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T. Hangan

ELASTIC STRIPS AND DIFFERENTIAL GEOMETRY

Abstract. The deformation energy $\mathcal{S}$ of an infinitely narrow strip cut down from the rectifying developable of a space curve was defined by Sadowsky in [6]. In the paper we determine the Euler-Lagrange equations of the functional $\mathcal{S}$ .One can interpret these equations as describing the motion of a material point in a plane endowed with a linear connection forced to satisfy a differential system of order three which recalls Newton's equations.

Introduction.

One handles the shape of a flat elastic Moebius band through numerical computations on the basis of a differential system of order 12 deduced from the equilibrium equations of an elastic rod, see [5]; we propose in what follows a system of order 6 which represents the Euler-Lagrange equations of a functional introduced by Sadowsky in 1930, see [6]. The Euler-Lagrange system of the more simple variational problem of Daniel Bernoulli which leads to elastic curves reduces, when length is preserved, to a differential equation of order three one can integrate; its geometry is developed in [3]. For Sadowsky's functional the geometric analysis of the corresponding Euler-Lagrange system produces a linear connection and two tensor fields of orders 4 and 2.

1. Sadowsky's functional

Let $\gamma : [0, L] \to \mathbf{E}^3$ be a space curve of class $\mathcal{C}^3$ parametrised by arc-length s. $\mathbf{E}^3$ denotes the oriented vector space $\mathbb{R}^3$ endowed with its canonical euclidean scalar product $\langle , \rangle$; the corresponding euclidean norm is denoted $\| . \|$ and " $\times$ " denotes the vector product. The mixed product $\langle a \times b, c\rangle$ will be denoted (a, b, c).One supposes that the curvature of γ, denoted $\kappa(s) = \left\| d^2\gamma/ds^2 \right\|$ is never zero so that the Frenet frame $(\mathrm{T}(s), \mathrm{N}(s), \mathrm{B}(s))$ at $\gamma(s)$ is well defined ; one has

$$\mathrm{T}(s) = \frac{d\gamma}{ds}, \qquad \mathrm{N}(s) = (\kappa(s))^{-1}\frac{d^2\gamma}{ds^2}, \qquad \mathrm{B}(s) = \mathrm{T}(s) \times \mathrm{N}(s).$$

The Frenet formulas write

$$\frac{d\mathrm{T}}{ds} = \kappa(s)\mathrm{N}(s), \qquad \frac{d\mathrm{N}}{ds} = -\kappa(s)\mathrm{T}(s) + \tau(s)\mathrm{B}(s), \qquad \frac{d\mathrm{B}}{ds} = -\tau(s)\mathrm{N}(s)$$

where $\tau(s)$ is the torsion of γ at $\gamma(s)$. Therefore

$$\frac{d^3\gamma}{ds^3} = -(\kappa(s))^2\mathrm{T}(s) + \frac{d\kappa}{ds}\mathrm{N}(s) + \tau(s)\kappa(s)\mathrm{B}(s).$$

and

$$(\frac{d\gamma}{ds}, \frac{d^2\gamma}{ds^2}, \frac{d^3\gamma}{ds^3}) = (\kappa(s))^2\tau(s).$$

We call

$$\mathcal{S}(\gamma) = \int_0^L (\kappa(s))^2(1 + (\frac{\tau(s)}{\kappa(s)})^2)^2 ds$$

"Sadowsky's functional" introduced in [6] as measure of the bending energy of an infinitely narrow strip with axis γ lying on the rectifying developable of γ. At the point $\gamma(s)$, the non zero principal curvature $\pi(s)$ of the rectifying developable of γ equals

$$\pi(s) = \kappa(s)(1 + (\frac{\tau(s)}{\kappa(s)})^2)$$

so that one has also

$$\mathcal{S}(\gamma) = \int_0^L (\pi(s))^2 ds.$$

Wunderlich [9] and G.Schwarz [7] defined closed space curves γ from the rectifying developable of which one can cut down along γ a Moebius strip. They have not computed the bending energy of these strips but one can observe these are not in equilibrium i.e. their bending energy is not a minimum under lengthpreserving deformation as they do not satify the Euler-Lagrange equations corresponding to $\mathcal{S}$.

2. Arclength preserving deformation.

I being an interval centered at $0 \in \mathbb{R}$, let $c : [0, L] \times I \to \mathbf{E}^3$ be a smooth family of curves of class $\mathcal{C}^3$ such that

$$c(s, 0) = \gamma(s), \qquad s \in [0, L].$$

One says that c is an arclength- preseving deformation of γ if along any curve $c_t : [0, L] \to \mathbf{E}^3$, $t \in I$, where

$$c_t(s) = c(s, t), \quad s \in [0, L],$$

the parameter s represents arc length, i.e.

$$\left\| \frac{dc_t(s)}{ds} \right\| = 1.$$

Curvature κ_t and torsion τ_t of the curve c_t satisfy then the equations

(1) $$(\kappa_t)^2 = \left\| c_t'' \right\|^2 = \left\langle c_t'', c_t'' \right\rangle$$

