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ON THE CONICS LYING IN A COMPLETE INTERSECTION

Abstract. In this paper we determine the dimension of the Hilbert scheme (also called Fano scheme) that parametrises all the conics lying in a complete intersection in a projective space or in a Grassmannian.

1. Introduction

Tennison proved in [7] the existence of smooth conics on the general quartic hypersurface $V(4)$ of $\mathbf{P}^4$ (which is a Fano threefold of the principal series, of index 1 and with Picard group isomorphic to $\mathbf{Z}$, according to Fano-Iskovskih classification, as in [4]), and then that the family of the conics on $V(4)$ is a two-dimensional scheme.

In this paper we work over the field of complex numbers $\mathbf{C}$ and we study the dimension of the Hilbert scheme (also called Fano scheme) $\mathcal{F}^2(X)$ parametrising the conics lying in a general n-dimensional complete intersection X in a projective space or in a Grassmannian, where conic means a rational curve of degree 2, which is necessarily plane. Using the tecnique of the incidence varieties as in [1], under the condition that X contains at least a conic, we are able to give a numerical formula for the dimension of $\mathcal{F}^2(X)$. In the case of a complete intersection in a projective space, the formula is $n-3+2\cdot\text{index}(X)$, where $\text{index}(X)$ is a numerical invariant which mesures the positivity of the anticanonical divisor of X (see section 2 for its formal definition). When X is a Fano surface or a Fano threefold contained in a projective space, the hypothesis that X contains a conic is superfluous, indeed these types of complete intersection always contain lines and conics. However, in general the existence of a line on a complete intersection X does not imply that X contains a conic, for example, the only rational curves lying on the general hypersurface $X \subset \mathbf{P}^{n+1}$ of degree $2n-1$, $n \geq 5$, are the lines (see [5]).

We also extend the result to complete intersections X in $G(k,m)$, the Grassmannian of k-dimensional subspaces of a vector space of dimension m embedded in the projective space $\mathbf{P}^{\binom{m}{k}-1}$ via the Plücker embedding, getting a bound for the dimension of the Hilbert scheme $\mathcal{F}^2(X)$ that parametrises the conics on X. The proof of this extension is analogous to that made in [1] for the lines lying on X. Also in this case it's necessary to assume the existence of at least a conic lying on the complete intersection. We can conclude that the dimension of $\mathcal{F}^2(X)$ is greater than or equal to $n-3+2\cdot\text{index}(X)$ (see section 3 for the exact definition of $\text{index}(X)$ in the case of a complete intersection X in a Grassmannian). Since we have no example in which there is strict inequality in the above formula for the dimension of $\mathcal{F}^2(X)$, we suspect that equality always holds.

The index of these varieties is linked to the geometry of the lines and the conics lying on them. A naive observation is that there is a similarity between the dimension of the two Hilbert schemes, in fact the dimension of the Hilbert scheme of the lines lying on the variety is $n-3+\text{index}(X)$.

2. The family of conics in a complete intersection in a projective space

In this section X denotes a complete intersection of type $(n_1,\dots,n_r)$ in the projective space $\mathbf{P}^{n+r}$, so $\dim X = n$ and $\operatorname{codim} X = r$, with $r \geq 1$ and $n \geq 2$. We also use the notation $V(n_1,\dots,n_r)$ for a complete intersection of type $(n_1,\dots,n_r)$, i.e. to denote the locus of common zeros of r general homogeneous polynomials of degrees $n_1,\dots,n_r$.

The *index* (also called *Fano index*) of the complete intersection X is defined as

$$\text{index}(X) := n + r + 1 - \sum_{i=1}^{r} n_i .$$

We recall that it holds $K_X \sim -\text{index}(X)\cdot H$, where K_X is the canonical divisor and H is the hyperplane divisor of X. Therefore it results:

$$\begin{array}{rcl} X \text{ is a Fano variety} & \Leftrightarrow & \text{index}(X) > 0 \\ X \text{ is Calabi-Yau variety} & \Leftrightarrow & \text{index}(X) = 0 \\ X \text{ is of general type} & \Leftrightarrow & \text{index}(X) < 0 \end{array}$$

We want to prove the following

THEOREM 1. *Let $X = V(n_1,\dots,n_r) \subset \mathbf{P}^{n+r}$ be a general complete intersection of dimension $n \geq 2$.*

1. *If $n-3+2\cdot\text{index}(X) < 0$, then X does not contain any conic.*
2. *If $n-3+2\cdot\text{index}(X) \geq 0$ and X contains at least a conic, then the Hilbert scheme $\mathcal{F}^2(X)$ that parametrises the conics lying on X is not empty of dimension exactly $n-3+2\cdot\text{index}(X)$.*

In order to obtain the above thesis we consider the following schemes:

- the scheme $\mathbf{Q}$ parametrising all complete intersections of type $(n_1,\dots,n_r)$ in $\mathbf{P}^{n+r}$, and we put $d_{\mathbf{Q}} := \dim \mathbf{Q}$;
- the (dense) open subset $\mathcal{P}$ of the multi-projective space

  $$\mathbf{P}\big(H^0(\mathbf{P}^{n+r},\mathcal{O}_{\mathbf{P}^{n+r}}(n_1))\big) \times \cdots \times \mathbf{P}\big(H^0(\mathbf{P}^{n+r},\mathcal{O}_{\mathbf{P}^{n+r}}(n_r))\big)$$

  which dominates $\mathbf{Q}$; obviously

  $$\dim \mathcal{P} = \sum_{i=1}^{r} \binom{n+r+n_i}{n+r} - r;$$

- the scheme $\mathcal{C}$ parametrising all plane curves of degree 2 in $\mathbf{P}^{n+r}$ which is a smooth irreducible projective variety of dimension $3(n+r)-1$, in fact $\mathcal{C} \cong \mathbf{P}(S^2\mathbf{E}^*)$, where $S^2\mathbf{E}^*$ is the dual of the second symmetric power of $\mathbf{E}$, the tautological bundle on the Grassmannian $G(3, n+r+1)$ of the 2-planes in $\mathbf{P}^{n+r}$ (see [7, page 715]), $\mathcal{C}$ contains the open dense subset $\overline{\mathcal{C}}$ parametrising all conics in $\mathbf{P}^{n+r}$ and obviously $\dim \overline{\mathcal{C}} = \dim \mathcal{C}$;

