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METRIC GEOMETRIES OVER THE SPLIT QUATERNIONS

Abstract. We give an overview of some recent results in hypersymplectic and para-quaternionic Kähler geometry, and introduce the notion of split three-Sasakian manifold. In particular, we discuss the twistor spaces and Swann bundles of para-quaternionic Kähler manifolds. These are used to classify examples with a fully homogeneous action of a semi-simple Lie group, and to construct distinct para-quaternionic Kähler metrics from indefinite real analytic conformal manifolds. We also indicate how the theory of toric varieties gives rise to constructions of hypersymplectic manifolds.

1. Introduction.

This is an expository article covering our recent work on construction and classification of some examples of pseudo-Riemannian manifolds whose geometry is based on the split quaternions. The two main themes are the use of twistor theory and of moment maps for group actions.

There are two main geometries that we wish to consider: hypersymplectic and para-quaternionic Kähler. Both geometries define Einstein metrics, with hypersymplectic manifolds being Ricci-flat, and have been used in various physical theories [24, 9, 13, 30]. The two geometries differ in that hypersymplectic manifolds come equipped with families of symplectic two-forms whereas para-quaternionic Kähler geometry is captured by a closed four-form. This means that techniques from symplectic geometry may be directly applied to hypersymplectic manifolds, while in the para-quaternionic case a more circuitous route must be taken. In the latter case, there are two different approaches one may use. The first is relatively well-established, and involves constructing a twistor space: a holomorphic manifold capturing the para-quaternionic Kähler geometry. However, this requires analyticity assumptions on the original geometry and is essentially only a local construction. A second approach is the so-called Swann bundle; this is globally defined and encodes the para-quaternionic Kähler geometry in a hypersymplectic structure. Here the problem is that this hypersymplectic metric has degeneracies which need to be taken into account. We discuss these two constructions and their interrelation and show how they may be used to classify para-quaternionic Kähler manifolds fully homogeneous under a semi-simple Lie group.

In the final two sections we turn to two other constructions. Firstly, in the hypersymplectic category we discuss how ideas of toric geometry may be used to produce examples starting from the action of a (usually compact) Abelian group on flat space. In the para-quaternionic Kähler case, we indicate how conformal geometry combined with twistor theory may be used to construct a wide variety of examples. Indeed each real analytic conformal manifold of indefinite signature, locally gives rise to distinct

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para-quaternionic Kähler metrics.

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2. The split quaternions.

The algebra $\mathbb{B}$ of split quaternions is a four-dimensional real vector space with basis $\{1, i, s, t\}$ satisfying

$$i^2 = -1, \quad s^2 = 1 = t^2, \quad is = t = -si.$$

This carries a natural indefinite inner product given by $\langle p, q\rangle = \operatorname{Re}\bar{p}q$, where $p = x + iy + su + tv$ has $\bar{p} = x - iy - su - tv$. We have $\|p\|^2 = x^2 + y^2 - s^2 - t^2$, so a metric of signature $(2, 2)$. This norm is multiplicative, $\|pq\|^2 = \|p\|^2\|q\|^2$, but the presence of elements of length zero means that $\mathbb{B}$ contains zero divisors.

The basis elements $1, i, s, t$ are not the only split quaternions with square ± 1. Using the multiplication rules for $\mathbb{B}$, one finds

$$p^2 = -1 \quad \text{if and only if} \quad p = iy + su + tv,\ y^2 - s^2 - t^2 = 1$$
$$p^2 = +1 \quad \text{if and only if} \quad p = iy + su + tv,\ y^2 - s^2 - t^2 = -1 \quad \text{or} \quad p = \pm 1.$$

The right $\mathbb{B}$-module $\mathbb{B}^n \cong \mathbb{R}^{4n}$ inherits the inner product $\langle \xi, \eta\rangle = \operatorname{Re}\bar{\xi}^T\eta$ of signature $(2n, 2n)$. The automorphism group of $(\mathbb{B}^n, \langle\cdot,\cdot\rangle)$ is

$$\mathrm{Sp}(n, \mathbb{B}) = \{\, A \in M_n(\mathbb{B}) : \bar{A}^T A = 1 \,\}$$

which is a Lie group isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$, the symmetries of a symplectic vector space $(\mathbb{R}^{2n}, \omega)$. In particular, $\mathrm{Sp}(1, \mathbb{B}) \cong \mathrm{SL}(2, \mathbb{R})$ is the pseudo-sphere of $\mathbb{B} = \mathbb{R}^{2,2}$. The Lie algebra of $\mathrm{Sp}(n, \mathbb{B})$ is

$$\mathfrak{sp}(n, \mathbb{B}) = \{\, A \in M_n(\mathbb{B}) : A + \bar{A}^T = 0 \,\},$$

so $\mathfrak{sp}(1, \mathbb{B}) = \operatorname{Im}\mathbb{B}$.

The group $\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})$ acts on $\mathbb{B}^n$ via

$$(A, p) \cdot \xi = A\xi\bar{p}. \tag{1}$$

This action is isometric with kernel $\mathbb{Z}/2 = \{\pm(1, 1)\}$, demonstrating that

$$\mathrm{Sp}(n, \mathbb{B})\mathrm{Sp}(1, \mathbb{B}) := \frac{\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})}{\mathbb{Z}/2}$$

is a subgroup of $O(2n, 2n)$.

Using the complex structure $\xi \mapsto -\xi i$, we may identify $\mathbb{B}^n$ with $\mathbb{C}^{n,n}$ via $\xi = z + ws$. In this context we see $\mathrm{Sp}(n, \mathbb{B})$ as a subgroup of $U(n, n)$ and note that it contains a compact n-dimensional torus $T^n = \{\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})\} \subset M_n(\mathbb{B})$. We will also make use of a second Abelian subgroup of rank n, namely $R^n = \{\mathrm{diag}(e^{s\phi_1}, \dots, e^{s\phi_n})\}$ isomorphic to $\mathbb{R}^n$.

Returning to equation (1), we have that as a representation of $\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})$, we may write $\mathbb{B}^n$ as the tensor product

$$\mathbb{B}^n = E_{\mathbb{R}} \otimes_{\mathbb{R}} H_{\mathbb{R}},$$

where $E_{\mathbb{R}} = \mathbb{R}^{2n}$ and $H_{\mathbb{R}} = \mathbb{R}^2$ are symplectic vector spaces with symplectic forms ω^E and ω^H. In this notation $\langle e \otimes h, e' \otimes h' \rangle = \omega^E(e, e')\omega^H(h, h')$.

3. Three split geometries.

Let M be a manifold of dimension m. The frame bundle $\mathrm{GL}(M)$ consists of all linear maps

$$u \colon \mathbb{R}^m \to T_x M.$$

It is a principal bundle with structure group $\mathrm{GL}(m, \mathbb{R})$, the group action being given by

$$(u \cdot A)(\mathbf{v}) = u(A\mathbf{v}). \tag{2}$$

For a closed Lie subgroup G of $\mathrm{GL}(m, \mathbb{R})$, a G-structure on M is a subbundle $G(M)$ of $\mathrm{GL}(M)$ that is a principal G-bundle under the action (2) for $A \in G$. In this situation, objects on $\mathbb{R}^m$ that are invariant under the action of G give rise to corresponding geometric structures on M.