(2) $$(c_t', c_t'', c_t''') = \kappa_t^2 \tau_t$$

where the prime denotes derivation with respect to s. The vector field along γ

$$W(s) = \frac{\partial c(s, t)}{\partial t} \mid t = 0$$

satisfies then the condition

$$\left\langle W^{'}(s), \mathrm{T}(s)\right\rangle = 0.$$

Expressed with respect to the Frenet frame, the field W writes

$$W(s) = u(s)\mathrm{T}(s) + v(s)\mathrm{N}(s) + w(s)\mathrm{B}(s)$$

where $v(s) = (\kappa(s))^{-1}u^{'}(s)$. Successive derivatives of W are

$$\begin{aligned}
W^{'} &= \left[\kappa u - \tau w + (\kappa^{-1}u^{'})^{'}\right]\mathrm{N} + \left[w^{'} + \tau\kappa^{-1}u^{'}\right]\mathrm{B}\\
W^{''} &= -\kappa\left[\kappa u - \tau w + (\kappa^{-1}u^{'})^{'}\right]\mathrm{T} +\\
&\left[(\kappa u - \tau w + (\kappa^{-1}u^{'})^{'})^{'} - \tau(w^{'} + \tau\kappa^{-1}u^{'})\right]\mathrm{N} +\\
&\left[(w^{'} + \tau\kappa^{-1}u^{'})^{'} + \tau(\kappa u - \tau w + (\kappa^{-1}u^{'})^{'})\right]\mathrm{B}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle W^{'''}, \mathrm{B}\right\rangle = 2\tau(\kappa u - \tau w + (\kappa^{-1}u^{'})^{'})^{'} + \tau^{'}(\kappa u - \tau w + (\kappa^{-1}u^{'})^{'}) +\\
(w^{'} + \tau\kappa^{-1}u^{'})^{''} - \tau^{2}(w^{'} + \tau\kappa^{-1}u^{'}).
\end{aligned}$$

We need these derivatives to get explicit expressions for the first variations of κ and τ i.e. for $\delta\kappa = (\partial\kappa_t/\partial t)\mid_{t=0}$, $\delta\tau = (\partial\tau_t/\partial t)\mid_{t=0}$ .From (1), (2) one gets

$$\delta\kappa = \left\langle \mathrm{N}, W^{''}\right\rangle \quad , \qquad \delta\tau = \left\langle \mathrm{B}, \kappa^{-1}W^{'''} - \kappa^{-2}\kappa^{'}W^{''} + \kappa W^{'}\right\rangle - \kappa^{-1}\tau\left\langle \mathrm{N}, W^{''}\right\rangle.$$

It will be convenient to introduce the ratio $\omega = \kappa^{-1}\tau$; its first variation $\delta\omega = (\partial\omega_t/\partial t)\mid_{t=0}$ is

$$\delta\omega = \left\langle \mathrm{B}, \kappa^{-2}W^{'''} - \kappa^{-3}\kappa^{'}W^{''} + W^{'}\right\rangle - 2\kappa^{-1}\omega\left\langle \mathrm{N}, W^{''}\right\rangle.$$

3. First variation of $\mathcal{S}$.

Denote

$$\delta\mathcal{S} = (\frac{\partial}{\partial t}\mathcal{S}(c_t))\mid_{t=0}$$

the first variation of $\mathcal{S}(\gamma)$ under arclength preserving deformation of γ. One gets

$$\begin{aligned}
\delta\mathcal{S} &= \int_0^L (2\kappa(1+\omega^2)^2\delta\kappa + \kappa^2 2(1+\omega^2)2\omega\delta\omega)ds\\
&= \int_0^L \Big\{2\kappa(1+\omega^2)(1-3\omega^2)\left\langle \mathrm{N}, W''\right\rangle\\
&+4\omega(1+\omega^2)\left\langle \mathrm{B}, W''' - \kappa^{-1}\kappa' W'' + \kappa^2 W'\right\rangle\Big\}\, ds.
\end{aligned}$$

Replacing the above derivatives of W one finds

$$\begin{aligned}\delta\mathcal{S} = \int_0^L \Big\{ & 2\kappa(1+\omega^2)^2(\kappa u - \kappa\omega w + (\kappa^{-1}u')')' + \\ & 4\kappa(1+\omega^2)\omega\omega'(\kappa u - \kappa\omega w + (\kappa^{-1}u')') + \\ & 4\omega(1+\omega^2)(w'+\omega u')'' + \\ & -4\omega(1+\omega^2)\kappa^{-1}\kappa'(w'+\omega u') + 2\kappa^2\omega(1+\omega^2)^2(w'+\omega u')\Big\}\, ds.\end{aligned}$$

Integration by parts gives finally

$$\delta\mathcal{S} = -2\int_0^L (wA(\kappa,\omega) + uB(\kappa,\omega))ds$$

with

$$\begin{aligned}A(\kappa,\omega) &= 2(\omega(1+\omega^2))''' + 2(\kappa^{-1}\omega\kappa'(1+\omega^2))'' + \\ &\quad \kappa\omega(1+\omega^2)^2\kappa' + \kappa^2(1+\omega^2)(1+3\omega^2)\omega', \\ B(\kappa,\omega) &= \omega\Big[2(\omega(1+\omega^2))''' + 2(\kappa^{-1}\omega\kappa'(1+\omega^2))''\Big] + \\ &\quad \omega'\Big[2(\omega(1+\omega^2))'' + 2(\kappa^{-1}\omega\kappa'(1+\omega^2))'\Big] + \\ &\quad (\kappa^{-1}(2\kappa\omega(1+\omega^2)\omega' + (1+\omega^2)^2\kappa')')' + (\omega^2\kappa^2(1+\omega^2)^2)' + \\ &\quad 2\kappa^2\omega(1+\omega^2)\omega' + \kappa(1+\omega^2)^2\kappa'.\end{aligned}$$

4. Differential equations of elastic strips.

In order for the strip represented by the curve γ to be in equilibrium it is necessary that $\delta\mathcal{S}=0$ for any field W ; this condition writes

$$A = B = 0$$

a system of two differential equations of order 3 in κ and ω as functions of the arclength s. In a developped form these equations become

$$\begin{gathered}\kappa^{-1}(1+\omega^2)\kappa''' + 2\omega\omega''' + \\ \kappa''(-\kappa^{-2}(1+\omega^2)\kappa' + 12\kappa^{-1}\omega\omega') + \omega''(6\kappa^{-1}\omega\kappa' + 8(1+\omega^2)^{-1}(1+3\omega^2)\omega') \\ -8\kappa^{-2}\omega\kappa'^2\omega' + 8\kappa^{-1}(1+\omega^2)^{-1}(1+3\omega^2)\kappa'\omega'^2 + 24\omega(1+\omega^2)^{-1}\omega'^3 + \\ \kappa(1+\omega^2)^2\kappa' + 3\kappa^2\omega(1+\omega^2)\omega' = 0\end{gathered}$$

and

$$\begin{gathered}2\kappa^{-1}\omega(1+\omega^2)\kappa''' + 2(1+3\omega^2)\omega''' + \\ \kappa''(-6\kappa^{-2}\omega(1+\omega^2)\kappa' + 4\kappa^{-1}(1+3\omega^2)\omega') + \omega''(2\kappa^{-1}(1+3\omega^2)\kappa' + 36\omega\omega') + \\ 4\kappa^{-3}\omega(1+\omega^2)\kappa'^3 - 4\kappa^{-2}(1+3\omega^2)\kappa'^2\omega' + 12\kappa^{-1}\omega\kappa'\omega'^2 + 12\omega'^3 + \\ \kappa\omega(1+\omega^2)^2\kappa' + \kappa^2(1+\omega^2)(1+3\omega^2)\omega' = 0.\end{gathered}$$