- the incidence varieties

$$\mathbf{I}' = \{(C, (V(n_1), \ldots, V(n_r))) \in \overline{\mathcal{C}} \times \mathcal{P} \mid C \subset V(n_1) \cap \cdots \cap V(n_r)\}$$

and

$$\mathbf{I} = \{(C, X) \in \overline{\mathcal{C}} \times \mathbf{Q} / C \subset X\}$$

with the natural projections

$$\begin{array}{ccc} \mathbf{I}' & \xrightarrow{pr_2'} & \mathcal{P} \\ {\scriptstyle pr_1'}\downarrow & & \\ \overline{\mathcal{C}} & & \end{array} \qquad (D1)$$

and

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{pr_2} & \mathbf{Q} \\ {\scriptstyle pr_1}\downarrow & & \\ \overline{\mathcal{C}} & & \end{array} \qquad (D2)$$

Now we need two lemmas:

LEMMA 1. *The incidence variety* $\mathbf{I}'$ *is irreducible and*

$$\dim \mathbf{I}' = 3n + r - 1 + \sum_{i=1}^{r} \binom{n+r+n_i}{n+r} - 2\sum_{i=1}^{r} n_i .$$

Proof. Let $C_0 \in \overline{\mathcal{C}}$ be any conic, then looking at diagram $(D1)$ the fiber over C_0 is

$$pr_1'^{-1}(C_0) = \{(C_0, Y) \in \overline{\mathcal{C}} \times \mathcal{P} \mid C_0 \subset Y\} \cong \{Y \in \mathcal{P} \mid Y \supset C_0\}.$$

The condition $C_0 \subset Y$ means that the conic C_0 must be contained in each of the r hypersurfaces $V(n_i)$, with $i = 1, \ldots, r$, but $C_0 \subset V(n_i)$ imposes $2n_i + 1$ linearly indipendent conditions for every $i = 1, \ldots, r$. Therefore $pr_1'^{-1}(C_0)$ is isomorphic to a product of linear subspaces in $\mathcal{P}$ of codimension $2n_1 + \cdots + 2n_r + r$, hence

$$\dim pr_1'^{-1}(C_0) = \sum_{i=1}^{r} \binom{n+r+n_i}{n+r} - 2\sum_{i=1}^{r} n_i - 2r.$$

Since $\overline{\mathcal{C}}$ is irreducible and pr_1' has irreducible fibers of constant dimension it follows the irreducibility of $\mathbf{I}'$. Moreover

$$\begin{aligned}\dim \mathbf{I}' &= \dim \overline{\mathcal{C}} + \dim pr_1'^{-1}(C_0)\\ &= 3n + r - 1 + \sum_{i=1}^{r} \binom{n+r+n_i}{n+r} - 2\sum_{i=1}^{r} n_i\,.\end{aligned}$$

□

LEMMA 2. *The incidence variety* $\mathbf{I}$ *is irreducible and*

$$\dim \mathbf{I} = 3n + 2r - 1 - 2\sum_{i=1}^{r} n_i + d_{\mathbf{Q}}.$$

Proof. We proceed in the same way as in Lemma 1, the problem is to guarantee that all the conditions imposed by the conic are independent. To this end we can consider the following commutative diagram with natural morphisms

$$\begin{array}{ccc} \mathbf{I}' & \xrightarrow{pr_2'} & \mathcal{P} \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbf{I} & \xrightarrow{pr_2} & \mathbf{Q} \end{array}$$

Since $\mathcal{P}$ and $\mathbf{Q}$ are irreducible and π is surjective with fibers of constant dimension, we have that for every $X_0 \in \mathbf{Q}$

$$\dim \pi^{-1}(X_0) = \sum_{i=1}^{r} \binom{n+r+n_i}{n+r} - r - d_{\mathbf{Q}}.$$

It's easy to see that $\pi^{-1}(X_0) \cong \pi'^{-1}(C, X_0)$ for every conic C contained in X_0, so $\dim \pi^{-1}(X_0) = \dim \pi'^{-1}(C, X_0)$, therefore

$$\dim \mathbf{I} = \dim \mathbf{I}' - \dim \pi'^{-1}(C, X_0) = 3n + 2r - 1 - 2\sum_{i=1}^{r} n_i + d_{\mathbf{Q}}.$$

Moreover, looking at the projection pr_1 we deduce that $\mathbf{I}$ is irreducible. □

Now the Hilbert scheme $\mathcal{F}^2(X)$ parametrising the conics lying in the complete intersection X is given by

$$\mathcal{F}^2(X) = pr_1(pr_2^{-1}(X))$$

which is obviously isomorphic to $pr_2^{-1}(X)$ (see diagram (D2)).

If the morphism pr_2 is surjective, then we have

$$\begin{aligned}\dim \mathcal{F}^2(X) &= \dim pr_2^{-1}(X) = \dim \mathbf{I} - \dim \mathbf{Q} \\ &= 3n + 2r - 1 - 2\sum_{i=1}^{r} n_i \\ &= n - 3 + 2 \cdot \operatorname{index}(X).\end{aligned}$$

Therefore, if $n-3+2\cdot\operatorname{index}(X) < 0$, then pr_2 cannot be surjective, so the general complete intersection X contains no conics. On the contrary, if $n-3+2\cdot\operatorname{index}(X) \geq 0$ and X contains at least a conic, then pr_2 is surjective and $\dim \mathcal{F}^2(X) = n - 3 + 2 \cdot \operatorname{index}(X)$, so we have proved Theorem 1.

