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**P. Sablonnière***

# UNIVARIATE SPLINE QUASI-INTERPOLANTS AND APPLICATIONS TO NUMERICAL ANALYSIS

**Abstract.** We describe some new univariate spline quasi-interpolants on uniform partitions of bounded intervals. Then we give some applications to numerical analysis: integration, differentiation and approximation of zeros.

## 1. Introduction

Univariate spline *quasi-interpolants* (abbr. QIs) can be defined as operators of the form

$$Qf = \sum_{j \in J} \mu_j(f) B_j$$

where $\{B_j, j \in J\}$ is the B-spline basis of some space of splines, say of degree $d$, on a bounded interval $I = [a, b]$ endowed with some partition $X_n$ of $I$ in $n$ subintervals. We denote by $\Pi_d$ the space of polynomials of total degree at most $d$. In general we impose that $Q$ is exact on the space $\Pi_d$, i.e. $Qp = p$ for all $p \in \Pi_d$. Some authors impose further that $Q$ is a projector on the space of splines itself (see e.g.[6], [11],[12]). As a consequence of this property, the approximation order is $O(h^{d+1})$ on smooth functions, $h$ being the maximum steplength of the partition $X_n$. The coefficients $\mu_j$ are local linear functionals which are in general of one of the following types:

(i) *differential type* : $\mu_j(f)$ is a linear combination of values of derivatives of $f$, of order at most $d$, at some point in $supp(B_j)$ (see e.g. [1], [2]). The associated quasi-interpolant is called a *differential quasi-interpolant* (abbr. DQI).

(ii) *integral type* : $\mu_j(f)$ is a linear combination of weighted mean values of $f$, i.e. of quantities $\int_a^b f w_j$ where $w_j$ can be, for example, a linear combination of B-splines (see e.g. [1], [2], [18]). The associated quasi-interpolant is called an *integral quasi-interpolant* (abbr. iQI).

(iii) *discrete type* : $\mu_j(f)$ is a linear combination of discrete values of $f$ at some points in the neighbourhood of $supp(B_j)$ (see e.g. [3], [11], [12], [14]). The associated quasi-interpolant is called a *discrete quasi-interpolant* (abbr. dQI).

The main advantage of QIs is that they have a direct construction without solving any system of linear equations. Moreover,they are local, in the sense that the value

*INSA and IRMAR, Rennes.

of $Qf(x)$ depends only on values of $f$ in a neighbourhood of $x$. Finally, they have a rather small infinity norm, so they are nearly optimal approximants.

In this paper, we only consider dQIs, neither DQIs nor iQIs. We also restrict our study to splines defined on *uniform partitions* of $I = [a, b]$. Our aim is to give *explicit formulas* for dQIs of degrees $2 \le d \le 5$ and some applications to three classical problems in numerical analysis: approximate integration and derivation and location of zeros of functions. The paper is organised as follows. In sections 2, we recall some facts about splines and quasi-interpolants. In section 3, we describe dQIs of degrees $2 \le d \le 5$ and we give their infinity norms and their approximation orders. In sections 4 and 5, we give the associated *quadrature formulas* and some numerical examples. In section 6, we give the *differentiation matrices* for quadratic and cubic splines with numerical examples. Finally, in section 7, we show, on a simple polynomial example, how quadratic dQIs can be applied to the *location of zeros* of functions.

## 2. Spline spaces on uniform partitions and dQIs

For $I = [a, b]$, we denote by $S_d(I, X_n)$ the space of splines of degree $d$ and class $C^{d-1}$ on the uniform partition $X_n = \{x_i = a + ih,\ 0 \le i \le n\}$ with meshlength $h = \frac{b-a}{n}$. A basis of this space is $\{B_j, j \in J\}$, with $J = \{1, 2, \ldots, n + d\}$. With these notations, $supp\,(B_j) = [x_{j-d-1}, x_j]$, and $\mathcal{N}_j = \{x_{j-d}, \ldots, x_{j-1}\}$ is the set of the $d$ interior knots in the support of $B_j$. As usual, we add multiple knots at the endpoints: $a = x_0 = x_{-1} = \ldots = x_{-d}$ and $b = x_n = x_{n+1} = \ldots = x_{n+d}$. We recall the represention of monomials in terms of symmetric functions of knots in $\mathcal{N}_j$ [6],[19].

