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BEYOND WARING'S PROBLEM FOR FORMS: THE BINARY DECOMPOSITION

Abstract. The study of the decomposition of a form of degree d in $n+1$ variables as the sum of forms involving $r \leq n$ variables is introduced. What is known for the sums of powers case $r=1$ is illustrated and new results are presented in the binary case $r=2$.

1. Introduction

Throughout this note S will denote the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. Given $f \in S_d$, a homogeneous form of degree d, the well known Waring's problem for forms deals with the study of expressions like: $f = {l_1}^d + \ldots + {l_s}^d$, where $l_1, \ldots, l_s \in S_1$ are linear forms. Noticing that ${l_i}^d$ is nothing more than a form in the univariate polynomial ring $\mathbb{C}[l_i]$, it is immediate to consider the following generalization: given $r \leq n$ and $f \in S_d$, study the decompositions of f of the form

$$f = f_1 + \ldots + f_s \tag{1}$$

where $f_i \in \mathbb{C}[y_{i1}, \ldots, y_{ir}]_d \subset S_d$ and the y_{ij}'s are linear forms, for $i = 1, \ldots, s$. In intuitive terms, we can consider the f_i's as forms in $n+1$ variables "essentially" involving r variables and (1) as a decomposition of f as the sum of forms involving a smaller number of variables. For $r=1$, the f_i's are just pure powers and (1) is a sums of powers decomposition of f. For $r=2$, the f_i's are called *binary forms* and (1) is called a *binary decomposition* of f. Notice that, e.g., $x_0^d + (x_1 + \cdots + x_n)^d$ is a binary form, while $\sum_{i=0}^n x_i^d$ is not a binary form for $n > 1$.

In the $r=1$ case, the sums of powers case, the decomposition (1) can be performed for any d when $n=1$ (see [2]) and for any n when $d=2$ (this is just the diagonalization of a symmetric matrix) or $d=3$ (see [3]). But there are not known algorithms for $n > 1$ and $d > 3$. In the more general case of $r \geq 2$ we can only try to exploit what we know in the sums of powers case, e.g. noticing that the sum of two pure powers is a binary form, but there are not dedicated procedures to compute (1). These remarks motivate our interest in a quantitative study of (1) and in particular in the investigation of the number of summands s. With this in mind we introduce $s_{min}(n,d) = \min\{s :$ exists (1) for a generic f in $S_d\}$, where the genericity assumption is made to have a behavior tamed enough to be studied in some generality. We can get an approximation of s_{min} by a parameters count: take general $f, f_1, \ldots, f_s$, i.e. with variable coefficients, and require the number of parameters in the left-hand side of (1) to be less or equal than the number of parameters in the right-hand side. This procedure gives an inequality and solving it we get $s_{exp}(n,d)$, i.e. the number of summands we expect in a decomposition of type (1) of a generic form of degree d in

$n+1$ variables. We remark that s_{exp} can be determined explicitly. Comparing this estimate with the number of summands appearing in the decomposition of a generic form of degree d in $n+1$ variables, it is easy to realize that $s_{min}(n,d) \geq s_{exp}(n,d)$. This inequality motivates our basic question: *(Q1) for which couples* (n,d) *does the equality* $s_{min}(n,d) = s_{exp}(n,d)$ *hold?* Question *(Q1)* addresses the problem of determining the minimal number of summands needed for the decomposition of a generic form. But, of course, there are *special* forms which can be decomposed using fewer summands. To study these forms we introduce the locus $\Sigma_s(n,d) \subseteq \mathbb{P}S_d$ of forms of degree d in $n+1$ variables which can be decomposed as in (1) using s summands. Hence it is natural to consider another question: *(Q2) what is the dimension of* $\Sigma_s(n,d)$*?* Actually, a parameters count gives an expected value for $\dim \Sigma_s(n,d)$. In this sense, question *(Q2)* is a generalization of question *(Q1)* .

In this note we will recall some known facts about the sums of powers case, $r=1$, and we will illustrate what it is known in the binary case, $r=2$ (for more details and proofs see [4]).

REMARK 1. To be rigorous, (1) represents a family of decompositions: one for each value of r. Hence we should mention r in the definitions, e.g., of s_{min} and s_{exp}. As different values of r will never appear in the same argument, we decide to keep the notation as simple as possible avoiding to mention r explicitly.