COROLLARY 1. *Let X be a Fano complete intersection of dimension 2, then $\mathcal{F}^2(X)$ is not empty and* $\dim \mathcal{F}^2(X) = 2 \cdot \operatorname{index}(X) - 1$.

Proof. First of all we recall that a smooth two-dimensional variety is Fano if and only if it is a Del Pezzo surface. It is well known that there are only three Del Pezzo surfaces which are complete intersection, namely the quadric $V(2)$ and the cubic $V(3)$ in $\mathbf{P}^3$ and the complete intersection of two quadrics $V(2,2)$ in $\mathbf{P}^4$. Therefore, let X be a Del Pezzo complete intersection surface, then X always contains at least one conic, indeed:

- if X is the quadric surface $V(2)$ in $\mathbf{P}^3$, each general plane of $\mathbf{P}^3$ cuts $V(2)$ properly in a conic (by Bertini's Theorem), so $\dim \mathcal{F}^2(X) = 3$;
- if X is the Del Pezzo surface of degree $d = 3, 4$ in $\mathbf{P}^d$ (i.e. if X is either $V(3)$ in $\mathbf{P}^3$ or $V(2,2)$ in $\mathbf{P}^4$), then one conic in X exists. In fact, X is the blow-up of the plane $\mathbf{P}^2$ in $9-d$ general points $P_1, \dots, P_{9-d}$ embedded with the linear system of the cubics through these points, then a general line of $\mathbf{P}^2$ through a P_i turns in a conic. In both cases we have $\dim \mathcal{F}^2(X) = 1$.

□

REMARK 1. In general, if $X = V(n_1, \dots, n_r) \subset \mathbf{P}^{n+r}$ contains a conic, then $Y = V(n_1, \dots, n_r) \subseteq \mathbf{P}^t$, $t \geq n + r$, contains a conic.

COROLLARY 2. *Let X be a Fano complete intersection of dimension 3, then $\mathcal{F}^2(X)$ is not empty and* $\dim \mathcal{F}^2(X) = 2 \cdot \operatorname{index}(X)$.

Proof. There are only six complete intersection Fano threefolds, namely the quadric $V(2)$, the cubic $V(3)$, and the quartic $V(4)$ in $\mathbf{P}^4$, the complete intersections $V(2,2)$ and $V(2,3)$ in $\mathbf{P}^5$, and $V(2,2,2)$ in $\mathbf{P}^6$.

If X is a Fano threefold of the form $V_{2g-2} \subset \mathbf{P}^{g+1}$, being g the genus of X (see [3, Proposition 1.6, page 488]), with $\operatorname{index}(X) = 1$, then X contains at least one conic. In fact, $Pic(X)$ is isomorphic to $\mathbf{Z}$ and X contains a line ([4, Theorem 4.4, page 487]). In particular, when:

- $X = V(4) \subset \mathbf{P}^4$ ([7]) or $X = V(2,3) \subset \mathbf{P}^5$ or $X = V(2,2,2) \subset \mathbf{P}^6$, then $\dim \mathcal{F}^2(X) = 2$.

Furthermore, for the other Fano threefolds, i.e. when $\text{index}(X) \geq 2$, it's easily verified the existence of at least a conic, it's enough to use Remark 1 and the proof of Corollary 1. So

- if $X = V(3) \subset \mathbf{P}^4$ or $X = V(2,2) \subset \mathbf{P}^5$, then $\dim \mathcal{F}^2(X) = 4$;
- if $X = V(2) \subset \mathbf{P}^4$, then $\dim \mathcal{F}^2(X) = 6$.

□

3. The family of conics in a complete intersection in a Grassmannian

In this section X denotes a complete intersection of type $(n_1, \ldots, n_r)$ and dimension n in the Grassmann variety $G(k, m)$ (vectorial notation), embedded by Plücker in $\mathbf{P}^{\binom{m}{k}-1}$, that is X is the intersection of the Grassmannian $G(k, m)$ with r general hypersurfaces $V(n_1), \ldots, V(n_r)$, of degrees $n_1, \ldots, n_r$ respectively, in the projective space $\mathbf{P}^{\binom{m}{k}-1}$. We have $\dim X = n$, $\operatorname{codim} X = r$ and $n + r = k(m - k) = \dim G(k, m)$, with $r \geq 1$ and $n \geq 2$.

The *index* of the complete intersection X in the Grassmannian $G(k, m)$ is defined as

$$\text{index}(X) := m - \sum_{i=1}^{r} n_i$$

(see [1, Proposition 1]). Since it holds $K_X \sim -\text{index}(X) \cdot H$, where K_X is the canonical divisor and H is the hyperplane divisor of X, then, like for a complete intersection in a projective space, the positivity or the vanishing of the index characterizes the fact that X is a Fano or a Calabi-Yau variety.

THEOREM 2. *Let $X \subset G(k, m)$ be a general complete intersection of type $(n_1, \ldots, n_r)$ and dimension $n \geq 2$. If $n - 3 + 2 \cdot \text{index}(X) \geq 0$ and X contains at least a conic, then the Hilbert scheme $\mathcal{F}^2(X)$ that parametrises the conics lying on X is not empty of dimension greater than or equal to $n - 3 + 2 \cdot \text{index}(X)$.*

Proof. Let $X = G(k, m) \cap V(n_1) \cap \cdots \cap V(n_r) \subset \mathbf{P}^{\binom{m}{k}-1}$ be the section of the Grassmannian $G(k, m)$ with $r = \dim G(k, m) - n$ general hypersurfaces of degrees $n_1, \ldots, n_r$ in $\mathbf{P}^{\binom{m}{k}-1}$.