$$e_r(x) = x^r = \sum_{j\in J} \theta_j^{(r)} B_j(x), \quad \theta_j^{(r)} = \binom{d}{r}^{-1} symm_r(\mathcal{N}_j), \quad 0 \le r \le d.$$

In particular, the *Greville points*

$$\theta_j = \theta_j^{(1)} = \frac{1}{d}\sum_{\ell=1}^{d} x_{j-\ell}$$

are the coefficients of $e_1 = \sum_{j\in J} \theta_j B_j$ and the vertices of the *control polygon* of $S = \sum_{j\in J} c_j B_j$ are the *control points* $\mathcal{P} = \{\tilde{c}_j = (\theta_j, c_j), j \in J\}$. The *Schoenberg-Marsden* operator is the simplest discrete quasi-interpolant which is exact on the space $\Pi_1$:

$$S_1 f = \sum_{j\in J} f(\theta_j)\, B_j, \quad S_1 p = p \quad \forall p \in \Pi_1.$$

A *discrete quasi-interpolant* (abbr. dQI) of degree $d$ is a spline operator of the form:

$$Q_d f = \sum_{j\in J} \mu_j(f)\, B_j$$

whose coefficients $\mu_j(f)$ are linear combinations of values of $f$ on either the set $T_n$ ( for $d$ even) or on the set $X_n$ (for $d$ odd), where

$$T_n = \{t_j, \, j \in J\}, \quad t_j = \frac{1}{2}(x_{j-2} + x_{j-1}), \qquad X_n = \{x_j, \quad 0 \le j \le n\}.$$

Therefore, for $d$ even, we set $f(T_n) = \{f_j = f(t_j), \quad j \in J\}$, and for $d$ odd, we set $f(X_n) = \{f_j = f(x_j), \quad 0 \le j \le n\}$. Moreover, $Q_d$ *is exact on* $\Pi_d$:

$$Q_d \, p = p \qquad \forall p \in \Pi_d.$$

For the construction of dQIs, i.e. for the determination of functionals $\{\mu_j(f), \, j \in J\}$, the exactness of $Q_d$ on $\Pi_d$ amounts to solve a system of linear equations for interior B-splines and a finite number of specific linear systems for boundary B-splines. The determinants of these systems being Vandermonde determinants, there is existence and unicity of dQIs with the above assumptions (see also [3] for more general cases). For the sake of completeness, we give below complete formulas for degrees $2 \le d \le 5$. Moreover, we give exact values or upper bounds of $\|Q_d\|_\infty$ and their approximation order on smooth functions. Actually, it is well known (see e.g. [6], chapter 5) that for any subinterval $I_k = [x_{k-1}, x_k]$, $1 \le k \le n$, and for any function $f$

$$\|f - Q_d f\|_{\infty, I_k} \le (1 + \|Q_d\|_\infty) d_{\infty, I_k}(f, \Pi_d)$$

where the distance of $f$ to polynomials is defined by

$$d_{\infty, I_k}(f, \Pi_d) = \inf\{\|f - p\|_{\infty, I_k}, \, p \in \Pi_d\}$$

Here, as usual, $\|f - p\|_{\infty, I_k} = max_{x \in I_k} |f(x) - p(x)|$. Therefore, for $f$ smooth enough, e.g. $f \in C^{d+1}(I)$, this implies that $\|f - Q_d f\|_\infty = O(h^{d+1})$.

