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LINEAR SYSTEMS OF PLANE CURVES WITH BASE POINTS OF BOUNDED MULTIPLICITY

1. Introduction

Let $\mathcal{L} = \mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ denote the linear system of degree d plane curves with k_i base points of multiplicity m_i for $i = 1, \ldots, s$, all in general position. The virtual and expected dimensions of $\mathcal{L}$ are respectively defined to be:

$$v(\mathcal{L}) := \binom{d+2}{2} - \sum k_i \binom{m_i+1}{2} - 1 \tag{1}$$

$$e(\mathcal{L}) := \max\{v(\mathcal{L}), -1\}. \tag{2}$$

The Harbourne-Hirschowitz conjecture gives geometric meaning to when multiple base points in general position fail to impose independent linear conditions on the space of degree d plane curves; i.e., when the dimension of $\mathcal{L}$ is greater than expected. The main result of [4] is a verification of this conjecture if the multiplicities of the base points are bounded by 7.

THEOREM 1. *If $m_i \leq 7$ for $i = 1, \ldots, s$, then $\dim \mathcal{L} > e(\mathcal{L})$ if and only if its base locus of $\mathcal{L}$ contains a multiple copy of a (-1)-curve.*

Theorem 1 follows from the lemma below, which reduces the proof the theorem to a finite, but very large, number of cases. Most of these cases are handled using a computer, the rest with ad hoc methods.

LEMMA 1. *For any positive integer M, there exists $D = D(M)$ with the following property: if the Harbourne-Hirschowitz conjecture is true for all $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ with $m_i \leq M$ for $i = 1, \ldots, s$ and $d < D(M)$, then it is true for all $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ with $m_i \leq M$ for $i = 1, \ldots, s$ and all values of d.*

The table below shows the first few values of $D(M)$.

M	2	3	4	5	6	7	8	9	10	...	N
$D(M)$	9	13	17	21	25	29	34	42	51	...	$O(N^2)$

Table 12.1: Values of $D(M)$ for $M = 2, \ldots, 10$

The proof Lemma 1 is similar to the proof of Theorem 4.1 in [3], which uses a degeneration of $\mathbb{P}^2$ into a reducible surface consisting of two rational components. This yields a recursive formula for the dimension of $\mathcal{L}$. For details, see [4].

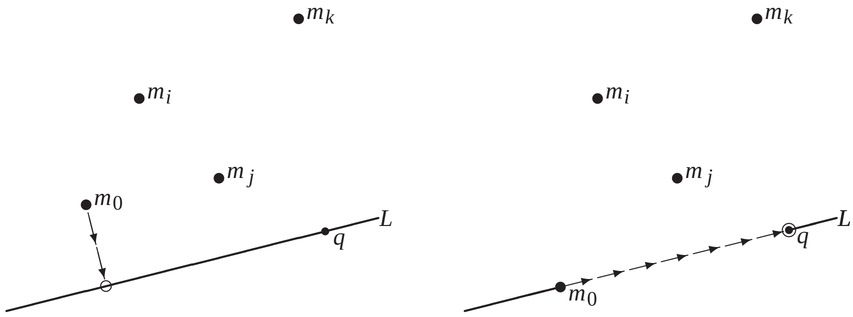

Figure 1: The two-step process of specializing the fat point m_0

2. Aligned ideals

The main algorithm used by the computer program arises from specializing the base points of $\mathcal{L}$ onto a fixed line $L \subseteq \mathbb{P}^2$, and then along the line onto a fixed point $q \in L$. After we specialize all of the base points of $\mathcal{L}$ in this manner, we are left with a linear system of plane curves with a rather exotic singularity at q, and our first goal is to describe the nature of these exotic singularities.

Choose coordinates in $\mathbb{P}^2$ such that

$$L = Z(Y), \tag{3}$$

$$q = Z(Y, Z). \tag{4}$$

In local coordinates $[x : y : 1]$, the line L is the x-axis and q is the point at infinity. Let α denote a strictly decreasing sequence of positive integers,

$$\alpha_1 > \alpha_2 > \cdots > \alpha_h > 0. \tag{5}$$

Let $\mathfrak{I}_d(\alpha)$ denote the d-th graded part of the ideal generated by the monomials $Y^i Z^{\alpha_i}$, for $i = 1, \dots, h$. Any ideal of this form will be called an *aligned ideal*.

