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# LECTURES ON THE STRUCTURE OF PROJECTIVE EMBEDDINGS

**Abstract.** In this we draw a picture of the "status of the art" in the theory of defective varieties, i.e. varieties whose secant spaces fill up a variety of dimension smaller than expected. Some links between the theory of defective varieties and other fields of algebraic geometry and mathematics are outlined. Several open problems and current researches on the subject are presented.

## 1. Introduction

These notes are concerned with projective algebraic geometry. They are not self contained: at least they do not start with the general definition of what a general projective variety is. For our scopes, it will be enough to say that a **projective variety** $X$ is just a subset of a projective space $\mathbb{P}^r$ defined by the vanishing of a set of **algebraic equations** (= homogeneous polynomials). For deeper details, we refer the reader to the massive literature on the subject. Let us just mention some classical books [36], [72], [46], [45], [47], whose foundations and methods are used freely through these lectures.

So, we aim to present some properties of the variety $X \subset \mathbb{P}^r$ and we agree that, from now on, our $X$ is **irreducible** (in the topological sense), **reduced** (a technical assumption, for which we refer to [46]) and furthermore we assume that $X$ is **non-degenerate**, which means that it lies in no proper hyperplane. Also, in order to avoid fuzzy behaviour with non–standard base fields, let us agree that everything is defined over the complex field $\mathbb{C}$. We advise that the theory may be considerably different (but not meaningless!) in positive characteristic or over non algebraically closed fields, like $\mathbb{R}$.

At the very beginning of this notes, for a while, let us consider the variety $X$ as an abstract object and its plongement in the projective space as a map $\iota : X \to \mathbb{P}^r$. When is $\iota$ sufficiently good? Of course it must be an embedding: roughly speaking, no pair of distinct points should be glued together. Generalizing this fact, sets of distinct points should go to independent sets, at least as soon as this is possible (this is the *secant* point of view). There is a differential version of this principle (the *tangent* point of view): sets of tangent vectors applied at distinct points should remain independent, when we replace $X$ with its image in $\iota$, as soon as this is compatible with the fact that we arrive into $\mathbb{P}^r$.

There are other characterization of what a *good plongement* is. Nevertheless all of them prove to be strictly connected each other. One of the main connections between the tangent and the secant point of view is the celebrated Terracini's lemma, the cornerstone of our theory.

It turns out that **defective** varieties, i.e. varieties whose embedding is not "as good as possible", are indeed special from many points of view. What catches our attention is the considerable number of remarks that prove how such defective varieties appear in several fields of Mathematics, even outside Geometry, in many unexpected manners.

The aim of these notes is to present the theory of defective varieties, starting from the geometric point of view, and its several connections with many geometrical problems, with interpolation, decomposition of products, and so on.

As the thory is nowadays rapidly developing in many directions, the account of the actual situation outlined here is necessarily partial and (hopefully!) immediately obsolete. On the other hand we believe that marking some fixed points in the theory could be useful as a reference for people who wants to approach the study of defective varieties, and also to suggest how one can broaden the range of their applications.

## 2. Secant varieties

### 2.1. The linearization problem

It is even difficult to determine a starting point for an introduction to the theory of secant varieties. At least because there are so many approaches, each one valid from some point of view, that the choice of a beginning for the tale seems rather arbitrary. Here we present the problem from the point of view of consecutive partial linearizations of a (non-linear) projective variety $X$.
So let us start with $X \subset \mathbb{P}^r$, which, once forever, is an irreducible, reduced, non–degenerate projective variety. Since $X$ is not contained in any proper hyperplane, the linear span of $X$ is $\mathbb{P}^r$ itself. So, except for the trivial case $X = \mathbb{P}^r$ (which we exclude hereafter), the variety $X$ is not linear.

A **partial linearization** is obtained by adding to $X$ all the points which are linearly spanned by points of $X$. We obtain in this way an increasing hierarchy of subsets of $\mathbb{P}^r$, which eventually end up with the whole projective space ($\mathbb{P}^r$ is obtained, at least, taking all points spanned by $r + 1$ points of $X$. In fact, we will see that it is obtained much earlier).

These subsets turn out to be quasi-projective, highly ruled varieties, whose structure gives important information on the projective embedding of $X$. For an application, let us just say by now that they are linked with the *hidden variables* of parameter spaces or with the decomposition of tensors. From our point of view, they represent natural intermediate steps between $X$ and the ambient space, in the sense of linear algebra, which is the ancestor of all our geometric investigations.

Our first easy exercise points out that the hierarchy is effective, unless $X$ is trivial.

EXERCISE 1. If $X \neq \mathbb{P}^r$, then $X$ cannot contain all the lines spanned by pairs $A, B \in X$.
If $X$ contains all the quoted lines, then it also contains any point $P$ spanned by a triple $A, B, C \in X$: indeed if $P = aA + bB + cC$, then the point $Q = bB + cC$ belongs to the line spanned by $B, C \in X$, hence it lies in $X$. But then observe that $P$ belongs to the line spanned by $A, Q$. Thus $P \in X$.

Going on by induction, it turns out that any point of $\mathbb{P}^r$ which is spanned by $k$ points of $X$ (any $k$) actually lies in $X$. But we yet observed that, since $X$ is non–degenerate, then $r+1$ general points of $X$ span $\mathbb{P}^r$.

Now we are ready to introduce the main objects of our investigation.

DEFINITION 1. *For any non–negative integer k define the $k$-**secant variety** $S_k(X)$ of $X$ to be the (reduced) closure of the set:*
$\{P \in \mathbb{P}^r : P$ *lies in the span of* $k+1$ *independent points of* $X\}$

We take the closure (in the Zariski topology) because we want $S_k(X)$ to be a projective variety itself. Indeed, even before taking the closure, the set we get is locally algebraic, because, roughly speaking, is obtained moving $k+1$ points of $X$ (which is algebraic) and taking the linear span, which is an algebraic procedure.
A more precise argument is contained in the following exercise, which should not be avoided also from those who are satisfied with the previous heuristic procedure, because it introduces some methods extensively used in the sequel.

EXERCISE 2. The set $\{P \in \mathbb{P}^r : P$ lies in the span of $k+1$ independent points of $X\}$ is locally algebraic in $\mathbb{P}^r$.
We need to introduce the formalism of Grassmannians. Let us indicate with $G(k,r)$ the Grassmannian of linear subspaces of (projective) dimension $k$ in $\mathbb{P}^r$. $G(k,r)$ is an algebraic variety, and we have an algebraic map

$$sp : U \to G(k,r)$$

(the *span map*) defined over the open subset $U$ which parametrizes independent $(k+1)$-tuples of points in the cartesian product $\mathbb{P}^r \times \cdots \times \mathbb{P}^r$ ($k+1$ times). The image of $sp$ restricted to $U \cap (X \times \cdots \times X)$ is thus a locally closed subset $Y$ of $G(k,r)$. Now consider the *incidence variety*

$$I(k,r) = \{(P,H) \in \mathbb{P}^r \times G(k,r) : P \in H\}$$

with the two projections $p_1 : I(k,r) \to \mathbb{P}^r$, $p_2 : I(k,r) \to G(k,r)$. Then $p_2^{-1}(Y)$ is algebraic, and $\{P \in \mathbb{P}^r : P$ lies in the span of $k+1$ independent points of $X\}$ coincides set-theoretically with $p_1(p_2^{-1}(Y))$. Hence it is locally closed. (The unexpert reader is strongly recommended to follow accurately these easy steps).

Using the formalism of the previous exercise, we may add some notation.

We indicate with $G_k(X)$ the *closure* of the image of $U \cap (X \times \cdots \times X)$ in $G(k,r)$, under the span map. So:

$G_k(X) =$ closure of $\{H \in G(k,r) : H$ is spanned by $k+1$ points of $X\}$.
$G_k(X)$ has no standard official name. We refer to it as the *k-th Grassmann secant variety* of $X$.

The inverse image $p_2^{-1}(G_k(X))$ is usually called the Grassmann *abstract* $k$-secant variety of $X$.

Finally let us set:

$$s_k(X) := \dim(S_k(X))$$

Of course the consecutive secant varieties of $X$ define a chain:

$$X \subset S_1(X) \subset \cdots \subset S_r(X) = \mathbb{P}^r \tag{1}$$

Exercise 1 essentially says that $X \neq S_1(X)$, unless $X = \mathbb{P}^r$. We may generalize it for higher secant spaces.

EXERCISE 3. The inclusions $S_k(X) \subset S_{k+1}(X)$ in (1) are proper, unless $S_k(X) = \mathbb{P}^r$.
Assuming that $S_k(X) = S_{k+1}(X)$, one shows $S_{k+1}(X) = S_{k+2}(X)$, thus by induction $S_k(X) = S_r(X) = \mathbb{P}^r$. Indeed if $P \in S_{k+2}(X)$, then $P$ is (a limit of) some sum $P = a_0P_0 + \cdots + a_{k+2}P_{k+2}$, with $P_i \in X$. But the point $Q = a_0P_0 + \cdots + a_{k+1}P_{k+1}$ belongs to $S_{k+1}$, hence by assumption it is (the limit of) a sum $b_0Q_0 + \cdots + b_kQ_k$, $Q_i \in X$. Then $P$ is a limit of a sum $b_0Q_0 + \cdots + b_kQ_k + a_{k+2}P_{k+2}$, hence it belongs to $S_{k+1}(X)$.

Consequently we have the **secant dimensional sequence** associated to $X$:

$$n = \dim(X) < s_1(X) < \cdots < s_k(X) < \cdots < s_K(X) = r \tag{2}$$

in which we implicitly define $K$ as the minimal integer for which $S_K(X) = \mathbb{P}^r$. This integer is an interesting invariant of the embedded variety $X$, to which we will refer as the **linearization constant** of $X$.

### 2.2. First motivations and examples

Let us play now an interlude. We feel urged to present here some example of interaction between the machinery introduced so far and everyday's mathematics.

A good starting point are those projective varieties which naturally represent tensor products. The most famous are Veronese varieties, Grassmannians and Segre products.

EXAMPLE 1. The **Veronese variety** $V(n, m)$ is defined in $\mathbb{P}^N$, $N = \binom{m+n}{n}$, as the set of points with homogeneous coordinates $(a_0^m, a_0^{m-1}a_1, a_0^{m-1}a_2, \ldots, a_n^m)$ for any choice of $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$. It corresponds to an algebraic image of $\mathbb{P}^n$ under a proper map, so it is an algebraic irreducible (and also smooth) subvariety of $\mathbb{P}^N$.

If one identifies $\mathbb{P}^n$ with the set of all linear forms in $n + 1$ variables (modulo scalar multiplication), then $\mathbb{P}^N$ can be identified with the set of forms of degree $m$ and $V(n, m)$ represents the subset of forms which decompose in a product of linear pieces.

With this representation in mind, one understands that the constant $K$ for $X = V(n, m)$ is the minimum such that a *general* form of degree $m$ in $n + 1$ variables is a linear combination of $K + 1$ completely decomposable forms.
One may appreciate this interpretation of $K$ comparing with the classical constants

introduced in number theory by Fermat and Waring, which compute the number of factors needed to express any large number as a sum of $m$-th powers.
We mention at this stage that a complete list of the values of $K$ for all pairs $n, m$, although intensively studied by classical geometers, has been obtained only (relatively) recently by Alexander and Hirschowitz (see [3]).

EXAMPLE 2. We do not introduce **Grassmannians** here: we just want to stress that $G(h, s)$, embedded via Plücker relations in $\mathbb{P}^N$, $N = \binom{s+1}{h+1} - 1$, is the analogue of the Veronese varieties, when the symmetric product is replaced by the wedge product.

Thus the constant $K$ for $X = G(h, s)$ is the minimum such that a *general* vector in $\Lambda^{h+1}\mathbb{C}^{s+1}$ is a linear combination of $K+1$ completely decomposable wedge products.

A complete list of the values of $K$ for all pairs $h, s$ is yet unknown! We have the list for $h = 1$, plus some uncomplete results for higher dimensional vector spaces. (see [12] for a survey).

EXAMPLE 3. Continuing in the path, consider the **Segre products** $S(a, b)$, defined in $\mathbb{P}^{ab+a+b}$, as the set of points with homogeneous coordinates $(x_0y_0, x_0y_1, \dots, x_ay_b)$ for any choice of $(x_0, \dots, x_a), (y_0, \dots, y_b) \neq (0, \dots, 0)$. It corresponds to an algebraic image of $\mathbb{P}^a \times \mathbb{P}^b$ under a proper map, so it is algebraic.

If one identifies $\mathbb{P}^{ab+a+b}$ with the set of all matrices $(a + 1) \times (b + 1)$ (modulo scalar multiplication), then $S(a, b)$ represents the subset of matrices which decompose in a product of two vectors, hence have rank 1. The constant $K$ represents here the minimum such that a *general* matrix is a linear combination of $K+1$ rank one matrices. Its value is well-known for any $a, b$.

