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# EXPONENTIAL INSTABILITY AND COMPLETE ADMISSIBILITY FOR SEMIGROUPS IN BANACH SPACES

**Abstract.** We associate a discrete-time equation to an exponentially bounded semigroup and we characterize the exponential instability of the semigroup in terms of the complete admissibility of the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$. As a consequence, we obtain that in certain conditions a $C_0$-semigroup is exponentially unstable if and only if the pair $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ is admissible with respect to an integral equation associated with it. We apply our results at the study of the exponential dichotomy of semigroups.

## 1. Introduction

In recent years an impressive progress has been made in the field of the asymptotic behaviour of evolution equations. There is an extensive literature concerning the asymptotic properties of semigroups of linear operators and their applications (see [1]-[3], [6], [8], [11], [17], [20], [21], [23], [27]). The possibility of reducing the non-autonomous case of the asymptotic behaviour of evolution families or of linear skew-product flows to the autonomous case of the properties of the evolution semigroups associated on diverse Banach function spaces, has proved to be an important source for interesting applications (see [3], [16]).

In this context the input-output conditions or the so-called *theorems of Perron type* became valuable tools in the study of the properties of evolution equations and their applications in control theory (see [3], [5], [7], [9], [10], [12]-[16], [24], [27]). These techniques have a long and impressive history that goes back to the work of Perron (see [22]). Perron's method has been successfully extended by Massera and Schäffer in [9] and by Daleckii and Krein in [5], respectively, in infinite dimensional spaces. Discrete-time theorems of Perron type for exponential stability have been proved by Przyluski in [24]. Discrete-time conditions of Perron type for uniform exponential dichotomy have been presented by Coffmann and Schäffer in [4] and by Henry in [7]. Recently, dichotomy has been expressed using discrete-time techniques of Perron type in [3], [12], [13], [27]. For other input-output conditions for exponential dichotomy we refer to [10] and [16].

In the past ten years beside stability and dichotomy, instability of evolution equations become one of the problems of special interest (see [11], [14], [16]). The aim of this paper is to obtain general input-output conditions for exponential instability of one parameter semigroups and to point out their relevance in the study of the exponential dichotomy.

Exponential instability of a semigroup will be related to the solvability of an equation associated with it. Using discrete-time methods, we establish the connections between the complete admissibility of the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ and the exponential instability of an exponentially bounded semigroup. As a consequence we obtain that a $C_0$-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially unstable and it can be extended to a group if and only if the pair $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ is admissible for it. We apply our results in order to obtain new characterizations for exponential dichotomy of semigroups.

## 2. Complete admissibility and exponential instability

The purpose of this section is to establish discrete-time characterizations for exponential instability of exponentially bounded semigroups.

Let $X$ be a real or a complex Banach space. Throughout this paper, the norm on $X$ and on $\mathcal{B}(X)$-the Banach algebra of all bounded linear operators on $X$, will be denoted by $||\cdot||$.

DEFINITION 1. *A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a semigroup if $T(0) = I$ and $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.*

DEFINITION 2. *A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be:*

$(i)$ *exponentially bounded if there are $M \geq 1$ and $\omega > 0$ such that*

$$||T(t)|| \leq Me^{\omega t}, \quad \forall t \geq 0;$$

$(ii)$ *$C_0$-semigroup if $\lim_{t \searrow 0} T(t)x = x$, for all $x \in X$.*

REMARK 1. Every $C_0$-semigroup is exponentially bounded (see [21]).

