{+\infty} \Lambda^{(m)}.$$

Here (see [1])

$$\Lambda^{(m)} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} d\tau_m\, g_m(A_{\tau_1}, \ldots, A_{\tau_m}), \tag{3}$$

where, for each $m \geq 2$, $g_m(A_{\tau_1}, \ldots, A_{\tau_m})$ is an homogeneous polynomial of first degree in each $A_{\tau_i}$. Moreover, it is a commutator polynomial in $A_{\tau_1}, \ldots, A_{\tau_m}$, i.e., it can be expressed as a linear combination of $A_{\tau_1}, \ldots, A_{\tau_m}$ and of their iterated commutators:

$$g_m(A_{\tau_1}, \ldots, A_{\tau_m}) = \sum_{\alpha=1}^{(2m-3)!!} b_{\upsilon_{1\alpha}} \ldots b_{\upsilon_{m\alpha}} w_\alpha(\tau_1, ..., \tau_m). \tag{4}$$

Each of $w_\alpha$ is an iterated Lie bracket of length $(m-1)$ of $A_{\tau_1}, \ldots, A_{\tau_m}$, $\upsilon_{ij}$ is the depth of $A_{\tau_i}$ in bracket $w_j$, $b_k = B_k/k!$, $k \geq 2$ and $B_k$ are Bernoulli numbers.

Following [1], one briefly explains the meaning of “depth of $A_{\tau_i}$ in $w_\alpha$”. Consider the symbols “ad" and $A_{\tau_1}$, ..., $A_{\tau_m}$ and all finite sequences of such symbols, which one calls *word* and denotes by $w$. One accepts repetitions of “ad” but no repetitions of $A_{\tau_1}$, ..., $A_{\tau_m}$ are allowed. A word is regular if by the introduction of suitable parenthesis it can be expressed as a commutator polynomial in $A_{\tau_1}$, ..., $A_{\tau_m}$ with the usual meaning of the symbol “ad”. Let $w$ be a regular word. Assume, $w = \upsilon_1 ... \upsilon_l A_\tau \widetilde{w}$. Then, depth of $A_\tau$ in $w$ is the number of regular words of the form $\upsilon_i ... \upsilon_l A_\tau$, $1 \leq i < l$. For more detailed description of polynomials $g_m$ see [1]. Another method to compute $\ln P^t$ using the so called chronological product is presented in [2, 3].

The series

$$\Lambda_{0,t}(A_\tau) = \sum_{m=1}^{\infty} \Lambda^{(m)}$$

is known to be absolutely convergent for $\int_0^t \|A_\tau\|\, d\tau \leq 0.44$, [1, Prop. 5.2].

### 2.3. Classical results on stability

Consider linear fast-oscillating system

$$\dot{x}(t) = A(k\,t)\,x(t) \tag{5}$$

where $x \in \mathbb{R}^n$, $A(t), t \geq 0$, is matrix-valued function continuous and 1-periodic with respect to $t$ and $k > 0$ is a large parameter.

A standard averaging result states that if all eigenvalues of corresponding averaged matrix $\int_0^1 A(t)\,dt$ are located in the open left complex half-plane then system (5) is asymptotically stable for all sufficiently large $k$.

Another stability condition uses monodromy matrix $P^1$ of time-periodic systems. If all eigenvalues of $P^1$ are located in the interior of the unit circle then the system is asymptotically stable; system is unstable if at least one eigenvalue lies outside the unit circle. In general it is difficult to compute spectrum of $P^1$.

It is more convenient to deal with the logarithm $\ln P^1$. In this case stability conditions can be formulated as: system (5) is asymptotically stable if all the eigenvalues of $\ln P^1$ are located in the open left complex half-plane and is unstable if at least one eigenvalue lies in the open right complex half-plane. System (5) is stable if all eigenvalues of $\ln P^1$ have non positive real part and purely imaginary eigenvalues are distinct.

## 3. Inverted double pendulum

Consider a mechanical system which consists of two mathematical inverted pendula (i.e., a pendulum in a gravitational field without friction and tension). Each pendulum is modeled by a mass point (the bob of mass $m_i$) and a massless beam of length $r_i$. The second pendulum is attached to the bob of the first one. One neglects the axial rotation of the beams, so there is one degree of freedom for each pendulum.