The situation becomes more complicate if we iterate the product process, considering the generalized Segre variety $X = S(a_1, \dots, a_m)$. In this case, the ambient space where $X$ lies represents the set of $m$-dimensional boxes (= tensors) of type $(a_1 + 1) \times \cdots \times (a_m + 1)$ and the Segre products represents the equivalent of "rank 1" boxes. So $K$ turns out to be here the minimum such that a *general* $m$-box is a linear combination of $K + 1$ boxes of rank 1. Worthless to say that very few facts are known about the value of $K$ for general $a_1, \dots, a_m$ (refer to [13]).

EXAMPLE 4. Now the reader can imagine several variations on the theme. We can mix the procedures obtaining tensors made with symmetric, alternating and general products. All these multilinear procedures give rise to huge projective spaces where decomposable objects are represented by algebraic subvarieties, for which the determination of the linearizing constant $K$ is relevant.

One may go further, taking weights for the variables involved in the products. One gets weighted projective spaces, which are realized as subvarieties of higher dimensional "ordinary" spaces, and still the linearization problem has considerable application for them. Also, one may consider some combinatorial subvarieties of the objects introduced in our previous examples, like Schubert cycles in Grassmannians or sparse tensors and ladders.

Our knowledge of the values of $K$ for all these varieties is faint.

EXAMPLE 5. Even for general algebraic varieties $X \subset \mathbb{P}^r$, the knowledge of the constant $K$ may have interesting applications.
Just to mention one of them: some probabilistic problems become easier to work with if one adds some "hidden variables" to the random variables directly involved in the process. In geometric terms, one plugs the original parameter space in a wider ambient space, as a non–linear subvariety $X$. Then the new points obtained taking linear combinations of the original ones must enter in the description of the process. These new points realize our secant varieties.

Coming back to our geometric birthplace, we cannot escape mentioning the applications of secant varieties to the simplification of the ambient space of the variety $X$.

EXAMPLE 6. For a given $X \subset \mathbb{P}^r$, one can try to realize the variety in a tinier projective space, with the most natural procedure of consecutive projections. However, one cannot expect for free that the projection from a general point $P \in \mathbb{P}^r - X$ sends isomorphically $X$ to $\mathbb{P}^{r-1}$. First of all, it is clear that if $P$ belongs to the line spanned by $A, B \in X$, then $A, B$ are patched together in the projection. Thus when $P \in S_1(X) - X$, the projection from $P$ is not one–to–one. It turns out that, in fact, the projection from $P \in \mathbb{P}^r - X$ is isomorphic to $X$ if and only if $P \notin S_1(X)$ (this is essentially due to the fact that tangent lines to $X$ lie in $S_1(X)$. See [46], §2.7).
Thus $X$ can be projected isomorphically to $\mathbb{P}^{r-1}$ from a general point of $\mathbb{P}^r$ exactly when $K > 1$.

EXAMPLE 7. Generalizing the previous procedure, if $P \notin S_2(X)$, then for any triple of independent points $A, B, C \in X$, the plane spanned by $A, B, C$ in $\mathbb{P}^r$ is sent to a plane in $\mathbb{P}^{r-1}$ under the projection from $P$. Thus $A, B, C$ remain independent after the projection.
It turns out that when $K > 2$, under the projection from a general point of $\mathbb{P}^r$, $X$ does not acquire any new trisecant line (of course, "old" trisecant lines are stable under projection). And one may easily generalize this principle for all $K$.

Our last observation, although elementary, is worthy of an explicit statement:

REMARK 1. $K$ is the maximum such that the projection from a general point $P \in \mathbb{P}^r$ preserves the independence of any set of $K + 1$ points of $X$.

### 2.3. A dimension contest

Trying to extimate the dimension of the objects introduced in section 2.1, one realizes immediately that it is easy to find, in any case, an upper bound.

Starting with $G_k(X)$, it is clear that it corresponds to (the closure of) an algebraic image of (an open dense subset of) $X \times \cdots \times X$ ($k + 1$ times). Hence:

$$\dim(G_k(X)) \leq \min\{(k+1)n, \dim(G(k,r))\} = \min\{kn + n, (k+1)(r-k)\}$$

where, as usual, we use $n$ for the dimension of $X$. We prove that equality always holds.

Let us show first two exercises on the connections between projections and secant varieties.

EXERCISE 4. Let $X$ be an irreducible, non–degenerate variety in $\mathbb{P}^r$, $X \neq \mathbb{P}^r$. Then the fibers of projection of $X$ from a general point $P \in X$ (internal projections) are generically finite.
This is an immediate consequence of exercise 1: the contrary would mean that $X$ contains a general secant line.

Observe that, indeed, in the exercise one could invoke the celebrated **trisecant lemma**, which says that the general internal projection is even birational. Since its validity is restricted to characteristic 0, and we do not need its full strength at this level, we preferred to give a direct argument.

EXERCISE 5. Let $X$ be an irreducible, non–degenerate variety in $\mathbb{P}^r$, $X \neq \mathbb{P}^r$. Fix a proper subvariety $Y \subset X$, of dimension $m$ and take $k < r - m$. Then a general subset of $k+1$ points on $X$ spans a $k$-space disjoint from $Y$.
Indeed everything is obvious when $k = 0$. For $k = 1$, take the projection of $X$ from a general point $P$. The image has dimension $n$ (by exercise 4) and does not coincide with the image of $Y$. This means exactly that the general secant line to $X$, passing through $P$, does not meet $Y$. The general case works similarly, by induction (details are left to the reader).

PROPOSITION 1. *$G_k(X)$ is irreducible, of dimension* $\min\{kn+n, (k+1)(r-k)\}$. *In particular, as soon as $k \leq r-n$, the intersection of $X$ with a general $(k+1)$-secant $k$-space is finite.*

*Proof.* The irreducibility is obvious, since $X$ is irreducible, so $X^{k+1}$ is, and the closure of an algebraic image of any dense open subset of $X^{k+1}$ is irreducible as well.
For the dimension, first assume $n > r - k$; then any $k$-plane $\pi$ meets $X$ in (at least) a curve; moving generically $k+1$ points of this curve, we see that $\pi \in G_k(X)$. It follows that $G_k(X)$ is the whole Grassmannian, hence its dimension is $(k+1)(r-k)$.
Assume now $n + k \leq r$ and assume $\dim G_k(X) < kn + n$. Since $G_k(X)$ is generically the image of a map $\phi_k : X^{k+1} \to G(k,r)$, and $\dim(X^{k+1}) = kn + n$, then necessarily $\phi$ cannot have finite fibers. Notice that the points of the fiber over some $\pi \in G_k(X)$ are contained in the intersection $\pi \cap X$. This is clearly impossible for $k = 1$: here $\pi$ is a line, hence it would coincide with the fiber, which means that any secant line to $X$ lies in $X$, i.e. $S_1(X) \subset X$, which contradicts the conclusion of exercise 1.
For general $k$, a similar contradiction is obtained by induction. Consider indeed a general point $P_0 \in X$ and let $X'$ be the image of $X$ in $\mathbb{P}^{r-1}$, under the projection from $P_0$. As above, this projection has finite general fibers. In particular $\dim(X') = n$, hence $k - 1 < (r-1) - \dim(X')$. Call $Y \subset X'$ the proper subset where the fibers are not finite. It follows by induction that the span $\pi$ of $k$ general points of $X'$ meets it in a finite set. Furthermore, the previous exercise implies that $\pi$ misses $Y$. Since $\pi$ is the

image of a general $k$-space, $(k+1)$-secant to $X$, the conclusion follows. □

Let us now turn to the dimension of secant varieties.

The incidence variety over the Grassmannian $G(k,r)$ is a $\mathbb{P}^k$-bundle. Thus the abstract secant variety, which corresponds to the bundle restricted to $G_k(X)$, has always dimension bounded by $\min\{(k+1)(r-k)+k, kn+n+k\}$.

$S_k(X)$ is the closure of an algebraic image of the abstract secant variety. Furthermore it lies in $\mathbb{P}^r$.

We get immediately:

COROLLARY 1. *The secant varieties $S_k(X)$ are irreducible.*

When we consider the dimension $s_k(X)$ of $S_k(X)$, we get another story. $s_k(X)$ is bounded by:

$$s_k(X) \le \min\{r, (k+1)n+k\}.$$

**The main point of all our lectures relies on the fact that the previous inequality may be strict.**

Examples will be discussed soon. The easiest one is the Veronese surface $V(2,2) \subset \mathbb{P}^5$ (in the notation of example 1), for which $s_1 = 4$. This has the amazing consequence that we need more squares of linear forms than expected (i.e. 3 instead of 2) to generate a general (homogeneous) form of degree 2 in 3 variables.

We arrive thus at the following basic:

DEFINITION 2. *We say that the variety $X$ is $k$-***defective** *when*

$$s_k(X) < \min\{r, (k+1)n+k\}.$$

*The difference $\delta_k = \min\{r, (k+1)n+k\} - s_k(X)$ is referred to as the $k$-th* **defect** *of $X$. We say that $X$ is defective when it is $k$-defective for some $k$. It is* **minimally** *$k$-defective if it is $k$-defective but not $(k-1)$-defective.*

In conclusion, it is implicit in the previous terminology that we will consider as *good* any embedding of $X$ in $\mathbb{P}^r$ for which the *secant varieties* have the *maximal dimension,* compatibly with the ambient space.

Let us remark an easy characterization of defectiveness, via the computation of secant spaces through points.

EXERCISE 6. $X$ is $k$-defective if and only if $S_k(X) \neq \mathbb{P}^r$ and the general point $P \in S_k(X)$ lies in infinitely many $(k+1)$-secant $k$-spaces.
Indeed $X$ is $k$-defective when the map from the $k$-th abstract secant variety to $\mathbb{P}^r$ is not surjective and its general fiber is not finite.

A proper map $Z \to Z'$ of projective varieties has **maximal rank** when it is either surjective, or it has finite general fibers. With this notation, $X$ is $k$-defective exactly when the map from the Grassmann abstract $k$-th secant variety to $\mathbb{P}^r$ is not of maximal rank.

EXERCISE 7. Assume that $X$ is $k$-defective and $S_{k+1}(X) \neq \mathbb{P}^r$. Then $X$ is also $(k+1)$-defective. Furthermore $\delta_k \leq \delta_{k+1}$.
Indeed since $X$ is $k$-defective, then it is contained in a positive-dimensional family.

We warn the reader that all the previous definition are standard in the recent literature, *except* for the definition of defect, for which we are going to see some variations.

### 2.4. Further exercises

EXERCISE 8. Prove that no variety is 0-defective.

EXERCISE 9. Prove that the invariant $K$ introduced in (2) is always smaller than $r-1$.

EXERCISE 10. For any $X$ and for any $a, b$, prove that $S_{a+b+1}(X) = S_a(S_b(X))$.

EXERCISE 11. Assuming that no Veronese variety is defective (which, by the way, is completely **false**!), extimate the minimal number of powers of linear forms necessary to generate a general form of degree $d$ in $m+1$ variables.

EXERCISE 12. Use the properties of rational cubics to prove that a general cubic form in 2 variables is combination of two cubes of linear forms.

EXERCISE 13. Find a quadratic form in 3 variables which is not a combination of two squares of linear forms. (It does not contradict the previous exercise!)

EXERCISE 14. Use Castelnuovo's formula for the genus of curves, to prove that curves in $\mathbb{P}^3$ are never defective.

## 3. The infinitesimal approach

### 3.1. Terracini's lemma

We said in the previous section that our main task concerns the computation of the dimension of secant varieties. Up to now, except for very particular cases, we found very few examples where the computation was effective.

Since secant varieties are irreducible (by corollary 1) and reduced (by construction), a possible way to compute consists in determining the dimension of the tangent space to $S_k(X)$ at a general point.

This is a general reduction step in Geometry: try to reduce the original problem in a problem concerning linear objects (as tangent spaces are).

The starting point in our reduction thus relies in determining the tangent space to secant varieties.

The result has been obtained by Terracini, at the beginning of last century. It not only opens the path for the computation of the invariants $s_k(X)$, but also provides a beautiful link between our setting and an apparently different characterization of *good* embeddings of projective varieties, with connections with the interpolation problem, inverse systems, algebraic products, and more.

Roughly speaking, the idea behind Terracini's lemma is simple. If $u \in S_k(X)$ is a general point, belonging to the linear span of the points $P_0, \ldots, P_k \in X$, then a tangent vector to $S_k(X)$ at $u$ can be interpreted as a direction in which $u$ can be moved infinitesimally, leaving it *inside* $S_k(X)$, i.e. in the span of some set of $k+1$ points of $X$, infinitesimally near the $P_i$'s. It should correspond then to an infinitesimal movement of the $P_i$'s inside $X$, hence to a set of tangent vectors to $X$ at the $P_i$'s.