Let $\overline{\mathcal{C}}$ be the scheme parametrising all conics in $\mathbf{P}^{\binom{m}{k}-1}$ and let $C \in \overline{\mathcal{C}}$ be a conic such that $C \subset X$. Then:

$$C \subset X \quad \Leftrightarrow \quad C \subset V(n_1) \cap \cdots \cap V(n_r) \quad \text{and} \quad C \subset G(k, m)$$

Let $\widetilde{\mathbf{S}}_1$ and $\widetilde{\mathbf{S}}_2$ be the two cycles in the cohomology ring of $\overline{\mathcal{C}}$ corresponding respectively to the subvariety $\mathbf{S}_1$ of the conics in $\mathbf{P}^{\binom{m}{k}-1}$ contained in the complete intersection $V(n_1, \ldots, n_r)$ and the subvariety $\mathbf{S}_2$ of the conics in $\mathbf{P}^{\binom{m}{k}-1}$ that are contained in

$G(k, m)$. The intersection $\mathbf{S}_1 \cap \mathbf{S}_2$ of the two subvarieties $\mathbf{S}_1$ and $\mathbf{S}_2$ represents the conics in $\mathbf{P}^{\binom{m}{k}-1}$ which are contained in X. In the cohomology ring of $\overline{\mathcal{C}}$ the intersection cycle $\tilde{\mathbf{S}}_1 \bullet \tilde{\mathbf{S}}_2$ corresponds to $\mathbf{S}_1 \cap \mathbf{S}_2$.

Let a_1 and a_2 be the codimensions in $\overline{\mathcal{C}}$ of $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. Since $\mathbf{S}_1 = \mathcal{F}^2(V(n_1) \cap \cdots \cap V(n_r))$, where $V(n_1) \cap \cdots \cap V(n_r)$ is a complete intersection in $\mathbf{P}^{\binom{m}{k}-1}$, we have that $\dim \mathbf{S}_1 = \dim \mathcal{F}^2(V(n_1) \cap \cdots \cap V(n_r)) = \left[\binom{m}{k} - 1 - r\right] - 3 + 2 \cdot \mathrm{index}(V(n_1) \cap \cdots \cap V(n_r))$ (see Theorem 1), so $a_1 = \dim \overline{\mathcal{C}} - \dim \mathbf{S}_1 = 2\sum_{i=1}^{r} n_i + r$.

Now, we deal with the subvariety $\mathbf{S}_2$, obtaining an upper bound on a_2. We can think of a conic lying on $G(k, m)$ as the image of a non constant morphism $\mathbf{P}^1 \to G(k, m)$ of degree 2, but this is equivalent to give a rank k vector bundle $\mathcal{F}$ on $\mathbf{P}^1$ together a surjective morphism $\mathcal{O}^m_{\mathbf{P}^1} \to \mathcal{F}$ (see [2, page 207]). But $Epi(\mathcal{O}^m_{\mathbf{P}^1}, \mathcal{F})$ is an open set in $Hom(\mathcal{O}^m_{\mathbf{P}^1}, \mathcal{F})$, which in turn is isomorphic to $H^0(\mathbf{P}^1, \mathcal{F} \otimes \mathcal{O}^m_{\mathbf{P}^1})$.

By Grothendieck's Theorem $\mathcal{F}$ is of type $\oplus_{i=1}^{k} \mathcal{O}_{\mathbf{P}^1}(\alpha_i)$, and since the image must be a conic, we have the condition

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 2.$$

Moreover we can assume $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$, hence we must have

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0, \quad \alpha_k = 2,$$

or

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{k-2} = 0, \quad \alpha_{k-1} = \alpha_k = 1,$$

that is

$$\mathcal{F} = \mathcal{F}' = \mathcal{O}^{k-1}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2) \qquad \text{or} \qquad \mathcal{F} = \mathcal{F}'' = \mathcal{O}^{k-2}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^2.$$

Thus we have two morphisms $\varphi' \colon Epi(\mathcal{O}^m_{\mathbf{P}^1}, \mathcal{F}') \to G(k, m)$, $\varphi'' \colon Epi(\mathcal{O}^m_{\mathbf{P}^1}, \mathcal{F}'') \to G(k, m)$ and $\mathbf{S}_2 = Im\varphi' \cup Im\varphi''$. In order to find a lower bound for $\dim \mathbf{S}_2$ we have to compute the dimension of $Im\varphi'$ and $Im\varphi''$.

By a straightforward computation we get

$$h^0(\mathcal{F}' \otimes \mathcal{O}^m_{\mathbf{P}^1}) = h^0(\mathcal{F}'' \otimes \mathcal{O}^m_{\mathbf{P}^1}) = (k+2)m.$$

Now in order to compute $\dim Im\varphi'$ and $\dim Im\varphi''$, we have to compute of the fibers φ' and φ'' so, we need to identify those surjective morphisms which give the same conic. First of all there is a three dimensional group acting on $\mathbf{P}^1$, so the dimension drops by three. Furthermore, there is the action of $Aut(\mathcal{F})$ on the vector bundle $\mathcal{F}$. Hence, the dimension of $\mathbf{S}_2$ is at least

$$h^0(\mathcal{F} \otimes \mathcal{O}^m_{\mathbf{P}^1}) - (\dim Aut(\mathbf{P}^1) + \dim Aut(\mathcal{F})) =$$

$$\begin{cases} (k+2)m - (k^2 + k + 2) & \text{if } \mathcal{F} = \mathcal{F}' \\ (k+2)m - (k^2 + 3) & \text{if } \mathcal{F} = \mathcal{F}'' \end{cases}$$

It follows that

$$\begin{aligned} a_1 + a_2 &\leq \dim \overline{\mathcal{C}} - k(m-k) - 2m + 3 + 2\sum_{i=1}^{r} n_i + r \\ &= \dim \overline{\mathcal{C}} - (n - 3 + 2 \cdot \operatorname{index}(X)) \end{aligned}$$

Therefore, if $n-3+2\cdot\operatorname{index}(X) \geq 0$ and X contains at least a conic, then $\mathcal{F}^2(X) \neq \emptyset$ and $\dim \mathcal{F}^2(X) \geq n - 3 + 2 \cdot \operatorname{index}(X)$. □

EXAMPLE 1. Let $X = V_3^{10}$, that is the intersection of the Grassmannian of lines of $\mathbf{P}^4$ with two general hyperplanes and a general quadric, or $X = V_3^{14}$, that is the intersection of the Grassmannian of lines of $\mathbf{P}^5$ with five general hyperplanes (both in the Plücker embedding), then $\dim \mathcal{F}^2(X) = 2$ (see [6]).