3.1. Hypersymplectic manifolds.

Take $m = 4n$, identify $\mathbb{R}^{4n}$ with $\mathbb{B}^n$ and take $G = \mathrm{Sp}(n, \mathbb{B}) \subset \mathrm{GL}(4n, \mathbb{R})$. An $\mathrm{Sp}(n, \mathbb{B})$-structure $\mathrm{Sp}_{\mathbb{B}}(M)$ on M defines a metric g of signature $(2n, 2n)$ by $g(u(\mathbf{v}), u(\mathbf{w})) = \langle \mathbf{v}, \mathbf{w} \rangle$. The right action of $-i$, s and t on $\mathbb{B}^n$ define endomorphisms I, S and T of $T_x M$ satisfying

$$I^2 = -1, \quad S^2 = 1 = T^2, \quad IS = T = -SI \tag{3}$$

and the compatibility equations

$$g(IX, IY) = g(X, Y), \quad g(SX, SY) = -g(X, Y) = g(TX, TY), \tag{4}$$

for $X, Y \in T_x M$. These properties mean that we obtain three 2-forms ω_I, ω_S and ω_T given by

$$\omega_I(X, Y) = g(IX, Y), \quad \omega_S(X, Y) = g(SX, Y), \quad \omega_T(X, Y) = g(TX, Y).$$

The manifold M is said to be *hypersymplectic* if the 2-forms ω_I, ω_S and ω_T are all closed:

$$d\omega_I = 0, \quad d\omega_S = 0, \quad d\omega_T = 0.$$

Adapting a computation of Atiyah & Hitchin [7] for hyperKähler manifolds, one finds that this implies that the endomorphisms I, S and T are all integrable. This means firstly that locally there are complex coordinates realising I. The integrability of S means that M is locally a product $M^S_+ \times M^S_-$ where for $\varepsilon = \pm 1$, TM^S_ε is the ε-eigenspace of S on TM. Note that these submanifolds M^S_ε are totally isotropic with respect to g, and that we obtain families of such splittings by considering the integrable endomorphisms $S_\theta = S\cos\theta + T\sin\theta$. The structures (M, g, S_θ) are sometimes called para-complex, see [15].

One also finds that I, S and T are parallel with respect to the Levi-Civita connection ∇ of g. Thus the holonomy group of M reduces to $\mathrm{Sp}(n, \mathbb{B})$. For this reason some authors refer to these structures as "neutral hyperKähler"; not to be confused with hyperKähler structures of signature $(4r, 4r)$. The name "hypersymplectic" is the terminology of Hitchin [23].

As the complexification of the action $\mathrm{Sp}(n, \mathbb{B})$ of $\mathbb{B}^n$ is the same as that of the complexification of $\mathrm{Sp}(n)$ action on $\mathbb{H}^n$, one may adapt computations from hyperKähler geometry, to show that hypersymplectic manifolds are Ricci-flat.

The basic example of a hypersymplectic manifold is $\mathbb{B}^n$. Identifying $\mathbb{B}^n$ with $\mathbb{C}^{n,n}$ as above one has $I(z, w) = (-zi, wi)$, $S(z, w) = (w, z)$ and one finds that

$$g = \mathrm{Re}\Big(\sum_{k=1}^{n} dz_k d\bar{z}_k - dw_k d\bar{w}_k\Big),$$

$$\omega_I = \frac{1}{2i}\sum_{k=1}^{n}(dz_k \wedge d\bar{z}_k + dw_k \wedge d\bar{w}_k),$$

$$\omega_S + i\omega_T = \sum_{k=1}^{n} dw_k \wedge d\bar{z}_k.$$

Note that $\omega_S + i\omega_T$ is a holomorphic $(2,0)$-form with respect to I; a fact which holds generally on hypersymplectic manifolds.

Many examples of hypersymplectic structures are known on Lie groups. Kamada [26] classified all the hypersymplectic structures on primary Kodaira surfaces. These are T^2-bundles over T^2 and may be regarded as nilmanifolds $\Gamma\backslash G$ for G a 2-step nilpotent Lie group. Examples on 2-step nilmanifolds in higher dimensions were obtained in [20]. In the non-compact realm, hypersymplectic structures on solvable Lie groups have been studied in [4, 5, 6], in particular the four-dimensional examples have been classified.

The work of Alekseevsky & Cortés [2] on classification of indefinite symmetric hyperKähler manifolds may be adapted to the present situation. We look for hypersymplectic manifolds with Abelian holonomy that are symmetric and simply-connected.

Consider $\mathbb{B}^n$ as the complex vector space $E = \mathbb{C}^{n,n}$ for I together with the complex symplectic form $\omega^E = \omega_S + i\omega_T$ and the real structure $s_E = S$. Note that $\omega^E(s_E X, s_E Y) = -\overline{\omega^E(X, Y)}$. Let $E = L_+ \oplus L_-$ be an s_E-invariant Lagrangian decomposition. Suppose R^+ is an s_E-invariant element in $S^4 L_+$, i.e., an invariant homogeneous polynomial of degree four on $L_+^* \cong L_-$, using ω^E to identify the dual space E^* with E.

One may then construct a simply connected symmetric hypersymplectic manifold M_{R^+} as G/K, where

$$\mathfrak{k} = \operatorname{span}\{i(R^+_{s_E A, B} - R^+_{A, s_E B}) : A, B \in E\}$$

is Abelian and

$$\mathfrak{g} = \mathfrak{k} + \mathbb{B}^n.$$

The Lie algebra structure of $\mathfrak{g}$ is defined as follows. Identify $\mathbb{B}$ as the real elements in $E \otimes H$, where $H = \mathbb{C}^{1,1}$ and the real structure is $\sigma = s_E \otimes s_H$. Then $\mathfrak{k}$ acts on $E \otimes H$ commuting with σ and

$$[A_1 \otimes h_1, A_2 \otimes h_2] = \omega^H(h_1, h_2) R^+_{A_1, A_2}.$$

Now take G to be the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and K to be the connected subgroup of G with Lie algebra $\mathfrak{k}$.

The space $E \otimes H$ carries a complex quaternionic structure with endomorphisms I, $J = iS$ and $K = iT$. Following through Alekseevsky & Cortés proofs [2], noting that a symmetric hypersymplectic manifold complexifies to a complex hyperKähler manifold, one finds, as also announced in [1]:

THEOREM 1. *Let M^{4n} be a simply-connected symmetric hypersymplectic manifold with Abelian holonomy. Then $M = M_{R^+}$, for some $R^+ \in (S^4 L_+)^{s_E}$ and some s_E-invariant Lagrangian decomposition $E = \mathbb{C}^{n,n} = L_+ \oplus L_-$.*

Note that M_{R^+} has no flat de Rham factor if and only if the elements $R^+_{A,B} C$ span L_+^*.

The smallest example of this construction is in real dimension 4. Here $E = \mathbb{C}^{1,1}$, so L_+ is a complex one-dimensional subspace, spanned by an element e that we can take to be s_E-invariant. Choose an imaginary $\tilde{e} \in L_-$ so that $\omega^E(e, \tilde{e}) = 1$ and fix a symplectic basis $\{h, \tilde{h}\}$ for H with $s_H h = h$ and $s_H \tilde{h} = -\tilde{h}$. Up to scale we have $R^+ = e^4$. The holonomy algebra $\mathfrak{h}$ is one-dimensional spanned by $E_3 = ie^2$, and $\mathbb{B}$ is spanned by $E_1 = i\tilde{e} \otimes h$, $E_2 = \tilde{e} \otimes \tilde{h}$, $E_4 = e \otimes h$ and $E_5 = ie \otimes \tilde{h}$. The Lie algebra $\mathfrak{g}$ has non-zero Lie brackets $[E_1, E_2] = E_3$, $[E_3, E_1] = E_4$, and $[E_3, E_2] = E_5$. The metric and symplectic forms are then given by

$$\begin{aligned} g &= E_1 \vee E_5 - E_2 \vee E_4, & \omega_I &= E_1 \wedge E_4 - E_2 \wedge E_5, \\ \omega_S &= E_1 \wedge E_5 - E_2 \wedge E_4, & \omega_T &= E_1 \wedge E_4 + E_2 \wedge E_5. \end{aligned}$$

As in all these examples, $\mathfrak{g}$ is three-step nilpotent.

3.2. Para-quaternionic Kähler manifolds.

Here we consider the larger structure group $\mathrm{Sp}(n, \mathbb{B})\mathrm{Sp}(1, \mathbb{B})$ acting on $\mathbb{B}^n = \mathbb{R}^{4n}$ via (1). Again we have metric of neutral signature $(2n, 2n)$, but now we cannot distinguish the endomorphisms I, S and T. Instead we have a bundle $\mathcal{G}$ of endomorphisms of TM that locally admits a basis $\{I, S, T\}$ satisfying (3) and (4). If $n > 1$, we say that M is *para-quaternionic Kähler* if its holonomy lies in $\mathrm{Sp}(n, \mathbb{B})\mathrm{Sp}(1, \mathbb{B})$. This is the same as requiring that the Levi-Civita connection preserve the bundle $\mathcal{G}$. In dimensions $4n \geqslant 12$, the computations of [32] show that this is equivalent to the global four-form Ω, locally defined by

$$\Omega = \omega_I \wedge \omega_I - \omega_S \wedge \omega_S - \omega_T \wedge \omega_T,$$

being closed, $d\Omega = 0$. The representation theoretic proof of the curvature properties of quaternionic Kähler manifolds [31] applies in this case to show that para-quaternionic Kähler manifolds are Einstein (see also [21]). For $n = 1$, one obtains similar properties by requiring M to be self-dual and Einstein.