DEFINITION 3. *A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be*

$(i)$ *exponentially stable if there are two constants $K, \nu > 0$ such that*

$$||T(t)x|| \leq Ke^{-\nu t}||x||, \quad \forall (t, x) \in \mathbb{R}_+ \times X;$$

$(ii)$ *exponentially unstable if there are two constants $K, \nu > 0$ such that*

$$||T(t)x|| \geq K\, e^{\nu t}||x||, \quad \forall (t, x) \in \mathbb{R}_+ \times X.$$

LEMMA 1. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on $X$. Then*

$(i)$ *$\mathbf{T}$ is exponentially stable if and only if there are $\delta > 0$ and $c \in (0, 1)$ such that $||T(\delta)x|| \leq c||x||$, for all $x \in X$;*

$(ii)$ *$\mathbf{T}$ is exponentially unstable if and only if there are $\delta > 0$ and $c > 1$ such that $||T(\delta)x|| \geq c||x||$, for all $x \in X$.*

*Proof.* It is immediate. □

We denote

$$l^\infty(\mathbb{N}, X) = \{s : \mathbb{N} \to X \mid \sup_{k\in\mathbb{N}} ||s(k)|| < \infty\}$$

which is a Banach space with respect to the norm $|||s||| = \sup_{k \in \mathbb{N}} ||s(k)||$.

DEFINITION 4. *The pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is said to be completely admissible for the semigroup* $\mathbf{T} = \{T(t)\}_{t \geq 0}$ *if for every* $s \in l^\infty(\mathbb{N}, X)$ *there exists a unique* $\gamma_s \in l^\infty(\mathbb{N}, X)$ *such that the pair* $(\gamma_s, s)$ *verifies the equation*

$$(E_d) \qquad \gamma_s(n+1) = T(1)\gamma_s(n) + T(1)s(n), \quad \forall n \in \mathbb{N}.$$

REMARK 2. If $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is a semigroup, consider the discrete-time linear control system

$$\begin{cases} x(n+1) = T(1)x(n) + T(1)s(n), & n \in \mathbb{N} \\ x(0) = x_0. \end{cases} \tag{2.1}$$

where $s \in l^\infty(\mathbb{N}, X)$ and $x_0 \in X$. For every initial condition $x_0$ and every input $s \in l^\infty(\mathbb{N}, X)$, the solution of the system (2.1) is given by

$$x(n; x_0, s) = T(n)x_0 + \sum_{k=0}^{n-1} T(n-k)s(k), \quad \forall n \in \mathbb{N}^*.$$

Then from Definition 4 we have that the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ is *completely admissible* for $\mathbf{T}$ if and only if for every $s \in l^\infty(\mathbb{N}, X)$ there is a unique initial condition $x_0 \in X$ such that $x(\cdot\,; x_0, s) \in l^\infty(\mathbb{N}, X)$.

LEMMA 2. *If the pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is completely admissible for the exponentially bounded semigroup* $\mathbf{T} = \{T(t)\}_{t \geq 0}$*, then* $T(1)$ *is invertible.*

*Proof.* Let $x \in X$ with $T(1)x = 0$. Consider the sequences $\gamma_1, \gamma_2 : \mathbb{N} \to X,\ \gamma_1(k) = 0, \gamma_2(k) = T(k)x$. Then $\gamma_1, \gamma_2 \in l^\infty(\mathbb{N}, X)$ and both verify the equation $(E_d)$ for $s = 0$. It follows that $\gamma_1 = \gamma_2$, so $x = \gamma_2(0) = \gamma_1(0) = 0$. It results that $T(1)$ is injective

Let $x \in X$ and let $s : \mathbb{N} \to X,\ \ s(n) = -\chi_{\{1\}}(n)x$. From hypothesis there is $\gamma \in l^\infty(\mathbb{N}, X)$ such that the pair $(\gamma, s)$ verifies the equation $(E^d_{\mathbf{T}})$. Then, denoting by

$$\delta : \mathbb{N} \to X, \quad \delta(n) = \begin{cases} \gamma(1) - x, & n = 0 \\ \gamma(n+1), & n \geq 1. \end{cases}$$

we have that $\delta \in l^\infty(\mathbb{N}, X)$ and an easy computation shows that the pair $(\delta, 0)$ verifies the $(E^d_{\mathbf{T}})$.