Solving this system with respect to the highest order derivatives one gets

$$
\begin{aligned}
\kappa''' &= \kappa''\left[\frac{1-3\omega^2}{\kappa(1+\omega^2)}\kappa' - \frac{8\omega(1+3\omega^2)}{(1+\omega^2)^2}\omega'\right] \\
&\quad -\omega''\left[\frac{4\omega(1+3\omega^2)}{(1+\omega^2)^2}\kappa' + \frac{4\kappa(2+3\omega^2+9\omega^4)}{(1+\omega^2)^3}\omega'\right] \\
&\quad +\frac{4\omega^2}{\kappa^2(1+\omega^2)}\kappa'^3 + \frac{4\omega(1+3\omega^2)}{\kappa(1+\omega^2)^2}\kappa'^2\omega' - \frac{4(2+9\omega^2+15\omega^4)}{(1+\omega^2)^3}\kappa'\omega'^2 \\
&\quad -\frac{12\omega\kappa(1+5\omega^2)}{(1+\omega^2)^3}\omega'^3 - \kappa^2(1+2\omega^2)\kappa' - 2\kappa^3\omega\frac{1+3\omega^2}{1+\omega^2}\omega'
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\omega''' &= \kappa''\left[\frac{2\omega}{\kappa^2}\kappa' - \frac{2(1-3\omega^2)}{\kappa(1+\omega^2)}\omega'\right] - \omega''\left[\frac{1-3\omega^2}{\kappa(1+\omega^2)}\kappa' + \frac{2\omega(5-3\omega^2)}{(1+\omega^2)^2}\omega'\right] \\
&\quad -\frac{2\omega}{\kappa^3}\kappa'^3 + \frac{2(1-\omega^2)}{\kappa^2(1+\omega^2)}\kappa'^2\omega' + \frac{2\omega(1+9\omega^2)}{\kappa(1+\omega^2)^2}\kappa'\omega'^2 - \frac{6(1-3\omega^2)}{(1+\omega^2)^2}\omega'^3 \\
&\quad +\frac{\omega\kappa}{2}(1+\omega^2)\kappa' - \frac{\kappa^2}{2}(1-3\omega^2)\omega'.
\end{aligned}
\tag{4}
$$

Circular helices, i.e. curves with constant curvature and torsion are evident solutions of this system.

To go forward lets make two remarks.

REMARK 1. The differential system which expresses the equilibrium of an elastic strip is of the general form

$$x''' = H_3(x'', x') + F_1(x'). \tag{5}$$

Here $s \in [0, L] \to x(s) = (\kappa(s), \omega(s))$ is a vectorial function of the independent variable s ,H_3 and F_1 are homogeneous functions of the derived vectors x'', x' and x' of degrees 3 and 1 respectively with coefficients depending on x i.e.

$$H_3(\lambda^2 x'', \lambda x') = \lambda^3 H_3(x'', x')\ ,\ F_1(\lambda x') = \lambda F_1.$$

REMARK 2. The motion of a material point in a potential field U is ruled by Newton's equations

$$x'' = H_2(x') + F \qquad , \ F = gradU.$$

The geometry of the space where the motion is achieved is described by the Christoffel symbols -Γ^i_{jk} where

$$H_2^i(x') = \sum_{j,k=1\ldots n} \Gamma^i_{jk} x^{j\,\prime} x^{k\,\prime}.$$

In analogy with this classical case we associate in what follows three differential objects to the third order system (5).

5. The connection associated to the system (5).

In componentwise form the system (5) on an $n-$dimensional manifold M writes

(6) $$x^{i\,'''} = A^i_{jk} x^{j\,''} x^{k\,'} + B^i_{jkl} x^{j\,'} x^{k\,'} x^{l\,'} + C^i_j x^{j\,'}, \qquad i = 1, 2, ..., n.$$

where one has to sum up when repeated indices appear. The invariance of the system (6) with respect to coordinate transformations implies the following:

REMARK 3. The functions
$$\Gamma^i_{jk} = \frac{1}{3} A^i_{jk}$$
transform like the coefficients of a linear connection Γ, see [8].

REMARK 4. The connection Γ decomposes as a sum $\Gamma = S + T$ where S is a symmetric connection (without torsion) and T is the torsion of Γ. Thus

$$S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \qquad , \qquad T^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}).$$

REMARK 5. Given a smooth curve, $s \in [0, L] \to c(s) \in M$, its third order jet $j^3_{c(s)}(c) = (c(s), c'(s), c''(s), c'''(s))$ at $c(s)$ defines a vector $a_3(c(s)) \in T_{c(s)}M$ with components

$$a^i_3(c) = c^{i'''} - 3\Gamma^i_{rs} c^{r''} c^{s'} - (\Gamma^i_{rs,t} - (\Gamma^i_{pr} + 2T^i_{pr})\Gamma^p_{st}) c^{r'} c^{s'} c^{t'}.$$

It will be called *acceleration of third order* of c at $c(s)$. In terms of the covariant derivation ∇ associated to Γ and of the torsion T of Γ, $a_3(c)$ writes

$$a_3(c) = \nabla_{c'}(\nabla_{c'} c') + 2T(\nabla_{c'} c', c').$$

REMARK 6. System (5) can be written

$$a_3(c(s)) = \Lambda_3(c'(s)) + C_1(c'))$$

where Λ_3 is a tensor field on M of type (1,3) symmetric in its three covariant indices and C_1 is a tensor field of type (1,1) i.e. a field of endomorphisms of the tangent bundle TM .In components , the field Λ expresses with the functions A^i_{jk} and B^i_{jkl} from (6).