The model example for para-quaternionic Kähler manifolds is the para-quaternionic projective space

$$\mathbb{B}P(n) = \frac{\mathrm{Sp}(n+1, \mathbb{B})}{\mathrm{Sp}(n, \mathbb{B})\mathrm{Sp}(1, \mathbb{B})} = \frac{\mathrm{Sp}(2n+2, \mathbb{R})}{\mathrm{Sp}(2n, \mathbb{R})\mathrm{SL}(2, \mathbb{R})}$$

studied by Blažić [12], described in Wolf [35] and which we will study more in the next section.

The curvature tensor R of a general para-quaternionic Kähler manifold may be written as

$$R = R_0 + kR_1, \tag{5}$$

where R_1 is the curvature tensor of $\mathbb{B}P(n)$ and R_0 commutes with the endomorphisms of $\mathcal{G}$. The constant k is zero if and only if g is Ricci-flat. If $k = 0$, then the holonomy algebra of g lies in $\mathfrak{sp}(n, \mathbb{B})$ and locally M admits a hypersymplectic structure. We will exclude this case in future discussion of para-quaternionic Kähler structures.

Looking at Berger's list [10], other symmetric para-quaternionic Kähler examples G/H with semi-simple symmetry group G may be found. Table 1.1 lists the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. This list is constructed by finding the examples where the projection of $\mathfrak{h}$ to the second factor in $\mathfrak{sp}(n, \mathbb{B}) + \mathfrak{sp}(1, \mathbb{B})$ is surjective. This gives all symmetric para-quaternionic Kähler spaces G/H with G semi-simple, since if the image of the projection is smaller than $\mathfrak{sp}(1, \mathbb{B})$, then equation (5) implies that $k = 0$ and the holonomy algebra lies in $\mathfrak{sp}(n, \mathbb{B})$. Alekseevsky & Cortés [3] have recently announced that this gives all symmetric para-quaternionic Kähler spaces.

We will see later that any para-quaternionic Kähler manifold homogeneous under a semi-simple symmetry group that acts maximally on $\mathcal{G}$ is one of these symmetric spaces.

3.3. Split three-Sasakian manifolds.

The para-quaternionic projective space $\mathbb{B}P(n)$ may be constructed from the flat hypersymplectic manifold $\mathbb{B}^{n+1}$ by the following procedure. Consider the positive pseudosphere $\mathcal{S}$ in $\mathbb{B}^{n+1}$:

$$\mathcal{S} = \{\, \xi \in \mathbb{B}^{n+1} : \|\xi\|^2 = 1 \,\}.$$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(n+2,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{su}(p+1,q+1)$	$\mathfrak{u}(p,q)+\mathfrak{su}(1,1)$
$\mathfrak{sp}(2n+2,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{so}^*(2n+4)$	$\mathfrak{so}^*(2n)+\mathfrak{so}^*(4)$
$\mathfrak{so}(p+2,q+2)$	$\mathfrak{so}(p,q)+\mathfrak{so}(2,2)$
$\mathfrak{g}_{2(2)}$	$\mathfrak{so}(2,2)$
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(6,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(6,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(3,3)+\mathfrak{su}(1,1)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(5,1)+\mathfrak{su}(1,1)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{so}(6,6)+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}^*(12)+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(10,2)+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)}+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-25)}+\mathfrak{sl}(2,\mathbb{R})$

Table 1.1: Lie algebras for symmetric para-quaternionic Kähler manifolds G/H with G semi-simple.

This carries a metric of signature $(2n-1, 2n+2)$ and $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(1,\mathbb{B})$ acts freely and properly on $\mathcal{S}$. The quotient $\mathcal{S}/\mathrm{SL}(2,\mathbb{R})$ is $\mathbb{B}P(n)$ and the projection is a pseudo-Riemannian submersion. The distribution $\mathcal{H}$ orthogonal to the $\mathrm{SL}(2,\mathbb{R})$-orbits is preserved by I, S and T and this is how one obtains the required bundle of endomorphisms $\mathcal{G}$ on $\mathbb{B}P(n)$. Being odd-dimensional, the tangent space of $\mathcal{S}$ can not be preserved by I. However using the restrictions of I, S and T to $\mathcal{H}$, and the vector fields generated by the $\mathrm{SL}(2,\mathbb{R})$-action, we see that $\mathcal{S}$ is an example of the following definition.

DEFINITION 1. *A pseudo-Riemannian manifold* $(\mathcal{S}, g)$ *is called* split three-Sasakian *if it admits three orthogonal Killing vector fields* ξ^1, ξ^2, ξ^3 *with length squared* 1, -1 *and* -1 *respectively, such that*

$$[\xi^1,\xi^2] = -\xi^3, \quad [\xi^2,\xi^3] = \xi^1, \quad [\xi^3,\xi^1] = -\xi^2$$

and the endomorphisms $\Phi_i = \nabla\xi^i$, $i = 1, 2, 3$, *satisfy*

$$(\nabla_X\Phi_i)Y = g(\xi^i, Y)X - g(X, Y)\xi^i.$$

Indeed, in this situation one finds that

$$\begin{gathered}
\Phi_1\xi^2 = \xi^3 = -\Phi_2\xi^1, \quad -\Phi_2\xi^3 = \xi^1 = \Phi_3\xi^2, \quad \Phi_3\xi^1 = \xi^2 = -\Phi_1\xi^3 \\
\Phi_1\Phi_2 Y - g(\xi^2, Y)\xi^1 = \Phi_3 Y = -\Phi_2\Phi_1 Y + g(\xi^1, Y)\xi^2 \\
-\Phi_2\Phi_3 Y + g(\xi^3, Y)\xi^2 = \Phi_1 Y = \Phi_3\Phi_2 Y - g(\xi^2, Y)\xi^3 \\
\Phi_3\Phi_1 Y - g(\xi^1, Y)\xi^3 = \Phi_2 Y = -\Phi_1\Phi_3 Y + g(\xi^3, Y)\xi^1,
\end{gathered}$$

so the Φ_i behave as the restrictions of I, S and T. Furthermore these structures are Einstein with scalar curvature $m(m-1)$, where $m = \dim \mathcal{S}$, and they extend to hypersymplectic structures on the cone $M = \mathcal{S} \times \mathbb{R}_{>0}$, by using the metric $dr^2 + r^2 g$ and

$$\begin{aligned} I &= \Phi_1 - g(\xi^1, \cdot) \otimes \psi + \tfrac{1}{r} dr \otimes \xi^1, \\ S &= \Phi_2 - g(\xi^2, \cdot) \otimes \psi + \tfrac{1}{r} dr \otimes \xi^2, \\ T &= \Phi_3 - g(\xi^3, \cdot) \otimes \psi + \tfrac{1}{r} dr \otimes \xi^3. \end{aligned}$$

In the case of $\mathcal{S}$, the hypersymplectic cone constructed here is the open set $\{\xi \in \mathbb{B}^{n+1} : \|\xi\|^2 > 0\}$ of vectors of positive norm. Conversely, if a cone $(M = \mathcal{S} \times \mathbb{R}_{>0}, dr^2 + r^2 g)$ is hypersymplectic then $\mathcal{S} \times \{1\}$ inherits a split three-Sasakian structure.

4. Associated geometries.

In this section we will concentrate on geometries associated to para-quaternionic Kähler structures. Twistor theory is the most well-developed aspect, but in indefinite signature it is only a local theory, that requires additional analyticity assumptions. In some situations a bridge for general para-quaternionic Kähler structures to the twistor theory is provided by the so-called Swann bundle, which is always defined but suffers from some degeneracies. First let us introduce some standard notation.