## 3. Discrete Quasi-Interpolants of degrees $2 \le d \le 5$

### 3.1. $C^1$ Quadratic dQI

For the $C^1$-quadratic dQI $Q_2 f = \sum_{j=1}^{n+2} \mu_j(f) \, B_j$, the coefficient functionals are easy to compute (details are given in [15],[16]):

$$\mu_1(f) = f_1, \, \mu_2(f) = \frac{1}{6}(-2f_1 + 9f_2 - f_3), \, \mu_{n+1}(f) = \frac{1}{6}(-f_n + 9f_{n+1} - 2f_{n+2}),$$

$$\mu_{n+2}(f) = f_{n+1}, \text{ and for} 3 \le j \le n$$

$$\mu_j(f) = \frac{1}{8}(-f_{j-1} + 10f_j - f_{j+1})$$

The exact value $\|Q_2\|_\infty = 1.4734$ has been computed in [16]. Therefore, for $f \in C^3(I)$ for example, we have the following error estimates

$$\|f - Q_2 f\|_{\infty, I_k} \le \frac{5}{2} d_{\infty, I_k}(f, \Pi_2) \text{ for } 1 \le k \le n \Longrightarrow \|f - Q_2 f\|_\infty = O(h^3).$$

### 3.2. $C^2$ Cubic dQI

For the $C^2$ cubic dQI $Q_3 f = \sum_{j=1}^{n+3} \mu_j(f)\, B_j$, the coefficient functionals are respectively:

$$\mu_2(f) = \frac{1}{18}(7f_0+18f_1-9f_2+2f_3),\ \mu_{n+2}(f) = \frac{1}{18}(2f_{n-3}-9f_{n-2}+18f_{n-1}+7f_n),$$

$$\mu_1(f) = f_0, \quad \mu_{n+3}(f) = f_n, \text{ and for} 3 \le j \le n+1$$

$$\mu_j(f) = \frac{1}{6}(-f_{j-3} + 8f_{j-2} - f_{j-1}).$$

As $|\mu_2|_\infty = |\mu_{n+2}|_\infty = 2$ and $|\mu_j|_\infty = \frac{5}{3}$ for $3 \le j \le n+1$, we obtain the upper bound $\|Q_3\|_\infty \le 2$. It is possible to improve that result by writing the operator in the "quasi-Lagrange" form:

$$Q_3 f = \sum_{j=1}^{n+3} f_j L_j$$

where the *fundamental functions* are linear combinations of B-splines, e.g. for $4 \le j \le n$, $L_j = \frac{1}{6}(-B_{j+3} + 8B_{j+2} - B_{j+1})$. It is well known that $\|Q_3\|_\infty$ is equal to the Chebyshev norm of the Lebesgue function:

$$\Lambda_3 = \sum_{j=1}^{n+3} |L_j|.$$

In each interval of the uniform partition, $\Lambda_3$ is bounded above by the cubic polynomial whose Bernstein-Bézier (abbr. BB-) coefficients are sums of absolute values of BB-coefficients of fundamental functions. This allows to see that the maximum of $\Lambda_3$ is attained in the interval $[x_1, x_2]$ and we obtain:

$$\|Q_3\|_\infty = \|\Lambda_3\|_\infty \approx 1.631.$$

From that we deduce for $f \in C^4(I)$ for example, we have the following error estimates

$$\|f - Q_3 f\|_{\infty,I_k} \le \frac{8}{3} d_{\infty,I_k}(f, \Pi_3) \text{ for } 1 \le k \le n \Longrightarrow \|f - Q_3 f\|_\infty = O(h^4).$$

### 3.3. Quartic dQI

For the $C^3$ quartic dQI $Q_4 f = \sum_{j=1}^{n+4} \mu_j(f)\, B_j$, the coefficient functionals are respectively:

$\mu_1(f) = f_1, \quad \mu_{n+4}(f) = f_{n+2},\ \mu_2(f) = \frac{17}{105} f_1 + \frac{35}{32} f_2 - \frac{35}{96} f_3 + \frac{21}{160} f_4 - \frac{5}{224} f_5,$
$\mu_3(f) = -\frac{19}{45} f_1 + \frac{377}{288} f_2 + \frac{61}{288} f_3 - \frac{59}{480} f_4 + \frac{7}{288} f_5,$