Before stating the lemma, let us introduce some useful pieces of notation.

DEFINITION 3. *For $P \in X$, we indicate with $T_P(X)$ the tangent space to $X$ at $P$. Brackets $<,>$ will be used to denote linear spans.*
*For $P_0, \ldots, P_k \in X$, we abbreviate $< T_{P_0}, \ldots, T_{P_k} >$ with $T_{P_0,\ldots,P_k}$.*

THEOREM 1. *(**Terracini's lemma** see [74]) Let $u$ be a general point of $S_k(X)$, belonging to the span $< P_0, \ldots, P_k >$ of the points $P_0, \ldots, P_k \in X$. Then the tangent space of $S_k(X)$ at $u$ is given by:*

$$T_u(S_k(X)) = T_{P_0,\ldots,P_k}. \tag{3}$$

*Proof.* We deeply use the assumption that we work in characteristic 0. Here we refer to the theorem of generic smoothness (see [46] III.10.5) to conclude that, for general points, the tangent space to $S_k(X)$ at $u$ is the image of the tangent space to the abstract secant variety at some preimage $(\pi, u)$ of $u$. Here $\pi$ is a $k$-space which meets $X$ at $P_0, \ldots, P_k$. For the same reason, the tangent space to $G_k(X)$ at $\pi$ is an image of the product $T_{P_0}(X) \times \cdots \times T_{P_k}(X)$.
The abstract secant variety is a $\mathbb{P}^k$ bundle over $G_k(X)$, which can be trivialized around $\pi$ by taking an identification for points in different linear spaces. A geometric way to do that uses a (general) projection of $\pi$ and the neighbouring $k$-spaces, from some fixed space $Q$ to a fixed $k$-space and identifies matching points.
So the tangent space to the abstract secant variety at $(\pi, u)$ decomposes in a sum $T \oplus T'$, where $T$ is the tangent space to $G_k(X)$ at $\pi$ and $T'$, the vertical space, corresponds canonically to the tangent space to $\pi$ and is mapped in $\mathbb{P}^r$ to $\pi$ itself. Now to prove the statement, it is enough to show that the composition:

$$T_{P_i}(X) \to T \to T \oplus T' \to \mathbb{P}^r$$

maps $T_{P_i}(X)$ to a space $T_i$ which, together with the line $L_i =< u, P_i >$, spans $< u, T_{P_i}(X) >$. Indeed it follows that $T_u(S_k(X))$ is spanned by $T_{P_0,\ldots,P_k}(X)$ and $< P_0, \ldots, P_k, u >= \pi$, but observe that, clearly, $\pi$ is yet contained in $T_{P_0,\ldots,P_k}(X)$.
In order to prove the claim, say for $i = 0$, fix the points $P_1, \ldots, P_k$ and take a tangent vector $\tau$ to $P_0$. Then $\tau$ implies a movement of the $k$-space $\pi$ which defines, via the

projection from $Q$, a movement of $u$ which is contained, for elementary reasons, in the span of $\pi$ and $\tau$. □

The unexpert reader will be surprised by the amount of consequences that such a natural result has and the consequences it spreads among apparently different fields of geometry.

Just to begin with, observe that the computation of the span of tangent spaces to $X$ is sometimes easy to perform, and allows us to show that some classical varieties are defective.

EXERCISE 15. A hyperplane $H$ contains the tangent space to $X$ at a smooth point $P$ if and only if the intersection $X \cap H$ has a singular point at $P$.
Indeed by assumption the tangent space to $T_P(X)$ has dimension $n = \dim(X)$ and the tangent space to $X \cap H$ at $P$ is $H \cap T_P(X)$. Moreover $P$ is singular in $H \cap X$ if and only if the tangent space to $T_P(X \cap H)$ has dimension greater than $\dim(X \cap H) = n - 1$. This clearly happens if and only if $H$ contains $T_P(X)$. □

EXAMPLE 8. Let us use Terracini's lemma to prove that the Veronese variety $V(2, 2) \subset \mathbb{P}^5$ of example 1 is 1-defective.
We need to prove that $s_1(X) = \dim S_1(X)$ is smaller than 5. Take a general point $u \in S_1(X)$ and assume it lies a line $L$ which meets $X$ at two points $P, Q$. Then, by Terracini's lemma, the tangent space to $S_1(X)$ at $u$ is the span $T_{P,Q}$ of the two tangent spaces to $X$ at $P, Q$. Since the span of linear spaces is linear, $\dim T_{P,Q} < 5$ if and only if there exists a hyperplane $H$ containing $T_{P,Q}$. By exercise 15 this happens exactly if and only if there exists a hyperplane $H$ with $H \cap V(2, 2)$ singular at $P, Q$.
Now, $V(2, 2)$ is the image of $\mathbb{P}^2$ in the map defined by the complete system of quadratic forms, which means that the intersections of $V(2, 2)$ with the hyperplanes of $\mathbb{P}^5$ are the images of plane conics. If $P_0, Q_0 \in \mathbb{P}^2$ are pre–images for $P, Q$, then the double line defined by $P_0, Q_0$ is a conic which is singular at $P_0, Q_0$ (in fact this is the unique conic with this property). Hence it corresponds to a hyperplane section of $X$ which is singular at $P, Q$. The proof is established: $\dim(T_{P,Q}) = 4 < 5$.

EXAMPLE 9. A similar computation shows that the Veronese variety $V(3, 2) \subset \mathbb{P}^9$ is 1-defective.
We need to prove that $s_1(X) = \dim S_1(X)$ is $2 \cdot 3 < 7$. Take a general point $u \in S_1(X)$ and fix points $P, Q \in X$ such that $u$ lies in the line $< P, Q >$. The tangent space to $S_1(X)$ at $u$ is the span $T_{P,Q}$ of the two tangent spaces to $X$ at $P, Q$. In order to prove that it is a linear space of dimension 6, it is sufficient to compute the dimension of the linear system of hyperplanes which contain it. Taking pre-images $P_0, Q_0 \in \mathbb{P}^3$ of $P, Q$, then we need to extimate the system of quadrics in $\mathbb{P}^3$ singular at $P_0, Q_0$. This is not, in general, an elementary task, as we shall discuss in a while.
In our specific case, however, observe that quadrics singular at $P_0, Q_0$ are singular along the line $L =< P_0, Q_0 >$, hence they split in a couple of planes along $L$. So the system has (projective) dimension 2. It turns out that $T_{P,Q}$ is contained in a system of hyperplanes of projective dimension 2, so it has dimension $9 - 3 < 7$, as claimed.

There are some features in the previous example, which will pop up continuosly in our computations. The first one is stressed here only for reference convenience:

REMARK 2. A linear subspace $\pi \subset \mathbb{P}^r$ has dimension $x$ if and only if it is contained in a linear system of hyperplanes of projective dimension $r - x - 1$.

The second one leads to a link between our theory of projective embeddings and interpolation on algebraic varieties. The principle is:
$T_{P_0,\dots,P_k}$ *is small exactly when there are many divisors, in the linear system cut on* $X$ *by the hyperplanes of* $\mathbb{P}^r$*, which are singular at the* $P_i$*'s.*

Next section explores the connection in more details.

### 3.2. Interpolation on varieties

Let $D$ be any divisor of the irreducible variety $X$. We use the symbol $|D|$ to indicate the *complete* linear system defined by $D$, considered either as a projective space, or as a collection of divisors.

The **interpolation problem** for any linear system $V \subset |D|$ consists in the study of the subsystem of divisors $E \in V$ which pass through fixed points $P_0, \dots, P_k$ with pre–assigned multiplicities $m_0, \dots, m_k$. We use the notation:

$$V(-m_0 P_0 - \cdots - m_k P_k)$$

for this subsystem. The first question we have concerns its dimension.

EXERCISE 16. When the points $P_0, \dots, P_k$ are general and all the multiplicities are 1, then the system $V(-P_0 \cdots - P_k)$ has dimension $\dim(V) - k - 1$.
Indeed notice that $V(-P_0 \cdots - P_k)$ has dimension one less than $V(-P_1 \cdots - P_k)$, except than when $P_0$ is a fixed point $V(-P_1 \cdots - P_k)$, which cannot happen when $P_0$ is general. □

A similar statement fails as soon as the $m_i$'s increase. There are varieties for which the dimension of $V(-m_0 P_0 \cdots - m_k P_k)$ is bigger than the expected one. This may happen even for the system $V$ of hyperplane sections of a smooth variety, and even if we have only one point involved! (see [77]).

For our matter, we are concerned with the case of interpolation for general points of multiplicity two and for the linear system of hyperplane sections.

Indeed, the linear system $V$ determines a map

$$\phi_V : X \to \mathbb{P}^r$$

where $r = \dim(V)$. Now, when $\phi_V$ is birational, after replacing $X$ with its image (does not matter if it is not an *isomorphic* image, for we are concerned with general points), the system $V$ becomes the system of hyperplane sections of $X \in \mathbb{P}^r$ and $V(-2P_0 \cdots - 2P_k)$ is exactly the subsystem of all hyperplane sections which are singular at the $P_i$'s.

EXERCISE 17. Let $V$ be a system of hyperplane sections for the reduced variety $X \subset \mathbb{P}^r$, of dimension $n$. Then the interpolation problem for one general point $P_0$ and multiplicity two, has always the same answer: $\dim(V(-2P_0)) = \dim(V) - (n+1)$. This is a direct consequence of exercise 15: the set of divisors in $V$ which are singular at $P_0$ corresponds to the set of hyperplanes containing $T_{P_0}(X)$. Since $P_0$ is smooth in $X$, then $\dim(T_{P_0}(X)) = n$ and we conclude. □

On the other hand, imposing singularity at two general points $P_0, P_1$, then we cannot immediately conclude that $\dim(V - 2P_0 - 2P_1) = \dim(V) - 2(n+1)$, as one would expect. The answer is indeed related to the dimension of the span $T_{P_0,P_1}(X)$, as explained in exercise 15.

It turns out that, discussing Terracini's lemma, we yet met a situation where the interpolation problem has an unexpected answer.

EXAMPLE 10. Take the Veronese variety $X = V(2,2) \subset \mathbb{P}^5$ and call $V$ the system of hyperplane sections. Fix two general points $P, Q \in X$. Then, by example 8, we know that the span of the two tangent spaces $T_{P,Q}(X)$ has dimension 4 and it determines one hyperplane. Thus $V(-2P_0 - 2P_1)$ has dimension 0, which is bigger than $\dim(V) - 2(n+1) = 5 - 2 \cdot 3 = -1$.

Let us state directly the following:

PROPOSITION 2. *On the variety $X \subset \mathbb{P}^r$ consider the linear system $V$ of hyperplane sections. Then $X$ is $k$-defective if and only if, for general points $P_0, \ldots, P_k \in X$, one has:*

$$\dim(V(-2P_0 - \cdots - 2P_k)) > \min(-1, r - (n+1)(k+1)).$$

*In this case, the difference* $\dim(V(-2P_0 - \cdots - 2P_k)) - \min(-1, r-(n+1)(k+1))$ *is the $k$-th defect of $X$.*

In interpolation theory, the linear system $V(-2P_0 - \cdots - 2P_k)$ is *special* when the previous inequality holds. So $X$ is $k$-defective if and only if $V(-2P_0 - \cdots - 2P_k)$ is special.

PROPOSITION 3. *Assume that $X$ is $k$-defective, with $k$-defect $\delta_k$. Assume $S_{k+1}(X) \neq \mathbb{P}^r$. Then $X$ is also $(k+1)$-defective.*
*If moreover $r \geq (k+2)n + k$, then the $(k+1)$-th defect $\delta_{k+1}$ is greater or equal than $\delta_k$.*

*Proof.* By assumptions and by Terracini's lemma, if $P_0, \ldots, P_k \in X$ are general points, then the span $T_{P_0,\ldots,P_k}$, which is the tangent space at a general point of $S_k(X)$, has dimension $\min(r, (k+1)n + k) - \delta_k$. Hence adding one general point $P_{k+1}$, the space $T_{P_0,\ldots,P_k,P_{k+1}}$, which is the span of $T_{P_0,\ldots,P_k}$ and $T_{P_{k+1}}$, has dimension at most $\min(r, (k+1)n+k) - \delta_k + n + 1$. This last number, by assumptions, is smaller than $r$, while it is clearly smaller than $(k+2)n + k + 1$. So $X$ is $(k+1)$-defective.

For the second assertion, observe that in our case

$$\delta_k = (k+1)n + k - \dim(T_{P_0,\dots,P_k})$$

$$\delta_{k+1} = (k+2)n + k - \dim(T_{P_0,\dots,P_k,P_{k+1}})$$

and the conclusion follows. □

The proposition justifies the reduction we will make in the study of defective varieties. Namely we are focused only on minimally defective ones.