From hypothesis we deduce that $\delta = 0$. In particular, this implies that $\gamma(1) = x$. But $\gamma(1) = T(1)\gamma(0)$. It follows that $x \in Im\, T(1)$. This shows that $T(1)$ is surjective and the proof is complete. □

THEOREM 1. *If the pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is completely admissible for the exponentially bounded semigroup* $\mathbf{T} = \{T(t)\}_{t \geq 0}$*, then* $\mathbf{T}$ *is exponentially unstable.*

*Proof.* Consider

$$D : l^\infty(\mathbb{N}, X) \to l^\infty(\mathbb{N}, X), \quad D(s) = \gamma_s.$$

It is easy to see that $D$ is a closed linear operator, so it is bounded.

Let $x \neq 0$ and let $\delta(n) = ||T(n)x||$, for all $n \in \mathbb{N}$. From Lemma 2 we obtain that $\delta(n) \neq 0$, for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we consider the sequences

$$s_n : \mathbb{N} \to X, \quad s_n(k) = -\frac{\chi_{\{0,\dots,n\}}(k)}{\delta(k)}\, T(k)x$$

and

$$\gamma_n : \mathbb{N} \to X, \quad \gamma_n(k) = \sum_{j=k}^{\infty} \frac{\chi_{\{0,\dots,n\}}(j)}{\delta(j)}\, T(k)x$$

where $\chi_{\{0,\dots,n\}}$ denotes the characteristic function of the set $\{0, \dots, n\}$. Then $s_n, \gamma_n \in l^\infty(\mathbb{N}, X)$. A simple computation shows that the pair $(\gamma_n, s_n)$ verifies the equation $(E_d)$, for every $n \in \mathbb{N}$. It follows that $Ds_n = \gamma_n$, for all $n \in \mathbb{N}$.

Let $L = ||D||$. Since $|||s_n||| = 1$, for every $n \in \mathbb{N}$, we deduce that

$$\sum_{j=k}^{n} \frac{1}{\delta(j)} \leq \frac{L}{\delta(k)}, \quad \forall n, k \in \mathbb{N}, n \geq k$$

so

$$\sum_{j=k}^{\infty} \frac{1}{\delta(j)} \leq \frac{L}{\delta(k)}, \quad \forall k \in \mathbb{N}.$$

Let

$$\alpha : \mathbb{N} \to \mathbb{R}, \quad \alpha(n) = \sum_{j=n}^{\infty} \frac{1}{\delta(j)}$$

and let $\nu > 0$ be such that $L \leq [1/(e^\nu - 1)]$. Then we have that

$$(e^\nu - 1)\alpha(n+1) \leq (e^\nu - 1)\alpha(n) \leq 1/\delta(n)$$

so $e^\nu\, \alpha(n+1) \leq \alpha(n)$, for all $n \in \mathbb{N}$. It follows that

$$\frac{1}{\delta(n)} \leq \alpha(n) \leq e^{-\nu n}\alpha(0) \leq \frac{L}{||x||}\, e^{-\nu n}, \quad \forall n \in \mathbb{N}^*.$$

Thus, we deduce that

$$\delta(n) \geq \frac{e^{\nu n}}{L}\, ||x||, \quad \forall n \in \mathbb{N}^*.$$

Let $p \in \mathbb{N}^*$ with $e^{\nu p} > L$. Taking into account that $p$ and $\nu$ do not depend on $x$, setting $c = e^{\nu p}/L$, we have that $c > 1$ and

$$||T(p)x|| \geq c\, ||x||, \quad \forall x \in X.$$

By applying Lemma 1 (ii) we deduce that **T** is exponentially unstable. $\square$

REMARK 3. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a semigroup on $X$ and there is $t_0 > 0$ such that $T(t_0)$ is invertible, then $T(t)$ is invertible, for all $t \geq 0$. Then it can be extended to a group (see [21]). In this case, if $S(t) := T(t)^{-1}$, for all $t \geq 0$, then $\mathbf{T}$ is exponentially unstable if and only if $\mathbf{S}$ is exponentially stable.