$\mu_4(f) = \frac{47}{315} f_1 - \frac{77}{144} f_2 + \frac{251}{144} f_3 - \frac{97}{240} f_4 + \frac{47}{1008} f_5,$
$\mu_{n+1}(f) = \frac{47}{315} f_{n+2} - \frac{77}{144} f_{n+1} + \frac{251}{144} f_n - \frac{97}{240} f_{n-1} + \frac{47}{1008} f_{n-2},$
$\mu_{n+2}(f) = -\frac{19}{45} f_{n+2} + \frac{377}{288} f_{n+1} + \frac{61}{288} f_n - \frac{59}{480} f_{n-1} + \frac{7}{288} f_{n-2},$
$\mu_{n+3}(f) = \frac{17}{105} f_{n+2} + \frac{35}{32} f_{n+1} - \frac{35}{96} f_n + \frac{21}{160} f_{n-1} - \frac{5}{224} f_{n-2},$

and for $5 \le j \le n$

$$\mu_j(f) = \frac{47}{1152}(f_{j-4} + f_j) - \frac{107}{288}(f_{j-3} + f_{j-1}) + \frac{319}{192} f_{j-2}$$

Let us give some details on the computation of functionals $\mu_k, k = 2, 3, 4$ . As $\mu_k(e_r) = \theta_k^{(r)}$ for $0 \le r \le 4$, we determine the five coefficients of the discrete functional

$$\mu_k(f) = \alpha_k f_1 + \beta_k f_2 + \gamma_k f_3 + \delta_k f_4 + \zeta_k f_5$$

as solutions of the three corresponding linear systems ($2 \le k \le 4$) of $5 \times 5$ linear equations

$$t_1^r \alpha_k + t_2^r \beta_k + t_3^r \gamma_k + t_4^r \delta_k + t_5^r \zeta_k = \theta_k^{(r)}, \quad 0 \le r \le 4$$

They have the same Vandermonde determinant $V_5(t_1, t_2, t_3, t_4, t_5) \neq 0$ since the $t_i's$ are distinct. Therefore they have unique solutions. The same technique is applied to the computation of other coefficient functionals.

As $|\mu_2|_\infty = |\mu_{n+3}|_\infty \approx 1.77$, $|\mu_3|_\infty = |\mu_{n+2}|_\infty \approx 2.09$, $|\mu_4|_\infty = |\mu_{n+1}|_\infty \approx 2.88$, $|\mu_j|_\infty \approx 2.49$ for $1 \le j \le 5$, we can conclude that $\|Q_4\|_\infty \le 2.88$ and that for $f \in C^5(I)$ for example, we have the following error estimates

$$\|f - Q_4 f\|_{\infty, I_k} \le 4\, d_{\infty, I_k}(f, \Pi_4) \text{ for } 1 \le k \le n \Longrightarrow \|f - Q_4 f\|_\infty = O(h^5).$$

### 3.4. Quintic dQI

For the $C^4$ quintic dQI $Q_5 f = \sum_{j=1}^{n+5} \mu_j(f)\, B_j$, the coefficient functionals are respectively:

$\mu_1 = f_0, \ \mu_{n+5} = f_n,$
$\mu_2 = \frac{163}{300} f_0 + f_1 - f_2 + \frac{2}{3} f_3 - \frac{1}{4} f_4 + \frac{1}{25} f_5$
$\mu_3 = \frac{1}{200} f_0 + \frac{103}{60} f_1 - \frac{73}{60} f_2 + \frac{7}{10} f_3 - \frac{29}{120} f_4 + \frac{11}{300} f_5$
$\mu_4 = -\frac{41}{400} f_0 + \frac{43}{60} f_1 + \frac{103}{120} f_2 - \frac{7}{10} f_3 + \frac{13}{48} f_4 - \frac{13}{300} f_5$
( symmetric formulas for $n + 2 \le j \le n + 4$), and for $5 \le j \le n$:

$$\mu_j = \frac{13}{240}(f_{j-5} + f_{j-1}) - \frac{7}{15}(f_{j-4} + f_{j-2}) + \frac{73}{40} f_{j-3}$$