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THE LENGTH OF THE FULL HIERARCHY OF NORMS

Abstract. We give upper and lower bounds for the length of the Full Hierarchy of Norms.

1. Introduction

The Hierarchy of Norms goes back to Moschovakis' proof of the First Periodicity Theorem and has been investigated by van Engelen, Miller and Steel in [5], and more recently, by Chalons [1] and Duparc [2] under the name "Steel hierarchy".

Duparc [2, Theorem 7] calculated the length of the hierarchy of Borel norms of length $\omega \cdot \delta < \omega_1$ to be $V_{\omega_1}(1+\delta)$. This should be compared to the height of the Borel Wadge hierarchy which is $V_{\omega_1}(2)$ by a theorem of William Wadge's [7].†

In the context of the Axiom of Determinacy, both the Wadge hierarchy and the Hierarchy of Norms are wellfounded (almost) linear quasi-orderings. It is well known that the length of the Wadge hierarchy is exactly $\Theta = \sup\{\alpha$; there is a surjection from $\mathbb{R}$ onto $\alpha\}$.

In this short paper, we comment on the length of the full Hierarchy of Norms which we shall call Σ. We can prove that $\Theta^2 \leq \Sigma < \Theta^+$.

2. Definitions & Basics

As usual in set theory, we identify the real numbers $\mathbb{R}$ with Baire space $\mathbb{N}^\mathbb{N}$ and use standard notation for Baire space. In particular, we write $x * y$ for the real defined by

$$x * y(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1, \end{cases}$$

and use the symbol $s^\frown x$ for the concatenation of the finite sequence s with the infinite sequence x. We also fix a listing of all continuous functions $\{\mathbf{g}_x : x \in \mathbb{R}\}$.

2.1. Set Theory without the Axiom of Choice

Since the main results of this paper will be in the context of the Axiom of Determinacy which contradicts the Axiom of Choice, let us briefly comment on some features of choiceless set theory. (We will be giving the exact axiomatic system for all results in order to avoid confusion.)

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†The function V_α is the αth Veblen function to the base ω_1.

It is the Axiom of Choice that guarantees the existence of lots of functions between ordinals and other sets, most notably, the real numbers $\mathbb{R}$, the powerset of the real numbers $\wp(\mathbb{R})$, and related sets. In ZF (without using the Axiom of Choice), the following are equivalent for a set X:

1. X is wellorderable (*i.e.*, X is in bijection with some ordinal),
2. there is an ordinal α and a surjection $f : \alpha \to X$, and
3. there is an ordinal β and an injection $g : X \to \beta$.

The question of existence of injections from ordinals into arbitrary sets and surjections from arbitrary sets into ordinals is much more subtle. Without the Axiom of Choice, the ordinals

$$\begin{aligned} \Omega &:= \sup\{\alpha\,;\ \text{there is an injection } f : \alpha \to \mathbb{R}\} \text{ and} \\ \Theta &:= \sup\{\alpha\,;\ \text{there is a surjection } f : \mathbb{R} \to \alpha\} \end{aligned}$$

can very well differ.

If the set of real numbers is not wellorderable, then $\Omega > \omega_1$ implies that there is an uncountable set of reals without the perfect set property;[‡] in particular, ZF + AD implies that $\Omega = \omega_1$. On the other hand, Θ can be rather large. In the literature on the Axiom of Determinacy, Θ plays an important rôle, and the following is known about it:[§]

PROPOSITION 1. *(*ZF*) There is no surjection from $\mathbb{R}$ onto Θ. If* AD *holds, then Θ is a fixed point of the $\aleph$-function, i.e., $\Theta = \aleph_\Theta$. If in addition $\mathbf{V} = \mathbf{L}(\mathbb{R})$, then Θ is regular.*

It is not decided by ZF + AD alone whether Θ is regular. It is consistent with both AD and the stronger $\mathsf{AD}_\mathbb{R}$ that Θ is singular (*cf.* [6]).

Without the Axiom of Choice, successor cardinals are not necessarily regular. The following is a well-known weak analogue of the pigeon hole principle for successor cardinals in ZF. We give its simple proof for the benefit of the reader who is less familiar with the ¬AC-context.

LEMMA 1 (PIGEON HOLE PRINCIPLE FOR SUCCESSOR CARDINALS). *(*ZF*) If κ is an infinite cardinal, then $\kappa^+ \to (\kappa)^1_\kappa$, i.e., for every function $f : \kappa^+ \to \kappa$ there is a set S of cardinality κ such that $f[S] = \{\alpha\}$ for some α.*

Proof. For each $\alpha < \kappa$, define $S_\alpha := \{\xi\,;\ f(\xi) = \alpha\}$. If one of the sets S_α has cardinality κ, we are done. Otherwise, for all $\alpha < \kappa$, o.t.$(S_\alpha) < \kappa$. Let $\pi_\alpha : S_\alpha \to \kappa$

‡ If $f : \omega_1 \to \mathbb{R}$ is an injection, then $X := \{f(\alpha)\,;\ \alpha < \omega_1\}$ is uncountable, but a perfect subset $P \subseteq X$ would give an injection from $\mathbb{R}$ into ω_1 making it wellorderable.