COROLLARY 2. *The smallest integer $K$ such that $S_K(X) = \mathbb{P}^r$ is also the smallest $K$ such that, for general points $P_0, \dots, P_K$, the hyperplane linear system satisfies*

$$V(-2P_0 - \cdots - 2P_K) = \emptyset.$$

### 3.3. Further examples and exercises

We collect in this section examples of defective varieties, discovered using Terracini's lemma. Some of them are trivial and their proofs are left to the reader as an exercise.

EXERCISE 18. Use the fact that no variety is 0-defective to prove that the tangent space at a general point has the same dimension as $X$.

EXERCISE 19. A cone $X$ of dimension 2 over a curve is 1-defective, as soon as $r \geq 5$.
Indeed the tangent spaces at two general points meet at the vertex of the cone, so they span at most a $\mathbb{P}^4$. □

EXERCISE 20. Generalize the previous example:
Any $n$-dimensional cone $X$ with vertex of dimension $m$ is 1-defective, when $r > 2n - m$.

The converse of the previous statement depends on the cone we work with:

EXERCISE 21. Find a cone $X$ as above, which is 1-defective even if $r = 2n - m$.

PROPOSITION 4. *Let $X$ be the cone over a variety $Y$, with vertex $W$. Fix general points $P_0 \dots, P_k$ in $X$ and let $Q_0, \dots, Q_k \in Y$ be their images in the projection from $W$. Then $T_{P_0,\dots,P_k}(X)$ is the span of $W$ and $T_{Q_0,\dots,Q_k}(Y)$.*

*Proof.* Indeed, for one points $P_0$, by generic smoothness ([46] III.10.5) the tangent space $T_{P_0}(X)$ maps onto $T_{Q_0}(Y)$ in the projection from $W$, hence $T_{P_0}(X) = < W, T_{Q_0}(Y) >$ for reasons of dimension. The general statement follows from elementary projective arguments. □

EXERCISE 22. Let $X \in \mathbb{P}^r$ be the cone over the variety $Y \subset \mathbb{P}^s$, with vertex at the linear space $W$ of dimension $r - s - 1$. Then $S_k(X) = \mathbb{P}^r$ if and only if $S_k(Y) = \mathbb{P}^s$.

EXERCISE 23. Let $Z \subset \mathbb{P}^r$ be the cone over a variety $Y \subset \mathbb{P}^s$, with vertex $W$ of dimension $r - s - 1$. Assume that $X \subset \mathbb{P}^r$ is an irreducible, non–degenerate subvariety of $Z$, which surjects onto $Y$ in the projection from $W$. Put $n = dim(X)$, $m = dim(Y)$. Pick a positive integer $k$ with $(n - m)(k + 1) > r - s$ and assume that $S_k(Y) \neq \mathbb{P}^s$. Then $X$ is $k$-defective.
Indeed if $P_0, \dots P_k$ are general points of $X$, which thus project to general points $Q_0, \dots, Q_k \in Y$ then $T_{P_0,\dots,P_k}(X)$ is contained in $T_{P_0,\dots,P_k}(Z)$. This last space does not cover $\mathbb{P}^r$, by assumption, and it has dimension at most $m(k+1)+k+(r-s)$ by the previous proposition. The conclusion follows since our numerical hypothesis implies $m(k + 1) + k - (r - s) < n(k + 1) + k$. □

This last exercise explains that many defective varieties are found as subvarieties of cones. It is not easy in general to determine all the subvarieties of the cones over a given $Y$. However, taking at least complete intersections, we find some non elementary class of defective objects.

EXERCISE 24. Let $Q$ be a smooth quadric in $\mathbb{P}^3$. Let $X$ be the image of $Q$ in the Veronese map which takes $\mathbb{P}^3$ to the variety $V(3, 2) \subset \mathbb{P}^9$. Then $X$ is not 1-defective but it is 2-defective.
Indeed first observe that the system $V$ of hyperplane sections of $X$ corresponds to the system cut by quadrics on $Q$. Hence $X$ spans $\mathbb{P}^8$ and indeed it corresponds to a hyperplane section of $V(3, 2)$.
Given two general points $Q_0, Q_1 \in Q$, calling $P_0, P_1$ their images in $X$, the system $V(-2P_0 - 2P_1)$ corresponds to the system of quadric sections of $Q$ which are singular at the $Q_i$'s. These sections are cut by quadrics $Q'$ which are tangent to $Q$ at the $P_i$'s. On the other hand, if $Q, Q'$ are tangent at $Q_0, Q_1$, i.e. they have the same tangent plane at the two points, then after replacing $Q'$ with a suitable linear combination $aQ + bQ'$, the equations of the tangent planes of $Q'$ at the $Q_i$'s vanish. Thus we may assume that our system is cut on $Q$ by quadrics which are singular at the $Q_i$'s. We yet computed the dimension of quadrics with two singular points: it is 2. Hence $V(-2P_0 - 2P_1)$ has dimension $2 = r - 2(n + 1)$, so $X$ is not 1-defective.
On the other hand, adding one general point $Q_2 \in Q$, which maps to $P_2$, then we have a quadric section of $Q$ which is singular at $Q_0, Q_1, Q_2$: it is the intersection of $Q$ with the plane $< Q_0, Q_1, Q_2 >$ counted twice. So $\dim(V(-2P_0 - 2P_1 - 2P_2)) = 0 > r - 3(n + 1) = -1$ and $X$ is 2-defective. □

EXERCISE 25. Generalize the previous example:
Let $Q$ be a smooth quadric in $\mathbb{P}^m$, $m > 2$. Let $X$ be the image of $Q$ in the Veronese map which takes $\mathbb{P}^m$ to the variety $V(m, 2) \subset \mathbb{P}^M$, $M = (m^2 + 3m + 2)/2 - 1$. Then $X$ is $(m - 1)$-defective.

The following lemma will be useful for many applications:

LEMMA 1. (**Linear lemma**) *Any set of m-planes in $\mathbb{P}^r$, such that any two of them meet in a $(m-1)$-plane, either is contained in some fixed $\mathbb{P}^{m+1}$ or has a $(m-1)$-plane for base locus.*

*Proof.* Call $H$ the $(m+1)$-plane spanned by two elements $L'$, $L''$ of the family. Assume that some $L$ in the set does not lie in $H$. Then $L$ meets $H$ in a $(m-1)$-plane, since it meets $L'$, $L''$ in a $(m-1)$-plane, and $L \cap H = L' \cap L''$. Any element of the set contained in $H$ must then contain $L \cap H$. Hence $L' \cap L''$ is the base locus. □

EXERCISE 26. When $X \subset \mathbb{P}^r$ has codimension 2, it is not defective.
We show indeed that $S_1(X) = \mathbb{P}^r$. Assume that $S_1(X) \neq \mathbb{P}^r$. Then for two general points $P, Q \in X$ we have $T_{P,Q} \neq \mathbb{P}^r$. So $T_P(X)$ and $T_Q(X)$ must meet in dimension $n-1$. Consider now the set of all tangent spaces to $X$ (at smooth points). By the Linear lemma, either they are contained in a fixed hyperplane (a contradiction, since $X$ spans $\mathbb{P}^r$) or they contain a fixed linear space $H$ of dimension $n-1$. But this last conclusion cannot hold in characteristic 0. Indeed otherwise, taking a general section of $X$ with a linear space $W$ of dimension 3, we get a curve $X \cap W$ which is irreducible ([46] III.7.9.1) and whose tangent lines all pass through the point $W \cap H$, which contradicts [46] IV.3.9. □

A generalization of the Linear lemma would be useful for our purposes. Unfortunately it seems rather hard and involved. See [55] for classical results on the subject.

Our last topic of this section concerns Grassmannians. In order to study secant varieties and tangent spaces to Grassmannians, let us recall the following characterization of the tangent spaces to a Grassmannian of lines at a point.

EXERCISE 27. In the Grassmannian of lines $G(1,s)$ fix a point $L$ and fix a $s-2$-space $\pi$ in $\mathbb{P}^s$ with $\pi \supset L$. The Schubert cycle $S_\pi$, of lines which meet $\pi$ is the intersection of $G(1,s)$ (in its Plücker embedding in $\mathbb{P}^r$, $r = (s^2+s-2)/2$) with a hyperplane tangent to $G(1,s)$ at $L$.
$S_\pi$ is clearly a hyperplane section of $G(1,s)$, so it is enough to prove that $L$ is a singular point of $S_\pi$. Fixing $L_\epsilon$ infinitesimally close to $L$, one uses linear algebra to show that the conditions for $L_\epsilon$ to meet $\pi$ are less than the expected value $(2s-3)-\dim(S_\pi)$. □

We refer to the hyperplane sections $S_\pi$ as above as *Schubert tangent sections* and the corresponding hyperplanes as *Schubert tangent hyperplanes*.

EXERCISE 28. The Grassmannian $G(1,s)$ is $k$-defective for all $k$ with $2k+1 \leq s-2$ and $(4s-2)k \geq s^2-s-2$.
Indeed fix $k+1$ general lines $L_0,\dots,L_k$ in $\mathbb{P}^s$. They span a linear subspace $\pi$ of dimension $2k+1 \leq s-2$. Fix a $(s-2)$-space $\Pi$ containing $\pi$. The Schubert cycle of lines meeting $\Pi$ represents a hyperplane section of $G(1,s)$ which is singular at all $L_i$'s. Hence $T_{L_0,\dots,L_k}(G(1,s)) \neq \mathbb{P}^r$. On the other hand, if $n = \dim(G(1,s)) = 2s-2$, then by assumptions $nk+n+k \geq r$. It follows $\dim(S_k(X)) < \min(r, nk+n+k) = r$ and the claim is proved. □

In particular, the Grassmannian of lines in $\mathbb{P}^5$ is 1-defective.

EXERCISE 29. It is highly unelementary to extend, with this method, similar result on Grassmannians of planes and higher dimensional varieties (but see [12]). The reader is invited to study the Grassmannians $G(2, s)$ of planes, for $s \leq 8$.

### 3.4. Tangential projections

Terracini's lemma suggests an interpretation of defective varieties using general tangential projections of varieties. It allows us to explain the hierarchical nature of defects and leads to an easy understanding of some results in the projective embedding of varieties.

We start stating explicitely another easy consequence of the theorem of generic smoothness:

LEMMA 2. *Let $X \subset \mathbb{P}^r$ be an irreducible variety and let $L$ be a linear space of dimension $s$, not containing $X$. Let $Y$ be the image of $X$ (with the reduced structure) in the projection from $L$ to some fixed general $\mathbb{P}^{r-s-1}$. Fix a general point $P \in X$ and call $Q$ its image on $Y$. Then the tangent space to $Y$ at $Q$ is the projection of $T_P(X)$ from $L$.*
*In particular, if $P$ is general, then:*

$$\dim(Y) = \dim(X) - \dim(T_P(X) \cap L) - 1.$$

*Proof.* Just see [46] III.10.5 and [46] III.10.4. □

We are going to use the result with $L$ = general tangent space to $X$. Observe that since $X$ is non–degenerate, it cannot contain $X$.

COROLLARY 3. *$X$ is 1-defective if and only if a general tangential projection sends $X$ to a variety $X_1 \subset \mathbb{P}^{r-n-1}$ which is different from $\mathbb{P}^{r-n-1}$ and has dimension smaller than $n = \dim(X)$.*

*Proof.* Fix general points $P_0, P_1 \in X$ and put $L = T_{P_0}(X)$, $Q$ = image of $P_1$ in the projection from $L$ ($Q \in X_1$). We know by Terracini's lemma that $X$ is 1-defective if and only if the span $T_{P_0,P_1}(X)$ does not fill $\mathbb{P}^r$ and has dimension smaller than $2n+1$. The latter condition means that $T_{P_1}(X)$ meets $L = T_{P_0}(X)$ at least at some point. Hence $X$ is 1-defective if and only if $T_Q(X_1)$ does not fill $\mathbb{P}^{r-n-1}$ and has dimension smaller than $n$. Since $Q$ is general in $X_1$, the claim follows. □

The previous result suggests a new way for a definition of defect:

DEFINITION 4. *Assume that $X$ is 1-defective. Call 1-st* **projection defect** *of $X$ the difference $p_1(X) = \dim(X) - \dim(X_1)$, $X_1$ being a general tangential projection of $X$.*

We warn the reader that the projection defect may be different from our former defect in some cases, namely when $r < 2n + 1$. In general we have:

EXERCISE 30. Prove that $X_1$ is non–degenerate in $\mathbb{P}^{r-n-1}$.