6. Differential geometric objects associated to system (3), (4).

In accord with Remark 3 , the linear conection Γ associated to system (3), (4) has components
$$\Gamma^\kappa_{\kappa\kappa} = \frac{1 - 3\omega^2}{3\kappa(1+\omega^2)}\ , \qquad \Gamma^\kappa_{\kappa\omega} = -\frac{8\omega(1+3\omega^2)}{3(1+\omega^2)^2}\ ,$$
$$\Gamma^\kappa_{\omega\kappa} = -\frac{4\omega(1+3\omega^2)}{3(1+\omega^2)^2},\ \Gamma^\kappa_{\omega\omega} = -\frac{4\kappa(2+3\omega^2+9\omega^4)}{3(1+\omega^2)^3}$$

$$\Gamma^{\omega}_{\kappa\kappa} = \frac{2\omega}{3\kappa^2}\ ,\ \Gamma^{\omega}_{\kappa\omega} = -\frac{2(1-3\omega^2)}{3\kappa(1+\omega^2)}\ ,\ \Gamma^{\omega}_{\omega\kappa} = -\frac{1-3\omega^2}{3\kappa(1+\omega^2)}\ ,\ \Gamma^{\omega}_{\omega\omega} = -\frac{2\omega(5-3\omega^2)}{3(1+\omega^2)^2}.$$

The components of the tensor of curvature of Γ , denoted R ,are

$$R^{\kappa}_{\kappa\kappa\omega} = R^{\omega}_{\omega\kappa\omega} = 0\ ,\ R^{\kappa}_{\omega\kappa\omega} = -\frac{4(1-3\omega^2)}{9(1+\omega^2)^3}\ ,\ R^{\omega}_{\kappa\kappa\omega} = \frac{4}{9\kappa^2(1+\omega^2)}.$$

The above particular form of R suggests to look for a volume -form invariant by parallel transport with respect to Γ. Indeed ,one has

PROPOSITION 1. *The volume form*

$$\Omega = (1+\omega^2)^3 d\kappa \wedge d\omega$$

is invariant by parallel transport with respect to Γ.

Concerning the torsion of Γ lets determine the 1-form $\Phi = \sum_j (\sum_{i=1}^n T^i_{ij}) dx^j$ obtained from T by tensorial contraction. As

$$T^{\kappa}_{\kappa\omega} = -\frac{2\omega(1+3\omega^2)}{3(1+\omega^2)^2} \quad , \quad T^{\omega}_{\kappa\omega} = -\frac{1-3\omega^2}{6\kappa(1+\omega^2)}$$

one has

$$\Phi = \frac{1-3\omega^2}{6\kappa(1+\omega^2)} d\kappa - \frac{2\omega(1+3\omega^2)}{3(1+\omega^2)^2} d\omega$$

so that

$$d\Phi = \frac{4\omega}{3\kappa(1+\omega^2)^2} d\kappa \wedge d\omega = \frac{4\tau}{3\pi^2} d\kappa \wedge d\omega.$$

PROPOSITION 2. *The exterior differential* $d\Phi$ *of the torsion 1-form* Φ *expresses in terms of the torsion* τ *of* γ *and of the principal nonzero curvature* π *of the rectifying developpable of* γ *.*

The components of the vectorial 1-form C_1 could be read on the equations (3) , (4):

$$C^{\kappa}_{\kappa} = -\kappa^2(1+2\omega^2)\ ,\ C^{\kappa}_{\omega} = -2\kappa^3\omega\tfrac{1+3\omega^2}{1+\omega^2}$$
$$C^{\omega}_{\kappa} = \tfrac{\kappa\omega}{2}(1+\omega^2)\ ,\ C^{\omega}_{\omega} = -\tfrac{\kappa^2}{2}(1-3\omega^2).$$

The basic invariants of C_1 are

$$\det C_1 = C^{\kappa}_{\kappa}C^{\omega}_{\omega} - C^{\kappa}_{\omega}C^{\omega}_{\kappa} = \frac{1}{2}\kappa^3\omega, \qquad Trace C_1 = C^{\kappa}_{\kappa} + C^{\omega}_{\omega} = -\frac{1}{2}(3\kappa^2+\tau^2)$$

and the roots λ_1, λ_2 of the characteristic equation of C_1 , i.e.

$$\lambda^2 - Trace C_1 \lambda + \det C_1 = 0$$

are

$$\lambda_1 = -\kappa^2, \quad \lambda_2 = -\frac{\kappa^2+\tau^2}{2}.$$

On the other hand , the components of the tensor Λ are more intricate and result after tedious calculations. These are

$$\Lambda^{\kappa}_{\kappa\kappa\kappa} = (\frac{2(1+3\omega^2)}{3\kappa(1+\omega^2)})^2, \quad \Lambda^{\omega}_{\kappa\kappa\kappa} = -\frac{4\omega(1+3\omega^2)}{9\kappa^3(1+\omega^2)}$$

$$\Lambda^{\kappa}_{\kappa\kappa\omega} = \frac{4\omega(1+3\omega^2)(5+21\omega^2)}{27\kappa(1+\omega^2)^3}, \quad \Lambda^{\omega}_{\kappa\kappa\omega} = \frac{1}{3}(\frac{9-38\omega^2-11\omega^4}{9\kappa^2(1+\omega^2)^2} - \frac{8\omega}{\kappa(1+\omega^2)^2})$$

$$\Lambda^{\kappa}_{\kappa\omega\omega} = \frac{4(1+3\omega^2)(1+51\omega^2+90\omega^4)}{27(1+\omega^2)^4}, \qquad \Lambda^{\omega}_{\kappa\omega\omega} = -\frac{4\omega(1+6\omega^2+45\omega^4)}{27\kappa(1+\omega^2)^3}$$

$$\Lambda^{\kappa}_{\omega\omega\omega} = -\frac{4\kappa\omega}{9(1+\omega^2)^5}(89+261\omega^2+567\omega^4+459\omega^6), \quad \Lambda^{\omega}_{\omega\omega\omega} = -\frac{8\omega^2(7+9\omega^2)}{9(1+\omega^2)^4}.$$

The operator $gradS$ was already described in [2].