As $|\mu_2|_\infty = |\mu_{n+4}|_\infty = 3.5$, $|\mu_3|_\infty = |\mu_{n+3}|_\infty \approx 3.92$, $|\mu_4|_\infty = |\mu_{n+2}|_\infty \approx 2.69$, and $|\mu_j|_\infty \approx 2.87$ for $5 \le j \le n$, we deduce that $\|Q_5\|_\infty \le 3.92$. Using a similar technique as for cubics, we find that $\|Q_5\|_\infty \approx 3.106$. Therefore, for $f \in C^6(I)$ for example, we have the following error estimates

$$\|f - Q_5 f\|_{\infty,I_k} \le 4.5\, d_{\infty,I_k}(f, \Pi_5) \text{ for } 1 \le k \le n \Longrightarrow \|f - Q_5 f\|_\infty = O(h^6).$$

## 4. Application to numerical integration

Newton-Cotes formulas are obtained by integrating interpolation polynomials (see e.g. [4],[7],[9]). In the same way, integrating spline quasi-interpolants give interesting quadrature formulas (abbr. QF) which are easily deduced from the above computations. We use the notations

$$\mathcal{I}(f) = \int_a^b f, \;\; \mathcal{I}_d(f) = \int_a^b Q_d f = \sum_{j\in J} \mu_j(f) \int_a^b B_j, \;\; E_d(f) = \mathcal{I}(f) - \mathcal{I}_d(f).$$

As $\int_a^b B_j = \frac{1}{d}(x_j - x_{j-d-1})$ and $\mu_j(f)$ are known explicitly, we can compute the following quadrature formulas. Moreover, as QIs give the best approximation order, we can conclude that $E_d(f) = O(h^{d+1})$ for $f \in C^{d+1}(I)$, where $h$ is the meshlength. Moreover, as for Newton-Cotes formulas, we get a higher approximation order for even degrees.

**(i) QF for quadratics**

$$\mathcal{I}_2(f) = \int_a^b Q_2 f = h \sum_{j=4}^{n-1} f_j + h \left[ \frac{1}{9}(f_1 + f_{n+2}) + \frac{7}{8}(f_2 + f_{n+1}) + \frac{73}{72}(f_3 + f_n) \right]$$

Error : for $f \in C^4(I)$, $E_2(f) = \mathcal{I}(f) - \mathcal{I}_2(f) = \frac{23}{5760}\, h^4 D^4 f(c) - \frac{1}{192}\, h^5 D^4 f(\bar{c})$
This result is proved in [16].

Error for Simpson: $E_2^*(f) = \mathcal{I}(f) - \mathcal{I}_2^*(f) = -\frac{1}{180}\, h^4 D^4 f(c)$
By comparing the two above errors, we see that the linear combination (extrapolation):

$$\tilde{\mathcal{I}}_2(f) = \frac{1}{55}(32\mathcal{I}_2(f) + 23\mathcal{I}_2^*(f))$$

is such that $\mathcal{I}(f) - \tilde{\mathcal{I}}_2(f) = O(h^5)$.

**(ii) QF for cubics**

$$\begin{aligned} \mathcal{I}_3(f) &= \int_a^b Q_3 f = h \sum_{j=4}^{n-4} f_j + h \Big[ \frac{23}{72}(f_0 + f_n) + \frac{4}{3}(f_1 + f_{n-1}) + \\ &+ \frac{19}{24}(f_2 + f_{n-2}) + \frac{19}{18}(f_3 + f_{n-3}) \Big] \end{aligned}$$

Error: $E_3(f) = \mathcal{I}(f) - \mathcal{I}_3(f) = O(h^4)$ for $f \in C^4(I)$. Numerical experiments show that this formula is not as good as the preceding one.

**(iii) QF for quartics**

$$\int_a^b Q_4 f = \mathcal{I}_4(f) = h \sum_{j=6}^{n-3} f_j + h\left[\frac{206}{1575}(f_1 + f_{n+2}) + \frac{107}{128}(f_2 + f_{n+1}) + \frac{6019}{5760}(f_3 + f_n)\right]$$
$$+h\left[\frac{9467}{9600}(f_4 + f_{n-1}) + \frac{13469}{13440}(f_5 + f_{n-2})\right]$$

Error: $\mathcal{I}(f) - \mathcal{I}_4(f) = O(h^6)$ for $f \in C^6(I)$. This is a remarkable formula, which can be compared to the Newton-Cotes formula of the same order. Numerical experiments show that the error for the former QF has also the opposite sign of the error for the latter, as in the quadratic case. The proof will be given elsewhere.

