Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, 4 (2005)
Control Theory and Stabil., I

L. Abrunheiro* – **M. Camarinha***

RIEMANNIAN CUBIC POLYNOMIALS

Abstract. This paper gives an analysis of the Riemannian cubic polynomials, with special interest in the Lie group $SO(3)$, based on the study of a second order variational problem. The corresponding Euler-Lagrange equation gives rise to an interesting system of nonlinear differential equations. Motivated by the problem of the motion of a rigid body, the reduction of the essential size and the separation of the variables of the system are obtained by means of invariants along the cubic polynomials.

1. Introduction

The question of generalizing cubic polynomials to non-Euclidean spaces has been the object of intensive investigation in the context of interpolation theory. It is well known that the cubic polynomials are solutions of a variational problem in the Euclidean space, whose Lagrangian function is the squared norm acceleration. A generalization of this notion to Riemannian manifolds was introduced in 1989 [8] and explored from a dynamic interpolation perspective in 1995 [5]. In the last years it has been developed an interesting geometric theory related to the cubic polynomials which is surprisingly close to the Riemannian theory of geodesics [5, 4]. An analytical theory for cubic polynomials (Ljusternik -Schnirelman theory, Morse theory) has also been established [6]. After that, it became important to find interesting cubic polynomials examples, namely the $SO(3)$ case.

The main idea of the present paper is a qualitative analysis of the cubic polynomials on $SO(3)$ and is motivated by the particular importance of the notion of cubic polynomials on Lie groups to the development of interpolation theory, for instance in trajectory planning for rigid body motion with applications to aeronautics and robotics. An analogous analysis, but for the particular case of the so-called null cubic polynomials, was produced in [9]. The paper starts with the definition of the generalized Riemannian cubic polynomials and the deduction of some invariants along the cubic polynomials. It follows a special analysis of the problem of the motion of a rigid body as a motivation to section 4, which is devoted to the essential case of $SO(3)$. In this specific situation, a reduction of the cubic polynomial equation and a property related with a reparametrization of the cubic polynomial are obtained.

2. Cubic polynomials on a Riemannian manifold

Consider a Riemannian manifold M, with Riemannian metric $\langle .,. \rangle$. Denote the symmetric connection on M, which is compatible with this metric, by ∇, and the covariant

*Work partially supported by ISR, FCT (Portuguese Science and Technology Foundation) project posi/sri/41618/2001.

derivative along a curve x in M by DX/dt where X is a vector field along x. Moreover, denote the curvature tensor field by R and the covariant differential of R by ∇R.

In order to generalize the notion of cubic polynomials to a Riemannian manifold, the following second order variational problem in M was formulated [8, 5]

$$\min_{x\in\Omega} \frac{1}{2}\int_0^T \left\langle \frac{D^2x}{dt^2}\,,\ \frac{D^2x}{dt^2}\right\rangle dt$$

where Ω is the class of C^1 piecewise smooth curves $x : [0, T] \to M$, satisfying

$$x(0) = p, \quad \frac{dx}{dt}(0) = \upsilon, \quad x(T) = q, \quad \frac{dx}{dt}(T) = u,$$

with $(p, \upsilon), (q, u) \in TM$ and $T \in \mathbb{R}^+$.

Cubic polynomials on M are smooth curves $x : [0, T] \to M$ that are solutions of the Euler-Lagrange equation

$$\frac{D^4x}{dt^4} + R\left(\frac{D^2x}{dt^2}\,,\ \frac{dx}{dt}\right)\frac{dx}{dt} = 0 \tag{1}$$

Consider x a cubic polynomial and denote the velocity vector field along x, $\dfrac{dx}{dt}$, by V.