EXERCISE 31. Let $X$ be 1-defective, with defect $\delta_1$ and projection defect $p_1$. Then:

$$\delta_1 = p_1 - \max(0, 2n + 1 - r).$$

Indeed call $m$ the dimension of $T_{P_0,P_1}(X)$ for a general choice of the points in $X$ ($2n + 1 \geq m > n$). The number $2n - m$ computes the dimension of the intersection between $T_{P_0}(X)$ and $T_{P_1}(X)$, so that, as above, $\dim(X_1) = n - \dim(T_{P_0}(X) \cap T_{P_1}(X)) - 1 = m - n - 1$.
Now if $X$ is defective, then $2n - m < r$ and $p_1 = 2n - m + 1$. On the other hand $\delta_1 = 2n + 1 - \min(r, 2n + 1) - m$. The conclusion follows.

Of course, using successive tangential projections, one has a pattern for a definition of successive projection defects. We give new pieces of notation.

Given the variety $X = X_0$, write $\pi_0 : X \to X_1$ for a general tangential projection. Taking a general tangential projection $X_1 \to X_2$, write $\pi_2 : X \to X_2$ for the composition of two general tangential projections. Continuing with the same procedure, define a sequence of tangential projections:

$$\pi_k : X = X_0 \to X_1 \to \cdots \to X_{k-1} \to X_k.$$

The process stops when $X_k$ is a linear space.

EXERCISE 32. $\pi_k$ is the projection of $X$ from the space $T_{P_0,\ldots,P_{k-1}}(X)$ for a general choice of the points $P_0, \ldots, P_{k-1} \in X$.

EXERCISE 33. If we set $m = \dim(T_{P_0,\ldots,P_{k-1}}(X))$, then $X_k$ is non–degenerate in the space $\mathbb{P}^{r-m-1}$. Hence $X_k$ is a linear space if and only if it coincides with $\mathbb{P}^{r-m-1}$.

EXERCISE 34. $X_k$ is a linear space if and only if $T_{P_0,\ldots,P_k}(X) = \mathbb{P}^r$.
$X_k$ is non–degenerate. Hence $X_k = \mathbb{P}^s$ is a linear space if and only if $s = r - m - 1$, with $m = \dim(T_{P_0,\ldots,P_{k-1}}(X))$. This happens if and only if the tangent space to $X_k$ at a general point $Q$ coincides with $\mathbb{P}^s$. This means that its pull–back in $\pi_k$ coincides with $\mathbb{P}^r$. Now just notice that if $P_k$ is a (general) pre-image of $Q$ in $X$, then the pull–back coincides with $T_{P_0,\ldots,P_k}(X)$.

COROLLARY 4. *$X$ is $k$-defective if and only if $\pi_k$ sends $X$ to a variety $X_k$ which is not linear and has dimension smaller than $n = \dim(X)$.*

*Proof.* Induction on $k$, the case $k = 1$ being corollary 3.
If $X_k = \mathbb{P}^s$ is a linear space, the previous example shows that it is not $k$-defective.
So assume that $X_k$ is not linear, and use induction. $\dim(X_k) < \dim(X)$ is equivalent to say that $\dim(T_Q(X_k)) < \dim(X)$, for a general point $Q \in X_k$. Taking a general

pre–image $P_k \in X$ for $Q$, one obtains:

$$\dim(T_{P_0,\dots,P_k}(X)) < \dim(T_{P_0,\dots,P_{k-1}}(X)) + n + 1 \leq nk + n + k$$

Since moreover $T_{P_0,\dots,P_k}(X)$ is not $\mathbb{P}^r$, then $X$ is $k$-defective in this case.

Assuming on the contrary $\dim(X_k) = \dim(X)$, then one gets $\dim(X_{k-1}) = n$ so that $X_{k-1}$ is not defective, by induction. Since it is not linear, then $\dim(T_{P_0,\dots,P_{k-1}}(X)) < r$, hence $\dim(T_{P_0,\dots,P_{k-1}}(X)) = n(k-1) + n + k - 1$. Thus:

$$\dim(T_{P_0,\dots,P_k}(X)) = \dim(T_{P_0,\dots,P_{k-1}}(X)) + n + 1 = nk + n + k$$

and we are done. □

DEFINITION 5. *Assume that $X$ is 1-defective. Call $k$-th* **projection defect** *of $X$ the difference $p_k(X) = \dim(X) - \dim(X_k)$, $X_k$ being a $k$-th general tangential projection of $X$.*

We have thus a non–decreasing chain of numbers:

$$p_0 = 0 \leq p_1 \leq \cdots \leq p_K$$

which starts being positive at the minimal $k$ such that $X$ is $k$-defective and stops at the invariant $K$ introduced in (2), for which $T_{P_0,\dots,P_K}(X)$ reaches $\mathbb{P}^r$.

The relationship between projection defects and our original defects becomes less elementary for $k > 1$. We exploit them in a remark.

REMARK 3. The tangent space $T_{P_0,\dots,P_k}(X)$ projects to $T_Q(X_k)$ in $\pi_k$ (here $Q$ is the image of $P_k$). Thus:

$$\dim(T_{P_0,\dots,P_{k-1}}(X)) + \dim(X_k) + 1 = \dim(T_{P_0,\dots,P_k}(X))$$

or, in other words:

$$p_k = n + 1 - \dim(T_{P_0,\dots,P_k}(X)) + \dim(T_{P_0,\dots,P_{k-1}}(X))$$

now recalling the definition of defect:

$$p_k = n + 1 - (\min(r, nk + n + k) - \delta_k) + \min(r, nk + k - 1) - \delta_{k-1}.$$

Let us consider the various cases.
If $r \geq nk + n + k$, then:

$$p_k = n + 1 - (nk + n + k - \delta_k) + nk + k - 1 - \delta_{k-1} = \delta_k - \delta_{k-1}.$$

On the other hand, when $r \leq nk + k - 1$:

$$p_k = n + 1 + \delta_k - \delta_{k-1}.$$

Finally in the remaining case $nk + k - 1 \leq r \leq nk + n + k$

$$p_k = n + 1 - (r - \delta_k) + nk + k - 1 - \delta_{k-1} = nk + n + k + \delta_k - \delta_{k-1}.$$

In particular, the case $r \geq nk + n + k$ should be fixed in mind, since some author defines $p_k$ as the natural defect. Also some authors (e.g. see [82] and [37]) defines the defect as $p_k - p_{k-1}$, which measures the drop of dimension introduced at the $k$-th tangential projection, i.e. the new defectivity introduced at level $k$.

Times has come to introduce some result. We are ready to use the previous machinery to prove some bounds on the defects.

THEOREM 2. *If $X$ is a curve, then it is never defective.*

*Proof.* It is an immediate consequence of corollary 4. Indeed if $X$ is $k$-defective, then $X_k$ is not linear, hence cannot be a point, and $\dim(X_k) < \dim(X)$. This is clearly impossible when $X$ is an irreducible curve. □

REMARK 4. Notice how the irreducibility condition plays in the previous claim. Indeed when $X$ is a reducible curve, its first secant variety $S_1(X)$ may have dimension $2 < 2n+1$. Indeed for $X$= union of 3 lines, meeting at a point, then $S_1(X)$ is the union of the three planes spanned by the pairs of lines.

It is clear that the tangent space $T_{P_0}(X)$ at a general point cannot contain $X$, which is non–degenerate. Hence $\dim(T_{P_0,P_1}(X)) \geq n+1$. We can say something more:

PROPOSITION 5. *If $r > n+1$, then* $\dim(T_{P_0,P_1}(X)) \geq n+2$.

*Proof.* Assume $\dim(T_{P_0,P_1}(X)) = n+1$. Then $\dim(T_{P_0}(X) \cap T_{P_1}(X)) = n-1$. By the Linear lemma 1 this means that all the tangent spaces are contained in the same $\mathbb{P}^{n+1}$ or they contain a fixed $\mathbb{P}^{n-1}$. The former case is impossible, since $X$ is non degenerate and $r > n+1$. The latter case cannot hold in characteristic 0, for the reasons explained in exercise 26. □

EXERCISE 35. If $n + 2k \leq r$ then $\dim(T_{P_0,\dots,P_k}(X)) \geq n + 2k$.
Induction on $k$, the case $k = 1$ being considered in the previous proposition. Assume then $\dim(T_{P_0,\dots,P_{k-1}}(X)) \geq n + 2k - 2$. If $T_{P_0,\dots,P_{k-1}}(X)$ coincides with $\mathbb{P}^r$, then our numerical assumption implies the claim. Otherwise, since $X$ is non-degenerate, then it cannot be contained in $T_{P_0,\dots,P_{k-1}}(X)$, thus if the strict inequality holds above, the claim follows. It remains to consider the case $\dim(T_{P_0,\dots,P_{k-1}}(X)) = n + 2k - 2$. Take the $(k-1)$-th tangential projection $X_{k-1}$. In this projection $T_{P_0,\dots,P_{k-1}}(X)$ maps to $T_Q(X_{k-1})$ and $T_{P_0,\dots,P_k}(X)$ maps to $T_{Q,Q'}(X_{k-1})$, where $Q$, $Q'$ are the images of $P_{k-1}$, $P_k$. Then the claim follows from the previous proposition.

EXERCISE 36. For any $k$, if $X$ is $k$-defective, then $\delta_k \leq (n-1)k$.

In general, we cannot expect to improve the previous bound on the defects, even for the first defect. Indeed:

EXAMPLE 11. Let $C$ be a non–degenerate curve in $\mathbb{P}^{r-n-1}$ and take the cone $X$ over $C$, with vertex $W$ of dimension $n-2$. $X$ has dimension $n$ and it is non–

degenerate in $\mathbb{P}^r$. Assume $r > n + 2k + 2$. Fix general points $P_0, \ldots, P_k \in X$ and call $Q_0, \ldots, Q_k$ their images in the projection from $W$. Then $T_{P_0,\ldots,P_k}(X)$ is the span of $W$ and $T_{Q_0,\ldots,Q_k}(X)$. Since $r - n - 1 > 2k + 1$ and $C$ is not $k$-defective by theorem 2, then $\dim(T_{Q_0,\ldots,Q_k}(X)) = 2k + 1$. Hence:

$$\dim(T_{P_0,\ldots,P_k}(X)) = 2k + n < r$$

and $X$ is $k$-defective, with $k$-th defect $\delta_k = (n-1)k$.

Clearly the variety $X$ of the previous example is rather singular along $W$. It turns out indeed that one can improve the bound considerably if one assumes the smoothness of $X$. This is a consequence of a deep result proved by Fulton and Lazarsfeld on the connectedness of some subvarieties of $X$. We will be back on the argument later on.

EXERCISE 37. Rephrase all the previous bounds for the projection defects.

### 3.5. Inverse systems

Occasionally, it turns out that some ad hoc technique computes the dimension of some secant variety, when $X$ has some well-defined structure. This is the case, for instance, of some varieties which parametrizes products.

In this section, we introduce an appropriate method for these objects and suggest how to extend it to arbitrary varieties.

We illustrate how the method works for Grassmannian. Consider the Grassmannian $X = G(h,s)$ of $h$-dimensional projective subspaces in $\mathbb{P}^s$, with its Plücker embedding in $\mathbb{P}^r$, $r = \binom{s+1}{h+1} - 1$. If $E = \mathbb{C}^{s+1}$ is the vector space which defines $\mathbb{P}^s$, then we may identify $\mathbb{P}^r$ with $\mathbb{P}(\Lambda^{h+1}E)$. Given $P \in G(h,s)$ and a basis $v_0, \ldots, v_h$ for $P$, the Plücker embedding maps $P$ to the point $v_0 \wedge \cdots \wedge v_h \in \mathbb{P}(\Lambda^{h+1}E) = \mathbb{P}^r$. As remarked above, $G(h,s)$ corresponds, from this point of view, to the set of decomposable vectors (i.e. wedge-monomials), while the secant variety $S_k(G(h,s))$ is the (closure of the) set of vectors which can be written as a sum of $k+1$ wedge-monomials.

Now consider the exterior algebra $\mathrm{Ext}(E) = \oplus_{i=0}^{\infty} \Lambda^i E$. Using the canonical pairing $\Lambda^i E \times \Lambda^{s+1-i} E \to \mathbb{C}$ one defines the perpendicular $Y^*$ of any homogeneous subspace $Y \subset \Lambda^i E$.

With this notation, we have the following interpretation of the tangent space to $X = G(h,s)$ at the point $P = v_0 \wedge \cdots \wedge v_h$:

EXERCISE 38. The perpendicular space $Y = T_P(X)^*$ is generated by the elements of degree $s - h$ in the ideal $(v_0, \ldots, v_h)^2 \subset \mathrm{Ext}(E)$.

