Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, 1 (2005)
Polynom. Interp. Proceed.

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A BOUND FOR SESHADRI CONSTANT ON $\mathbb{P}^2$

Abstract. This is a summary of the talk given in Workshop on Polynomial Approximation and Projective Embeddings, Torino, September 17-18, 2003. The talk was based on the article, published electronically on 28 July 2003 in Math. Nachr. **257**.

1. Theorems

Let C be a curve in $\mathbb{P}^2$ passing through general points $P_1, \dots, P_r$ with multiplicities $m_1, \dots, m_r$. The conjecture stated by Nagata in [4] says that for $r > 9$ it holds: $d := \deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^r m_i$. The conjecture is still open except for the case when r is a square, cf. [4], [3].

The above question may be restated using Seshadri constants (cf. [2]). The multiple point Seshadri constant of the line bundle $\mathcal{O}_{\mathbf{P}^2}(1)$, is defined as follows. For $P_1, \dots, P_r$, pairwise distinct points on $\mathbb{P}^2$ we define $\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1); P_1, \dots, P_r) := \inf_C \left\{ \frac{\deg C}{\sum_{i=1}^r m_i} \right\}$, where C is a curve through $P_1, \dots, P_r$ with $\mathrm{mult}_{P_i} C = m_i$, $i = 1 \dots r$. So, Nagata's conjecture says that $\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1), P_1, \dots, P_r) = \frac{1}{\sqrt{r}}$ for general points $P_1, \dots, P_r$. We prove:

THEOREM 1. *For $P_1, \dots, P_r$, $r > 9$, general points on $\mathbb{P}^2$, we have*
$\varepsilon(\mathcal{O}_{\mathbf{P}^2}(1), P_1, \dots, P_r) \geq \frac{1}{\sqrt{r + \frac{1}{12}}}$.

The theorem given below gives an ampleness criterion, crucial in the proof of Theorem 1.

THEOREM 2. *Let $\pi : X \to \mathbf{P}^2$ be a blowing up of $\mathbb{P}^2$ in $r > 9$ general points, $P_1, \dots, P_r$, with exceptional divisors $E_1, \dots, E_r$. Let $H := \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$. Consider a line bundle on X of the form $L := dH - k \sum_{i=1}^r E_i$, where d and k are natural numbers with $d \geq 3k + 1$. Assume that $r \leq \frac{d^2}{k^2} - \frac{1}{12}$. Then L is ample on X.*

2. About proofs

Let us say a few words about the proof of Theorem 2. This proof is based upon three results. First of them is the result of Ciliberto and Miranda, [1]:

RESULT 1. Consider a linear system on $\mathbb{P}^2$ of curves of degree p passing through $r > 9$ general points with multiplicity exactly m. Then, for $m \leq 12$ such a system has the expected dimension.

The second result is by Xu, [6]:

RESULT 2. Let $P_1, \ldots, P_{r-1}, P$ be general points in $\mathbb{P}^2$ and let C be a reduced and irreducible curve of degree p passing through P_i's with multiplicities $\mathrm{mult}_{P_i} C = m_i$, for $i = 1, \ldots, r-1$ and through P with multiplicity $m \geq 2$. Then $p^2 - \sum_{i=1}^{r-1} m_i^2 - m_j \geq m^2 - m + 1$.

The next result was proved by Szemberg, [5].

RESULT 3. Let $P_1, ..., P_r$ be general points on $\mathbb{P}^2$, let a curve C be reduced, irreducible and submaximal (i.e. such that $\frac{\deg C}{\sum_{i=1}^r \mathrm{mult}_{P_i} C} < \sqrt{\frac{1}{r}}$.) Then C is almost homogeneous (i.e. all but at most one multiplicities at P_i's equal).

Having these three results, the idea of the proof is simple. As $L^2 > 0$ we have to prove that for every reduced and irreducible curve C it holds $LC > 0$. To obtain this, assume that $C = pH - \sum_{i=1}^r m_i E_i$, with $m_1 \geq ... \geq m_r \geq 0$. The first step of our proof is to check, using Result 1, that for C with $m_r > 12$, $LC > 0$ holds. Next, if for C we have $p\sqrt{r} \geq \sum_{i=1}^r m_i$, then of course $LC > 0$. Thus, assume that $p\sqrt{r} < \sum_{i=1}^r m_i$ (so C is a submaximal curve for $\mathcal{O}_{\mathbf{P}^2}(1)$). We have to check that for every such curve C, $LC > 0$; from Result 3 it follows that it is enough to consider C almost homogeneous. From Result 1 it follows that we can exclude C homogeneous. This way we are left with: $C = pH - \sum_{i=1}^{r-1} m_1 E_i - m_r E_r$ or $C = pH - \sum_{i=2}^r m_r E_i - m_1 E_1$. Analyzing carefully the cases we prove the theorem.

Now we present the proof of Theorem 1: let C be a curve of degree p passing through $P_1, ..., P_r$ with multiplicities $m_1, ..., m_r$ (and at least one m_i is greater than zero). From the definition of Seshadri constants it follows that we have to prove that $\frac{p}{\sum_{i=1}^r m_i} \geq \frac{1}{\sqrt{r+\frac{1}{12}}}$. To show this, take L as in Theorem 2, with given r and k and with minimal possible $d_k = \left\lceil k\sqrt{r+\frac{1}{12}} \right\rceil$. Then, $LC = d_k p - k\sum_{i=1}^r m_i > 0$, as L is ample. So, $p > \frac{k}{d_k} \sum_{i=1}^r m_i$. Taking the limit with $k \to \infty$ we get $\frac{p}{\sum_{i=1}^r m_i} \geq \frac{1}{\sqrt{r+\frac{1}{12}}}$.

REMARK 1. The presented bound is already meaningly improved by the recent results of Harbourne and Roé, cf. arXiv.org: math/0309064.

Acknowledgement. I would like to thank the Organizers of the 'School and Workshop on Polynomial Interpolation and Projective Embeddings' for their warm hospitality and for creative and inspiring atmosphere during the School. I would like also to acknowledge the financial support of EAGER and UJ grant DBN-414/CRBW/K-V-4/2003.

References

[1] CILIBERTO C. AND MIRANDA R., *Linear systems of plane curves with base points of equal multiplicity*, Trans.Amer.Math.Soc. **352** (2000), 4037–4050.

[2] DEMAILLY J.-P., *Singular Hermitian metrics on positive line bundles*, Lect. Notes Math. **1507**, Springer 1992, 87–104.

[3] HARBOURNE B., *On Nagata's conjecture*, J. Algebra **236** (2001), 692–702.

[4] NAGATA M., *On the 14-th problem of Hilbert*, Am. J. Math. **81** (1959), 766–772.

[5] SZEMBERG T., *Global and local positivity of line bundles*, Habilitationsshift, Essen 2001.

[6] XU G., *Ample line bundles on smooth surfaces*, J. reine angew. Math. **469** (1995), 199–209.

AMS Subject Classification: 14C20.

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