LEMMA 1. *[2] The following expression is invariant along the cubic polynomial*

$$\left\langle \frac{D^2V}{dt^2}\,,\ V\right\rangle - \frac{1}{2}\left\langle \frac{DV}{dt}\,,\ \frac{DV}{dt}\right\rangle.$$

Proof. The result follows from the integration of the inner product of (1) with V. □

LEMMA 2.

$$\frac{d}{dt}\left[\left\langle \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2}\right\rangle - \left\langle \frac{D^3V}{dt^3}, \frac{DV}{dt}\right\rangle\right] = \left\langle (\nabla_V R)\left(\frac{DV}{dt}, V\right)V, \frac{DV}{dt}\right\rangle$$

Proof. Use the tensor curvature property $\langle R(X,Y)Z, W\rangle = \langle R(W,Z)Y, X\rangle$ and the definition of the covariant differentiation of the curvature tensor R, that is,

$$\begin{aligned}&(\nabla_X R)(Y,Z)W \\ &\quad = \nabla_X [R(Y,Z)W] - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W,\end{aligned}$$

to get

$$\frac{d}{dt}\left\langle R\left(\frac{DV}{dt}V\right), \frac{DV}{dt}\right\rangle - \left\langle (\nabla R)\left(\frac{DV}{dt}, V\right)V, \frac{DV}{dt}\right\rangle$$
$$= 2\left\langle R\left(\frac{DV}{dt}, V\right)V, \frac{D^2V}{dt^2}\right\rangle.$$

Moreover, notice that

$$\frac{d}{dt}\left\langle \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2}\right\rangle = 2\left\langle \frac{D^3V}{dt^3}, \frac{D^2V}{dt^2}\right\rangle$$

In order to complete the proof, it is sufficient to make the inner product of (1) with $\frac{D^2V}{dt^2}$ and apply the above equalities in the obtained equation. □

REMARK 1. In Riemannian locally symmetric manifolds, the lemma 2 gives a second invariant along the cubic polynomial,

$$\left\langle \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2}\right\rangle - \left\langle \frac{D^3V}{dt^3}, \frac{DV}{dt}\right\rangle.$$

3. Motion of a rigid body

The Lie group $SO(3)$ is the configuration space for the motion of a rigid body with no external forces and fixed centre of mass. This problem is an interesting motivation to the study of cubic polynomials on $SO(3)$, since the motion can be described as motion along geodesics on the Lie group $SO(3)$ provided with a left-invariant Riemannian metric ([1], appendix 2).

The motion of a rigid body is described by the equation

$$\frac{d}{dt}(\mathrm{J}\,y) = (\mathrm{J}\,y) \times y \tag{2}$$

where $y = (y_1, y_2, y_3)^t$ is the angular velocity and J is the diagonal matrix whose elements are the principal moments of inertia in the body. The following two invariants are satisfied

$$\langle \mathrm{J}\,y\,,\ y\rangle = c \qquad \text{and} \qquad \langle \mathrm{J}^2\,y\,,\ y\rangle = m \tag{3}$$

with c and m real constants.

PROPOSITION 1. *The rigid body motion equation (2) reduces to*

$$\frac{d^2y}{dt^2} = D_1 y + D_2 \begin{bmatrix} {y_1}^3 \\ {y_2}^3 \\ {y_3}^3 \end{bmatrix}$$

with D_1 and D_2 diagonal matrices.

Proof. Use the property $u \times v = \frac{\mathrm{J}}{|\mathrm{J}|}[(\mathrm{J}u) \times (\mathrm{J}v)]$ to rewrite (2) as $\frac{dy}{dt} = \frac{1}{|\mathrm{J}|}(\mathrm{J}^2\, y) \times (\mathrm{J}\, y)$. Now differentiate this, use (2) and apply the cross product property

$$u \times (v \times w) = \langle u, w\rangle\, v - \langle u, v\rangle\, w \tag{4}$$

to obtain

$$\frac{d^2y}{dt^2} = \langle \mathrm{J}\, y\,,\ y\rangle\, \mathrm{J}^{-1}\, y \,+\, \frac{1}{|\mathrm{J}|}\,\langle \mathrm{J}^2\, y\,,\ y\rangle\, \mathrm{J}\, y \,-\, \left(\langle y\,,\ y\rangle \,+\, \frac{1}{|\mathrm{J}|}\,\langle \mathrm{J}^2\, y\,,\ \mathrm{J}\, y\rangle\right) y$$

To conclude the proof use in the above equation the following property and the invariants (3)

$$\langle \mathrm{J}^2 y\,,\ \mathrm{J}\, y\rangle = |\mathrm{J}|\langle y\,,\ y\rangle - \mathrm{tr}(\mathrm{J}_{\mathrm{cof}})\langle \mathrm{J}y\,,\ y\rangle + \mathrm{trJ}\langle \mathrm{J}^2 y\,,\ y\rangle$$

where $|.|$ denotes the matrix determinant, tr the matrix trace and $._{\mathrm{cof}}$ the cofactor matrix. □

REMARK 2. The proposition 1 gives an alternative proof of the well known result (see [7]) that the rigid body equations can be solved in terms of the Jacobian elliptic functions. Indeed, each coordinate of y is a solution of the differential equation $(dz/dt)^2 = az^4 + bz^2 + c$ [10].

