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WHITE SURFACES AND THEIR TRISECANT LINES

Abstract. This note deals with the number n of trisecant lines passing through a generic point of a White surface S. Either $n = 6$ or $n = \infty^1$ and S is a Segre polygonal surface.

Let X be a surface in $\mathbb{P}^5$ with isolated singularities. Consider the projection π_l from a generic line l in $\mathbb{P}^5$, and denote by B the image of the singular locus of X. The image of X by π_l is a surface S in $\mathbb{P}^3$, such that $X \setminus B$ has a 1-dimensional locus of double points and a finite number of triple points. Since l is chosen generically, the fiber of π_l over a triple point p of $S \setminus B$ consists of 3 distinct points p_1, p_2, p_3.

What is their possible postulation, that is to say what is the dimension of their linear span?

Modern interest for questions of this sort stems out of the work of Pink- ham and Lazarsfeld ([10],[9])which proves the Castelnuovo-Mumford regularity conjecture for smooth surfaces. One could expects that for a generic choice of l this span $< p_1, p_2, p_3 >$ has dimension 3. This can be proved to be equivalent to say that the dimension of the trisecant lines locus of S is at most 3-dimensional. The existence of surfaces X with a 4-dimensional trisecant line locus therefore provide counter-example to this naive belief. We will say that such a surface possess an excess of trisecant lines. The only smooth example known is the Ferrara surface [5], that is to say a special Enriques surface of degree 10 in $\mathbb{P}^5$. Moreover Conte and Verra [3] have shown that this characterizes speciality for Enriques surfaces embedded in $\mathbb{P}^5$ by their Fano embedding. Historically the first example of surface with an excesss of trisecant lines known is a singular surface of degree 10 constructed by Segre in 1924. Let C be a plane conic and $l_1, \ldots, l_6$ be six tangent lines to C general enough so they meet two by two in 15 points P. The linear system of quintics passing by those points P is regular and define a surface S_0 of degree 10 in $\mathbb{P}^5$, that we'll call Segre polygonal surface. This surface belongs to the family of White surfaces ([12],[6] and [7]), that is to say surfaces of P^5 image of $\mathbb{P}^2$ by the rational map ϕ associated to the linear system of quintics passing by a group P of 15 distinct points not lying on any quartic curve. White surfaces are surfaces of degree 10 with possibly isolated 4-fold singularities all image of lines joingning 5 base points. Remark also that polygonal surfaces, i.e. White surfaces for which P is the group of points two by two cut out by six lines, are projective degenerations of Enriques surfaces of $\mathbb{P}^5$. Dobler has shown in his thesis that Segre's polygonal surface is a degeneration of special Enriques surfaces and that it is the only polygonal surface with an excess of trisecant lines. It is therefore natural to ask if there exists other White surfaces with an excess of trisecant lines. To tackle this problem one should notice that a trisecant line passing by a generic point $\phi(q)$ of S corresponds to a pair of points Π for which the linear system of quintics passing by $P + \Pi$ is 1-irregular. Using ideas of Gambier [6], especially his explicit construction of pairs of points associated to a generic point one can show the following [1]

THEOREM 1. *Let S be a White surface in $\mathbb{P}^5$ and $\phi(q)$ a generic point on it.*

1. *There pass at least one trisecant line to S by $\phi(q)$.*
2. *The surface S has an excess of trisecant lines exactly if S is a Segre polygonal surface.*
3. *If S is not a Segre polygonal surface, there pass exactly six trisecant lines to S by $\phi(q)$*

Let us point out that (1) was shown by Dobler in case S is smooth and (2) was already proved in 1882 by H.Krey [8] for a generic White surface. As a corollary of (1) one can show that the generic point of the principal componant of the the Hilbert scheme of 18 points in $\mathbb{P}^2$ irregular in degree 5 corresponds to 18 points not lying on any curve of degree 4. This improves in this very particular case a result of M.A. Coppo [4].

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