**(iv) QF for quintics**

$$\mathcal{I}_5(f) = \int_a^b Q_5 f = h \sum_{j=6}^{n-6} f_j + h\left[\frac{157}{480}(f_0 + f_n) + \frac{961}{720}(f_1 + f_{n-1}) + \frac{133}{180}(f_2 + f_{n-2})\right] +$$
$$h\left[\frac{271}{240}(f_3 + f_{n-3}) + \frac{1393}{1440}(f_4 + f_{n-4}) + \frac{361}{360}(f_5 + f_{n-5})\right]$$

Error: $E_5(f) = \mathcal{I}_(f) - \mathcal{I}_5(f) = O(h^6)$ for $f \in C^6(I)$. Numerical experiments show that this formula is not as good as the preceding one.

## 5. Numerical examples

We compare numerical results on QF applied to the computation of

$$\mathcal{I}(f_1) = \int_{-1}^{1} \frac{1}{1+16x^2}\,dx \quad \text{and} \quad \mathcal{I}(f_2) = \int_{-1}^{1} e^{-x} sin(5\pi x)\,dx.$$

**(i) QF/dQI degrees 2 and 3**

$E_2(f) = \mathcal{I}(f) - \mathcal{I}_2(f) = O(h^4)$, $E_3(f) = \mathcal{I}(f) - \mathcal{I}_3(f) = O(h^4)$ for $f \in C^4(I)$

**Simpson's QF**

$$E_2^*(f) = \mathcal{I}(f) - \mathcal{I}_2^*(f) = O(h^4) \text{ for } f \in C^4(I)$$

EXAMPLE 1. $\mathcal{I}(f_1)$

| $n$ | $E_2^*$ | $E_2$ | $E_3$ |
|---|---|---|---|
| 128 | 0.73(−9) | −0.55(−9) | −0.44(−8) |
| 256 | 0.45(−10) | −0.33(−10) | −0.26(−9) |
| 512 | 0.28(−11) | −0.21(−11) | −0.15(−10) |
| 1024 | 0.18(−12) | −0.13(−12) | −0.95(−12) |

EXAMPLE 2. $\mathcal{I}(f_2)$

| $n$ | $E_2^*$ | $E_2$ | $E_3$ |
|---|---|---|---|
| 128 | 0.14(−6) | −0.11(−6) | −0.92(−6) |
| 256 | 0.90(−8) | −0.67(−8) | −0.52(−7) |
| 512 | 0.56(−9) | −0.41(−9) | −0.31(−8) |
| 1024 | 0.73(−9) | −0.52(−9) | −0.37(−8) |

**(ii) QF/dQI degree 4**

$$E_4(f) = \mathcal{I}(f) - \mathcal{I}_4(f) = O(h^6) \text{ for } f \in C^6(I).$$

**Newton-Cotes QF of degree 4:** $E_4^*(f) = \mathcal{I}(f) - \mathcal{I}_4^*(f) = O(h^6)$ for $f \in C^6(I)$.

EXAMPLE 3. $\mathcal{I}(f_1)$

| $n$ | $E_4$ | $E_4^*$ |
|---|---|---|
| 128 | −0.83(−12) | 1.10(−12) |
| 256 | −0.12(−13) | 0.24(−13) |
| 512 | −0.18(−15) | 0.37(−15) |
| 1024 | −0.29(−17) | 0.59(−17) |

EXAMPLE 4. $\mathcal{I}(f_2)$

| $n$ | $E_4$ | $E_4^*$ |
|---|---|---|
| 128 | 0.23(−7) | −0.68(−7) |
| 256 | 0.44(−9) | −1.04(−9) |
| 512 | 0.73(−11) | −1.62(−11) |
| 1024 | 0.12(−12) | −0.25(−12) |