4. Cubic polynomials on $SO(3)$

Consider the Lie group $SO(3)$ equipped with a bi-invariant Riemannian metric and the corresponding Lie algebra $(so(3), [.,.])$. Let $\hat{.}$ denotes the isomorphism between the Lie algebras $(\mathbb{R}^3, \times)$ and $(so(3), [.,.])$.

THEOREM 1. *[9] A smooth curve $x : I \to SO(3)$ is a cubic polynomial if and only if it satisfies*

$$(dL_{x^{-1}})_x \frac{dx}{dt} = \hat{v} \tag{5}$$

$$\frac{d^3v}{dt^3} + v \times \frac{d^2v}{dt^2} = 0 \tag{6}$$

The smooth curves $v : I \to \mathbb{R}^3$, solutions of the equation (6), are called *Lie quadratics*.

LEMMA 3. *If v is a Lie quadratic, then the following invariants along v are* *satisfied:*

$$\langle\frac{d^2v}{dt^2}\,,\ v\rangle - \frac{1}{2}\langle\frac{dv}{dt}\,,\ \frac{dv}{dt}\rangle = I_1 \tag{7}$$

$$\langle\frac{d^2v}{dt^2}\,,\ v\times\frac{dv}{dt}\rangle + \frac{1}{2}\langle v\times\frac{dv}{dt}\,,\ v\times\frac{dv}{dt}\rangle = I_2 \tag{8}$$

$$\langle\frac{d^2v}{dt^2}\,,\ \frac{d^2v}{dt^2}\rangle = I_3 \tag{9}$$

Proof. The result follows from the integration of the inner product of the equation (6) with v, $v\times\dfrac{dv}{dt}$ and $\dfrac{d^2v}{dt^2}$, respectively. □

Observe that (7) and (8) correspond to the invariants presented in Section 2.

PROPOSITION 2. *Let v be a Lie quadratic. Then $f = \langle v, v\rangle$ satisfies*

$$f^{(4)} + ff'' - \frac{3}{4}(f')^2 - 2I_1 f - 6(I_2 + I_3) = 0. \tag{10}$$

Furthermore, if $y = f(t)$ is a solution of the differential equation (10), then the equation (6) reduces to

$$\frac{d^5v}{dt^5} + f\,\frac{d^3v}{dt^3} + \frac{3}{2}\,f'\,\frac{d^2v}{dt^2} - (\frac{1}{6}\,f'' + \frac{2}{3}\,I_1)\,\frac{dv}{dt} - \frac{1}{3}\,f^{(3)}\,v = 0.$$

Proof. From (7) it follows $\dfrac{d^2}{dt^2}\langle v, v\rangle = 3\langle\dfrac{dv}{dt}, \dfrac{dv}{dt}\rangle + 2I_1$. On the other hand, (8) and (9) imply

$$\frac{d^2}{dt^2}\langle\frac{dv}{dt}, \frac{dv}{dt}\rangle = \langle\frac{dv}{dt}, v\rangle^2 - \langle v, v\rangle\langle\frac{dv}{dt}, \frac{dv}{dt}\rangle + 2(I_2 + I_3).$$

Conjugating the above equalities the differential equation (10) yields. In order to complete the proof, it suffices to differentiate twice the equation (6) and use (4). □

REMARK 3. Note that (6) is equivalent to $\dfrac{d^2v}{dt^2} + v\times\dfrac{dv}{dt} = C$, $C \in \mathbb{R}^3$. The *null Lie quadratics* ($C = 0$) were studied in [9] and Proposition 2 was proved to this case ([9], Lemma 3).