For the handling of arclength-preserving deformations see [4].

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Theodor HANGAN, Laboratoire de Mathématiques, Université de Haute Alsace, 4, rue des Frères Lumière, 68093 Mulhouse Cedex, FRANCE
e-mail: `t.hangan@uha.fr` *Lavoro pervenuto in redazione il 08.10.2004.*

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L. Rempulska – Z. Walczak

THE STRONG APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY OPERATORS OF SZÁSZ-MIRAKYAN AND BASKAKOV TYPE

Abstract. In this note we define certain linear operators of the Szász-Mirakyan and Baskakov type in the space of differentiable functions.

We introduce the strong differences of functions and these operators and we give the Jackson type theorems for them. These theorems show that the order of the strong approximation depends on differential properties of function f and it depends not on the power $q > 0$ given in the formula of strong difference.

The generalized Bernstein polynomials in the space of differentiable functions were examined in [5].

1. Introduction

1.1.

The strong approximation connected with trigonometric Fourier series was investigated in several papers published in last 50 years. This problem was examined also in the monograph [6].

For example: let $S_n(f;\cdot)$ and $\sigma_n(f;\cdot)$ be the n-th sum and the n-th $(C,1)$-mean of Fourier series of a 2π-periodic function f continuous on $\mathbb{R}$. Then we have

$$\sigma_n(f;x) - f(x) = \frac{1}{n+1}\sum_{k=0}^{n}(S_k(f;x) - f(x))\,, n \in \mathbb{N}_0 = \{0,1,2,\dots\}, x \in \mathbb{R}.$$

The n-th strong $(C,1)$-mean of this series is defined by the formula

$$H_n^q(f;x) := \left\{\frac{1}{n+1}\sum_{k=0}^{n}|S_k(f;x) - f(x)|^q\right\}^{1/q}, n \in \mathbb{N}_0, x \in \mathbb{R},$$

where q is a fixed positive number. It is obvious that

$$|\sigma_n(f;x) - f(x)| \leq H_n^1(f;x)$$

and

$$H_n^q(f;x) \leq H_n^p(f;x), \qquad 0 < q < p < \infty,$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

1.2.

In [1], [2] and [7] (also [3], [4]) were examined approximation properties of the Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)$$

and the Baskakov operators

$$V_n(f;x) := \sum_{k=0}^{\infty}\binom{n-1+k}{k}x^k(1+x)^{-n-k}f\left(\frac{k}{n}\right),$$

$x \in [0,\infty]$, $n = 1,2,\ldots$, for functions f continuous on the interval $[0,\infty]$.

The results given in [2] show that for every r-th times ($r \geq 2$) differentiable function f we have

$$|S_n(f;x) - f(x)| = O_x\left(n^{-1}\right),$$

$$|V_n(f;x) - f(x)| = O_x\left(n^{-1}\right),$$

for $n \in \mathbb{N}$ and every $x \geq 0$, i.e. the order of approximation of f by $S_n(f)$ and $V_n(f)$ is independent on differential properties of functions f if $r \geq 2$.

1.3.

In this paper we shall introduce the certain class of linear operators of the Szász-Mirakyan and Baskakov type

$$L_{n,r}(f;A;x) = \sum_{k=0}^{\infty}a_{nk}(x)\sum_{j=0}^{r}\frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x-\frac{k}{n}\right)^j,$$

in the space of r-th times differentiable functions f.

For these operators we shall define the strong differences

$$H_{n,r}^q(f;A;x) = \left\{\sum_{k=0}^{\infty}a_{nk}(x)\left|\sum_{j=0}^{r}\frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x-\frac{k}{n}\right)^j - f(x)\right|^q\right\}^{1/q}$$

with $q > 0$ and we shall prove that

$$H_{n,r}^q(f;A;x) = o_x\left(n^{-r/2}\right) \qquad \text{as } n \to \infty,$$

at every $x \geq 0$ and $q > 0$.

We can verify that the formula (6) of $L_{n,0}(f)$ contains the Szász-Mirakyan and Baskakov operators $S_n(f)$ and $V_n(f)$.

From results given in Sections 2 and 3 we can deduce that introduced operators $L_{n,r}$, $r \geq 2$, have better approximation properties than classical Szász-Mirakyan and Baskakov operators $S_n(f)$ and $V_n(f)$. The order of approximation of r-th times differentiable function f by $L_{n,r}(f)$ improves if r grows. Moreover we shall observe that the order of approximation is independent on $q > 0$ and if $q = 1$, then the result on strong approximation implies identical result for ordinary approximation of f by $L_{n,r}(f)$.

2. Definitions and preliminary properties

2.1.

Let C_B be the space of all real-valued functions f uniformly continuous and bounded on $\mathbb{R}_0 = [0, \infty)$ with the norm

(1) $$\| f \| \equiv \| f(\cdot) \| := \sup_{x \in R_0} |f(x)|.$$

For $f \in C_B$ we shall consider the modulus of continuity

(2) $$\omega(f; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|, \qquad t \geq 0,$$

where $\Delta_h f(x) := f(x+h) - f(x)$. It is known ([3]) that $\omega(f; \lambda t) \leq (1+\lambda)\omega(f; t)$ for $\lambda, t \geq 0$ and $lim_{t \to 0+} \omega(f; t) = 0$ for every $f \in C_B$.

Let $r \in \mathbb{N}_0$ and let C_B^r be the class of all functions $f \in C_B$ having the derivatives $f', \ldots, f^{(r)} \in C_B$. The norm in C_B^r is given by (1) ($C_B^0 \equiv C_B$).