**(iii) QF/dQI degree 4:** $E_4(f) = \mathcal{I}(f) - \mathcal{I}_4(f) = O(h^6)$ for $f \in C^6(I)$.

**QF/dQI degree 5:** $E_5(f) = \mathcal{I}(f) - \mathcal{I}_5(f) = O(h^6)$ for $f \in C^6(I)$, EXAMPLE 5. $\mathcal{I}(f_1)$

| $n$ | $E_4$ | $E_5$ |
|---|---|---|
| 128 | $-0.83(-12)$ | $0.95(-11)$ |
| 256 | $-0.12(-13)$ | $0.14(-12)$ |
| 512 | $-0.18(-15)$ | $0.21(-14)$ |
| 1024 | $-0.29(-17)$ | $0.32(-16)$ |

EXAMPLE 6. $\mathcal{I}(f_2)$

| $n$ | $E_4$ | $E_5$ |
|---|---|---|
| 128 | $0.23(-7)$ | $-0.27(-6)$ |
| 256 | $0.44(-9)$ | $-0.50(-8)$ |
| 512 | $0.73(-11)$ | $-0.83(-10)$ |
| 1024 | $0.12(-12)$ | $-0.13(-11)$ |

## 6. Application to numerical differentiation

Differentiating interpolation polynomials leads to classical finite differences for the approximate computation of derivatives. Therefore, it seems natural to approximate derivatives of $f$ by derivatives of $Q_d f$ as long as it is possible, i.e. up to the order $d-1$. The general theory will be developed elsewhere. Here we only give results for the first derivative and $d = 2, 3$. We evaluate $(Q_d f)' = \sum_{j\in J} \mu_j(f) B_j'$ at points $T_n$ for $d$ even and at points $X_n$ for $d$ odd.

### (i) Differentiation matrix for quadratics

The derivation matrix $\mathcal{D}_2 \in \mathbb{R}^{(n+2)\times(n+2)}$ is defined as follows: setting $y \in \mathbb{R}^{n+2}$ for the vector with components $y_j = f(t_j)$, $j \in J$ and $y' \in \mathbb{R}^{n+2}$ for the vector with components $y_j' = (Q_2 f)'(t_j)$, $j \in J$, we simply write:

$$y' = \frac{1}{h}\mathcal{D}_2 y$$

$$\mathcal{D}_2 = \begin{pmatrix} -8/3 & 3 & -1/3 & 0 & 0 & 0 & \dots & 0 & 0 \\ -7/6 & 11/16 & 13/24 & -1/16 & 0 & 0 & \dots & 0 & 0 \\ 1/6 & -3/4 & 1/48 & 5/8 & -1/16 & 0 & \dots & 0 & 0 \\ 0 & 1/16 & -5/8 & 0 & 5/8 & -1/16 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/16 & -5/8 & 0 & 5/8 & -1/16 & 0 \\ 0 & 0 & \dots & 0 & 1/16 & -5/8 & -1/48 & 3/4 & -1/6 \\ 0 & 0 & \dots & 0 & 0 & 1/16 & -13/24 & -11/16 & 7/6 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1/3 & -3 & 8/3 \end{pmatrix}$$

**(ii) Differentiation formula for cubics**

The derivation matrix $\mathcal{D}_3 \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined as follows: setting $y \in \mathbb{R}^{n+1}$ for the vector with components $y_j = f(x_j), 0 \le j \le n$ and $y' \in \mathbb{R}^{n+1}$ for the vector with components $y'_j = (Q_3 f)'(x_j), 0 \le j \le n$, we obtain:

$$y' = \frac{1}{h}\mathcal{D}_3 y$$

$$\mathcal{D}_3 = \begin{pmatrix} -11/6 & 3 & -3/2 & 1/3 & 0 & 0 & \dots & 0 & 0 \\ -1/3 & -1/2 & 1 & -1/6 & 0 & 0 & \dots & 0 & 0 \\ 1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 & \dots & 0 & 0 \\ 0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 \\ 0 & 0 & \dots & 0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 \\ 0 & 0 & \dots & 0 & 0 & 1/6 & -1 & 1/2 & 1/3 \\ 0 & 0 & \dots & 0 & 0 & -1/3 & 3/2 & -3 & 11/6 \end{pmatrix}$$