Let M^{4n} be para-quaternionic Kähler. Let F denote the $\mathrm{Sp}(n, \mathbb{B})\mathrm{Sp}(1, \mathbb{B})$-frame bundle of M. Locally this admits a double cover $\tilde{F}$ that is a principal bundle with structure group $\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})$. If we have a representation $\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B}) \to \mathrm{Aut}(V)$ of this group on a vector space V we may define a vector bundle on M as

$$\frac{\tilde{F} \times V}{\mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})},$$

where $(u, v) \cdot (A, p) = (u \cdot (A, p), (A^{-1}, p^{-1}) \cdot v)$, for $u \in \tilde{F}$, $v \in V$, $A \in \mathrm{Sp}(n, \mathbb{B})$ and $p \in \mathrm{Sp}(1, \mathbb{B})$. We will denote the resulting vector bundle by V. Note that V is globally defined precisely when $(-1, -1) \in \mathrm{Sp}(n, \mathbb{B}) \times \mathrm{Sp}(1, \mathbb{B})$ acts trivially in the original representation.

Two fundamental examples of this construction are the local bundles E and H associated to the standard representations of $\mathrm{Sp}(n, \mathbb{B})$ on $\mathbb{B}^n \cong \mathbb{C}^{n,n}$ and of $\mathrm{Sp}(1, \mathbb{B}) \cong \mathbb{C}^{1,1}$ on $\mathbb{B}$, respectively. Equation (1) shows that

$$E \otimes H = TM \otimes \mathbb{C}. \tag{6}$$

We have already seen these bundles in the context of hypersymplectic manifolds in §3.1. As there, E and H carry complex symplectic forms ω^E and ω^H and real structures s^E and s^H. One may define metrics on these bundles by $g^E(\cdot, \cdot) = \omega^E(s^E \cdot, \cdot)$, etc.

4.1. The twistor space.

The essential construction is provided by Bailey & Eastwood [8], amongst others, in a rather more general context, where specific details related to real structures are not completely specified.

Suppose that the para-quaternionic Kähler manifold M is real analytic, meaning that there is an atlas for which the change of coordinates is real analytic and that in these local coordinates the metric g and the bundle $\mathcal{G}$ are real analytic. In this situation we may consider the complexification $\mathcal{M}$ of M. Here we have that g extends uniquely to a holomorphic metric $g^{\mathbb{C}}$ and that $\mathcal{G}$ admits local bases $\{I, S, T\}$ that extend holomorphically to complex linear transformations on $T\mathcal{M}$ satisfying (3).

On $\mathcal{M}$, equation (6) becomes $T\mathcal{M} = E \otimes H$. For each non-zero $h \in H_x$, we define

$$T_{[h]} = E_x \otimes h \subset T_x\mathcal{M}.$$

Multiplying h by a non-zero $\lambda \in \mathbb{C}$ defines the same subspace of $T_x\mathcal{M}$, so the set of such spaces at x is parameterised by $[h] \in \mathbb{P}(H_x) = \mathbb{C}P(1)$. Note that $\mathbb{P}(H)$ is a globally defined two-sphere bundle over $\mathcal{M}$. If $\|h\|^2 \neq 0$ then $T_{[h]}$ may realised as the $+i$-eigenspace of an almost complex structure of the form $aI + bS + cT$, where $a^2 - b^2 - c^2 = 1$.

A maximal submanifold Σ of $\mathcal{M}$ is called an *α-surface* if

$$T_y\Sigma = T_{[h]}$$

for some $[h] \in \mathbb{P}(H_y)$ and each $y \in \Sigma$. One proves that an α-surface is totally geodesic in $\mathcal{M}$ and from this one sees that the family $\mathbb{P}_x$ of α-surfaces through a given point $x \in \mathcal{M}$ is parameterised by $\mathbb{C}P(1)$. Note that $T_{[h]}$ is totally isotropic with respect to $g^{\mathbb{C}}$, so the geodesics here are null, albeit of a special type, since $T_{[h]}$ is parallel along the curve.

DEFINITION 2. *The* twistor space Z *of* M *is the set of all α-surfaces in* $\mathcal{M}$.

In general, this space will not be topologically well-behaved. However, if we assume that $\mathcal{M}$ is geodesically convex, which is always the case locally, then one can prove that Z is a complex manifold.

The twistor space Z has several additional properties. Firstly, the *twistor lines* $\mathbb{P}_x$ have normal bundle $2n\mathcal{O}(1)$. Two such twistor lines $\mathbb{P}_x$ and $\mathbb{P}_y$ intersect when x and y lie on a common α-surface; Z is covered by twistor lines. Secondly, Z carries a *real structure* σ given as the pull-back of the real structure s^H on $H \to M$ and the complex conjugation map in $\mathcal{M}$ that fixes M. This real structure on Z has fixed points, for example for $x \in M$ and $\|h\|^2 = 0$. The σ-invariant twistor lines are precisely those $\mathbb{P}_x$ with x in the real manifold M.

The third property of Z is the existence of a complex contact form θ. This may be defined as follows. A tangent vector in Z at an α-surface Σ corresponds to a normal field $J = j_E \otimes j_H$ such that $[J, X] = 0$ for all $X \in T_{[h]}$. Putting $j = j_E\omega^H(j_H, h)$ we

get a one-to-one correspondence between elements of $T_\Sigma Z$ and sections j of $E \to \Sigma$ satisfying
$$\omega^E(\nabla_{\cdot\otimes h} j, \cdot) = \mu\omega^E$$
for some constant $\mu \in \mathbb{C}$ depending on j. The complex contact form θ is defined by
$$\theta_\Sigma(j) = \mu.$$
This is a non-vanishing form provided M is not (locally) hypersymplectic. Since $\mu \mapsto \lambda^2\mu$ when we replace h by λh, θ is a holomorphic one-form on Z taking values in a line bundle L such that $L|_{\mathbb{P}_x} \cong \mathcal{O}(2)$. The contact property of the form θ follows from the fact that $\theta \wedge (d\theta)^n$ vanishes at no point of Z. The form is compatible with the real structure in the sense that $\sigma^*\theta = -\bar{\theta}$, and each twistor line $\mathbb{P}_x$ is transverse to $\ker\theta$.

The power of the twistor construction is that it may be inverted.

THEOREM 2. *Let Z be a complex manifold of dimension $2n + 1$ with a real structure σ and a complex contact form θ such that $\sigma^*\theta = -\bar{\theta}$. If there is a rational curve $\mathbb{P} = \mathbb{C}P(1)$ in Z such that*

1. *$\mathbb{P}$ has normal bundle $2n\mathcal{O}(1)$ in Z,*
2. *$\sigma(\mathbb{P}) = \mathbb{P}$,*
3. *σ has fixed points on $\mathbb{P}$,*
4. *$\theta|_{\mathbb{P}}$ is non-zero,*

then Z is the twistor space of a para-quaternionic Kähler manifold M of dimension $4n$ that is not hypersymplectic.

This result is proved in much the same way as LeBrun's inverse twistor construction for quaternionic Kähler manifolds [29], taking care of the slightly different properties of the real structure. The manifold M is obtained as the parameter space of rational curves in Z satisfying the above four conditions.

The basic example of this construction is provided by taking $Z = \mathbb{C}P(2n + 1)$, with real structure $\sigma[z, w] = [\bar{w}, \bar{z}]$ and complex contact form $\theta = i(z^T dw - w^T dz)$. Rational curves with normal bundle $2n\mathcal{O}(1)$ are simply the projectivisations of two-dimensional linear subspaces of $\mathbb{C}^{2n+2}$. The contact form is non-degenerate on a σ-invariant rational curve $\mathbb{P}$ if and only if we can write $\mathbb{P} = \mathbb{P}(\mathrm{span}((z, w), (\bar{w}, \bar{z}))$ with $\|(z, w)\|^2 > 0$. The resulting para-quaternionic Kähler manifold M is $\mathbb{B}P(n)$.

4.2. The positive Swann bundle

Let us return to an arbitrary para-quaternionic Kähler manifold M^{4n}. We noted above that the bundle H is only defined locally due to a $\mathbb{Z}/2$-ambiguity. This may be resolved by considering the global bundle, the *Swann bundle*, given as
$$\mathcal{U}(M) = (H \setminus 0)/(\mathbb{Z}/2),$$

where $\mathbb{Z}/2$ acts as multiplication by ± 1 on the fibre of H and 0 is the zero section. This is no longer a vector bundle, but rather has fibre

$$(\mathbb{R}^4 \setminus \{0\})/\pm 1 = \mathbb{R}P(3) \times \mathbb{R}_{>0}$$

over M.