REMARK 4. The proposition 2 motivates interesting problems, such as numerical methods applications or integrability study of the corresponding Hamiltonian system. The cubic polynomials may be expressed as extremals of sub-Riemannian optimal control problems on $TSO(3)$, whose Hamiltonian function is exactly the first invariant (7) (see [3] for details)

$$\langle\frac{d^2v}{dt^2}\,,\ v\rangle - \frac{1}{2}\langle\frac{dv}{dt}\,,\ \frac{dv}{dt}\rangle.$$

It is well known from the theory of geodesics, that only the linear reparametrization preserves the curve as a geodesic. This result follows from the fact that the velocity vector field length is an invariant along the geodesic. The invariant (8) has a similar role in the analysis of the freedom for cubic polynomial reparametrization. However, for the cubic polynomial, the study of the degenerated case turns out to be particularly interesting.

The following proposition analyses the freedom for cubic polynomial reparametrization.

PROPOSITION 3. *Let $x : I \to SO(3)$ be a cubic polynomial and $\upsilon : I \to \mathbb{R}^3$ the corresponding Lie quadratic. If the invariant (8) along υ is no zero, a reparametrization of x is a cubic polynomial if and only if it is linear.*

Proof. It is enough to note that, if $y : I' \to SO(3)$ is a smooth curve obtained by a reparametrization of x, $y = x \circ s$, and $w : I' \to \mathbb{R}^3$ is the corresponding Lie curve defined by $(dL_{y^{-1}})_y \dfrac{dy}{dt} = \hat{w}$, then

$$\begin{aligned}&\langle \frac{d^2 w}{dt^2}\,,\ w \times \frac{dw}{dt}\rangle + \frac{1}{2}\langle w \times \frac{dw}{dt}\,,\ w \times \frac{dw}{dt}\rangle\\ &= (s')^6 \left(\langle \frac{d^2 \upsilon}{dt^2}\,,\ \upsilon \times \frac{d\upsilon}{dt}\rangle + \frac{1}{2}\langle \upsilon \times \frac{d\upsilon}{dt}\,,\ \upsilon \times \frac{d\upsilon}{dt}\rangle \right).\end{aligned}$$

□

References

[1] ARNOLD V.I., *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics **60**, Springer-Verlag, New York 1989.

[2] CAMARINHA M., *The geometry of cubic polynomials on Riemannian manifolds*, Ph. D. Thesis in Pure Mathematics, University of Coimbra, Portugal 1996.

[3] CAMARINHA M., CROUCH P. AND SILVA LEITE F., *Hamiltonian structure of generalized cubic polynomials*, Preprints of the Workshop of Lagrangian and Hamiltonian Methods for Nonlinear Control, Princeton, New Jersey, USA, (2000), 13–18.

[4] CAMARINHA M., CROUCH P. AND SILVA LEITE F., *On the geometry of Riemannian cubic polynomials*, Differential Geom. Appl. **15** (2001), 107–135.

[5] CROUCH P. AND SILVA LEITE F., *The dynamic interpolation problem on Riemannian manifolds, Lie groups and symmetric spaces*, J. Dynam. Control Systems **I2** (1995), 177–202.

[6] GIAMBÒ R., GIANNONI F. AND PICCIONE P., *An analytical theory for Riemannian cubic polynomials*, IMA J. Math. Control Inform. **19** (2002), 445–460.

[7] HERMANN R., *Differential geometry and the calculus of variations*, Academic Press Inc., New York 1968.

[8] NOAKES L., HEINZINGER G. AND PADEN B, *Cubic splines on curved spaces*, IMA J. of Math. Control Inform. **6** (1989), 465–473.

[9] NOAKES L., *Null cubics and Lie quadratics*, J. Math. Phys. **44** 3 (2003), 1436–1448.

[10] PRASOLOV V. AND SOLOVYEV Y., *Elliptic functions and elliptic integrals*, Translations of Mathematical Monographs **170**, American Math. Society, Providence R.I. 1997.

AMS Subject Classification: 53B20, 58E10, 49S05.

Lígia ABRUNHEIRO, Instituto Superior de Contabilidade e Administração, Universidade de Aveiro, R. Ass. Hum. dos Bombeiros de Aveiro Apartado 58, 3811-953 Aveiro, PORTUGAL
e-mail: `ligia.abrunheiro@isca.ua.pt`

Margarida CAMARINHA, Departamento de Matemática, Universidade de Coimbra, FCTUC Apartado 3008, 3001-454 Coimbra, PORTUGAL
e-mail: `mmlsc@mat.uc.pt`