THEOREM 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on $X$. Then, $\mathbf{T}$ is exponentially unstable and it can be extended to a group if and only if the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ is completely admissible for $\mathbf{T}$.*

*Proof. Necessity.* Let $s \in l^\infty(\mathbb{N}, X)$. Define

$$\gamma_s : \mathbb{N} \to X, \quad \gamma_s(n) = -\sum_{j=n}^{\infty} T(j-n)^{-1} s(j)$$

where for every $k \in \mathbb{N}$, $T(k)^{-1}$ denotes the inverse of the operator $T(k)$. We observe that the pair $(\gamma_s, s)$ verifies the equation $(E_d)$. To prove the uniqueness of $\gamma_s$, it is sufficient to show that if $\gamma \in l^\infty(\mathbb{N}, X)$ and

$$\gamma(n+1) = T(1)\gamma(n), \quad \forall n \in \mathbb{N} \tag{2.2}$$

then $\gamma = 0$. Indeed, since $\mathbf{T}$ is exponentially unstable, from relation (2.2) it follows that

$$||\gamma(n)|| = ||T(n)\gamma(0)|| \geq Ke^{\nu n}||\gamma(0)||, \quad \forall n \in \mathbb{N} \tag{2.3}$$

where $K, \nu > 0$ are given by Definition 3 (ii). Since $\gamma \in l^\infty(\mathbb{N}, X)$, from relation (2.3) it follows that $\gamma(0) = 0$. Then from relation(2.2) we obtain that $\gamma = 0$.

In conclusion, we deduce that the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ is completely admissible for $\mathbf{T}$.

*Sufficiency.* It follows from Theorem 1, Lemma 2 and Remark 3. □

## 3. Exponential instability of $C_0$-semigroups

In what follows, as consequences of the results in the previous section we deduce necessary and sufficient conditions for exponential instability of $C_0$-semigroups in terms of the solvability of an integral equation.

Let $X$ be a real or a complex Banach space and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a $C_0$-semigroup on $X$.

In what follows, we denote by $C_b(\mathbb{R}_+, X)$ the linear space of all continuous functions $u : \mathbb{R}_+ \to X$ with the property that $\sup_{t\geq 0} ||u(t)|| < \infty$. With respect to the norm

$$|||u||| := \sup_{t\geq 0} ||u(t)||$$

$C_b(\mathbb{R}_+, X)$ is a Banach space.

DEFINITION 5. *The pair $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ is said to be completely admissible for* $\mathbf{T}$ *if for every $u \in C_b(\mathbb{R}_+, X)$, there exists a unique function $f_u \in C_b(\mathbb{R}_+, X)$ such that the pair $(f_u, u)$ verifies the integral equation*

$$(E_c) \qquad f_u(t) = T(t-s)f_u(s) + \int_s^t T(t-\tau)u(\tau)\,d\tau, \quad \forall t \geq s \geq 0.$$

THEOREM 3. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a $C_0$-semigroup on $X$. If the pair $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ is completely admissible for* $\mathbf{T}$*, then* $\mathbf{T}$ *is exponentially unstable.*

*Proof.* Let $M, \omega > 0$ be such that $||T(t)|| \leq Me^{\omega t}$, for all $t \geq 0$.

Let $\beta : [0, 1] \to [0, 2]$ be a continuous function with compact support such that $supp\ \beta \subset (0, 1)$ and $\int_0^1 \beta(\tau)\,d\tau = 1$.

We prove that the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ is completely admissible for $\mathbf{T}$. Indeed, let $s \in l^\infty(\mathbb{N}, X)$. We define the function

$$v : \mathbb{R}_+ \to X, \quad v(t) = T(t-[t])s([t])\beta(t-[t]).$$

Then

$$||v(t)|| \leq 2Me^{\omega}||s([t])||, \quad \forall t \geq 0$$

so $v \in C_b(\mathbb{R}_+, X)$. From hypothesis there is $h \in C_b(\mathbb{R}_+, X)$ such that the pair $(h, v)$ verifies the equation $(E_c)$. In particular, for every $n \in \mathbb{N}$ we have that

$$h(n+1) = T(1)h(n) + \int_n^{n+1} T(n+1-\tau)v(\tau)\,d\tau = T(1)h(n) + T(1)s(n).$$

Setting $\gamma_s(n) = h(n)$, for all $n \in \mathbb{N}$, we have that $\gamma_s \in l^\infty(\mathbb{N}, X)$ and the pair $(\gamma_s, s)$ verifies the equation $(E_d)$.