Thus we get the following characterization:

PROPOSITION 6. *If $P_0 = v_{0,0} \wedge \cdots \wedge v_{0,h}, \ldots, P_k = v_{k,0} \wedge \cdots \wedge v_{k,h}$ are general points of $X = G(h,s)$, then for $u$ general in $< P_0, \ldots, P_k >$, the tangent space*

*$T_u(S_k(X))$ is perpendicular to*

$$W = [(v_{0,0}, \dots, v_{0,h})^2 \cap \cdots \cap (v_{k,0}, \dots, v_{k,h})^2]_{s-h}.$$

*Proof.* The perpendicular to the span of $< T_{P_0}(X), \dots, T_{P_k}(X) >= T_u(S_k(X))$ is the intersection of the perpendicular spaces to each $T_{P_i}(X)^*$. □

Now we provide an example in which the dimension of the perpendicular space can be effectively computed:

THEOREM 3. *(**Catalisano, Geramita, Gimigliano**) When $(h+1)(k+1) \le s+1$ and $h \ge 2$, the k-th secant variety to $G(h,s)$ has the expected dimension*

$$\dim(S_k(G(h,s))) = \min(r, (k+1)(h+1)(s-h)+k).$$

*Proof.* One uses the reduction of proposition 6.
The condition $(h+1)(k+1) \le s+1$ implies that we can choose the points $P_0, \dots, P_k$ from a canonical basis $e_0, \dots, e_s$ of $E$, using separate variables:

$$P_0 = e_0 \wedge \cdots \wedge e_h,\ P_1 = e_{h+1} \wedge \cdots \wedge e_{2h+1}, \dots P_k = e_{kh+k} \wedge \cdots \wedge e_{kh+k+h}.$$

Notice that the natural action of the linear group on $G(h,s)$ is transitive over the $(k+1)$-tuples of points, in our setting. So any $(k+1)$-tuple of points can be reduced to the previous one.
Now one finds a basis of

$$[(e_0, \dots, e_h)^2 \cap \cdots \cap (e_{kh+k}, \dots, e_{kh+k+h})^2]_{s-h}$$

by taking all the products of $s-h$ elements of the basis in which two elements sits in $\{e_0, \dots, e_h\}$, two elements sits in $\{e_{h+1}, \dots, e_{2h+1}\}$ and so on.
The proof follows from an easy dimension count. □

A completely similar approach works for symmetric tensor products, which are the algebraic analogue of Veronese varieties, and general tensor products, which are the algebraic analogue of Segre products of projective spaces.

In the former case, however, the results one can easily obtain in this way are weaker than the classical results obtained using the techniques we will explain in the next chapters.

In the latter case, the use of perpendicular spaces is harder, because we need to work with several different vector spaces. We refer to the paper [13] for a reference on results in progress. We just cite:

THEOREM 4. *(see [13], Proposition 3.3) Let $X = \mathbb{P}^{n_0} \times \cdots \times \mathbb{P}^{n_t}$, with its Segre embedding in $\mathbb{P}^r$, $r = (n_0+1)\cdots(n_t+1)-1$. Fix an integer k such that*

$$\prod_{i=1}^{t}(n_i+1) - \sum_{i=1}^{t}(n_i) + 1 \le k \le \min(n_0, \prod_{i=1}^{t}(n_i+1) - 1).$$

*Then $X$ is k-defective.*

REMARK 5. One of the main facilities in the proof of the two previous results relies in the number $k+1$ of points, which is small with respect to the intrinsic coordinates of $X$. This implies that under the action of the linear group, one can assume that $P_0, \dots, P_k$ are independent points with very special coordinates, so that the span of their tangent spaces is easy.
Similar simplifi cations apply as soon as $X$ is $(k+1)$-homogeneous, i.e. there exists a group acting on $X$, so that any $(k+1)$-tuples of independent points of $X$ lie in the same orbit.

For Grassmannians the natural pairing allows to translate our question on the dimension of secant varieties to a question on the intersection of suitable perpendicular spaces. When $X$ is arbitrary, one may create a similar procedure, introducing the pairing by means of the theory of **inverse systems**.

Identify $\mathbb{P}^r$ with a set of forms of given degree $d$ in a given number of variables $j$ (modulo scalar multiplication). Then consider the polynomial ring

$$A = \mathbb{C}[d_0, \dots, d_j]$$

as a ring of differentials on the forms which represent points of $\mathbb{P}^r$.
In other words, any $P \in \mathbb{P}^r$ represents a form in $j$ variables, over which the elements of $A$ act as sequences of partial derivatives.

DEFINITION 6. *We say that $d \in A$ and $f \in \mathbb{P}^r$ are* **apolar** *when $d(f) = 0$.*
*For any subset $X \subset \mathbb{P}^r$, the* **apolar set** *$X^\perp$ is the set of all $d \in A$ such that $d(f) = 0$ for all $f \in X$.*
*For any subset $D \subset A$, the* **inverse system** *of $D$ is the set $D^{-1} = \{f \in \mathbb{P}^r : d(f) = 0\}$ for all $d \in D$.*

Of course this defi nition is invariant when $f$ is multiplied by some scalars, hence it is well–posed.

EXERCISE 39. If $f, g \in \mathbb{P}^r$ belong to $D^{-1}$, then all the points in $< f, g >$ also belong to $D^{-1}$, hence $D^{-1}$ is always a linear space.

EXERCISE 40. The $k$-th secant variety $S_k(X)$ is then also the closure of:
$S_k(X) = \{f :$ there are $f_0, \dots, f_k \in X$ with $f \in ((f_0, \dots, f_k)^\perp)^{-1}\}$.

One should wonder why it is interesting to give such a complicate characterization of secant varieties. Apart from the fact that the theory of inverse systems has several interesting applications, one should consider that apolarity is indeed an algebraic condition, hence one has a way to reproduce algebraically the construction of secant varieties.

See [48], [9], [24] for applications to the study of the dimension of some secant varieties.

## 4. Degenerate subvarieties

Probably some readers are wondering that up to now, we put down on the paper several introductory facts, but no really ultimate results (except for theorem 2, perhaps). We ask the readers to patient only few more lines. Indeed we must still introduce our last tool, then the investigation could start and unelementary results will come at hand.

### 4.1. The Infinitesimal Bertini's Theorem

The infinitesimal Bertini's Theorem links defectivity with set of points at which the general tangent space $T_{P_0,\dots,P_k}(X)$ is actually tangent to $X$.

THEOREM 5. **Infinitesimal Bertini's Theorem** *Let $X$ be an irreducible variety and let $Y$ be a reduced, irreducible algebraic subvariety of some linear system $V$ of divisors in $X$. Let $y \in Y$ be a general point, let $S := S_y$ be a component of the singular locus of the divisor $H$ parametrized by $y$, not contained in Sing($X$). Then the projective tangent space to $Y$ at $y$ in $V$ is contained in $V(-S)$.*

Before we sketch a proof for the theorem, let us justify its nickname.

Bertini's classical theorems say that, in characteristic 0, if a linear system has a moving singular point, then this point is contained in the base locus, off Sing($X$). In the previous setting, the family $Y$ of divisors we work with is not, in general, a linear system: it can be any subfamily of the linear system $V$. So we cannot conclude that a moving singularity lies in the base locus. On the other hand, any variety $Y$ "becomes linear" when it is replaced with a tangent space at some of its points. The theorem says that the tangent space contains the moving singularities (off Sing($X$)). So the infinitesimal movements of $H$ in the family have indeed the moving singularities in the base locus.

*Proof.* Let $v$ be any tangent vector to $Y$ at $y$ and let $s \in H^0(H, N_{H,X})$ be the section of the normal bundle $N_{H,X}$ of $H$ in $X$ corresponding to the first order deformation $H_s$ of $H$ determined by $v$. Since $y \in Y$ is general, if $x \notin \mathrm{Sing}(X)$ is any singular point of $H$, there is a first order deformation of $x$ along which $H_\sigma$ stays singular.
Fix coordinates $z_1, \dots, z_n$ on $X$ centered at the smooth point $x$, let $f(z_1, \dots, z_n) = 0$ be the equation of $H$ in these coordinates and let $f(z_1, \dots, z_n) + \epsilon g(z_1, \dots, z_n) = 0$ be the equation of $H_\sigma$. The Taylor expansions of $f$ and $g$ give us:

$$f(z_1, \dots, z_n) = \sum_{i=2}^{\infty} f_i(z_1, \dots, z_n)$$

$$g(z_1, \dots, z_n) = \sum_{i=0}^{\infty} g_i(z_1, \dots, z_n)$$

where the $f_i, g_i$'s are homogeneous polynomials of degree $i$.
Any infinitesimal deformation $\xi$ of $x$ is given by $z_i = \epsilon a_i$, $i = 1, \ldots, n$.

Notice that $H_\sigma$ is singular at $\xi$ if and only if the partial derivatives of $f + \epsilon g$ vanish at $(\epsilon a_1, \ldots, \epsilon a_n)$. Since $f_1 \equiv 0$ by assumption, we can find $\xi$ with $H_\sigma$ is singular at $\xi$ if and only if the derivatives of $\epsilon g(\epsilon a_1, \ldots, \epsilon a_n)$ vanish, i.e. if and only if $g_0 \equiv 0$. This means precisely that the section of the normal bundle associated with $\sigma$ vanishes at any point of $S$ not in $\mathrm{Sing}(X)$. The claim follows. □

The classical analysis of the structure of defective varieties is based on the previous theorem and the following consequence, pointed out by Terracini:

COROLLARY 5. (Terracini) *Assume that $X$ is $k$-defective. Then for a general choice of the points $P_0, \ldots, P_k \in X$, a general hyperplane $H$ of $\mathbb{P}^r$ which contains the space $T_{P_0,\ldots,P_k}(X)$, is indeed tangent along a subvariety $\Sigma(H)$ of positive dimension.*

*Proof.* We follow the modern proof introduced by Ciliberto and Hirschowitz in [26]. Let $X_0^{k+1}$ be the open subset of $X^{k+1}$ described by $(k+1)$-tuples formed by independent points. Call $V$ the linear system of hyperplane divisors in $X$. Consider now the closure $I \subset V \times X_0^{k+1}$ of the **singular incidence correspondence**:

$$\{(H \cap X, S) \in V \times X_0^{k+1} : S \subset \mathrm{Sing}(H \cap X)\}$$

and let $p_1 : I \to V$ and $p_2 : I \to X_0^{k+1}$ be the two projections.
Since $X$ is $k$-defective, for a general choice of the points $T_{P_0,\ldots,P_k}(X) \neq \mathbb{P}^r$, so it lies in some hyperplane. This means that $p_2$ dominates $X_0^{k+1}$. Its general fiber is a projective space of dimension

$$w = \dim(V(-2P_0 - \cdots - 2P_k)) > r - (k+1)(n+1)$$

by proposition 2. Then there is only one irreducible component $J$ of $I$ which dominates $X_0^{k+1}$ and one has:

$$\dim(J) > r - k - 1.$$

Consider now the family of divisors on $X$ given by $Y = p_1(J) \subset V$. If $(H \cap X, S) \in J$ is general, with $S = (P_0, \ldots, P_k)$, then there is a component $\Sigma$ of the locus where $H$ is tangent to $X$, which intersects the set of smooth points of $X$. Hence we may apply the infinitesimal Bertini's theorem and conclude that the projective tangent space to $Y$ at $H \cap X$ is contained in $V(-\Sigma) \subset V(-S)$. Since $S$ is formed by $k+1$ general points of $X$, then $\dim(V(-S)) = r - k - 1$, hence:

$$\dim(Y) \leq \dim(V(-\Sigma)) \leq \dim(V(-S)) = r - k - 1 \tag{4}$$

Hence the map $J \to Y$ has positive dimensional fibers. This means that, for fixed general $H$ containing $T_{P_0,\ldots,P_k}(X)$, there exists a positive dimensional subset $\Sigma \subset H \cap X$ such that $H \cap X$ is singular along $\Sigma$. □

Setting some new piece of notation, one can be much more precise on the structure of $\Sigma$.

DEFINITION 7. *For a hyperplane $H$ of $\mathbb{P}^r$, denote with $Z(H)$ the set of points $\{Q \in X - Sing(X) : T_Q(X) \subset H\}$.*
*If $P_0, \dots, P_k \in X$ are general and $H$ is a general hyperplane which contains $T_{P_0,\dots,P_k}$, call* **contact locus** *or entry locus of $H$ the union $\Sigma(H) = \Sigma_{P_0,\dots,P_k}(X)$ of all the components of $Z(H)$ which contain some $P_i$.*
*For a general choice of the points, the dimension of the contact locus $\Sigma(H)$ is an invariant which depends only on $X$. $\nu_k(X) = \dim(\Sigma(X))$ is called the $k$-th* **singular defect** *of $X$.*

The previous corollary can be rephrased as follows:

If $X$ is $k$-defective, then the singular defect $\nu_k(X)$ is positive.

With a simple refinement of the previous argument, one can find much more precise information on the contact locus of a defective variety. The main (classical) remark says that the contact locus tends to be considerably degenerate. This remark, together with Terracini's lemma, is the main tool which makes possible the investigation of the world of defective varieties.