§ *Cf.* [3, p. 396 *sqq.*].

be the Mostowski collapse of S_α. Clearly, $\kappa^+ = \bigcup_{\alpha<\kappa} S_\alpha$. We define

$$F: \begin{array}{lll} \kappa^+ & \to & \kappa \times \kappa \\ \xi & \mapsto & \langle f(\xi), \pi_{f(\xi)}(\xi)\rangle. \end{array}$$

Then F is an injection of κ^+ into $\kappa \times \kappa$ which is a contradiction to the definition of κ^+. □

In the following, we will call a surjection $\varphi : \mathbb{R} \to \alpha$ a **norm**. We call $\mathrm{lh}(\varphi) := \alpha$ the **length of** φ. By Proposition 1,

$$\Theta = \{\alpha\,;\, \exists\varphi\,(\,\varphi \text{ is a norm } \& \ \alpha = \mathrm{lh}(\varphi)\,)\}.$$

For each norm φ, we can define a prewellordering $\leq_\varphi$ on $\mathbb{R}$, defined by

$$x \leq_\varphi y :\Longleftrightarrow \varphi(x) \leq \varphi(y),$$

and furthermore identify the norm with the set $X_\varphi := \{x * y\,;\, x \leq_\varphi y\} \subseteq \mathbb{R}$.

2.2. The Wadge Hierarchy

The Wadge ordering on sets of reals, defined by

$$A \leq_{\mathrm{W}} B :\Longleftrightarrow \text{ there is a continuous } f \text{ such that } f^{-1}[B] = A$$

defines one of the most fundamental complexity hierarchies of descriptive set theory. From $\leq_{\mathrm{W}}$, we derive the Wadge degrees

$$[A]_{\mathrm{W}} := \{B\,;\, A \leq_{\mathrm{W}} B \ \&\ B \leq_{\mathrm{W}} A\}.$$

If $\mathcal{D}_{\mathrm{W}}$ denotes the set of the Wadge degrees, we call the ordering $\langle\mathcal{D}_{\mathrm{W}}, \leq_{\mathrm{W}}\rangle$ the **Wadge hierarchy**.

The following facts about the Wadge hierarchy are well-known:

PROPOSITION 2 (WADGE'S LEMMA). *($\mathsf{ZF} + \mathsf{AD}$) For sets $A, B \subseteq \mathbb{R}$, we either have $A \leq_{\mathrm{W}} B$ or $\mathbb{R}\backslash B \leq_{\mathrm{W}} A$. Thus the Wadge hierarchy is almost linear (except for antichains of length two).*

THEOREM 1 (MARTIN-MONK THEOREM). *($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) The Wadge hierarchy is wellfounded.*

Now, using Theorem 1 under the assumption of $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$, we can assign ordinals called the **Wadge rank** to sets of reals by

$$|A|_{\mathrm{W}} := \mathrm{height}(\langle\{B\,;\, B <_{\mathrm{W}} A\}, \leq_{\mathrm{W}}\rangle).$$

THEOREM 2 (WADGE). *The height of the Wadge hierarchy is Θ.*

For each $\alpha < \Theta$, we define

$$\begin{aligned} \wp_\alpha &:= \{A\,;\ |A|_{\mathrm{W}} = \alpha\}, \text{ and} \\ \wp_{\leq\alpha} &:= \{A\,;\ |A|_{\mathrm{W}} \leq \alpha\}. \end{aligned}$$

PROPOSITION 3. *($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) For each $\alpha < \Theta$, there is a surjection $f : \mathbb{R} \to \wp_{\leq\alpha}$.*

Proof. Fix $A \in \wp_\alpha$. Then the function $x \mapsto \mathbf{g}_x^{-1}[A]$ is a surjection from $\mathbb{R}$ onto $\wp_{\leq\alpha}$. □

COROLLARY 1. *($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$) Let $\alpha < \Theta$ be fixed. Suppose that $\langle \mathcal{A}_\gamma\,;\ \gamma < \Theta\rangle$ is a sequence such that $\mathcal{A}_\gamma \subseteq \wp_{\leq\alpha}$ for all $\gamma < \Theta$. Then there are $\gamma_0 \neq \gamma_1$ such that $\mathcal{A}_{\gamma_0} \cap \mathcal{A}_{\gamma_1} \neq \varnothing$.*

Proof. If not, the function

$$A \mapsto \min\{\gamma\,;\ A \in \mathcal{A}_\gamma\}$$

is a surjection from a subset of $\wp_{\leq\alpha}$ onto Θ. Together with the surjection from Proposition 3, this yields a surjection from $\mathbb{R}$ onto Θ which contradicts Proposition 1. □

3. The Hierarchy of Norms

For two norms φ and ψ, we say that φ is **FPT-reducible to** ψ (for "**F**irst **P**eriodicity **T**heorem"; in symbols: $\varphi \leq_{\mathrm{FPT}} \psi$) if there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have

$$\varphi(x) \leq \psi(F(x)).$$

FPT-reducibility can be expressed in game terms: Look at the two-player perfect information game where player I plays x, player II plays y and is allowed to pass provided he plays infinitely often, and player II wins if and only if $\varphi(x) \leq \psi(y)$. We call this game $\mathbf{G}_{\leq}(\varphi, \psi)$; then $\varphi \leq_{\mathrm{FPT}} \psi$ if and only if player II has a winning strategy in $\mathbf{G}_{\leq}(\varphi, \psi)$.