Assume that the pivot of first pendulum is subject to an arbitrary fast oscillation $\delta s(k\,t)$ where $\delta > 0$ is a small fixed parameter, $k$ can be arbitrarily large and, in any case, $\delta k > 1$. Assume also that $s(t)$ is a $C^2(\mathbb{R})$ 1-periodic function and that $\dot{s}(0) = 0$. Let $m_i$ and $r_i$ be the mass and length of each pendulum. Consider a coordinate system with the origin at the pivot of the first pendulum. Let $\theta_i; i = 1, 2$ to be the angle between each pendulum and the positive part of vertical axis and $g$ is the acceleration due to gravity.

From now on $\theta = (\theta_1, \theta_2)$. One uses standard matrix notation: $0_n$ will denote square matrix of order $n$ with zero entries and $I_n$ the identity matrix of order $n$.

The double inverted pendulum admits four equilibrium points: $(0, 0)$, $(\pi, 0)$, $(0, \pi)$ and $(\pi, \pi)$. Upper position corresponds to $\theta = (0, 0)$. The (forced) Hamiltonian $H^\ell$, corresponding to the linearized equations of the motion near this equilibrium point, is

$$H^\ell(t, p, \theta) = \frac{1}{2}\Big(\langle p, C\,p\rangle + \langle B(t)\,\theta, \theta\rangle\Big) \tag{6}$$

with $B(t) = -[g\,A + \delta k^2 \ddot{s}(kt)\,A]$, and $A$ and $C$ are constant matrices depending only on the parameters $m_1,\ m_2,\ r_1,\ r_2$:

$$A = \begin{pmatrix} (m_1 + m_2)r_1 & 0 \\ 0 & m_2 r_2 \end{pmatrix}, \quad C = \begin{pmatrix} (r_1^2 m_1)^{-1} & -(r_1 r_2 m_1)^{-1} \\ -(r_1 r_2 m_1)^{-1} & (m_1 + m_2)(r_2^2 m_2 m_1)^{-1} \end{pmatrix}.$$

Hamiltonian equations in matrix form are

$$\begin{pmatrix} \dot\theta \\ \dot p \end{pmatrix} = \begin{pmatrix} 0_2 & C \\ g\,A + \delta k^2 \ddot{s}(kt)\,A & 0_2 \end{pmatrix} \begin{pmatrix} \theta \\ p \end{pmatrix}. \tag{7}$$

Setting $\tau = k\,t$, denoting $z = (\theta, p)^T$ and $\dot z = dz/d\tau$, one transforms (7) into linear time-variant system

$$\dot z(\tau) = k^{-1} \begin{pmatrix} 0_2 & C \\ g\,A + \delta k^2 \ddot{s}(\tau)\,A & 0_2 \end{pmatrix} z(t)$$

or

$$\dot z(\tau) = Q(\tau) z(\tau). \tag{8}$$

To establish stability result one has to study monodromy matrix $P^1$ for linear time-varying system (8) corresponding to the Hamiltonian (6). One accomplishes this by chronological calculus variational formula introduced in Section 2.1. The main work consists of the study of the series expansion for $\ln P^1$. One studies the convergence of the series, estimates the rest term of its truncation, and derive stability result from the spectral information regarding this truncation.

### 3.1. Monodromy matrix

Apply variational formula (2) to the system (8) written as

$$\dot z(\tau) = [k^{-1} M + \delta k N(\tau)]\, z(\tau)$$

where

$$M = \begin{pmatrix} 0_2 & C \\ g\,A & 0_2 \end{pmatrix} \qquad \text{and} \qquad N(\tau) = \begin{pmatrix} 0_2 & 0_2 \\ \ddot{s}(\tau)\,A & 0_2 \end{pmatrix}.$$

One obtains

$$\begin{aligned} \overrightarrow{\exp} \int_0^1 \left(k^{-1} M + \delta k N_\tau\right) d\tau &= \overrightarrow{\exp} \int_0^1 \left(\overrightarrow{\exp} \int_0^\tau \operatorname{ad}(\delta k N_\sigma) d\sigma\; k^{-1} M\right) d\tau \,\circ \\ &\quad \circ \overrightarrow{\exp} \int_0^1 \delta k N_\tau\, d\tau \\ &= \overrightarrow{\exp} \int_0^1 D_\tau\, d\tau \circ \overrightarrow{\exp} \left(\delta k \int_0^1 N_\tau\, d\tau\right). \end{aligned}$$

From 1−periodicity of $s(\cdot)$ it follows that $\int_0^1 N_\tau\, d\tau = 0_4$. Besides

$$\overrightarrow{\exp}\left(\delta k \int_0^1 N_\tau\, d\tau\right) = \mathrm{e}^{\delta k \int_0^1 N_\tau\, d\tau} = I_4.$$