One may define geometric structures on the total space of $\mathcal{U}(M)$ using the geometry of the frame bundle F as follows (cf. [33]). Let ϑ denote the canonical $\mathbb{B}^n$-valued one-form on F, defined by $\vartheta_u(v) = u^{-1}(\pi_* v)$ for $u \in F$, $v \in T_u F$ and where $\pi \colon F \to M$ is the projection. The Levi-Civita connection ∇ on M is represented by a connection one-form $\omega_+ + \omega_-$ on F with values in $\mathfrak{sp}(n, \mathbb{B}) \oplus \mathfrak{sp}(1, \mathbb{B})$. As ∇ is torsion-free, we have $d\vartheta = -\omega_+ \wedge \vartheta - \vartheta \wedge \omega_-$, using (1). Pull these forms back to $\tilde{F} \times \mathbb{B}$ and let $x \colon \tilde{F} \times \mathbb{B} \to \mathbb{B}$ be projection to the second factor. Writing $r^2 = x\bar{x}$ and $\alpha = dx - x\omega_-$ one finds that the $\mathrm{Im}\mathbb{B}$-valued two-forms $\alpha \wedge \bar{\alpha}$ and $x\bar{\vartheta}^T \wedge \vartheta\bar{x}$ descend first to H and then to $\mathcal{U}(M)$. If M is not (locally) hypersymplectic one has $d\omega_- + \omega_- \wedge \omega_- = c\bar{\vartheta}^T \wedge \vartheta$ for some non-zero constant c. Putting

$$\omega_I i + \omega_S s + \omega_T t = \alpha \wedge \bar{\alpha} + cx\bar{\vartheta}^T \wedge \vartheta\bar{x}$$

provides $\mathcal{U}(M)$ with three closed non-degenerate two forms away from $r^2 = 0$. This structure on $\mathcal{U}(M) \setminus \{r^2 = 0\}$ is hypersymplectic with metric $\mathrm{Re}(\alpha \otimes \bar{\alpha} + cr^2\bar{\vartheta}^T \otimes \vartheta)$. On the *positive Swann bundle*

$$\mathcal{U}_+(M) := \{\, w \in \mathcal{U}(M) : r^2(w) > 0 \,\}$$

we may write this hypersymplectic metric as

$$dr^2 + r^2(g_{\mathcal{S}^3_+} + c\pi^* g_M).$$

We see that the metric is conical and so

$$\mathcal{S}_+(M) := \{\, w \in \mathcal{U}(M) : r^2(w) = 1 \,\}$$

inherits a split three-Sasakian structure. Also, the metric on M may be recovered up to homothety from $\mathcal{U}_+(M)$ by restricting to the complement of the span of X, IX, SX and TX, where $X = r^2\partial/\partial r^2$. For $M = G/H$ a symmetric space from Table 1.1, we have that $\mathcal{S}_+(M) = G/K$, where $\mathfrak{h} = \mathfrak{k} + \mathfrak{sl}(2, \mathbb{R})$ as a direct sum of Lie algebras.

When M is real analytic, we may try to relate $\mathcal{U}(M)$ to the twistor space Z. Firstly, there is a map $\mathcal{U}(M) \to Z$ given by sending a point $\pm h$ over $x \in M$ to the α-surface through x tangent to $[h]$. This is in fact a map from the bundle $\mathbb{P}(H) \to M$ to Z. Notice that these are both complex spaces of dimension $2n + 1$. If $[s^H h] \neq [h]$, then locally the α-surface only meets M in x. Since $[h]$ is uniquely determined by $T_{[h]} = E \otimes h$ at x, this shows that the map $\mathbb{P}(H) \to Z$ is locally bijective away from real points. We remark that $\mathbb{P}(\mathcal{U}_+(M))$ is the 'twistor space' Z_+ considered for example in [25], also referred to as a 'reflector space' when the base M is four-dimensional.

5. Nilpotent orbits.

Having introduced a number of constructions related to para-quaternionic geometry, we now wish to give an application to studying and constructing examples with large symmetry groups. We first begin with a twistor-theoretic construction of a family of examples.

Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. The complexification $G^{\mathbb{C}}$ acts on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ via the adjoint action. Each adjoint orbit $\mathcal{O} = G^{\mathbb{C}} \cdot X \subset \mathfrak{g}^{\mathbb{C}}$ carries a complex symplectic form $\omega_{\mathcal{O}}$ of Kirillov, Kostant and Souriau, given by

$$\omega_{\mathcal{O}}([A, X], [B, X]) = \langle X, [A, B] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the negative of the Killing form. In the case where G is semi-simple, one may check that $X \in \mathfrak{g}^{\mathbb{C}}$ is nilpotent, i.e., $(\mathrm{ad}_X)^k = 0$ for some k, if and only if $\lambda X \in \mathcal{O}$ for all $\lambda \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$. For a fixed X this action of $\mathbb{C}^*$ on X may then be realised by a one-parameter subgroup of $G^{\mathbb{C}}$. Indeed the Jacobson-Morosov theorem gives the existence of an $\mathfrak{sl}(2, \mathbb{C})$-subalgebra containing X with standard basis $\{H, X, Y\}$ satisfying

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

The $\mathbb{C}^*$-action on X is then given by $\mathrm{Ad}_{\exp(tH/2)}$. In particular, X is itself tangent to the orbit, and we may consider

$$\theta = iX \lrcorner \omega_{\mathcal{O}}, \qquad \theta([X, A]) = i\langle X, A \rangle.$$

As $\omega_{\mathcal{O}}$ scales under the $\mathbb{C}^*$-action and is closed, we have $d\theta = i\omega_{\mathcal{O}}$ and θ descends to a complex contact form on the projectivised orbit $\mathbb{P}(\mathcal{O})$. The nilpotent elements A in the subalgebra $\mathfrak{sl}(2, \mathbb{C})$ define a rational curve $\mathbb{P}$ in the projectivised orbit $\mathbb{P}(\mathcal{O})$. Using the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$ one may show that the normal bundle of $\mathbb{P}$ is a direct sum of copies of $\mathcal{O}(1)$ [34]. The space $\mathbb{P}(\mathcal{O})$ also carries a real structure σ induced by complex conjugation in $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$.

To apply the inverse twistor construction Theorem 2 to $\mathbb{P}(\mathcal{O})$ we need real points and real twistor lines on which θ is non-zero. The real points $[A]$ are exactly those that can be represented by $A \in \mathcal{O} \cap \mathfrak{g}$ in the real subalgebra, so this intersection needs to be non-empty. As A is necessarily nilpotent and so null this forces the Killing form on $\mathfrak{g}$ to be indefinite and so G is non-compact. Once we have such a real A, we may apply the Jacobson-Morosov theorem in the real algebra $\mathfrak{g}$ to construct a real twistor line through A. The complex contact form will necessarily be non-degenerate on this line, since the Killing form is non-degenerate on the $\mathfrak{sl}(2, \mathbb{C})$-subalgebra. We thus have the following result.

THEOREM 3. *Let G be a non-compact semi-simple Lie algebra and $\mathcal{O}$ a nilpotent orbit of $G^{\mathbb{C}}$. The space $\mathbb{P}(\mathcal{O})$ is the twistor space of a para-quaternionic Kähler manifold M if and only if $\mathcal{O}$ is the complexification of a nilpotent orbit in the real algebra $\mathfrak{g}$. The Lie group G acts on M preserving the para-quaternionic Kähler structure.*

Descriptions of real nilpotent orbits may be found in e.g. [14]: for classical algebras these are given by signed Young diagrams; for exceptional algebras one uses weighted Dynkin diagrams. The final statement of the theorem, follows from the fact that the G-action preserves the twistor space data.

The above examples may be used in the study of para-quaternionic Kähler manifolds M^{4n} with semi-simple symmetry group G. Assume that G acts almost effectively. Let us say that M is *fully homogeneous* if G acts transitively on $M = G/H$ preserving the para-quaternionic Kähler structure, and the isotropy algebra $\mathfrak{h} \subset \mathfrak{sp}(n, \mathbb{B}) + \mathfrak{sp}(1, \mathbb{B})$ projects surjectively on to the second factor. The group G then acts on the frame bundle F of M, and hence on the positive Swann bundle $\mathcal{U}_+(M)$. This lifted action preserves the hypersymplectic structure of $\mathcal{U}_+(M)$. The assumption that the action on M is full implies that the G-action on $\mathcal{U}_+(M)$ is of cohomogeneity one, i.e., the largest orbits are of codimension one.