DEFINITION 8. *If $V$ is a linear system on $X$ and $X' \subset X$ is a subvariety, then the number $h_V(X') = \dim(V) - \dim(V(-X'))$ is called the* **number of conditions** *imposed by $X'$ to $V$.*

THEOREM 6. *Assume that $X$ is $k$-defective, with $\dim(S_k(X)) = s_k$. Choose general points $P_0, \dots, P_k \in X$, a general hyperplane $H$ of $\mathbb{P}^r$ which contains the space $T_{P_0,\dots,P_k}(X)$ and call $\Sigma(H)$ the contact locus of $H$. Then the number of conditions $h_V(\Sigma(H))$ imposed by $\Sigma(H)$ to the system $V$ of hyperplane sections satisfies:*

$$k + 1 \le h_V(\Sigma(H)) \le 1 + s_k - (n - \nu_k)(k + 1) \tag{5}$$

*(remind that $\nu_k(X) = \dim(\Sigma(H))$). In other words:*

$$\dim(< \Sigma(H) >) \le s_k - (n - \nu_k)(k + 1)$$

*Proof.* Just repeat the proof of the corollary. Taking the notation, by definition of defect one can write precisely the dimension $w$ of a general fiber of $p_2$:

$$w = \dim(V(-2P_0 - \cdots - 2P_k)) \ge r - s_k - 1$$

so that $\dim(J) = r - s_k - 1 + n(k + 1)$. Now, by definition, the general fibre of the restriction of $p_1$ to $J$ has dimension $(k + 1)\nu_k$. In conclusion we have:

$$r - s_k - 1 + n(k + 1) = \dim(J) = \dim(p_1(J)) + (k + 1)\nu_k$$

while as above

$$\dim(p_1(J)) \le \dim(V(-\Sigma)) \le r - k - 1$$

which yields the assertion. $\square$

In normal words, the results above say:
if $X$ is $k$-defective, then a general hyperplane which is tangent at $k+1$ points of $X$ is indeed tangent along a positive dimensional subvariety $\Sigma(H)$, which imposes not many conditions to the hyperplanes, hence it is degenerate.

EXERCISE 41. If $V$ is the hyperplane linear system on $X$ and $X'$ is a subvariety, then $h_V(X') = w < r$ if and only if $X'$ is degenerate, contained in some $\mathbb{P}^w$.
Indeed observe that $V(-X')$ has a base formed by $r+1-w$ elements and $X'$ sits in the base locus of $V(-X')$, which is the intersection of the elements of a base.

EXAMPLE 12. Let $X$ be the cone over a curve $C \subset \mathbb{P}^4$, with vertex at a point. Then $X \subset \mathbb{P}^5$ is 1-defective. As explained in exercise 19, the intersection of two general tangent spaces is the vertex, hence the defect $\delta_1$ is 1. If $P_0, P_1 \in X$ are general points, then any hyperplane $H$ which is tangent to $X$ at $P_0, P_1$ is indeed tangent to $X$ along the two lines $L_0, L_1$ of the ruling passing through $P_0, P_1$. Hence the contact locus $\Sigma(H)$ is the (plane conic) $L_0 \cup L_1$ and $\nu_1(X) = 1$, $h_V(\Sigma(H)) = 3$. Formula (5) reads:

$$2 \leq h_V(\Sigma(H)) \leq 2 \cdot (1+1) - 1$$

and works.

EXERCISE 42. Repeat the previous example for general cones.

On the other hand, the reader is invited to reflect immediately on the fact that when a variety $X$ has a positive $k$-th singular defect, then it is *not necessarily* $k$-defective.

A first example is provided by cones $X$: imposing to a hyperplane the tangency at one point, we get tangency along a line. So $\nu_0(X) > 0$. But $X$ is not 0-defective (which does not make any sense!).

For a more sophisticated example, we need two elementary results:

EXERCISE 43. Fix two disjoint linear spaces $L, R \subset \mathbb{P}^s$. Let $X$ be a subvariety of $R$ and call $W$ the cone over $X$, with vertex $L$. Fix a linear space $L'$ containing $L$. Then the projection of $W$ from $L'$ coincides with the projection of $X$ from $L' \cap R$.

EXERCISE 44. Let $L$ be a linear space and let $L' \subset L$ be a subspace. Then the projection of $X$ from $L'$ is contained in the cone over the projection of $X$ from $L$, with vertex at the projection of $L$ from $L'$.

EXAMPLE 13. Let $W \subset \mathbb{P}^6$ be the cone, with vertex at a point $P$, over the Veronese surface $V(2,2) \subset \mathbb{P}^5$. Let $X$ be the intersection of $W$ with a general hypersurface of degree bigger than 1. We will see later on that $X$ is not 1-defective.
At a general point $P_0 \in X$, the tangent space $T_{P_0}(X)$ is a hyperplane in $T_{P_0}(W)$, which in turn is $< P, T_Q(V(2,2)) >$, $Q$ be the point of $V(2,2)$ corresponding to $P_0$ in the projection from $P$. By exercise 43, the projection of $W$ from $T_{P_0}(W)$ coincides with the projection of $V(2,2)$ from $T_Q(V(2,2))$, which is a plane curve, since $V(2,2)$ is defective. Hence by exercise 44 the projection of $W$ from $T_{P_0}(X)$ is a cone $Z$ over a

plane curve. $Z$ contains the projection of $X$ from $T_{P_0}(X)$. Since $X$ is not 1-defective, then $Z$ coincides with this projection.
Now fix another general point $P_1 \in X$ and call $A$ its image in $Z$. The tangent space to the cone $Z$ at $A$ contains the tangent spaces along a positive dimensional subvariety $Z' \subset Z$. Hence $T_{P_0,P_1}(X)$ contains the tangent spaces to $X$ along the counterimage of $Z'$. It follows that any hyperplane which contains $T_{P_0,P_1}$ is indeed tangent along a positive dimensional subvariety, i.e. $\nu_1(X) > 0$.

### 4.2. An example: the Veronese surface

Let us visit again the Veronese surface $X = V(2,2)$, corresponding to the image of $\mathbb{P}^2$ in $\mathbb{P}^5$, via the linear system of conics.

We know that $X$ is 1-defective, with defect $\delta_1(X) = 1$.

In the interpretation suggested above, we get that a general tangential projection of $X$ to $\mathbb{P}^2$ must be a curve. This is clear, indeed. Tangential projection means that we restrict our linear system to the system of conics with one fixed double point $P$. The linear system is then composed with the pencil of lines through $P$. Thus it contracts $\mathbb{P}^2$ to a curve, which is the image of a line not passing through $P$ under the complete $g^2_2$. In other words, the image $X_1$ of $X$ in a general tangential projection is a plane conic. In particular the projection defect $p_1$ is equal to $\delta_1$. Compare with remark 3 (here $r = 5$ is equal to $nk + n + k$).

$X$ has a singular defect. A general tangent line to $X_1$ meets it in a double point. Since this line is the image of a space $T_{P_0,P_1}$ under the projection from $T_{P_0}$ (with $P_0, P_1 \in X$ general points), it turns out that $T_{P_0,P_1} = H$ is a hyperplane, tangent to $X$ exactly along a fibre of the projection $X \to X_1$. As explained above, these fibers are the image in $\mathbb{P}^5$ of lines through $P$, hence conics in $\mathbb{P}^5$. So the singular defect is $\nu_1 = 1$ and a general contact variety is a (plane) conic $\Sigma$.

The conditions imposed by $\Sigma$ to the hyperplanes of $\mathbb{P}^5$ are $h_V(\Sigma) = 3$. Formula (5) reads:

$$2 \leq h_V(\Sigma(H)) \leq 2 \cdot (1+1) - 1$$

and works.

Notice that, as the points $P_0, P_1$ vary, the corresponding contact variety of the hyperplane $T_{P_0,P_1}$ determine a family of conics in $X$. This family is indeed a linear system and this linear system determines an isomorphism between the Veronese surface $X$ and $\mathbb{P}^2$.

### 4.3. The Gauss map

It turns out that the definition of contact locus has some clones which are relevant for the study of defective varieties. The first one starts from a remark concerning the intersection of $X$ with the tangent space $T_{P_0,\dots,P_k}$.

DEFINITION 9. *For a general choice of the points $P_0, \dots, P_k$, consider the set $T$ of regular points $Q \in X$ such that $T_Q(X) \subset T_{P_0,\dots,P_k}(X)$. Call $\Gamma_{P_0,\dots,P_k}(X)$ the union of all the components of the closure of $T$ which contains one of the points $P_i$'s. $\Gamma_{P_0,\dots,P_k}(X)$ will be referred to as the $k$-th* **tangential contact locus**. *We call* **tangential singular defect** *the dimension $\gamma_k(X)$ of $\Gamma_{P_0,\dots,P_k}(X)$, for a general choice of the points.*

It is clear that all the hyperplanes $H$ which are tangent to $X$ at $P_0, \dots, P_k$ contain $T_{P_0,\dots,P_k}(X)$; hence for all such hyperplanes $H$ one has

$$\Gamma_{P_0,\dots,P_k}(X) \subset \Sigma(H).$$

More precisely one gets the following characterization:

EXERCISE 45. $\Gamma_{P_0,\dots,P_k}(X)$ is the intersection of $\Sigma(H)$ for all hyperplanes $H$ which are tangent to $X$ at $P_0, \dots, P_k$.
Enough to remind that $T_{P_0,\dots,P_k}(X)$ is the intersection of all the hyperplanes which are tangent to $X$ at $P_0, \dots, P_k$.

Using the classical Bertini's theorem, one gets:

PROPOSITION 7. *For $P_0, \dots, P_k$ general, if $H$ is a hyperplane which contains $T_{P_0,\dots,P_k}$, then $\Sigma(H) \subset X \cap T_{P_0,\dots,P_k}(X)$.*

*Proof.* The set of hyperplanes containing $T_{P_0,\dots,P_k}(X)$ cuts on $X$ a linear system of hyperplane sections, whose moving singular part must be contained, by Bertini's theorem, in the base locus of the system. Since $\Sigma(H)$ lies in the singular locus of the divisor $H \cap X$ and the base locus is exactly $X \cap T_{P_0,\dots,P_k}(X)$, the claim follows. □

Thus we have, for a general set of points $P_0, \dots, P_k \in X$ and for a hyperplane $H$ which is tangent to $X$ at all the $P_i$'s, a sequence of inclusions:

$$\Gamma_{P_0,\dots,P_k}(X) \subset \Sigma(H) \subset T_{P_0,\dots,P_k}(X) \cap X \tag{6}$$

and $\gamma_k(X) \leq \nu_k(X)$.

THEOREM 7. *Assume that $X$ is $k$-defective. Then for a general choice of the points $P_0, \dots, P_k \in X$, the tangential contact locus $\Gamma_{P_0,\dots,P_k}(X)$ has positive dimension.*

*Proof.* Fix general points $P_0, \dots, P_k$. Let $X_k$ be the tangential projection of $X$ from the space $T_{P_1,\dots,P_k}(X)$. Then we know from corollary 3 that $X \to X_k$ has positive dimensional fibers. If $Q$ is the image of $P_0$ in the projection, then $T_{P_0,\dots,P_k}(X)$ maps to $T_Q(X_k)$, hence it is tangent to $X$ along the fiber of the projection over $Q$. □

PROPOSITION 8. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, $k$–defective threefold. For a general choice of $P_0, \dots, P_k \in X$ and a general choice* *of the hyperplane* $H \in \mathcal{H}(-2P_0 - \cdots - 2P_k)$*, the contact loci* $\Gamma = \Gamma_{P_0,\dots,P_k}$ *and* $\Sigma = \Sigma(H) = \Sigma_{P_0,\dots,P_k}(H)$ *are equidimensional. Furthermore either they are irreducible or they consist of* $k+1$ *irreducible components, one through each of the points* $P_0, \dots, P_k$*.*

*Proof.* Let us give the proof only for $\Sigma(H)$, the other proof being similar.
First of all, let us move slightly the points $P_i$'s on $\Sigma(H)$ to a new set of points $\{Q_0, \dots, Q_k\}$. Then $Q_0, \dots, Q_k$ are also general points on $X$. Furthermore $T_{X,P_0,\dots,P_k}$ contains the tangent spaces to $X$ at the points $Q_i$'s, so for reasons of dimension, it coincides with $T_{X,Q_0,\dots,Q_k}$. Then we may assume that $P_0, \dots, P_k$ are smooth points for $\Sigma(H)$, hence there is only one irreducible component of $\Sigma(H)$ through each one of the points $P_0, \dots, P_k$. By monodromy we may exchange the points $P_i$ and therefore all components of $\Sigma(H)$ have the same dimension.
Assume now that there is a component of $\Sigma$ which contains more than one of the $P_i$'s, say $P_0$ and $P_1$. By monodromy, we can let $P_0$ stay fixed and we can move $P_1$ to any one of the points $P_i$, $i > 1$. Then we see that also $P_0$ and $P_i$, $i > 1$, stay on an irreducible component of $\Sigma$. Since $P_0$ sits on only one irreducible component of $\Sigma$, then this component has to contain all the points $P_0, \dots, P_k$ and therefore it has to coincide with $\Sigma(H)$. □

EXERCISE 46. If $X$ is a surface, whose general $k$-th contact locus has positive dimension, then the first inclusion in (6) is an equality and the second is a local, set-theoretical equality.
Indeed $T_{P_0,\dots,P_k}(X) \cap X$ is a divisor in $X$. As $H$ moves in the linear system of hyperplanes tangent to $X$ at the $P_i$'s, every $\Sigma_{P_0,\dots,P_k}(H)$ is a divisor, so it is composed with some components of $T_{P_0,\dots,P_k}(X) \cap X$. Finally observe that moving $H$, $\Sigma_{P_0,\dots,P_k}(H)$ is fixed thus it coincides with $\Gamma_{P_0,\dots,P_k}$, by the previous result.