On the other hand,

$$\begin{aligned} D_\tau &= \overrightarrow{\exp}\left(-\delta k\,\mathrm{ad} \int_0^\tau N_\sigma\, d\sigma\right) k^{-1} M = \mathrm{e}^{-\delta k\,\mathrm{ad} \int_0^\tau N_\sigma\, d\sigma}\, k^{-1} M \\ &= \left(I_4 + (-\delta k)\,\mathrm{ad} \int_0^\tau N_\sigma\, d\sigma + \frac{(-\delta k)^2}{2}\,\mathrm{ad}^2 \int_0^\tau N_\sigma\, d\sigma + \dots\right) k^{-1} M. \qquad (9) \end{aligned}$$

By direct computations

$$\mathrm{ad}\left(\int_0^\tau N_\sigma\, d\sigma\right) M = \begin{pmatrix} -CA\dot{s}(\tau) & 0_2 \\ 0_2 & AC\dot{s}(\tau) \end{pmatrix},$$
$$\mathrm{ad}^2\left(\int_0^\tau N_\sigma\, d\sigma\right) M = \begin{pmatrix} 0_2 & 0_2 \\ -2ACA\dot{s}^2(\tau) & 0_2 \end{pmatrix}$$

and

$$\mathrm{ad}^j\left(\int_0^\tau N_\sigma\, d\sigma\right) M = 0_4, \qquad \text{for } j \geq 3.$$

Therefore series in (9) ends at the second order term and monodromy matrix can be represented as

$$P^1 = \overrightarrow{\exp} \int_0^1 D_\tau\, d\tau$$

where

$$D_\tau = \begin{pmatrix} \delta\dot{s}(\tau)CA & k^{-1}C \\ k^{-1}g\,A - \delta^2 k\,\dot{s}^2(\tau)A\,C\,A & -\delta\dot{s}(\tau)AC \end{pmatrix}. \qquad (10)$$

Obviously the matrix $P^1$ is uniquely determined, but the representation of $P^1$ as a chronological exponential $\overrightarrow{\exp} \int_0^1 D_\tau\, d\tau$ is not unique. Our construction of $D_\tau$ allows to establish stability results basing on the properties of the averaging of $D_\tau$.

From what was said in Section 2.3, the system can not be asymptotically stable since $\ln P^1$ is traceless. It can be stable if all eigenvalues are imaginary numbers (two pairs of conjugate imaginary numbers).

### 3.2. Asymptotic expansion for the logarithm of monodromy matrix

Introduce the logarithm

$$\Lambda_{0,t}(D_\tau) = \ln \overrightarrow{\exp} \int_0^t D_\tau\, d\tau.$$

As it was mentioned in Section 2.2 $\Lambda_{0,t}(D_\tau)$ admits an expansion

$$\Lambda_{0,t}(D_\tau) = \sum_{m=1}^{\infty} \Lambda^{(m)} \tag{11}$$

where $\Lambda^{(m)}$ is defined by (3).

Since $w_\alpha$ in (4) is a Lie bracket of length $(m-1)$, one arranges an estimate for $[D_{\tau_m}, \ldots, [D_{\tau_2}, D_{\tau_1}] \ldots]$ where $D_\tau$ is defined by (10). Assume that $|\dot{s}(\tau)| \leq \mu$ for all $\tau \in [0,1]$.

LEMMA 1. *There exist positive constants $\sigma$ and $a$ such that, for all $\bar{\tau} = (\tau_1, \ldots, \tau_m) \in [0,1]^m$, iterated Lie bracket of length $m$, $m > 2$, $[D_{\tau_m}, \ldots, [D_{\tau_2}, D_{\tau_1}] \ldots]$ can be represented as*

$$\delta^m \begin{pmatrix} \sigma_{11}^m(\bar{\tau})\,(CA)^m + \varepsilon^2\, C_{11}^m & \varepsilon\, \sigma_{12}^m(\bar{\tau}) =, C(AC)^{m-1} + \varepsilon^3\, C_{12}^m \\ \varepsilon^{-1}\, \sigma_{21}^m(\bar{\tau}) A\,(CA)^m + \varepsilon\, C_{21}^m & = \sigma_{11}^m(\bar{\tau})\,(AC)^m - \varepsilon^2\,(C_{11}^T)^m \end{pmatrix},$$

*where $\varepsilon = (\delta k)^{-1}$, and $|\sigma_{ij}^m(\bar{\tau})| < \sigma^m$ and $\|C_{ij}^m\| < a^m$ for ; $i, j = 1, 2$.*

Proof is done by induction on length of the Lie bracket.