To prove the uniqueness of $\gamma_s$ it is sufficient to show that if $\gamma \in l^\infty(\mathbb{N}, X)$ is such that

$$\gamma(n+1) = T(1)\gamma(n), \quad \forall n \in \mathbb{N} \tag{3.1}$$

then $\gamma = 0$.

Let $\gamma \in l^\infty(\mathbb{N}, X)$ which verifies the relation (3.1). Consider the function $g : \mathbb{R}_+ \to X$, $g(t) = T(t-[t])\gamma([t])$. From relation (3.1) we deduce that $g$ is continuous and

$$g(t) = T(t-\tau)g(\tau), \quad \forall t \geq \tau \geq 0. \tag{3.2}$$

Since the pair $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ is completely admissible for $\mathbf{T}$, from relation (3.2) it follows that $g = 0$, so $\gamma = 0$. Thus, we deduce that the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ is completely admissible for $\mathbf{T}$ and using Theorem 1 we obtain the conclusion. □

THEOREM 4. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a* $C_0$*-semigroup on* $X$. *Then,* $\mathbf{T}$ *is exponentially unstable and it can be extended to a group if and only if the pair* $(C_b(\mathbb{R}_+, X), C_b(\mathbb{R}_+, X))$ *is completely admissible for* $\mathbf{T}$.

*Proof. Necessity.* Let $u \in C_b(\mathbb{R}_+, X)$ and let

$$f : \mathbb{R}_+ \to X, \quad f(t) = -\int_t^\infty T(\tau - t)^{-1} u(\tau)\, d\tau.$$

Then, $f \in C_b(\mathbb{R}_+, X)$ and the pair $(f, u)$ verifies the equation $(E_c)$. Using similar arguments as in the necessity of Theorem 2 we obtain the uniqueness of $f$.

*Sufficiency.* It follows from Theorem 3, Lemma 2 and Remark 3. □

## 4. An application for the case of exponential dichotomy of semigroups

In what follows we will apply our results in order to characterize the exponential dichotomy of semigroups in Banach spaces.

Let $X$ be a real or a complex Banach space and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on $X$.

DEFINITION 6. $\mathbf{T}$ *is said to be exponentially dichotomic if there exist a projection* $P \in \mathcal{B}(X)$ *and two constants* $K \geq 1$ *and* $\nu > 0$ *such that:*

$(i)$ $T(t)P = PT(t)$*, for all* $t \geq 0$*;*

$(ii)$ $T(t)_| : Ker\, P \to Ker\, P$ *is an isomorphism, for all* $t \geq 0$*;*

$(iii)$ $||T(t)x|| \leq Ke^{-\nu t}||x||$*, for all* $x \in Im\, P$ *and all* $t \geq 0$*;*

$(iv)$ $||T(t)x|| \geq \frac{1}{K} e^{\nu t}||x||$*, for all* $x \in Ker\, P$ *and all* $t \geq 0$.

DEFINITION 7. *Let* $Y$ *be a linear subspace of* $X$. $Y$ *is said to be* $\mathbf{T}$*- invariant if* $T(t)Y \subset Y$*, for all* $t \geq 0$.

We consider the linear subspace

$$X_1 = \{x \in X : \sup_{t\geq 0} ||T(t)x|| < \infty\}.$$

We suppose that $X_1$ is a closed linear subspace which has a $\mathbf{T}$-invariant (closed) complement $X_2$ such that $X = X_1 \oplus X_2$. Let $P$ be the projection corresponding to the above decomposition, i.e. $Im\, P = X_1$ and $Ker\, P = X_2$.

