Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63, 1 (2005)
Polynom. Interp. Proceed.

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A CONJECTURE ON SPECIAL LINEAR SYSTEMS OF $\mathbb{P}^3$

Abstract. In this note we deal with linear systems of $\mathbb{P}^3$ through fat points. We consider the behavior of these systems under a cubo-cubic Cremona transformation that allows us to produce a class of special systems which we conjecture to be the only ones.

1. Introduction and statement of the problem

In what follows we assume that the ground field is algebraically closed of characteristic 0. Consider r points in general position on $\mathbb{P}^n$, to each one of them associate a natural number m_i called the *multiplicity* of the point. We will denote by $\mathcal{L} = \mathcal{L}_n(d, m_1, \ldots, m_r)$ the linear system of hypersurfaces of degree d through the r points with the given multiplicities. Define the *virtual dimension* of the system as $v(\mathcal{L}) = \binom{d+n}{n} - \sum \binom{m_i+n-1}{n} - 1$ and its *expected dimension* by $e = \max\{v, -1\}$. Observe that $e \leq \dim(\mathcal{L})$ and that the inequality may be strict if the conditions imposed by the points are dependent. In this case we say that the system is *special*. The problem of classifying special systems has been completely solved in the case $m_1 = \ldots = m_r = 2$ (see [1]) and it has been largely studied for linear systems on the plane (see [3, 4, 10]). The main conjecture on the structure of special planar systems has been formulated in [6, 7]. In this note we report about some recent results in the case of $\mathbb{P}^3$. In [8] we gave a counterexample to a conjecture (see [2]) about the structure of special linear systems of $\mathbb{P}^3$. Starting from this idea in [9] we analyzed the behavior of linear systems under a cubo-cubic Cremona transformation of $\mathbb{P}^3$. This allowed us to construct a class of special linear systems which we conjecture to be all the possible ones.
Throughout this note we will denote by X the blow up of $\mathbb{P}^3$ along the r fixed points and by $E_i \cong \mathbb{P}^2$ the exceptional divisors.
From the Riemann-Roch formula on a smooth threefold, we obtain that $v(\mathcal{L}) = (\mathcal{L}(\mathcal{L} - K_X)(2\mathcal{L} - K_X) + c_2(X)\mathcal{L})/12$. If the linear system can be written as $\mathcal{L} = F + \mathcal{M}$, where F is a fixed divisor of $\mathcal{L}$ and $\mathcal{M}$ is the residual system, then the above formula implies:

$$v(\mathcal{L}) = v(\mathcal{M}) + v(F) + \frac{F\mathcal{M}(\mathcal{L} - K_X)}{2}. \tag{1}$$

All the results described in this note can be found in [9].

2. Cubic Cremona transformations and linear systems

It is possible to consider the behavior of linear systems under a birational transformation of $\mathbb{P}^3$. In particular we need to consider a transformation which sends linear systems through points into systems through points. Consider the system $\mathcal{L}_3(3, 2^4)$; by putting the four double points in the fundamental ones, the associated rational map is $\mathrm{Cr} : (x_0 : x_1 : x_2 : x_3) \to (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1})$. This birational map induces the following action on a linear system $\mathcal{L}$:

$$\mathrm{Cr}\,(\mathcal{L}) = \mathcal{L}_3(d + k, m_1 + k, \ldots, m_4 + k, m_5, \ldots, m_r), \tag{2}$$

where $k = 2d - \sum_{i=1}^{4} m_i$. By using this transformation, it is easy to see that if $2d < m_1 + m_2 + m_3$, then the plane through the first three points is a fixed component of the system. Observe that $\dim \mathrm{Cr}\,(\mathcal{L}) = \dim \mathcal{L}$ but in general the virtual dimensions of the two systems may be different as stated in the following:

PROPOSITION 1. *Let $\mathcal{L}$ be a linear system such that $2d \geq m_i + m_j + m_k$ for any choice of $\{i, j, k\} \subset \{1, 2, 3, 4\}$ then*

$$v(\mathrm{Cr}\,(\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \binom{1 + t_{ij}}{3} - \sum_{t_{ij} \leq -2} \binom{1 - t_{ij}}{3}, \tag{3}$$

where $t_{ij} = m_i + m_j - d$.