2. The sums of powers decomposition

Some particular instances of the sums of powers decomposition were classically studied by Clebsch, Darboux, London, Sylvester, Terracini and others (see [4] and the references there). Particular attention was devoted to the investigation of *defective* couples: a couple (n,d) is said to be defective if $s_{min}(n,d) \neq s_{exp}(n,d)$, i.e. if the generic form of degree d in $n+1$ variables can not be written as the sum of the expected number of powers of linear forms. A straightforward computation shows that $(n,2)$ is defective for all n and it can be shown that other defective couples exist, namely $(n,d) = (2,4), (3,4), (4,4), (4,3)$ (see [5]). All these defective couples were known since the beginning of the last century, but it was quite difficult to prove that no other defective couples exist. Finally, this was done by Alexander and Hirschowitz in 1995 (see [1]):

THEOREM 1. *A generic form of degree* d *in* $n+1$ *variables is the sum of* $s_{exp}(n,d) = \lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$ *sums of powers of linear forms, unless* $(n,d) = (2,4), (3,4), (4,4), (4,3)$ *and* $(n,2)$ *for all* n.

This Theorem completely answer to question *(Q1)* . Actually, the result by Alexander and Hirschowitz also gives a complete answer to question *(Q2)* showing that the expected behavior is the right one with few exceptions. More precisely, they determine $\dim \Sigma_s(n,d)$ which turns out to be the expected one unless few exceptions.

3. The binary decomposition

We are not aware of classical attempts to study the binary decomposition of forms and we will briefly illustrate the main results contained in [4], namely we will show what it is known for $n = 2, 3$. As before, a couple (n, d) will be said to be defective if $s_{min}(n, d) \neq s_{exp}(n, d)$, i.e. if the generic form of degree d in $n + 1$ variables can not be written as the sum of the expected number of binary forms. In the three variable case, $n = 2$, it is possible to determine a formula for s_{min} and using it we can show:

THEOREM 2. *For the binary decomposition of forms in three variables ($n = 2$), the only* <u>*not*</u> *defective couples are obtained for $d = 2, 3, 4, 5, 6$ and 8.*

We stress the sharp contrast with the sums of powers case where there are only few defective couples. The previous Theorem settles question *(Q1)* for the binary decomposition of forms in three variables. Concerning question *(Q2)* we can only show that $\Sigma_2(2, d)$ has dimension 1 less than expected for $d > 3$. The four variables case, $n = 3$, seems to be more complex and there are not known defective couples (actually it is conjectured that defective couples do not exist). In this case, the only results concern question *(Q1)* :

THEOREM 3. *For the binary decomposition of forms in four variables ($n = 3$), there are not defective couples for $d \leq 5$.*

4. Final remarks

For the purpose of this note, it is convenient to collect some interesting facts in the following remarks.

The sums of power decomposition of forms in two variables ($r = 1, n = 1$) is quite easy and question *(Q1)* was already answered by Sylvester. The binary decomposition of forms in three variables ($r = 2, n = 2$) can be successfully studied and a complete answer to question *(Q1)* is given in Theorem 2, while the problem is still open when $n > 2$. These facts suggests that the bigger the gap $n-r$ the more difficult the problem. Moreover, it is reasonable to think that the study of decompositions of type (1) for $r = n$ could be successfully attempted.

Although quite algebraic in their presentation, the decompositions of type (1) have a deep geometric nature. In the case $r = 1$, questions *(Q1)* and *(Q2)* can be naturally expressed in terms of the higher secant varieties of the Veronese $V_{n,d} = \nu_d(\mathbb{P}^n)$, where ν_d is the d-uple embedding. For example, to answer question *(Q2)* one has to determine $\dim \mathrm{Sec}^s(V_{n,d})$, i.e. the dimension of the variety of s-secant $\mathbb{P}^{s-1}$ to the Veronese. To give the same kind of geometric interpretation in the case $r = 2$, one needs to introduce the variety of binary forms $X_{n,d}$ which parameterizes the binary forms of degree d in $n + 1$ variables.

The variety of binary forms $X_{n,d}$ can be easily constructed from the Veronese $V_{n,d}$: consider a rational normal curve of degree d, $C \subset V_{n,d}$, and take its linear span $\langle C \rangle$;

making C to vary and taking the union of the linear spaces $\langle C \rangle$ one obtains $X_{n,d}$. Theorems 2 and 3 are obtained by studying $\mathrm{Sec}^s(X_{n,d})$ in terms of the Veronese, using a sort of Terracini's Lemma. It would be interesting to generalize this procedure: given a variety Y construct a new variety Z by taking "distinguished" subvarieties of Y and the union of their linear spans (this is an attempt to generalize the notion of higher secant variety where the "distinguished" subvarieties are just finite sets of points). Is there any analogous of Terracini's Lemma relating the geometry of $\mathrm{Sec}^s(Z)$ to the geometry of Y?

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