REMARK 4. $T(t)P = PT(t)$, for all $t \geq 0$.

REMARK 5. $(i)$ If $T_s(t) = T(t)_{|Im\ P}$, for all $t \geq 0$, then $\mathbf{T}_s = \{T_s(t)\}_{t\geq 0}$ is an exponentially bounded semigroup on $Im\ P$.

(ii) If $T_u(t) = T(t)_{|Ker\ P}$, for all $t \geq 0$, then $\mathbf{T}_u = \{T_u(t)\}_{t\geq 0}$ is an exponentially bounded semigroup on $Ker\ P$.

DEFINITION 8. *The pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is said to be admissible for* $\mathbf{T}$ *if for every* $s \in l^\infty(\mathbb{N}, X)$ *there exists* $\gamma \in l^\infty(\mathbb{N}, X)$ *such that the pair* $(\gamma, s)$ *verifies the discrete-time equation*

$$\gamma(n+1) = T(1)\gamma(n) + s(n), \quad \forall n \in \mathbb{N}. \tag{4.1}$$

In what follows we establish the connections between the exponential dichotomy of the semigroup $\mathbf{T}$ and the admissibility of the pair $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$.

PROPOSITION 1. *If the pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is admissible for* $\mathbf{T}$*, then for every* $s \in l^\infty(\mathbb{N}, X)$ *there is a unique* $\gamma \in l^\infty(\mathbb{N}, X)$ *such that the pair* $(\gamma, s)$ *verifies the equation* (4.1) *and* $\gamma(0) \in Ker\ P$.

*Proof.* Let $s \in l^\infty(\mathbb{N}, X)$. Then there is $\tilde{\gamma} \in l^\infty(\mathbb{N}, X)$ such that the pair $(\tilde{\gamma}, s)$ verifies the equation (4.1). Then for

$$\gamma : \mathbb{N} \to X, \quad \gamma(n) = \tilde{\gamma}(n) - T(n)P\tilde{\gamma}(0)$$

we have that $\gamma \in l^\infty(\mathbb{N}, X)$ and $\gamma(0) = (I - P)\tilde{\gamma}(0) \in Ker P$. Moreover, it is easy to see that the pair $(\gamma, s)$ verifies the equation (4.1).

To prove the uniqueness of $\gamma$, let $\delta \in l^\infty(\mathbb{N}, X)$ with $\delta(0) \in Ker P$ such that the pair $(\delta, s)$ verifies the equation (4.1).

Setting $\alpha = \gamma - \delta$, it follows that $\alpha(n+1) = T(1)\alpha(n)$, for all $n \in \mathbb{N}$ and $\alpha(0) \in Ker\ P$. Since $\alpha \in l^\infty(\mathbb{N}, X)$ we deduce that

$$||T(n)\alpha(0)|| = ||\alpha(n)|| \le |||\alpha|||, \quad \forall n \in \mathbb{N}$$

so $\alpha(0) \in Im\ P$. This implies that $\alpha(0) = 0$, so $\delta = \gamma$. □

THEOREM 5. *If the pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is admissible for* $\mathbf{T}$*, then* $T(t)_| : Ker\ P \to Ker\ P$ *is an isomorphism, for all* $t \ge 0$ *and there are* $K_1, \nu_1 > 0$ *such that*

$$||T(t)x|| \ge \frac{1}{K_1} e^{\nu_1 t}||x||, \quad \forall t \ge 0, \forall x \in Ker\ P.$$