LEMMA 2. *Series* (11) *is absolutely convergent if*

$$|\delta| < \frac{0.4432}{\hat{\sigma}\, a}$$

*where $\hat{\sigma} = (\sigma^m + 1)^{1/m}$ and $a$, $\sigma$ are the constants introduced in Lemma* 1.

*Proof.* Proof is carried out by proving several inequalities.

For a matrix $A = (a_{ij})_{i,j=1}^n$, $a_{ij} \in \mathbb{R}$, one defines the norm by

$$\|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|.$$

In what follows $[D_{\tau_m}, \ldots, [D_{\tau_2}, D_{\tau_1}] \ldots]$ is denoted by $D_m$.

From Lemma 1 and since $\varepsilon < 1$

$$\begin{aligned}
\|D_m\| &< \delta^m \max\{\sigma^m \|CA\|^m + \varepsilon^2 \|C_{11}^m\| + \varepsilon^{-1}\sigma^m \|A\|\,\|CA\|^m + \varepsilon\,\|C_{21}^m\|; \\
&\qquad \varepsilon\,\sigma^m \|C\|\|AC\|^{m-1} + \varepsilon^3 \|C_{12}^m\| + \sigma^m \|AC\|^m + \varepsilon^2 \|(C_{11}^T)^m\|\} \\
&< \delta^m a^m \max\{\sigma^m + \varepsilon^2 + \varepsilon^{-1}\sigma^m + \varepsilon; \varepsilon\,\sigma^m + \varepsilon^3 + \sigma^m + \varepsilon^2\} \\
&< \delta^m a^m \max\{\sigma^m(\varepsilon^{-1}+1) + \varepsilon(\varepsilon+1);\ \sigma^m(\varepsilon+1) + \varepsilon^2(\varepsilon+1)\}, \\
&< \delta^m a^m [\sigma^m(\varepsilon^{-1}+1) + \varepsilon(1+\varepsilon)] < 2\,\delta^m a^m \varepsilon^{-1}(\sigma^m + 1) \\
&< 2\,\varepsilon^{-1}\delta^m a^m \hat{\sigma}^m,
\end{aligned}$$

where $\hat{\sigma}^m > \sigma^m + 1$. Therefore,

$$\|D_m\| \le 2\,\varepsilon^{-1}\,\delta^m\,a^m\,\hat{\sigma}^m. \tag{12}$$

Now one sets an upper bound for $\|g_m(D_{\tau_1},\dots,D_{\tau_m})\|$. From the definition (4) of $g_m$ it follows that

$$\|g_m(D_{\tau_1},\dots,D_{\tau_m})\| \le \left(\sum_{\alpha=1}^{(2m-1)!!} |b_{\upsilon_{1\alpha}}\dots b_{\upsilon_{m\alpha}}|\right)\|D_m\| \le \chi_m\,\|D_m\|, \tag{13}$$

where $\chi_m$ are constants determined by the Taylor expansion of the function

$$\chi(z) = \frac{z}{2}(1-\cot\frac{z}{2}) + 2 = \sum_{\alpha=0}^{\infty} |b_\alpha| z^\alpha, \qquad z \in \mathbb{C}.$$

Recall that this Taylor expansion converges for $|z| \le 2\pi$.

From (3), (12) and (13) it follows that

$$\begin{aligned}
\|\Lambda^{(m)}\| &\le \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{m-1}} d\tau_m\,\|g_m(D_{\tau_1},\dots,D_{\tau_m})\| \\
&\le \chi_m\left(\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{m-1}} d\tau_m\,\|D_m\|\right) \\
&\le 2\,\varepsilon^{-1}\,\chi_m\,\delta^m\,a^m\,\hat{\sigma}^m\left(\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{m-1}} d\tau_m\right) \\
&\le 2\,\varepsilon^{-1}\,\chi_m\,\delta^m\,a^m\,\hat{\sigma}^m\,\frac{1}{m!}.
\end{aligned}$$

In [1] it is established that, for each $\gamma \in (0, 2\pi)$

$$\chi_m \le (m-1)!\frac{\gamma}{2}\left(\frac{2M(\gamma)}{\gamma}\right)^m$$

where $M(\gamma) = \max_{z\in\mathbb{C},|z|=\gamma} |\chi(z)|$. Therefore

$$\|\Lambda^{(m)}\| < 2\,\varepsilon^{-1}\,(m-1)!\frac{\gamma}{2}\left(\frac{2M(\gamma)}{\gamma}\right)^m \delta^m\,a^m\,\hat{\sigma}^m\,\frac{1}{m!} < \varepsilon^{-1}\,\gamma\left(\frac{2M(\gamma)}{\gamma}\,\delta\,a\,\hat{\sigma}\right)^m.$$

