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# ON SMOOTH SURFACES IN $\mathbb{P}^4$ CONTAINING A PLANE CURVE AND AN APPLICATION

**Abstract.** We consider smooth surfaces in $\mathbb{P}^4$ and we prove that, under cerain hyphotheses, this surfaces actually contain a plane curve. Then we prove that the degree of such surfaces is bounded. This yields a result on codimension two smooth subcanonical subvarieties in $\mathbb{P}^n$, $n \geq 5$ giving further evidence to Hartshorne conjecture in codimension two.

This is a short summary of the contents of two papers, see [1] and [2].
We work over an algebraically closed field of characteristic zero.
The main results are:

THEOREM 1. *Let $\Sigma \subset \mathbb{P}^4$ be an hypersurface of degree s with a (s-2)-uple plane, then the degree of smooth surfaces $S \subset \Sigma$ with $q(S) = 0$ is bounded.*

THEOREM 2. *Let $S \subset \mathbb{P}^4$ be a smooth surface with $q(S) = 0$ and lying on a quartic hypersurface $\Sigma$, such that $Sing(\Sigma)$ has dimension two, then $d = deg(S) \leq 40$.*

As an application to codim. two subvarieties in $\mathbf{P}^n$ we have:

THEOREM 3. *Let $X \subset \mathbf{P}^n$, $n \geq 5$, be a smooth codimension two subcanonicalsubvariety, lying on a hypersurface $\Sigma$ of degree $s$ having alinear subspace $K$ of codimension two and multiplicity $(s-2)$.Then $X$ is a complete intersection.*

The proofs of the above results can be found in [1] and rest on a careful inspection of the geometric set up.

The assumptions of theorems 1 and 2 may be explained by next lemma.

LEMMA 1. *If $S \subset \Sigma \subset \mathbb{P}^4$ is a smooth surface, $\Sigma$ a degree $s$ hypersurface with a (s-2)-uple plane, then $S$ contains a plane curve or $h^0(\mathcal{I}_S(2)) \neq 0$.*

From now on we suppose $h^0(\mathcal{I}_S(2)) = 0$ and thus $S$ contains a plane curve.
As for theorem 3, recall that, by Lefschetz's theorem, if $X \subset \mathbf{P}^n$, $n \geq 4$, is a codimension two subvariety contained in a hypersurface $\Sigma$, if $X$ is not a complete intersection, then $dim(X \cap Sing(\Sigma)) \geq n-4$. We then consider a very particular situation: we assume the singular locus of $\Sigma$ is as large as possible (codim. two) but the simplest possible (a linear subspace).

REMARK 1. (i) Theorem 2 is of some interest for the classification of non general type surfaces in $\mathbb{P}^4$, since it is known that such surfaces lie on hypersurfaces of low degree.

The idea of studying surfaces containing a plane curve is due to the fact that all known rational surfaces contain a plane curve (this has been observed by Catanese and Hulek). One could wonder if this can be generalized to non general type surfaces. The answer in negative, indeed there are sections of the Horrocks-Mumford bundle that do not contain any plane curve.
(ii) Theorem 3 gives further evidence to Hartshorne conjecture in codim. two.
(iii) It is easy to show that the assumption $q(S) = 0$ implies that all hyperplane sections of $S$ are linearly normal in $\mathbf{P}^3$. It follows that all hyperplane sections of $\Sigma$ have to be linearly normal too.
In the case of quartic hypersurfaces with $dim(Sing(\Sigma)) = 2$, this implies that $Sing(\Sigma)$ is a plane or a union of planes. This explains the difference between the hypotheses of theorems 1 and 2. Moreover this also explains why we did not start considering hypersurfaces with a linear subspace of codim. two and multiplicity $(s - 1)$. Indeed the $\mathbf{P}^3$ section of such hypersurfaces is not linearly normal.

Let us fix some notations. Let $S$ be a smooth surface in $\mathbb{P}^4$, let $P$ be a plane curve contained in $S$, $p = deg(P)$, and $\Pi$ be the plane containing $P$. We suppose $P$ is the 1-dim. part of $S \cap \Pi$.
We denote by $\delta$ the linear system cut out on $S$, residually to $P$, by the hyperplanes containing $\Pi$, it turns out that $\delta = |H - P|$. Let $\mathcal{B}$ be the base locus of $\delta$. We call $Y_H \in \delta$ the element cut by $H$ and $C_H = S \cap H = Y_H \cup P$.

LEMMA 2. *The curve $P$ is reduced and the base locus $\mathcal{B}$ of $\delta$ is empty or 0-dimensional and contained in $\Pi$. The general $Y_H \in \delta$ is smooth out of $\Pi$ and doesn't have any component in $\Pi$.*

*Proof.* Clearly $\mathcal{B} \subset \Pi$. Let $P_1$ be an irreducible component of $P$, $P_1 \subset \mathcal{B}$. Then for all $H$ containing $\Pi$, $C_H = H \cap S$ is singular along $P_1$. It follows that $T_x S \subset H$, $\forall$ $x \in P_1$ and $\forall$ $H \supset \Pi$ ($S$ is smooth). We get $T_x S = \Pi$, $\forall x \in P_1$, but this contradicts Zak's theorem (see [4]) which states that the Gauss map is finite. The same argument shows that $P$ is reduced. We conclude by Bertini's theorem. $\square$

REMARK 2. Since $\delta$ is a pencil and $dim(\mathcal{B}) \leq 0$, it follows that $deg(\mathcal{B}) = (H - P)^2 = d - 2p + P^2$.
It turns out also that $\mathcal{B}$ is the residual scheme of $S \cap \Pi$ with respect to $P$.

For the proof of 3 we also need the following results.

LEMMA 3. *Let $X \subset \boldsymbol{P}^5$ be a smooth subcanonical 3-fold of degree $d$, then ifd $\leq$ 25, $X$ is a complete intersection.*

LEMMA 4. *With usual notations, if $S \subset \mathbb{P}^4$ is subcanonical with $\omega_S \cong \mathcal{O}_S(a)$, $P \subset S$ a plane curve:*
*(i) $deg(P) \leq a + 3$;*
*(ii) If $\mathcal{R} = \emptyset$, then $S$ is a complete intersection.*

**References**

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