**(iii) Some numerical results**

Again we use the two functions $f_1(x) = \dfrac{1}{1+16x^2}$ and $f_2(x) = e^{-x} sin(5\pi x)$ on the interval $I = [-1, 1]$. For $p = 1, 2$, we set $\varepsilon_p = \max_{v\in V} |f'_p(v) - (Q_d f_p)'(v)|$ where $V = T_n$ (resp. $V = X_n$) for $d = 2$ (resp. $d = 3$) and $\varepsilon^*_p = \max_{v\in V} |f'_p(v) - \delta f_p(v)|$ where $\delta f_p(v)$ is the classical centered approximation of $f'_p(v)$ of order 2 (with standard modifications at the endpoints).

For *quadratics*, we obtain the following results

| $n$ | $\varepsilon_1$ | $\varepsilon_1^*$ | $\varepsilon_2$ | $\varepsilon_2^*$ |
|---|---|---|---|---|
| 64 | 0.014009 | 0.047853 | 0.016143 | 0.046317 |
| 128 | 0.003138 | 0.012079 | 0.003674 | 0.011606 |
| 256 | 0.000767 | 0.003036 | 0.000872 | 0.002902 |
| 512 | 0.000190 | 0.000759 | 0.000212 | 0.00725 |
| 1024 | 0.0000475 | 0.0001899 | 0.000052 | 0.000181 |

We see that the orders are all $O(h^2)$. However, the errors for the derivatives of the quadratic QI ($\varepsilon_1$ and $\varepsilon_2$) are between 3 and 4 times less than the errors for the centered finite differences ($\varepsilon_1^*$ and $\varepsilon_2^*$).

For *cubics*, we obtain the following results

| $n$ | $\varepsilon_1$ | $\varepsilon_1^*$ | $\varepsilon_2$ | $\varepsilon_2^*$ |
|---|---|---|---|---|
| 64 | 3.0(−3) | 4.7(−2) | 1.0(−2) | 4.7(−2) |
| 128 | 2.0(−4) | 1.2(−2) | 1.4(−3) | 1.2(−2) |
| 256 | 1.3(−5) | 3.0(−3) | 1.8(−4) | 2.9(−3) |
| 512 | 8.0(−7) | 7.6(−4) | 2.4(−5) | 7.2(−4) |
| 1024 | 5.0(−8) | 1.9(−4) | 3.0(−6) | $1.8E(-4)$ |

Of course, $\varepsilon_1^*$ and $\varepsilon_2^*$ are both $O(h^2)$ and $\varepsilon_1$ and $\varepsilon_2$ are both at least $O(h^3)$. However, for the function $f_1$, a superconvergence phenomenon occurs because we have $\varepsilon_1 = O(h^4)$ instead of $O(h^3)$. We shall study this kind of results in a further paper.

## 7. Approximating zeros of a function by those of a quadratic dQI

Let $f$ be a continuous function defined on $I = [a, b]$. In order to locate the zeros of $f$ in this interval, we approximate $f$ by its $C^1$ quadratic dQI $g = Q_2 f$ and we compute the *exact zeros* of $g$: this is quite possible because $g$ is piecewise quadratic. The complete study will be done elsewhere. Here we take a simple example: we want to approximate the zeros of the Legendre polynomial $P_8(x)$ in the interval $I = [-1, 1]$. The five zeros of $P_8$ are respectively $\{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$, with

$$x_1 = .1834346425,\ x_2 = .5255324099,\ x_3 = .7966664774,\ x_4 = .9602898565.$$

The following array gives the errors $\varepsilon_k = x_k - \bar{x}_k,\ 1 \le k \le 4$ where $\bar{x}_k$ is the zero of $g$ nearest to $x_k$.

| $n$ | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|---|---|
| 16 | .000543 | .003784 | .013753 | −.007841 |
| 32 | −.000043 | .000210 | .000556 | −.001017 |
| 64 | −.000013 | −.000012 | .000043 | .000026 |

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Paul Sablonnière, Centre de mathématiques, INSA de Rennes, 20 avenue des Buttes de Coësmes, CS 14315, F-35043-RENNES Cédex, France
e-mail: `psablonn@insa-rennes.fr`

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