Series (11) converges absolutely if

$$\left|\,\delta\,a\,\hat{\sigma}\,\right| < \max_{\gamma\in(0,2\pi)} \frac{\gamma}{2\,M(\gamma)} = 0.4432.$$

□

### 3.3. Estimate for the rest term $\Lambda - \Lambda^{(1)}$

One would like to estimate the rest term $\Lambda - \Lambda^{(1)}$. Obviously,

$$\Lambda - \Lambda^{(1)} = \sum_{m=2}^{\infty} \Lambda^{(m)}$$

and by Lemma 2 the latter series is convergent for sufficiently small $\delta > 0$. The following result holds.

LEMMA 3. *For series* (11)*, there exists a constant* $\xi > 0$ *such that,*

$$\Lambda - \Lambda^{(1)} = \delta^2 R$$

*where*

$$R = \begin{pmatrix} R_{11} + \varepsilon^2 Q_{11} & \varepsilon R_{12} + \varepsilon^3 Q_{12} \\ \varepsilon^{-1} R_{21} + \varepsilon Q_{21} & -R_{11}^T - \varepsilon^2 Q_{11}^T \end{pmatrix},$$

*and* $\|R_{ij}\|,\ \|Q_{ij}\| < \xi;\ i, j = 1, 2.$

Notice that the block structure of $\Lambda^{(m)}$ has been established in Lemma 1. Let one set an estimate for the left-upper $2 \times 2$ block. One gets

$$\begin{aligned} \|\Lambda_{11}^{(m)}\| &< \frac{\delta^m}{m!} \chi_m \|\sigma_{11}^m(\bar{\tau})\,(CA)^m + \varepsilon^2 C_{11}^m\| \\ &< \frac{\delta^m}{m} \frac{\gamma}{2} \left(\frac{2M(\gamma)}{\gamma}\right)^m (\sigma^m \|CA\|^m + \varepsilon^2 \|C_{11}^m\|) \\ &< \frac{\delta^m}{m} \frac{\gamma}{2} \left(\frac{2M(\gamma)}{\gamma}\right)^m a^m (\sigma^m + \varepsilon^2) \\ &< \delta^2 \frac{\gamma}{2} \left(\frac{2M(\gamma)}{\gamma} a\right)^2 \left[\left(\delta\, a\, \sigma \frac{2M(\gamma)}{\gamma}\right)^{m-2} + \varepsilon^2 \left(\delta\, a \frac{2M(\gamma)}{\gamma}\right)^{m-2}\right]. \end{aligned}$$

Thus the series $\sum_{m=2}^{\infty} \Lambda_{11}^{(m)}$ converges absolutely if $|\delta\, a\, \sigma| < \frac{\gamma}{2\, M(\gamma)}$ and $|\delta\, a| < \frac{\gamma}{2\, M(\gamma)}$. By Lemma 2, both conditions are true since $\sigma < \bar{\sigma}$ and $\bar{\sigma} > 1$.

The proof for other blocks is similar.

### 3.4. Stability of the first-order averaging

Eigenvalues of $\Lambda^{(1)}$ perform a crucial role in establishing stability conditions.

Computing $\Lambda^{(1)}$, which is the first averaging of $D_\tau$, one obtains

$$\Lambda^{(1)} = \int_0^1 D_\tau \, d\tau = \begin{pmatrix} 0_2 & k^{-1} C \\ k^{-1} g\, A - \delta^2 k\, \bar{s} A\, C\, A & 0_2 \end{pmatrix}$$

with

$$\bar{s} = \int_0^1 \dot{s}^2(\tau)\, d\tau > 0.$$

Obviously $\Lambda^{(1)}$ is a Hamiltonian matrix of dimension 4.