Regarding $\mathcal{U}_+(M)$ as a symplectic manifold with respect to $\omega_a = \omega_I$, ω_S or ω_T, we may find a *moment map* $\mu_a : \mathcal{U}_+(M) \to \mathfrak{g}$. The defining property of μ_a is that it is a G-equivariant map such that

$$d\langle \mu_a, X\rangle = \xi_X \lrcorner \omega_a, \tag{7}$$

where $X \in \mathfrak{g}$ and ξ_X is the vector field on $\mathcal{U}_+(M)$ generated by X. The general theory of moment maps guarantees the existence of μ_a when $\mathfrak{g}$ is semi-simple. However, in the special case of the Swann bundle one can find explicit formulæ: the map $\mu : \mathcal{U}(M) \to \mathfrak{g} \otimes \mathbb{B}$ given by

$$\langle \mu, X\rangle = -X \lrcorner (x\omega_- \bar{x})$$

may be written as $\mu = \mu_I i + \mu_S s + \mu_T t$ and one may now prove that the components μ_a, $a = I, S, T$ restrict to moment maps for the corresponding ω_a on $\mathcal{U}_+(M)$.

Consider the complex map

$$\mu^c = \mu_S + i\mu_T : \mathcal{U}(M) \to \mathfrak{g}^{\mathbb{C}}.$$

This restricts to a moment map with respect to $\omega_S + i\omega_T$ for the infinitesimal complex action of $G^{\mathbb{C}}$ on $\mathcal{U}_+(M)$. As G acts with orbits of dimension $4n + 3$ on $\mathcal{U}_+(M)$, the $G^{\mathbb{C}}$ orbits must be of real dimension $4n + 4$, since they are complex and so even-dimensional. As $G^{\mathbb{C}}$ is semi-simple, the equivariance property of μ^c ensures that μ^c has rank $4n + 4$ on $\mathcal{U}_+(M)$ and is thus an étale map from $\mathcal{U}_+(M)$ to a G-invariant open subset of an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}^{\mathbb{C}}$.

Standard theory for such moment maps shows that μ^c pulls-back the complex symplectic form $\omega_{\mathcal{O}}$ to the restriction of $\omega_S + i\omega_T$ to the $G^{\mathbb{C}}$-orbit on $\mathcal{U}_+(M)$. Since this orbit is of full dimension we have that $\mu^c|_{\mathcal{U}_+(M)}$ is locally a complex symplectomorphism.

On $\mathcal{U}_+(M)$ we have a scaling action given by multiplication by real numbers in the fibre. This clearly commutes with μ^c, so the orbit $\mathcal{O}$ is invariant under scaling and hence is a nilpotent orbit. As μ^c is equivariant the real group G acts on $\mathcal{O}$ with cohomogeneity one. It is an interesting open problem to determine the cohomogeneity of complex coadjoint orbits under the action of real forms G. In the case of G compact, much is known: if the orbit is cohomogeneity one then it is a *minimal orbit*, that is, the closure of the orbit is simply $\mathcal{O} \cup \{0\}$ and $\mathcal{O}$ is the orbit of a highest root vector; furthermore cohomogeneity decreases strictly as one passes to orbits lying in the closure of a given orbit [17]. In our situation we have some extra information that is crucial. Examining the definition of μ^c on all of $\mathcal{U}(M)$ and using the property that the action is full, shows that μ^c vanishes nowhere. Also, the real structure on $G^{\mathbb{C}}$ pulls back to multiplication by s on $\mathcal{U}(M)$. But this action has fixed points. Again examining the condition of fullness shows that there are fixed points lying in a $G^{\mathbb{C}}$-orbit of full dimension. Using a detailed understanding of the differential of μ^c one may show that $\mathcal{O} \cap \mathfrak{g}$ is non-empty.

Classifying nilpotent orbits $\mathcal{O}$ that meet $\mathfrak{g}$ and are of cohomogeneity one is a tractable problem. Work at an $X \in \mathcal{O} \cap \mathfrak{g}$, find a real $\mathfrak{sl}(2, \mathbb{C})$-subalgebra containing X and use the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathfrak{g}^{\mathbb{C}}$. One sees that the cohomogeneity is one only when $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{k}^{\mathbb{C}} + r\mathbb{C}^2$, where $\mathfrak{k}^{\mathbb{C}}$ commutes with $\mathfrak{sl}(2, \mathbb{C})$ and $\mathbb{C}^2$ is the standard representation. This is exactly the description of a minimal nilpotent orbit for $\mathfrak{g}^{\mathbb{C}}$. One finds that the group action is almost effective only if $\mathfrak{g}$ is simple. In these cases, the highest root orbits are explicitly known in $\mathfrak{g}^{\mathbb{C}}$ and standard tables list those orbits meeting a given real form. Writing $\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{k}^{\mathbb{C}}$, one finds that $(\mathfrak{g}, \mathfrak{h})$ are exactly as in Table 1.1. Given $M = G/H$ a symmetric space from Table 1.1, one may check that $\mu_c(\mathcal{U}(M))$ is the corresponding minimal nilpotent orbit with complex symplectic form $\omega_{\mathcal{O}}$. Conversely, to show that a fully homogeneous M is locally G/H with its symmetric para-quaternionic Kähler structure, it is now sufficient to verify that the hypersymplectic structure on $\mu^c(\mathcal{U}_+(M)) \subset \mathcal{O}$ is unique, since the vertical distribution of the fibration $\mathcal{U}_+(M) \to M$ is the para-quaternionic span of the scaling action. Using the cohomogeneity one property, this may be proved using the techniques of [16].

THEOREM 4. *A para-quaternionic Kähler manifold M fully homogeneous under the action of semi-simple Lie group G is symmetric and described by Table 1.1.*

These examples are analytic and their twistor spaces are the projectivisations $\mathbb{P}(\mathcal{O})$ of the corresponding nilpotent orbit.

6. Toric constructions.

In Kähler geometry a rich collection of examples may be constructed as toric varieties: these are manifolds of real dimension $2n$ whose geometry is invariant under the Hamiltonian action of an n-dimensional torus with generic orbits of dimension n. These examples have the advantage that much of their geometry is determined by combinatorics of the moment map. Often examples may be constructed as symplectic quotients of flat space $\mathbb{C}^d$ by a torus action.

In the hypersymplectic situation it is thus natural to look at which geometries may be constructed by exploiting a symmetry group acting on $\mathbb{B}^d = \mathbb{C}^{d,d}$. In §2, we already noted that there is a d-dimensional torus T^d that acts on $\mathbb{C}^{d,d}$ and this preserves the hypersymplectic structure described in §3.1. Following Guillemin [22] and Bielawski

& Dancer [11], a subtorus of N of T^d may be described as follows. Let $\{e_1, \dots, e_d\}$ denote the standard basis for $\mathbb{R}^d$. Consider a linear map

$$\beta\colon \mathbb{R}^d \to \mathbb{R}^n, \qquad \beta(e_k) = u_k,$$

for some u_k vectors in $\mathbb{R}^n$. Then $\mathfrak{n} = \ker\beta$ is a linear subspace of $\mathbb{R}^d$. Regarding the latter as the Lie algebra of T^d, we see $\mathfrak{n}$ as the Lie algebra of an Abelian subgroup N of T^d. This subgroup is closed, and hence compact, if the vectors u_k are integral, i.e., lie in the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. More precisely, we take N to be the kernel of $\exp\circ\beta\circ\exp^{-1}\colon T^d \to T^n$.