*Proof.* From hypothesis we deduce that the pair $(l^\infty(\mathbb{N}, Ker\ P), l^\infty(\mathbb{N}, Ker\ P))$ is admissible for the semigroup $\mathbf{T}_u$. In particular, we have that for every $s \in l^\infty(\mathbb{N}, Ker\ P)$ there is $\gamma \in l^\infty(\mathbb{N}, Ker P)$ such that

$$\gamma(n+1) = T_u(1)\gamma(n) + T_u(1)s(n), \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Moreover, using Proposition 1 we deduce that the sequence $\gamma$ which verifies (4.2) is uniquely determined. It follows that the pair $(l^\infty(\mathbb{N}, Ker\ P), l^\infty(\mathbb{N}, Ker\ P))$ is completely admissible for $\mathbf{T}_u$. Then, from Theorem 2 we obtain the conclusion. □

THEOREM 6. *If the pair* $(l^\infty(\mathbb{N}, X), l^\infty(\mathbb{N}, X))$ *is admissible for* $\mathbf{T}$*, then there are* $K_2, \nu_2 > 0$ *such that*

$$||T(t)x|| \le K_2\, e^{-\nu_2 t}||x||, \quad \forall t \ge 0, \forall x \in Im\ P.$$

*Proof.* Let $x \in ImP$. Consider the sequence

$$s : \mathbb{N} \to X, \quad s(n) = T(n)x.$$

Since $x \in ImP$, we have that $s \in l^{\infty}(\mathbb{N}, X)$. From Proposition 1 it follows that there is $\gamma \in l^{\infty}(\mathbb{N}, X)$ with $\gamma(0) \in Ker P$ such that the pair $(\gamma, s)$ verifies the equation (4.1). Then we have that

$$\gamma(n) = T(n)\gamma(0) + nT(n-1)x, \quad \forall n \in \mathbb{N}^*.$$

From Theorem 5, there are $K_1, \nu_1 > 0$ such that

$$\frac{1}{K_1} e^{\nu_1 n} ||\gamma(0)|| \le ||T(n)\gamma(0)|| \le ||\gamma(n)|| + n||T(n-1)x|| \le$$

$$\le |||\gamma||| + n|||s|||, \quad \forall n \in \mathbb{N}^*.$$

This inequality implies that $\gamma(0) = 0$, so

$$\gamma(n) = nT(n-1)x, \quad \forall n \in \mathbb{N}^*.$$

Hence we obtain that

$$\sup_{n \in \mathbb{N}} (n+1)||T(n)x|| < \infty, \quad \forall x \in ImP.$$

From the uniform boundedness principle it follows that there is $M > 0$ such that

$$(n+1)||T(n)x|| \le M\, ||x||, \quad \forall n \in \mathbb{N}, \forall x \in ImP.$$

This shows that there is $p \in \mathbb{N}^*$ such that

$$||T_s(p)x|| \le \frac{1}{2}\, ||x||, \quad \forall x \in ImP.$$

By applying Lemma 1 for the semigroup $\mathbf{T}_s$ we obtain the conclusion. □

The main result of this section is:

THEOREM 7. **T** *is exponentially dichotomic if and only if the pair* $(l^{\infty}(\mathbb{N}, X), l^{\infty}(\mathbb{N}, X))$ *is admissible for* **T**.

*Proof. Necessity.* Let $P$ be the projection given by Definition 6. For every $k \in \mathbb{N}$, let $T(k)_|^{-1}$ denote the inverse of the operator $T(k)_| : Ker\ P \to Ker\ P$. If $s \in l^{\infty}(\mathbb{N}, X)$ we define the sequence $\gamma : \mathbb{N} \to X$ by

$$\gamma(n) = \chi_{\mathbb{N}^*}(n) \sum_{k=1}^{n} T(n-k)Ps(k-1) - \sum_{k=n+1}^{\infty} T(k-n)_|^{-1}(I-P)s(k-1).$$

Then $\gamma \in l^{\infty}(\mathbb{N}, X)$ and the pair $(\gamma, s)$ verifies the equation (4.1). This shows that the pair $(l^{\infty}(\mathbb{N}, X), l^{\infty}(\mathbb{N}, X))$ is admissible for **T**.

*Sufficiency.* It follows from Theorem 5 and Theorem 6, taking $K = max\{K_1, K_2\}$ and $\nu = min\{\nu_1, \nu_2\}$. □

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