Denoting $\delta^2 \bar{s} = \gamma$ and $k^{-1} g\, A - \gamma\, k\, ACA = \Sigma$ one computes the characteristic polynomial of $\Lambda^{(1)}$:

$$\det(\lambda I_4 - \Lambda^{(1)}) = \lambda^4 + p_2 \lambda^2 + p_0 \tag{14}$$

where

$$p_2 = -\operatorname{tr}(k^{-1}\Sigma\, C) = -k^{-2} g\ \operatorname{tr}(A\, C) + \gamma\ \operatorname{tr}(A\, C)^2 \tag{15}$$

and

$$p_0 = \det(k^{-1}\Sigma\, C) = \det(k^{-2} g\, I_2 - \gamma\ A\, C)\det(A\, C). \tag{16}$$

As one knows $\Lambda^{(1)}$ is a Hamiltonian matrix; its characteristic polynomial (14) is biquadratic. If $\Lambda^{(1)}$ is stable and possesses two distinct pairs of conjugate imaginary nonzero eigenvalues, then $p_2$, $p_0$ must be positive. Therefore there must hold:

$$p_2 = -\operatorname{tr}(k^{-1}\Sigma\, C) > 0 \tag{17}$$

and

$$p_0 = \det(k^{-1}\Sigma\, C) > 0. \tag{18}$$

Inequality (17) is equivalent to (see equation (15))

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\, d\tau > k^{-2}\, g\, \frac{\operatorname{tr}(A\, C)}{\operatorname{tr}(A\, C)^2}. \tag{19}$$

To study the sign of $\det(k^{-1}\Sigma C)$ in (18) note that $\det(A\, C) > 0$. Then from (16), the sign of $\det(k^{-1}\Sigma C)$ depends only on the sign of $\det\left(k^{-2} g\, I_2 - \gamma\, A\, C\right)$, which is a quadratic polynomial in $\gamma$:

$$\det\left(k^{-2} g\, I_2 - \gamma\, A\, C\right) = \gamma^2 \det(A\, C) - \gamma\ k^{-2}\, g \operatorname{tr}(A\, C) + k^{-4} g^2. \tag{20}$$

The discriminant of this latter polynomial equals

$$\operatorname{tr}^2(A\, C) - 4\det(A\, C) > 0.$$

Therefore this polynomial has two real roots

$$\gamma_\pm = k^{-2}\, g\, \frac{-\operatorname{tr}(A\, C) \pm \sqrt{\operatorname{tr}^2(A\, C) - 4\det(A\, C)}}{2\det(A\, C)},$$

and the inequality (18) holds either for

$$\delta^2 \int_0^1 \dot{s}^2(\tau) d\tau > \gamma_+, \tag{21}$$

or for

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\, d\tau < \gamma_-. \tag{22}$$

Inequality (19) is incompatible with (22), while (21) implies (19). Therefore (19)-(22)-(21) can be reduced to a single inequality (equivalent to (21))

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\, d\tau > k^{-2} g \frac{\upsilon (r_1 + r_2) + \sqrt{\upsilon^2 (r_1 + r_2)^2 - 4\upsilon\, r_1\, r_2}}{2\upsilon}, \tag{23}$$

where

$$\frac{m_1 + m_2}{m_1} = \upsilon,\ \det(A\, C) = \frac{\upsilon}{r_1 r_2} > 0,\ \operatorname{tr}(A\, C) = \upsilon \frac{r_1 + r_2}{r_1 r_2} > 0. \tag{24}$$

The following fact holds for the polynomial (14).

LEMMA 4. *The discriminant $p_2^2 - 4p_0$ of the biquadratic polynomial (14) is nonnegative for any choice of the parameters $m_1, m_2, r_1, r_2$ of the system. It is positive if the condition (23) holds.*

*Proof.* To simplify the notation denote $\operatorname{tr}(A\, C), \operatorname{tr}(A\, C)^2$ by $t_1, t_2$ and $\det(A\, C)$ by $d$, correspondingly. Recall that $\delta^2 \bar{s} = \gamma$, and put $\eta = k^{-2} g$. Then from formulae (15), (16) and (20) one concludes

$$\begin{aligned} p_2^2 - 4\, p_0 &= (-\eta\, t_1 + \gamma\, t_2)^2 - 4(\gamma^2 d - \gamma\, \eta\, t_1 + \eta^2)\, d \\ &= \eta^2 (t_1^2 - 4\, d) - 2\gamma\, \eta\, t_1 (t_2 - 2\, d) + \gamma^2 (t_2 - 2\, d)(t_2 + 2\, d). \end{aligned}$$

For any $(2 \times 2)$-matrix $N$ the identity $\operatorname{tr} N^2 = (\operatorname{tr} N)^2 - 2 \det N$ holds. Taking $N = AC$ one concludes with the identity $t_2 = t_1^2 - 2d$. Therefore $t_1^2 - 4\, d = t_2 - 2\, d$ and one obtains:

$$p_2^2 - 4\, p_0 = (t_1^2 - 4\, d)(\gamma\, t_1 - \eta)^2. \tag{25}$$

Substituting expressions (24) for $t_1 = \operatorname{tr}(A\, C)$ and $d = \det(A\, C)$ one gets

$$t_1^2 - 4\, d = \frac{\upsilon}{r_1^2\, r_2^2} \left( \upsilon\, r_1^2 + \upsilon\, r_2^2 + 2(\upsilon - 2)\, r_1\, r_2 \right).$$

Since $\upsilon > 1$ for $m_2 > 0$ (non triviality condition), the latter expression is strictly positive for all non vanishing $r_1, r_2$ and hence the right hand side of (25) is nonnegative.