We may write the moment maps (7) for this action of N on $\mathbb{C}^{d,d}$ as follows:

$$\mu_I(z, w) = \sum_{k=1}^{d} \tfrac{1}{2}(|z_k|^2 + |w_k|^2)\alpha_k + c_1,$$

$$\mu_S + i\mu_T(z, w) = \sum_{k=1}^{d} i\bar{z}_k w_k \alpha_k + c_2 + ic_3,$$

where α_k is the orthogonal projection of e_k to $\mathfrak{n}$. The vectors c_j lie in $\mathfrak{n}$, so we may choose scalars $\lambda_k^{(j)}$ such that $c_j = \sum_{k=1}^{d} \lambda_k^{(j)}\alpha_k$. Using the description of $\mathfrak{n}$ as the kernel of β, one may show that (z, w) lies in $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_S^{-1}(0) \cap \mu_T^{-1}(0)$ if and only if there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{C}^n$ such that

$$\langle a, u_k\rangle = \tfrac{1}{2}(|z_k|^2 + |w_k|^2) + \lambda_k^{(1)}, \tag{8}$$

$$\langle b, u_k\rangle = i\bar{z}_k w_k + \lambda_k^{(c)}, \tag{9}$$

for $k = 1, \dots, d$ and $\lambda_k^{(c)} = \lambda_k^{(2)} + i\lambda_k^{(3)}$.

In principle, the work of Hitchin [23] now tells us that $M = \mu^{-1}(0)/N$ is hypersymplectic. However, there are a number of conditions that need to be satisfied to guarantee this:

(F) N should act freely and properly on $\mu^{-1}(0)$,

(S) the rank of $d\mu$ should be $3\dim\mathfrak{n}$ at each point of $\mu^{-1}(0)$,

(D) $\mathfrak{G} \cap \mathfrak{G}^\perp = \{0\}$ at each point of $\mu^{-1}(0)$,

where $\mathfrak{G}_p = \{\, X_p : X \in \mathfrak{n} \,\}$. As N is compact, the action is automatically proper, so we only need to consider freeness in condition (F). Conditions (F) and (S) guarantee that the quotient M is a smooth manifold. When (F) and (S) are satisfied, the symplectic forms ω_I, ω_S and ω_T descend to closed two-forms on M and in this case, condition (D) is equivalent to the induced forms defining a non-degenerate hypersymplectic structure. When (F) is satisfied, smoothness of the quotient follows from non-degeneracy (D).

Since conditions (F), (S) and (D) are so important for the quotient construction it is interesting that they may in fact be understood by considering combinatorics of the moment map for the residual action of $T^n = T^d/N$ on M. To describe these results, let us first consider the case of smallest dimension and take N to be the trivial subgroup.

For the action of T^1 on $\mathbb{C}^{1,1}$, $u_1 = 1$ and the left-hand sides of (8) and (9) are simply $a \in \mathbb{R}$ and $b \in \mathbb{C}$. However, $|z|^2 + |w|^2 \geqslant 2|\bar{z}w|$ implies a constraint on the possible pairs (a, b) and we find that the image of $\mu = \mu_I i + \mu_S s + \mu_T t$ is

$$\{ (a, b) \in \mathbb{R} \times \mathbb{C} : a - c_1 \geqslant |b - (c_2 + ic_3)| \};$$

a solid cone in $\mathbb{R}^3$. Considering the orbits of T^1, we see that the T^1-action is free except where $a - c_1 = 0 = b - (c_2 + ic_3)$ and that there are two T^1-orbits corresponding to (a, b) precisely when $a - c_1 > |b - (c_2 + ic_3)|$.

In general, let N be a compact Abelian subgroup of T^d and write $M = \mu^{-1}(0)/N$, as above, even when this quotient is singular. We have a map

$$\phi \colon M \to \mathbb{R}^{3n}, \qquad \phi(z, w) = (a, b),$$

where a and b are as in equations (8) and (9). When the quotient M is smooth and hypersymplectic, ϕ is simply the moment map for the action of T^n on M. However, we emphasise that ϕ is defined even when conditions (F), (S) and (D) are not satisfied.

Define

$$a_k = \langle a, u_k \rangle - \lambda_k^{(1)}, \qquad b_k = \langle b, u_k \rangle - \lambda_k^{(c)},$$

where $\lambda_k^{(c)} = \lambda_k^{(2)} + i\lambda_k^{(3)}$. As $u_1, \dots, u_d$ span $\mathbb{R}^n$, the numbers $a_1, \dots, a_d$ and $b_1, \dots, b_d$ determine a and b. Motivated by the four-dimensional example, we introduce the solid convex cones K_k, their sides W_k and 'vertices' V_k given by

$$\begin{aligned} K_k &= \{ (a, b) \in \mathbb{R}^n \times \mathbb{C}^n : a_k \geqslant |b_k| \}, \\ W_k &= \{ (a, b) \in \mathbb{R}^n \times \mathbb{C}^n : a_k = |b_k| \}, \\ V_k &= \{ (a, b) \in \mathbb{R}^n \times \mathbb{C}^n : a_k = 0 = |b_k| \}. \end{aligned}$$

For a given $x = \phi(z, w)$, let

$$J = \{ k : x \in V_k \}, \qquad L = \{ \ell : x \in W_\ell \}.$$

PROPOSITION 1. *The image of the moment map ϕ is the convex set*

$$K = \bigcap_{i=1}^{d} K_k \subset \mathbb{R}^n.$$

The induced map $\tilde{\phi} \colon M/T^n \to K$ is finite-to-one with the preimage of (a, b) containing $2^{d-|L|}$ orbits of T^n.

From this result we may obtain some first topological information about the quotient M.

THEOREM 5. *Let $M = \mu^{-1}(0)/N$ with $N \leqslant T^d$ given by integral vectors $u_1, \dots, u_d \in \mathbb{R}^n$. Then*

1. *M is connected if and only if $W_k \cap K \neq \varnothing$ for each $k = 1, \dots, d$,*
2. *M is compact if and only if the convex polyhedra*

$$\{\, s \in \mathbb{R}^n : \langle s, u_k \rangle \geqslant \lambda_k,\ k = 1, \dots, d \,\}$$

   are bounded for each choice of $\lambda_1, \dots, \lambda_d \in \mathbb{R}$.

Turning to conditions (F), (S) and (D), freeness may be determined using the techniques of Delzant [19] and Guillemin [22] for Kähler metrics on toric varieties. Since (F) and (D) imply (S), the following result suffices to determine when we obtain smooth hypersymplectic structures on M.

PROPOSITION 2. *The freeness condition* (F) *is satisfied at each $(z, w) \in \mu^{-1}(0)$ if and only if at each $x \in K$ the vectors $(u_k : x \in V_k)$ are contained in a $\mathbb{Z}$-basis for the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.*

The non-degeneracy condition (D) *fails at some point $(z, w) \in \mu^{-1}(0)$ if and only if there exist scalars $\zeta_1, \dots, \zeta_d$ not all zero and a vector $s \in \mathbb{R}^n$ such that $\sum_{k=1}^{d} \zeta_k u_k = 0$ and*

$$4\zeta_k^2(a_k^2 - |b_k|^2) = \langle s, u_k \rangle, \qquad \text{for } k = 1, \dots, d,$$

where $\phi(z, w) = (a, b)$.

The smoothness condition (S) in the presence of a locally free action of N may be stated in terms of injectivity of certain linear maps $\Lambda_{(a,b)}$ depending on $(a, b) \in K$. To be precise, for a subset $P \subset \{1, \dots, d\}$, let $\mathbb{R}_P$ be the subspace of $\mathbb{R}^d$ spanned by e_k for $k \in P$. Now write $\mathfrak{n}_{L,J}$ for the kernel of the map $\mathbb{R}_{L\setminus J} \to \mathbb{R}^n/\mathrm{Im}(\beta|_{\mathbb{R}_J})$ induced by β. Then condition (S) holds only if

$$\Lambda_{(a,b)} \colon \mathfrak{n}_{L,J} \otimes (\mathbb{R} \times \mathbb{C}) \to \mathbb{C}_{L\setminus J}, \qquad \Lambda_{(a,b)}(c_k, d_k) = (a_k d_k + b_k c_k)$$

is injective.

While these conditions may seem rather technical, they have some useful immediate consequences. For example, this last version of condition (S) holds trivially at points where L is empty. This is true of points of the *combinatorial interior* of K, which is defined to be the set

$$\mathrm{CInt}(K) = K \setminus \bigcup_{k=1}^{d} W_k.$$

THEOREM 6. *If the combinatorial interior of K is non-empty, then a dense open subset of $M = \mu^{-1}(0)/N$ carries a smooth hypersymplectic structure.*

On the other hand, there are simple situations in which conditions (S) and (D) fail. If more than $3n$ of the W_k's meet at $x \in K$ then the smoothness condition (S) can not hold at the common point. Similarly, if $n + 1$ of the W_k's meet at an $x \in K$ then condition (D) fails. This indicates that one can expect examples where the quotient is smooth, but the induced hypersymplectic structure has degeneracies. In fact, one can show that if the quotient M is compact, then the degeneracy locus is always non-empty. This occurs for example when taking the quotient by the diagonal circle in T^d.