2.2.

Denote by Ω the set of all infinite matrices $A = [a_{nk}(x)]_{n \in \mathbb{N},\, k \in \mathbb{N}_0}$, $N \in \{1, 2, \ldots\}$, of functions $a_{nk} \in C_B$ having the following properties:

(i) $a_{nk}(x) \geq 0$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$,

(ii) $\sum_{k=0}^{\infty} a_{nk}(x) = 1$ for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$,

(iii) for every $p \in \mathbb{N}$ the series $\sum_{k=0}^{\infty} k^p a_{nk}(x)$ is uniformly convergent on $\mathbb{R}_0$ and its sum $\Phi_p(\cdot; A)$ is function depending on p and A such that $(1 + x^p)^{-1} \Phi_p(x; A)$ belongs to the space C_B.

(iv) for every $p \in \mathbb{N}$ there exists a positive constant $M_1(p, A)$ depending on p and A such that the function

$$T_{n,2p}(x; A) := \sum_{k=0}^{\infty} a_{nk}(x) \left(\frac{k}{n} - x \right)^{2p}, \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N},$$

satisfies the condition

$$\sup_{x\in R_0}(1+x^{2p})^{-1}T_{n,2p}(x;A)\le M_1(p,A)\cdot n^{-p},\qquad n\in\mathbb{N}.$$

DEFINITION 1. *Let the matrix $A\in\Omega$ and let C_B^r be a space with $r\in\mathbb{N}_0$. For $f\in C_B^r$ we define the operators*

(3) $$L_{n,r}(f;A;x):=\sum_{k=0}^{\infty}a_{nk}(x)F_r\left(\frac{k}{n},x\right),\qquad n\in\mathbb{N},\quad x\in\mathbb{R}_0,$$

where

(4) $$F_r(t,x):=\sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j,\qquad t,x\in\mathbb{R}_0.$$

For these operators we introduce the strong differences with the power $q>0$ as follows:

(5) $$H_{n,r}^q(f;A;x):=\left\{\sum_{k=0}^{\infty}a_{nk}(x)\left|F_r\left(\frac{k}{n},x\right)-f(x)\right|^q\right\}^{1/q},\ x\in\mathbb{R}_0,n\in\mathbb{N}.$$

In particular we have

(6) $$L_{n,0}(f;A;x):=\sum_{k=0}^{\infty}a_{nk}(x)f\left(\frac{k}{n}\right)$$

and

(7) $$H_{n,0}^q(f;A;x):=\left\{\sum_{k=0}^{\infty}a_{nk}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right|^q\right\}^{1/q},$$

for every $f\in C_B$, $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$.

From formulas (3), (4) and (6) we deduce that

(8) $$L_{n,r}(1;A;x)=1,\qquad n\in\mathbb{N},\ x\in\mathbb{R}_0,\ r\in\mathbb{N}_0,$$

and for $f\in C_B^r$

(9) $$L_{n,r}(f;A;x)=L_{n,0}\left(F_r(t,x);A;x\right).$$

The formulas (8), (9) and (5) imply that for $f\in C_B^r$ we have

(10) $$L_{n,r}(f;A;x)-f(x)=L_{n,0}\left(F_r(t,x)-f(x);A;x\right),$$

$$(11)\qquad H^q_{n,r}(f;A;x) = \left(L_{n,0}\left(\left|F_r\left(\frac{k}{n},x\right) - f(x)\right|^q;A;x\right)\right)^{1/q},$$

$$(12)\qquad \left|L_{n,r}(f;A;x) - f(x)\right| \le H^1_{n,r}(f;A;x),$$

and by the Hölder inequality and (8)

$$(13)\qquad H^q_{n,r}(f;A;x) \le H^p_n(f;A;x) \qquad \text{if} \quad 0 < q < p < \infty,$$

for all $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $r \in \mathbb{N}_0$.

2.3.

In this paper we shall denote by $M_k(\alpha, \beta)$, $k \in \mathbb{N}$, suitable positive constants depending only on indicated parameters α, β.

Now we shall give two main lemmas.

By (6), (8) and (1) we immediately obtain

LEMMA 1. *For every $A \in \Omega$ and $f \in C_B$ we have*

$$\|L_{n,0}(f;A;\cdot)\| \le \|f\|, \qquad n \in \mathbb{N}.$$

LEMMA 2. *Let $A \in \Omega$ and $r \in \mathbb{N}$. Then*

$$(14)\qquad |L_{n,r}(f;A;x)| \le \|f\| + \frac{2}{r!}\|f^{(r)}\|\left(T_{n,2r}(x;A)\right)^{1/2},$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, where $T_{n,2r}(\cdot;A)$ is defined in (iv). Further we have

$$(15)\qquad \sup_{x\in\mathbb{R}_0}(1+x^r)^{-1}|L_{n,r}(f;A;x)| \le \|f\| + M_2\|f^{(r)}\|n^{-r/2},$$

for $n \in \mathbb{N}$, where $M_2 = M_1(r,A)\cdot\frac{2}{r!}$.

The inequalities (14) *and* (15) *and formulas* (3) *and* (4) *show that $L_{n,r}(f;A)$is well defined for every $f \in C^r_B$ and the function $(1+x^r)^{-1}L_{n,r}(f;A;\cdot)$ belongs to the space C_B.*

Proof. Similarly as in [5] we apply the following modified Taylor formula of $f \in C^r_B$ at a fixed point $t \in \mathbb{R}_0$:

$$(16)\qquad f(x) = \sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}I_r(x,t), \qquad x \in \mathbb{R}_0,$$

where

$$(17)\qquad I_r(x,t) := \int_0^1 (1-u)^{r-1}\left\{f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right\}du.$$

By (9), (4), (16) and (17) it follows that

$$L_{n,r}(f;A;x) = L_{n,0}\left(f(x) - \frac{(x-t)^r}{(r-1)!} I_r(x,t); A; x\right), \qquad x \in \mathbb{R}_0, n \in \mathbb{N},$$

which by (6), (8) and (1) implies that

$$|L_{n,r}(f;A;x)| \le \|f\| + \frac{1}{(r-1)!} L_{n,0}\left(|x-t|^r |I_r(x,t)|; A; x\right).$$