If condition (21) holds, then $\gamma > \eta \frac{r_1 + r_2}{2}$ and (since $\upsilon > 1$)

$$\gamma\, t_1 - \eta = \gamma\, \upsilon \frac{r_1 + r_2}{r_1 r_2} - \eta > \gamma \frac{r_1 + r_2}{r_1 r_2} - \eta > \eta \left( \frac{(r_1 + r_2)^2}{2 r_1 r_2} - 1 \right) = \eta \frac{r_1^2 + r_2^2}{2 r_1 r_2} > 0.$$

Therefore $p_2^2 - 4p_0$ defined by (25) is positive provided that (21) holds. $\square$

### 3.5. Stability of double inverted pendulum

Finally, one is able to establish the main result regarding the stability of double pendulum, or, the same of matrix $\Lambda$.

THEOREM 1. *For each $\epsilon > 0$ there exist $\delta_0 > 0, k_0 > 0$ such that upper equilibrium position $(\theta_1, \theta_2) = (0, 0)$ of inverted double pendulum is stable if $0 < \delta < \delta_0, k > k_0$ and*

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\, d\tau > k^{-2} g \, \frac{\upsilon(r_1 + r_2) + \sqrt{\upsilon^2(r_1 + r_2)^2 - 4\, \upsilon\, r_1 r_2}}{2\upsilon} + \epsilon \tag{26}$$

*and unstable if $0 < \delta < \delta_0, k > k_0$ and*

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\, d\tau < k^{-2} g \, \frac{\upsilon(r_1 + r_2) + \sqrt{\upsilon^2(r_1 + r_2)^2 - 4\, \upsilon\, r_1 r_2}}{2\upsilon} - \epsilon. \tag{27}$$

*Proof.* From Lemma 3, $\Lambda = \Lambda^{(1)} + \delta^2 R$ and one sees that

$$\det\left(\lambda I - (\Lambda^{(1)} + \delta^2 R)\right) = \det(\lambda I - \Lambda^{(1)}) + r(\lambda)$$

where $r(\lambda) = \rho_2 \lambda^2 + \rho_0$, $\rho_2 = \mathcal{O}(\delta(\delta^2 + k^{-2}))$, $\rho_0 = \mathcal{O}(\delta(\delta^4 + k^{-4}))$, as $\delta + k^{-1} \to 0$. The characteristic polynomial of $\Lambda$ can be written as

$$\det(\lambda I - \Lambda) = \lambda^4 + q_2 \lambda^2 + q_0 \tag{28}$$

and, under condition (26), is close to the characteristic polynomial $\lambda^4 + p_2 \lambda^2 + p_0$ of $\Lambda^{(1)}$. Namely, $p_0 \neq 0$, $p_2 \neq 0$ and $q_2 = p_2 (1 + \mathcal{O}(\delta))$, $q_0 = p_0 (1 + \mathcal{O}(\delta))$ as $\delta \to 0$.

Assume that condition (26) holds. Then $p_0 > 0$, $p_2 > 0$ and $p_2^2 - 4p_0 > 0$. Evidently for sufficiently small $\delta_0$, and for $\delta < \delta_0$ and $k$ satisfying (26) one gets $q_2 > 0$, $q_0 > 0$ and $q_2^2 - 4q_0 > 0$. Therefore $\Lambda$ is stable.

If condition (27) holds, then $p_0 < 0$ and for sufficiently small $\delta_0$, also $q_0 < 0$ and one gets instability of $\Lambda$. □

## 4. Spherical case

### 4.1. Simple spherical pendulum

Consider a simple inverted pendulum under the same settings as in Section 3. Assume that motion takes place on 3D-space.

As before, let $m$ and $r$ be, respectively, mass and length of the pendulum. Let $\theta$ be the angle between the pendulum and its projection on the $xOz$ plane. Denote by $\phi$ the angle between the latter projection and the positive part of vertical axis $z$. The system has two degrees of freedom: $\theta$ and $\phi$. Let $q$ denote $(\theta, \phi)$.