We write $\varphi \equiv_{\mathrm{FPT}} \psi$ for $\varphi \leq_{\mathrm{FPT}} \psi$ & $\psi \leq_{\mathrm{FPT}} \varphi$, define FPT-degrees by

$$[\varphi]_{\mathrm{FPT}} := \{\psi\,;\ \psi \equiv_{\mathrm{FPT}} \varphi\},$$

denote the set of FPT-degrees by $\mathcal{D}_{\mathrm{FPT}}$, and call the structure

$$\langle \mathcal{D}_{\mathrm{FPT}}, \leq_{\mathrm{FPT}}\rangle$$

the **Full Hierarchy of Norms**.[¶]

LEMMA 2. *(ZF) If φ and ψ are norms and* $\mathrm{lh}(\psi) < \mathrm{lh}(\varphi)$, *then* $\psi <_{\mathrm{FPT}} \varphi$.

[¶]"Full" in order to distinguish it from Duparc's variants with bounded length and/or bounded complexity.

Proof. Let x be such that $\varphi(x) \geq \mathrm{lh}(\psi)$. The strategy "play x regardless of what your opponent does" is winning for player I in $\mathbf{G}_{\leq}(\varphi, \psi)$ and for player II in $\mathbf{G}_{\leq}(\psi, \varphi)$. □

LEMMA 3. *(ZF) If φ and ψ are norms and $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi) = \alpha+1$, then $\varphi \equiv_{\mathrm{FPT}} \psi$.*

Proof. There are x and y such that $\varphi(x) = \psi(y) = \alpha$. Then "play x" is a winning strategy for player II in $\mathbf{G}_{\leq}(\psi, \varphi)$ and "play y" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi, \psi)$. □

The following theorem is implicitly contained in Moschovakis' proof of the First Periodicity Theorem (*cf.* [4, 6B]):

THEOREM 3 (MOSCHOVAKIS). *(ZF + AD + DC($\mathbb{R}$)) The relation $\leq_{\mathrm{FPT}}$ is a prewellordering. Thus, $\langle \mathcal{D}_{\mathrm{FPT}}, \leq_{\mathrm{FPT}} \rangle$ is a wellordering.*

We write

$$\begin{aligned} |\varphi|_{\mathrm{FPT}} &:= \text{o.t.}(\langle\{\psi\,;\,\psi <_{\mathrm{FPT}} \varphi\}, \leq_{\mathrm{FPT}}\rangle), \text{ and} \\ \Sigma &:= \text{o.t.}(\langle \mathcal{D}_{\mathrm{FPT}}, \leq_{\mathrm{FPT}}\rangle). \end{aligned}$$

It is the goal of this paper to give upper and lower bounds for Σ. In analogy to the classes $\wp_\alpha$ and $\wp_{\leq\alpha}$, we define for $\xi < \Sigma$:

$$\begin{aligned} \Phi_\xi &:= \{\varphi\,;\,|\varphi|_{\mathrm{FPT}} = \xi\}, \text{ and} \\ \Phi_{\leq\xi} &:= \{\varphi\,;\,|\varphi|_{\mathrm{FPT}} \leq \xi\}. \end{aligned}$$

The two hierarchies are related and yet notably different. First, let us note that the classes Φ_ξ are much larger than the classes $\wp_\alpha$:

PROPOSITION 4. *(ZF + AD + DC($\mathbb{R}$)) If $0 < \xi < \Sigma$, then there is a surjection from Φ_ξ onto $\wp(\mathbb{R})$.*

Proof. Let $\varphi \in \Phi_\xi$. For $x \in \mathbb{R}$, let $x^+(n) := x(n+1)$. For each $A \in \wp(\mathbb{R})$, we define a norm

$$\varphi_A(x) := \begin{cases} \varphi(x^+) & \text{if } x(0) = 0, \\ 1 & \text{if } x(0) \neq 0 \text{ and } x^+ \in A, \text{ and} \\ 0 & \text{if } x(0) \neq 0 \text{ and } x^+ \notin A. \end{cases}$$

We note that "play 0 and after that copy" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi, \varphi_A)$, and "if the first move is 0, copy; if the first move is not 0, then play an arbitrary real x such that $\varphi(x) \geq 1$" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi_A, \varphi)$, so we have that $\varphi_A \in \Phi_\xi$. The function

$$\psi \mapsto \begin{cases} A & \text{if } \psi = \varphi_A, \\ \varnothing & \text{otherwise} \end{cases}$$

is a surjection from Φ_ξ onto $\wp(\mathbb{R})$. □

COROLLARY 2. *($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$) If $0<\xi<\Sigma$ and $\alpha<\Theta$ arbitrary, then $\Phi_\xi \not\subseteq \wp_{\leq\alpha}$.*

Proof. Suppose $\Phi_\xi \subseteq \wp_{\leq\alpha}$, then there is a surjection from $\wp_{\leq\alpha}$ onto Φ_ξ, hence onto $\wp(\mathbb{R})$ by Proposition 4. Now Proposition 3 gives us a surjection from $\mathbb{R}$ onto $\wp(\mathbb{R})$, which of course contradicts Cantor's Theorem. □

4. Lower and upper bounds for Σ

LEMMA 4 (DIAGONAL LEMMA). *(ZF) If $\lambda<\Theta$ is a limit ordinal and φ is a norm of length λ, then there is a norm φ^+ of length λ such that $\varphi <_{\mathrm{FPT}} \varphi^+$.*

Proof. For a function φ define φ^+ by

$$\varphi^+(x) := \begin{cases} \varphi(\mathbf{g}_{x^+}(x))+1 & \text{if } x(0)\neq 0, \text{ and} \\ \varphi(x^+) & \text{otherwise.} \end{cases}$$

Note that φ^+ is a norm with $\mathrm{lh}(\varphi^+)=\mathrm{lh}(\varphi)=\lambda$. Towards a contradiction, let F witness $\varphi \geq_{\mathrm{FPT}} \varphi^+$. Let z be a code for F, *i.e.*, $F=\mathbf{g}_z$. Then

$$\begin{aligned} \varphi(F(\langle 1\rangle^\frown z)) &\geq \varphi^+(\langle 1\rangle^\frown z) \\ &= \varphi(\mathbf{g}_z(\langle 1\rangle^\frown z))+1 \\ &= \varphi(F(\langle 1\rangle^\frown z))+1 \\ &> \varphi(F(\langle 1\rangle^\frown z)). \end{aligned}$$

Contradiction. □

We say that φ is **embedded** in ψ if there is some x such that we have $\psi(x*y)=\varphi(y)$ for all y.