But for $f \in C_B^r$, $r \in N$, we have

$$|I_r(x,t)| \le 2\|f^{(r)}\| \int_0^1 (1-u)^{r-1} du = \frac{2}{r}\|f^{(r)}\|, \qquad x \in \mathbb{R}_0,$$

and further

$$|L_{n,r}(f;A;x)| \le \|f\| + \frac{2}{r!}\|f^{(r)}\| L_{n,0}\left(|x-t|^r; A; x\right).$$

Applying the Hölder inequality and (8) and (iv) for A, we get

$$L_{n,0}\left(|x-t|^r; A; x\right) \le \left(L_{n,0}\left((x-t)^{2r}; A; x\right)\right)^{\frac{1}{2}} \equiv \left(T_{n,2r}(x;A)\right)^{\frac{1}{2}}, \tag{18}$$

for $x \in \mathbb{R}_0$, $n \in N$ and $r \in \mathbb{N}$. From the above follows (14).

Using the inequality given in (iv) to (14), we immediately obtain (15) and we complete this proof. □

3. Theorems and corollaries

3.1.

First we shall consider the strong differences $H^q_{n,0}(f;A)$.

THEOREM 1. *Suppose that* $A \in \Omega$, $q > 0$ *and* $f \in C_B^1$. *Then*

$$H^q_{n,0}(f;A;x) \le \|f'\| \left(T_{n,2s}(x;A)\right)^{\frac{1}{2s}}, \qquad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{19}$$

where

$$s = \begin{cases} q & \text{if} \quad q \in \mathbb{N} \\ [q]+1 & \text{if} \quad 0 < q \notin \mathbb{N} \end{cases}$$

and $[q]$ *is the integral part of* q.

Proof. a) Let $q \in \mathbb{N}$. Then for $f \in C_B^1$ we can write

$$|f(t) - f(x)|^q = \left|\int_x^t f' du\right|^q \le \|f'\|\, |t-x|^q, \qquad t, x \in \mathbb{R}_0.$$

From this and by (7), (10), (11) and (18) we get

$$H^q_{n,0}(f;A;x) \le \| f'\| \left(L_{n,0}\left(|t-x|^q;A;x\right)\right)^{\frac{1}{q}} \le \| f'\| \left(T_{n,2q}\,(x;A)\right)^{\frac{1}{2q}},$$

$x \in \mathbb{R}_0$, $n \in \mathbb{N}$.

b) If $0 < q \notin N$, then $s = [q]+1$ belongs to N and $q < s$. Then by (13) we can write

$$H^q_{n,0}(f;A;x) \le H^s_{n,0}(f;A;x), \qquad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{20}$$

and by (19) for $s \in N$ we get also (19) for $0 < q \notin \mathbb{N}$. □

THEOREM 2. *Suppose that* $A \in \Omega$ *and* $q > 0$. *Then there exists* $M_3 = M_3(q,A) = const. > 0$ *such that*

$$\sup_{x \in R_0} (1+x)^{-1} H^q_{n,0}(f;A;x) \le M_3\, \omega\left(f;\frac{1}{\sqrt{n}}\right), \qquad n \in \mathbb{N}. \tag{21}$$

Proof. For $f \in C_B$ we consider the Stieklov function

$$f_h(x) := \frac{1}{h}\int_0^h f(x+u)du, \qquad x \in \mathbb{R}_0, \quad h > 0.$$

From this and by (1) and (2) we get

$$\| f_h - f\| \le \omega\,(f;h)\,, \tag{22}$$

$$\| f_h'\| \le h^{-1}\omega\,(f;h)\,, \tag{23}$$

for $h > 0$, i.e. $f_h \in C^1_B$ if $f \in C_B$.

Let $q \ge 1$. By the inequality

$$|f(t)-f(x)| \le |f(t)-f_h(t)| + |f_h(t)-f_h(x)| + |f_h(x)-f(x)|$$

and by the Minkowski inequality and (6)-(8) we get

$$H^q_{n,0}(f;A;x) \le \left(L_{n,0}\left(|f(t)-f_h(t)|^q;A;x\right)\right)^{\frac{1}{q}} +$$

$$+\left(L_{n,0}\left(|f_h(t)-f_h(x)|^q;A;x\right)\right)^{\frac{1}{q}} + |f_h(x)-f(x)| := W_1(x)+W_2(x)+W_3(x).$$

Lemma 1 and (22) imply that

$$\|W_1(\cdot)\| \le \| f_h - f\| \le \omega\,(f;h)\,,$$

$$\|W_3(\cdot)\| \le \omega\,(f;h)\,, \qquad h > 0,\ n \in \mathbb{N}.$$

By Theorem 1 and (23) and (iv) for A we have

$$W_2(x) \equiv H^q_{n,0}(f_h;A;x) \le \| f_h'\| \left(T_{n,2q}\,(x;A)\right)^{\frac{1}{2q}} \le$$

$$\leq (M_1(q, A))^{1/(2q)}\, h^{-1}\omega\,(f; h)\, \frac{1+x}{\sqrt{n}},$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $h > 0$. From the above we deduce that

$$(1+x)^{-1}H^q_{n,0}(f; A; x) \leq M_3(q; A)\omega\,(f; h)\left(1 + h^{-1}n^{-1/2}\right), \tag{24}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $h > 0$. Setting $h = 1/\sqrt{n}$ in (24), we obtain the desired estimation (21) for $q \in \mathbb{N}$.

If $0 < q \notin \mathbb{N}$, then we apply (20) with $s = [q] + 1$. By (20) and (21) for $H^s_{n,0}(f; A; \cdot)$ we immediately obtain (21) for $0 < q \notin \mathbb{N}$. Thus the proof is completed. □

3.2.