Proceeding as for planar case, an equilibrium point for system is $q = 0$. In a neighborhood of this equilibrium, the four-dimensional Hamiltonian system of first order equations arising from linearized Hamiltonian is, in matrix form,

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & (m\,r^2)^{-1}\,I_2 \\ m\,r\,I_2\,[g + k^2\ddot{s}(k\,t)] & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

or

$$\dot{z}(t) = \begin{pmatrix} 0 & C \\ A\,g + \delta\,k^2\ddot{s}(k\,t)\,A & 0 \end{pmatrix} z(t) = Q(t)\,z(t) \tag{29}$$

with $z(t) = (q, p)^T$, $A = m\,r\,I_2$ and $C = (m\,r^2)^{-1}\,I_2$.

System (29) is analogous to system (8) and one may apply the approach of previous sections. So, for simple spherical pendulum equilibrium position $q = 0$ is stable whenever, c.f. condition (23),

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\,d\tau > k^{-2}\,g\,\frac{\operatorname{tr}(A\,C) + \sqrt{\operatorname{tr}^2(A\,C) - 4\det(A\,C)}}{2\det(A\,C)} + \epsilon.$$

One has $\operatorname{tr}(A\,C) = 2\,r^{-1}$ and $\det(A\,C) = r^{-2}$. Stability condition arises in a very simple form.

THEOREM 2. *For each $\epsilon > 0$ there exist $\delta_0 > 0$, $k_0 > 0$ such that upper equilibrium position $(\theta, \phi) = 0$ of simple spherical pendulum is stable if* $0 < \delta < \delta_0$, $k > k_0$ *and*

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\,d\tau > k^{-2}\,g\,r + \epsilon$$

*and unstable if* $0 < \delta < \delta_0$, $k > k_0$ *and*

$$\delta^2 \int_0^1 \dot{s}^2(\tau)\,d\tau < k^{-2}\,g\,r - \epsilon.$$

### 4.2. Double spherical pendulum

Consider a double pendulum as presented in Section 3 describing a 3-D motion. Let $\theta_i$; $i = 1, 2$ and $\phi_i$; $i = 1, 2$ be as described in Section 4.1.

Proceeding as in two previous cases one linearizes Hamiltonian system at the equilibrium point and obtains eight equations on variables $\theta_1$, $\theta_2$, $\phi_1$, $\phi_2$, $p_{\theta_1}$, $p_{\theta_2}$, $p_{\phi_1}$ and $p_{\phi_2}$ which take the form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0_2 & C \\ -B(t) & 0_2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

where

$$C = \begin{pmatrix} G & 0_2 \\ 0_2 & G \end{pmatrix}, \quad \text{with} \quad G = \frac{1}{m_1 m_2 r_1^2 r_2^2} \begin{pmatrix} r_2^2 & -m_2\,r_1\,r_2 \\ -m_2\,r_1\,r_2 & (m_1 + m_2)\,r_1^2 \end{pmatrix},$$

and

$$B(t) = \begin{pmatrix} -A & 0_2 \\ 0_2 & -A \end{pmatrix} [g + \delta\, k^2 \ddot{s}(k\,t)], \quad \text{with} \quad A = \begin{pmatrix} (m_1 + m_2)\, r_1 & 0 \\ 0 & m_2\, r_2 \end{pmatrix}.$$

This system of eight equations can be re-written as two decoupled systems of dimension four such that each system structure is analogous to system (7)

$$\begin{pmatrix} \dot{\theta} \\ \dot{p}_\theta \end{pmatrix} = \begin{pmatrix} 0 & G \\ g\,A + \delta\, k^2 \ddot{s}(k\,t)\,A & 0 \end{pmatrix} \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} \tag{30}$$

and

$$\begin{pmatrix} \dot{\phi} \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} 0 & G \\ g\,A + \delta\, k^2 \ddot{s}(k\,t)\,A & 0 \end{pmatrix} \begin{pmatrix} \phi \\ p_\phi \end{pmatrix}. \tag{31}$$

Decoupled form (30)-(31) of spherical inverted double pendulum shows that one can study independently its projections on $x\,O\,y$ (system on variables $\phi_i;\ i = 1, 2$) and $y\,O\,z$ (system on variables $\theta_i;\ i = 1, 2$). Therefore stability conditions for spherical inverted double pendulum coincides with those presented in Theorem 1.

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