In particular this implies that $v(\mathrm{Cr}\,(\mathcal{L})) \geq v(\mathcal{L})$ if the degree of $\mathrm{Cr}\,(\mathcal{L})$ is smaller than the one of $\mathcal{L}$. This means that as far as $2d < m_1 + m_2 + m_3 + m_4$ we can perform a Cremona transformation decreasing the degree and the multiplicities of the system. If at some step we get a system such that $2d < m_1 + m_2 + m_3$, then we remove the plane from the base locus. After a finite number of steps we get a system with $2d \geq m_1 + m_2 + m_3 + m_4$, which we say to be in *standard form*.

3. Conjecture

To each linear system $\mathcal{L}$ we associate the 1-cycle $\Gamma(\mathcal{L}) := \sum_{t_{ij} \geq 1} t_{ij} l_{ij}$, where $t_{ij} = m_i + m_j - d$ and l_{ij} is the line through p_i and p_j. Observe that by definition $H^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = H^0(\mathcal{L})$, since each line $l_{ij} \in \Gamma(\mathcal{L})$ is contained in $\mathrm{Bs}\,(\mathcal{L})$ with multiplicity at least t_{ij}.

PROPOSITION 2. *The relation between the Euler characteristic of the two sheaves is given by:*

$$\chi(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = \chi(\mathcal{L}) + \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3}.$$

This implies that for each $\Gamma = \sum \alpha_{ij} l_{ij}$ with $2 \leq \alpha_{ij} \leq t_{ij}$ we have

$$\dim \mathcal{L} - v(\mathcal{L}) \geq \sum \binom{\alpha_{ij}+1}{3} - h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma).$$

It is possible to prove that if Γ is just a multiple line then $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) = 0$ and the system is special.

The preceding discussion gives us a class of special systems in standard form. We can construct another such class in the following way. Let $Q = \mathcal{L}_3(2, 1^9)$ be a quadric through nine points and suppose that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$. By formula 1 we have that $v(\mathcal{L}) < v(\mathcal{L} - Q)$ which implies that $\mathcal{L}$ is special. By assuming the Harbourne-Hirschowitz conjecture to be true for planar systems with 10 points, it is possible to prove that if $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$ then Q is a fixed component of $\mathcal{L}$.

CONJECTURE 2. A linear system $\mathcal{L}$ in standard form is special if and only if one of the following holds:

(i) there exists a quadric Q such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$;

(ii) at least one of the coefficients of $\Gamma(\mathcal{L})$ is bigger then 1.

Assuming that this conjecture and the Harbourne-Hirschowitz (for systems through 10 points) are true we can remove all the quadrics of step (i) from the base locus of $\mathcal{L}$. Then the residual system $\mathcal{L}'$ is still in standard form and

$$\dim \mathcal{L} = v(\mathcal{L}') + \sum_{t'_{ij} \geq 2} \binom{t'_{ij}+1}{3}$$

assuming that $h^2(\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')}) = 0$.

We conclude this note with two propositions about *homogeneous* linear systems, i.e. the systems $\mathcal{L}$ for which $m_1 = \ldots = m_r = m$.

PROPOSITION 3. *The system $\mathcal{L}$ is empty for $d \leq 2m - 1$ and $r \geq 8$.*

By assuming Conjecture 2 and Harbourne-Hirschowiz conjecture for linear systems on $\mathbb{P}^2$ with 10 points, we can also prove the following:

PROPOSITION 4. *If $d \geq 2m$ the system $\mathcal{L}$ is special if and only if $r = 9$ and $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$.*

Therefore if the system $\mathcal{L}$ has more than 9 fixed points (or exactly 8 points) then it is not special. If it has 9 fixed points, it is special if and only if d satisfies the inequalities of the preceding proposition. If $r \leq 7$ and $d \geq 2m$, the system can not be special. Finally, if $r \leq 7$ and $d \leq 2m - 1$, by applying a finite number of Cremona transformations we reduce to a system in standard form.

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AMS Subject Classification: 14C20.

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