Non-trivial non-compact examples of this construction without degeneracies may be given in all dimensions as follows. Take $d = n+1$, put $u_k = e_k$ for $k = 1, \dots, n$ and let $u_{n+1} = e_1 + \dots + e_n$. Take all the $\lambda_k^{(i)}$ to be zero apart from $\lambda^{(1)} = -\lambda < 0$. Then K is the product $\mathcal{K}_1 \times \dots \times \mathcal{K}_n \subset \mathbb{R}^{3n}$, where $\mathcal{K}$ is the solid cone in $\mathbb{R}^3$ given by $a \geqslant |b|$. As λ is strictly positive, one gets that K lies in the interior of K_{n+1} and so W_{n+1} does not meet K. One may now directly check that conditions (F), (S) and (D) hold for this configuration of $n+1$ cones in $\mathbb{R}^{3n}$ and that resulting hypersymplectic quotient M is smooth and non-degenerate. Topologically M is a disjoint union of two copies of $\mathbb{R}^{4n}$, however the induced metric is not flat.

Finally, let us mention that one may consider quotients of $\mathbb{B}^d$ by non-compact subgroups of $R^d = \{\mathrm{diag}(e^{s\phi_1}, \dots, e^{s\phi_d})\}$. In this case, condition (F) is harder to work with as one has to ensure that the group action on $\mu^{-1}(0)$ is proper. At this stage we do not know of any non-degenerate metrics arising globally from such a construction.

7. Conformal geometry.

In this section we will see that LeBrun's construction [29] of quaternionic Kähler manifolds using conformal geometry can be extended to the para-quaternionic setting starting from a far wider variety of initial geometries. More precisely, we can show:

THEOREM 7. *Let X be a real-analytic indefinite pseudo-Riemannian manifold of dimension $k+2$. Then we can construct a para-quaternionic Kähler manifold M of dimension $4k$ from X. Different conformal manifolds X give rise to distinct para-quaternionic Kähler manifolds M.*

This thus extends LeBrun's $\mathcal{H}$-space of four-manifolds [27] to higher dimensions.

The construction is modelled on the following example. Let M be the symmetric para-quaternionic Kähler manifold

$$M = \frac{SO(p+2, q+2)}{SO(2,2) \times SO(p,q)} \tag{10}$$

with $k = p+q \geqslant 2$. Geometrically M is the Grassmannian $\mathrm{Gr}_4^{2,2}(\mathbb{R}^{p+2,q+2})$ of oriented four-planes on which the inner product on $\mathbb{R}^{p+2,q+2}$ restricts to an inner product of split signature. It is an open subset in the set $\tilde{Gr}_4(\mathbb{R}^{k+4})$ of all oriented four-planes, whose boundary consists of planes on which the quadratic form degenerates. The generic boundary point is a four-plane containing one null line which is orthogonal to all lines in that four-plane. If $pq \neq 0$, there are two disjoint hypersurfaces at infinity: both are isomorphic to Grassmann bundles of three-planes that fibre over the space X of null lines in $\mathbb{R}^{p+2,q+2}$; one consists of three-planes of Lorentzian signature $(1,2)$, the other of three-planes of signature $(2,1)$. (If $pq = 0$, then only one of these hypersurfaces will occur).

The set X of null lines in $\mathbb{R}^{p+2,q+2}$ is a quadric in $\mathbb{R}P^{k+3}$, which can be identified with the pseudo-conformal compactification of $\mathbb{R}^{p+1,q+1}$ obtained by adding a null-cone at infinity. Under this identification, the natural action on X of the isometry group $SO(p+2,q+2)$ of M, corresponds to conformal transformations of $\mathbb{R}^{p+1,q+1}$.

The complexification of X is a quadric $\mathfrak{X} \subset \mathbb{C}P^{k+3}$, whose conformal structure is given by requiring the null geodesics to be the straight lines in $\mathfrak{X}$. In other words the set $\mathfrak{N}$ of null geodesics in $\mathfrak{X}$ is the Grassmannian $\mathrm{Gr}_2^0(\mathbb{C}^{k+4})$ of totally isotropic two-planes in $\mathbb{C}^{k+4}$. But the twistor space Z of M is a subset of the projectivised minimal orbit in $\mathfrak{so}(k+4,\mathbb{C})$, which can also be identified with $\mathrm{Gr}_2^0(\mathbb{C}^{k+4})$. Thus the twistor space of M is a subset of $\mathfrak{N}$.

This motivates the following construction. Let (X,h) be a real-analytic pseudo-Riemannian manifold of signature $(p+1,q+1)$, $k = p+q \geqslant 2$, and let $\mathfrak{X}$ be its complexification. The set $\mathfrak{N}$ of null-geodesics in $\mathfrak{X}$ is (under certain convexity conditions) a complex manifold of dimension $2k+1$, see [28]. The real structure on $\mathfrak{X}$ sends null geodesics to null geodesics, and so induces a real structure σ on $\mathfrak{N}$ whose fixed point set corresponds to the real null geodesics in X. Moreover, $\mathfrak{N}$ is naturally equipped with a contact structure θ induced from the canonical one-form on $T^*\mathfrak{X}$.

The set C of null geodesics through a point in $\mathfrak{X}$ which are tangent to some non-degenerate three-plane $V \subset \mathfrak{X}$ is called a *conic section*. It is a conic obtained as the intersection of a projective plane with a quadric, so will be a rational curve $\mathbb{C}P(1)$. The $(4k-1)$-dimensional set of all conic sections is a Grassmann bundle $\mathfrak{M}_0 = \mathrm{Gr}_3^t(T\mathfrak{X})$ of non-degenerate three-planes in $T\mathfrak{X}$. Although the conic sections are not twistor lines, $\mathfrak{M}_0$ is part of a $4k$-dimensional family of rational curves $\mathbb{P}$, most of which are twistor lines satisfying $\theta|_{\mathbb{P}} \neq 0$. If V is the complexification of a Lorentzian three-plane in TX, then $\sigma(C) = C$ and σ has fixed points on C corresponding to the real null geodesics. Thus there are real twistor lines in $\mathfrak{N}$ satisfying the conditions of Theorem 2. The set M of all real twistor lines is now a para-quaternionic Kähler manifold with hypersurfaces at infinity given by the Grassmann bundles $\mathrm{Gr}_3^{2,1}(TX)$ and $\mathrm{Gr}_3^{1,2}(TX)$ of TX.

Because everything in the construction agrees with the flat case to low order, the asymptotics of the metric near the hypersurface at infinity will mirror those of the model case (10). But in the latter case, we can concretely calculate the metric. The tangent space T_PM at a split-signature four-plane P can be identified with the set of linear maps from P to its orthogonal space $P^\perp$. The metric g on M is then given by $g(X,Y) = \mathrm{T}r(X^*Y)$ for $X, Y \in T_PM$, where * denotes the adjoint with respect to the induced inner products on P and $P^\perp$.

Consider the case when $p > 0$. Let t be a defining function for the hypersurface $\mathrm{Gr}_3^{2,1}(\mathbb{R}^{p+1,q+1}) \times \mathbb{R}^{p+1,q+1}$ at infinity in the model. Write the standard metric h on $\mathbb{R}^{p+1,q+1}$ as $h = h^\parallel + h^\perp$, where $h^\parallel$, $h^\perp$ denote respectively the parts parallel and perpendicular to the indefinite three-plane. Let $\hat{h}$ denote the standard symmetric metric on $Gr_3^{2,1}(\mathbb{R}^{p+1,q+1})$. Some cumbersome calculations in certain preferred coordinates then show, that the metric near infinity, i.e., where $t=0$, can be written in the form

$$g = \hat{h} - t^{-2}(dt^2 - h^\parallel + h^\perp).$$

This expression allows one to reconstruct the initial conformal structure on X from $t^2 g$. (A similar expression is valid near the other hypersurface when $q > 0$.) Hence different conformal manifolds $(X, [h])$ actually give rise to different para-quaternionic Kähler manifolds, providing a wealth of such examples.

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