Now we shall prove analogue of Theorem 2 for $f \in C^r_B$ with $r \in \mathbb{N}$.

THEOREM 3. *Let $A \in \Omega$, $r \in \mathbb{N}$ and $q > 0$. Then there exists $M_4 = M_4(q, r, A) = const. > 0$ such that for every $f \in C^r_B$ we have*

$$\sup_{x\in\mathbb{R}_0}\left(1 + x^{r+1}\right)^{-1} H^q_{n,r}(f; A; x) \leq M_4 n^{-r/2}\omega\left(f^{(r)}; n^{-1/2}\right) \tag{25}$$

for all $n \in \mathbb{N}$.

Proof. First let $q \in \mathbb{N}$. Similarly as in the proof of Lemma 2 we apply the Taylor formula (16) and (17) to (11). Then we get

$$H^q_{n,r}(f; A; x) \leq \frac{1}{(r-1)!}\left(L_{n,0}\left(|x-t|^{rq}\,|I_r(x,t)|^q\,; A; x\right)\right)^{\frac{1}{q}}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. By (17) and (2) and the properties of $\omega\,(f; \cdot)$ we can write

$$|I_r(x,t)| \leq \int_0^1 (1-u)^{r-1}\omega\left(f^{(r)}; u|x-t|\right) du \leq$$

$$\leq \omega\left(f^{(r)}; |x-t|\right)\int_0^1 (1-u)^{r-1} du \leq$$

$$\leq \frac{1}{r}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\left(\sqrt{n}|x-t| + 1\right), \quad x, t \in \mathbb{R}_0,\ n \in \mathbb{N}.$$

Consequently,

$$H^q_{n,r}(f; A; x) \leq \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\left\{\sqrt{n}\left(L_{n,0}\left(|x-t|^{q(r+1)}; A; x\right)\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(L_{n,0}\left(|x-t|^{rq}; A; x\right)\right)^{\frac{1}{q}}\right\}$$

and by (18) and the property (iv) it follows that

$$(26) \qquad H^q_{n,r}(f; A; x) \le$$

$$\le \frac{1}{r!}\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)\left\{\sqrt{n}\left(T_{n,2q(r+1)}(x; A)\right)^{\frac{1}{2q}} + \left(T_{n,2qr}(x; A)\right)^{\frac{1}{2q}}\right\} \le$$

$$\le M_5(q, r, A)\omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right)n^{-r/2}\left(2 + x^r + x^{r+1}\right)$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. From (26) we immediately obtain (25) for $q \in \mathbb{N}$.

If $0 < q \notin \mathbb{N}$, then by (13) we have

$$(27) \qquad H^q_{n,r}(f; A; x) \le H^s_{n,r}(f; A; x), \qquad s = [q] + 1.$$

Now applying (25) for $H^{[q]+1}_{n,r}(f; A; \cdot)$ to (27), we obtain (25) for $0 < q \notin \mathbb{N}$. □

3.3.

Theorem 2, Theorem 3 and (12) imply the following corollaries:

COROLLARY 1. *For* $A \in \Omega$, $q > 0$ *and* $f \in C^r_B$ *with* $r \in \mathbb{N}_0$ *we have*

$$\lim_{n\to\infty} n^{r/2} H^q_{n,r}(f; A; x) = 0 \qquad \text{at every} x \in \mathbb{R}_0.$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 \ge 0$.

COROLLARY 2. *If* $A \in \Omega$, $q > 0$, $f \in C^r_B$ *with* $r \in \mathbb{N}_0$ *and* $f^{(r)} \in Lip\,\alpha$, $0 < \alpha \le 1$, *then*

$$\sup_{x\in\mathbb{R}_0}\left(1 + x^{r+1}\right)^{-1} H^q_{n,r}(f; A; x) = O(n^{-(r+\alpha)/2}).$$

COROLLARY 3. *Let* $A \in \Omega$ *and* $r \in \mathbb{N}_0$. *Then there exists* $M_6 = M_6(r, A) = const. > 0$ *such that for every* $f \in C^r_B$ *there holds inequality*

$$\sup_{x\in\mathbb{R}_0}\left(1 + x^{r+1}\right)^{-1}\left|L_{n,r}(f; A; x) - f(x)\right| \le M_6 n^{-r/2}\omega\left(f^{(r)}; n^{-1/2}\right), \ n \in \mathbb{N}.$$

3.4.

Finally, we can state that

1) Corollary 3 shows that the operators $L_{n,r}(f)$, $r \ge 1$, defined by (3) have better approximation properties than $L_{n,0}(f)$. The order of approximation of $f \in C^r_B$, $r \ge 1$, by $L_{n,r}(f)$ improves if r grows.

2) From Theorems 1-3 and Corollaries 1,2 results that the order of strong approximation of $f \in C^r_B$ by $L_{n,r}(f)$ also improves if r grows. Moreover we can observe that the order of strong approximation with the power $q > 0$ is not dependent on q.

3) The inequality (12) shows that introduce strong approximation of f by $L_{n,r}(f)$ is more general than ordinary approximation.

4) The definition (6) of $L_{n,0}(f; A)$ contains the Szász-Mirakyan and Baskakov operators $S_n(f)$ and $V_n(f)$ ([1-4, 7]) given in Section 1.2 and associated with the matrices A_S and A_V on the elements

$$A_S : \qquad a_{nk}(x) = e^{-nx} \frac{(nx)^k}{k!},$$

$$A_V : \qquad a_{nk}(x) = \binom{n-1+k}{k} x^k (1+x)^{-n-k}.$$

It is easily verified that A_S and A_V belong to the set Ω.

5) The definition (3) of $L_{n,r}(f)$ with $r \in N$ contains also generalized Szász-Mirakyan and Baskakov operators $S_{n,r}(f)$ and $V_{n,r}(f)$ in the space of r-th times differentiable functions investigated in [8].

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Lucyna REMPULSKA, Zbigniew WALCZAK, Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, POLAND
e-mail: `lrempuls@math.put.poznan.pl`
e-mail: `zwalczak@math.put.poznan.pl`[2ex]
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