Aligned ideals are monomial ideals by definition, but not all monomial ideals are aligned—for example, $\langle X^2, Y^2 \rangle$ is not an aligned ideal. We can visualize α by creating a $(d+1) \times (d+1)$ triangle of boxes which represent the monomial basis for degree d polynomials, with Y^d representing the box in the top corner, Z^d and X^d respectively representing the bottom left and bottom right corner boxes, and the rest of the monomials distributed among the boxes in the usual manner. The polynomials in $\mathfrak{I}_d(\alpha)$ are generated by the monomials corresponding to all but the right-most α_i boxes in the i-th row from the bottom. If we shade in the boxes corresponding to monomials which do not lie in $\mathfrak{I}_d(\alpha)$, we see the "shape" of α appearing in the bottom right corner of boxes. For example, in Figure 2, the aligned ideal $\mathfrak{I}_3(3, 2)$ consists of cubics with a cusp point at q and is generated by the monomials Y^3, $Y^2 Z$, $Y Z^2$, $X Y^2$, and Z^3.

These box diagrams are particularly convenient for denoting how an aligned ideal will change as we impose an additional m-fold point and then specialize the m-fold point

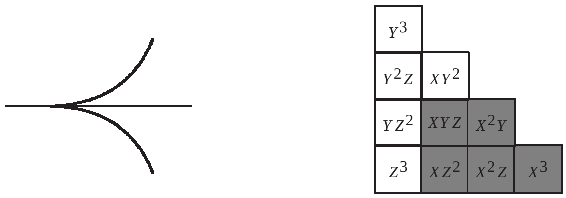

Figure 2: The aligned ideal $\mathfrak{I}_3(3, 2)$

onto L and q. The box diagram for the new aligned ideal has up to $\binom{m}{2}$ more shaded boxes than to the original one. The algorithm below exactly determines which boxes become shaded in this process.

1. Fill in the lowest $m + 1 - j$ unshaded boxes in the j-th column from the left, for $j = 1, \ldots, m$, with dots. If there are not enough unshaded boxes in a column, we use as many dots as we need and discard the rest.
2. Slide each row of dots as far to the right as possible within the white boxes.
3. Shade in the dotted boxes.

The two horizontal rows of the Figure 3 below demonstrate this algorithm performed for $m = 3$ and two different aligned ideals. In the first row, $\mathfrak{I}_4(2, 1)$ becomes $\mathfrak{I}_4(5, 3, 1)$. In the second row, $\mathfrak{I}_4(3, 2, 1)$ becomes $\mathfrak{I}_4(5, 4, 2)$ while discarding a dot in the first step.

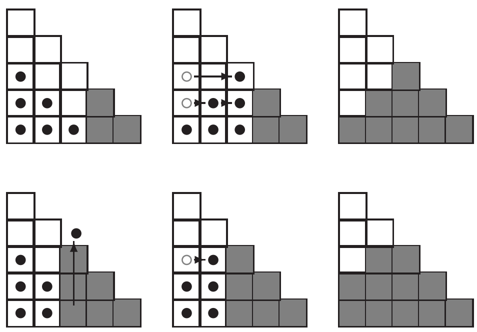

Figure 3: A couple of examples

By upper-semicontinuity, the dimension of $\mathcal{L}$ is bounded above by the dimension of the aligned ideal which arises from specializing all of the multiple points onto L and q; this is exactly one less than the number of white boxes left after we repeat the box diagram algorithm with all the multiplicities. Clearly, if it is possible to iterate the box diagram with all of the m_i (with multiplicity k_i) without losing any dots, then every multiple point imposes them maximum possible number of linear conditions, and so $\mathcal{L}$ is non-special.

3. The proof of the Theorem 1

To prove the Harbourne-Hirschowitz conjecture for $M \leq 7$, we programmed a computer to enumerate all linear systems $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ of degree 29 or less, with points of multiplicity 7 or less. There $125, 220, 076$ of these, almost all of which were shown to satisfy the Harbourne-Hirschowitz conjecture by the box diagram algorithm:

	125,220,076	total systems
-	124,850,912	are empty via the box diagram algorithm
-	366,691	are empty by Bezout's theorem (see [4])
-	2,013	contain multiple (-1)-curves in the base locus
-	418	are empty via a degeneration $\mathbb{P}^2$ (see [4])
	42	systems remain

The remaining 42 linear systems are found in Table 2 of [4]. A large number of these are either homogeneous or satisfy $d < m_1 + m_2 + m_3$ for m_1, m_2, and m_3 representing the three highest multiplicities of the points in the base locus; thus, the speciality of the linear system is equivalent to the speciality of one of lower degree via a quadratic Cremona transformation. The rest of the systems are handled case by case in [4] using elementary techniques.

References

[1] Bocci C. and Miranda R., *Topics on interpolation problems in Algebraic Geometry*, preprint.

[2] Ciliberto C. and Miranda R., *Degenerations of planar linear systems*, J. Reine Agnew. Math **501** (1998), 191–220.

[3] Ciliberto C. and Miranda R., *Linear systems of plane curves with base points of equal multiplicity*, Trans. Amer. Math. Soc. **352** 9 (2000), 4037–4050.

[4] Yang S., *Linear systems in $\mathbb{P}^2$ with base points of bounded multiplicity*, arXiv:math.AG/0406591.

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