LEMMA 5. *(ZF) If φ is embedded in ψ, then $\varphi \leq_{\mathrm{FPT}} \psi$.*

Proof. Let x witness that φ is embedded in ψ. Then "Play x on your even moves and copy the moves of player I on your odd moves" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi,\psi)$: If player I plays y, player II answers $x*y$ and wins since $\varphi(y)=\psi(x*y)$. □

LEMMA 6. *(ZF) If $\lambda<\Theta$ is a limit ordinal and $\alpha<\Theta$, then there is a $<_{\mathrm{FPT}}$-increasing sequence $\langle\varphi_\nu\,;\,\nu<\alpha\rangle$ of norms such that for all $\nu<\alpha$, we have $\mathrm{lh}(\varphi_\nu)=\lambda$.*

Proof. Let $\alpha<\Theta$, and fix a surjection $f:\mathbb{R}\to\alpha$ and a norm $\varphi:\mathbb{R}\to\lambda$. We define norms φ_ν by induction, beginning with $\varphi_0:=\varphi$.

Assume that φ_ξ is defined for $\xi<\nu$ and let

$$\varphi^*_\nu(x*y) := \begin{cases} \varphi_{f(x)}(y) & \text{if } f(x)<\nu, \\ \varphi(y) & \text{otherwise.} \end{cases}$$

Note that for $\xi < \nu$, φ_ξ is embedded in φ^*_ν. Thus, by Lemma 5, $\varphi_\xi \leq_{\mathrm{FPT}} \varphi^*_\nu$.

Let $\varphi_\nu := (\varphi^*_\nu)^+$. Then for all $\xi < \nu$, we have $\varphi_\xi <_{\mathrm{FPT}} \varphi_\nu$ by the Diagonal Lemma 4. $\square$

In the following, let $\langle \lambda_\alpha\,;\ \alpha < \Theta\rangle$ be the strictly increasing enumeration of all limit ordinals in Θ (the inverse of the Mostowski collapse).

THEOREM 4. *(*$\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$*) Let* $\alpha < \Theta$ *and let* φ *be a norm of length* $\lambda_\alpha + 1$. *Then* $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \alpha$.

Proof. We prove the claim by induction on α. For $\alpha = 0$, the claim is trivial. Let α be the least counterexample as witnessed by φ (of length $\lambda_\alpha + 1$).

Case 1. Let $\alpha = \gamma + 1$, and let ψ be a norm of length $\lambda_\gamma + 1$. By minimality of α, $|\psi|_{\mathrm{FPT}} \geq \Theta \cdot \gamma$. Since α was a counterexample, $|\varphi|_{\mathrm{FPT}} = \Theta \cdot \gamma + \zeta$ for some $\zeta < \Theta$. We apply Lemma 6 to get a $<_{\mathrm{FPT}}$-increasing sequence $\langle \psi_\eta\,;\ \eta < \zeta + 2\rangle$ such that all ψ_η have length λ_α. Lemma 2 yields that $\psi <_{\mathrm{FPT}} \psi_\eta <_{\mathrm{FPT}} \varphi$ (for all η), but $|\psi_{\zeta+1}|_{\mathrm{FPT}} \geq \Theta \cdot \gamma + \zeta + 1 > \Theta \cdot \gamma + \zeta = |\varphi|_{\mathrm{FPT}}$. Contradiction.

Case 2. If α is a limit ordinal, then $\Theta \cdot \alpha = \bigcup_{\gamma<\alpha} \Theta \cdot \gamma$. By induction hypothesis, we have $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \gamma$ for all $\gamma < \alpha$, so $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \alpha$, so α was no counterexample. $\square$

COROLLARY 3. *(*$\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$*)* $\Theta^2 \leq \Sigma$.

THEOREM 5. *(*$\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$*)* $\Sigma < \Theta^+$.

Proof. Towards a contradiction, suppose that $\Phi_\xi \neq \varnothing$ for all $\xi < \Theta^+$. Define $w : \Theta^+ \to \Theta$ by

$$w(\xi) := \min\{\alpha\,;\ \exists \varphi \in \Phi_\xi\ (|X_\varphi|_{\mathrm{W}} = \alpha)\}.$$

By the Pigeon Hole Principle 1, we find $\alpha \in \Theta$, $S \subseteq \Theta^+$ and $b : \Theta \to S$ such that b is a bijection and for all $\xi \in S$, we have $w(\xi) = \alpha$.

For $\xi \in S$, we define

$$H_\xi := \{X_\varphi\,;\ \varphi \in \Phi_\xi\ \&\ |X_\varphi|_{\mathrm{W}} = \alpha\} \subseteq \wp_\alpha.$$

Then $\langle H_{b(\gamma)}\,;\ \gamma \in \Theta\rangle$ is a sequence of subsets of $\wp_\alpha$ as in Corollary 1, and so there are $\gamma_0 \neq \gamma_1$ such that $H_{b(\gamma_0)} \cap H_{b(\gamma_1)} \neq \varnothing$. But if $X_\varphi \in H_{b(\gamma_0)} \cap H_{b(\gamma_1)}$, then $|\varphi|_{\mathrm{FPT}} = b(\gamma_0) \neq b(\gamma_1) = |\varphi|_{\mathrm{FPT}}$ which is absurd. $\square$

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