EXERCISE 47. If $X$ is a defective threefold, then the first inclusion in (6) is an equality.
Indeed $T_{P_0,\dots,P_k}(X) \cap X$ is a divisor in $X$ and $\Gamma_{P_0,\dots,P_k}$ has dimension at least one. None of them depends on the hyperplane $H \supset T_{P_0,\dots,P_k}$. Hence also $\Sigma(H)$ is fixed, for general $H$.

Let us go back to the exercise 46. We know that the general $k$-th contact loci $\Sigma$ and $\Gamma$ are positive-dimensional, but $X$ is not $k$-defective. See example 13. So one is led to the following:

DEFINITION 10. *We say that* $X$ *is* $k$**-weakly defective** *(resp.* **tangentially** $k$**-weakly defective***) when for a general choice of* $P_0, \dots, P_k \in X$ *and the hyperplane* $H \supset T_{P_0,\dots,P_k}$*, the contact locus* $\Sigma_{P_0,\dots,P_k}(H)$ *(resp. tangential contact locus* $\Gamma_{P_0,\dots,P_k}$*) are positive dimensional.*

It is clear now that:

defective → tangentially weakly defective → weakly defective.

On the other hand, these concepts are disjoint. There are examples of varieties which are weakly defective but not tangentially weakly defective.

The study of tangentially defective varieties has interest from his own point of view. Usually tangentially defective varieties are referred to as *varieties with degenerate Gauss map*.

DEFINITION 11. *The* **Gauss map** *of $X$ is the rational map $g(X) : X \to G(n,r)$ which sends a general (smooth) point $P \in X$ to the point of the Grassmannian which parametrizes the tangent space $T_P(X)$.*
*The* **dual variety** *of $X$ is the (closure of the) variety $X^v \subset (\mathbb{P}^r)^v$ of the dual projective space which parametrizes hyperplanes which contain some tangent space to $X$ at some smooth point.*

EXERCISE 48. The image of the Gauss map $g(X)$ has dimension smaller than $X$ if and only if $X$ is tangentially 0-weakly defective.

EXERCISE 49. The dual variety of $X$ has dimension at most $r-1$. It has dimension $r-1$ if and only if $X$ is not 0-weakly defective.
Indeed consider the singular incidence variety:

$$I = \{(P, H) \in X \times (\mathbb{P}^r)^v : H \supset T_P(X)\}$$

Since the fiber of the first projection of this variety over Reg$(X)$ has fixed dimension, then $\dim(I) = r-1$. $X^v$ is the image of $I$ in the second projection. So $\dim(X^v) < r-1$ if and only if a general hyperplane which is tangent to $X$ at some point $P$, is indeed tangent to $X$ in infinitely many points.

The theories of varieties for which the Gauss map has image of dimension $< n$ (varieties with *degenerate Gauss map*) and varieties with "small" dual variety, are indeed classically considered and also studied from a modern point of view.

PROPOSITION 9. *The set $\Sigma$ of points where a general tangent hyperplane is tangent to $X$ is a linear subvariety of $X$.*
*The general fibers of the Gauss map are linear subspaces.*

*Proof.* The first assertion is just an application of (5) for $k = 0$. Indeed $\nu_0$ is exactly the dimension of $\Sigma$ and $h(\Sigma) = \nu_0 + 1$, i.e. $\Sigma$ is contained in a $\mathbb{P}^{\nu_0}$.
The second assertion follows now, since the general fiber of the Gauss map is just the intersection of several loci $\Sigma$. □

The main result about the dimension of Gauss image is due to Zak, as a consequence of Fulton–Hansen connectedness principle that will be discussed later, in section 5:

THEOREM 8. *The dimension of the image of the Gauss map is at least $n - b - 1$, where $b \geq -1$ is the dimension of the singular locus of $X$.*
*In particular, when $X$ is smooth, then the Gauss map is birational.*

A classification of surfaces with degenerate Gauss map is classical (see e.g. [44]):

THEOREM 9. *Surfaces whose images in the Gauss map are curves are either cones or the union of tangent lines to a fixed curve.*

A similar result for higher dimensional varieties seems quite challenging. Several partial results are known. See [2] and [54] for an account of the theory and many extremal examples, essentially obtained taking the 1-secant variety to some extremal varieties.

For varieties with small dual variety, we refer to [83].

### 4.4. Varieties with many degenerate subvarieties

An important principle in Algebraic geometry, still to be fully explored and understood, suggests that a general projective variety $X$ of small codimension contains few irreducible subvarieties of special type. This is the philosophy behind the Noether-Lefschetz theorem on general hypersurfaces and more recent results on the geometric genus of their subvarieties (see e.g. [29], [80], [81], [23], [30]).

For our scopes, the principle suggests that there are only "few" degenerate subvarieties in $X$, unless $X$ has a very particular structure. This is more or less the base for our classification of defective varieties: since they have indeed a family of degenerate objects passing through general $(k+1)$-tuples of points, we may hope to use the family to derive complete information on $X$ itself.

The study of varieties covered by degenerate subvarieties is however a subject interesting by itself and intensively studied in classical projective geometry. We give in this section some hint on our knowledge on the field, with particular focus on results which are relevant for the classification of defective varieties.

Let us start with two classical facts:

PROPOSITION 10. *Assume that $X$ contains a family of plane curves, with the property that there exists an element of the family through any pair of general points of $X$. Then either $X$ is a projective space, or the plane curves have degree* 2.

*Proof.* If the curves are lines, then there is a line in $X$ which connects a general pair of points. This is a classical characterization of projective spaces. Assuming that the plane curves have degree 3 or bigger, we get a contradiction. Namely, by assumption, any secant line to $X$ lies in the plane of some curve of the family, so it is at least trisecant. This is excluded by the trisecant lemma. □

THEOREM 10. *(**Severi**) The only surfaces in $\mathbb{P}^r$, $r \leq 5$ with a 2-dimensional linear system of generically irreducible conics are the Veronese surface $V(2,2) \subset \mathbb{P}^5$ and its projection to lower dimensional spaces.*

*Proof.* The first step is to reduce to the case in which the self intersection of the system

is 1. Clearly we have infinitely many divisors of the system through a general point, so the self intersection is positive. Since the general divisor of the system is irreducible, if the self intersection is at least 2, then two general planes defined by the conics meet in a line. By lemma 1, either $r = 3$ or these planes have a fixed line. The latter case is impossible, since we have a divisor of the system through two general points of $X$. The former implies that the system is cut by the family of planes through a point. Hence $X$ has degree 2, and it is the projection of a Veronese surface from a secant line.
So, assume that two general elements of the system meet at one point. This implies that the map $\phi : X \to \mathbb{P}^2$ induced by the linear system is birational. It sends conics to lines. Thus, if $\phi'$ is its inverse, then $\phi'$ sends lines to conics. It follows that $X$ is the image of $\mathbb{P}^2$ in a linear system of conics. □

The previous characterization of the Veronese surface can be strenghtened. Indeed, we need not to assume that the 2-dimensional family of conics is a linear system: it is a consequence of our setting. Let us explain in details.

DEFINITION 12. *We say that a family $\mathcal{F}$ of $X$ is $k$-***filling** *if for a general choice of $k$ points of $X$, there exists an element of the family passing through the points.*
*We say that the family is an* **involution** *of dimension $k$ if for a general choice of $k$ points of $X$, there exists exactly one element of the family passing through them.*

Notice that $k$-filling implies that, writing $m$ for the dimension of the elements of $\mathcal{F}$, then:

$$\dim(\mathcal{F}) \geq k(n - m) \tag{7}$$

and the map from the total space of $k$-th cartesian products of elements of $\mathcal{F}$ to $X^k$ is dominant.

We have an involution when the previous map is birational, which implies that the equality holds in (7).

Clearly, any linear system of divisors is an involution. Conversely there are involutions which are not linear systems: just take any family of dimension 2 composed with a pencil. It turns out that, essentially, families composed with pencils are the only involutions which are not linear systems. This fact was classically observed by Castelnuovo and Humbert for divisors on curves, and extended to higher dimension in [16], §5.

THEOREM 11. *(***Chiantini-Ciliberto***) Let $X$ be a reduced, irreducible variety of dimension $n > 1$. Let $\mathcal{F}$ be a $k$-dimensional involution of divisors on $X$, which has no fixed divisor and whose general divisor $D$ is reduced. Then either $\mathcal{F}$ is a linear system or it is composite with a pencil.*

*Proof.* We just give a sketch of the proof, referring the interested reader to [16]. Since the problem is birational, we may assume $X$ is smooth. We argue by induction on $k$, the case $k = 1$ being classically known.
Suppose $k = 2$ and the general divisor $D \in \mathcal{F}$ is irreducible. If $P$ is a general point

of $X$, then $\mathcal{F}(-P)$ is an 1-dimensional involution with no fixed divisor. $\mathcal{F}(-P)$ is a linear system of dimension 1. Indeed one can prove that for a general choice of $P$, then the divisors in $F(-P)$ have no fixed tangent directions at the point. So $\mathcal{F}(-P)$ sends the blowing up of $X$ at $P$ to a curve which is dominated by the exceptional divisor: this curve is rational.
$\mathcal{F}(-P)$ is different from $\mathcal{F}(-Q)$, if $P$ and $Q$ are two general points of a general divisor in $\mathcal{F}$. This immediately implies that the natural map $Y \to Pic(X)$ (here $Y$ is the space which parametrizes the family) is constant, i.e. $\mathcal{F}$ is contained in a linear system. We want to prove now that $\mathcal{F}$ itself is a linear system.

Take $D, D' \in \mathcal{F}$ general divisors. Let $Z$ be the scheme-theoretic intersection of $D$ and $D'$. First we notice that $Z \neq \emptyset$. Otherwise we would have $\dim|D| \leq 1$, contrary to the fact that $\dim|D| \geq \dim \mathcal{F} = m \geq 2$. Furthermore we claim that, as $D'$ varies in $\mathcal{F}$, $Z$ describes a dense Zariski subset of $D$. Otherwise, since $D$ is irreducible, $Z$ would stay fixed. By blowing up $Z$ we would then reduce to the case $Z = \emptyset$ which we excluded already. This implies that we can choose a general point $P$ on $X$ in such a way that it lies on $D \cap D'$. Hence $D$ and $D'$ are connected by the $(m-1)$-dimensional linear system $\mathcal{F}(-P)$ inside $\mathcal{F}$. This proves that $\mathcal{F}$ itself is a linear system.
Suppose now the general divisor $D \in \mathcal{F}$ is reducible and $\dim(\mathcal{F}) > 1$. Let $P$ be a general point on $X$. Suppose $\mathcal{F}(-P)$ has no fixed divisor. By induction there is a pencil $f : X \to C$ and an involution $\mathcal{E}$ on $C$ such that $\mathcal{F}(-P) = f^*\mathcal{E}$. Since all divisors in $\mathcal{F}(-P)$ contain $P$, then all fibres of $f$ contain $P$. Let $d > 1$ be the degree of divisors in $\mathcal{E}$. Then a general divisor $D$ in $\mathcal{F}(-P)$ would consist of $d$ fibres of $f$, all passing through $P$. Hence $D$ would be singular at $P$, against the generality of $P$ and $D$. In conclusion $\mathcal{F}(-P)$ has a fixed divisor. This implies the claim. □

The study of varieties $X$ covered by degenerate subvarieties is a classical subject of investigation and the mess of results obtained in this theory is so big that we do not even try to give a short account here.

The starting case is given of course by ruled varieties, and this yet measures the task of classifying such objects.

Classically, the main tool was introduced by C. Segre in [68]: the spaces spanned by the degenerate subvarieties describe a family of linear spaces whose **foci** of any order determine linear systems, useful to reconstruct a canonical image of $X$.

For surfaces, we have the theorem 10 of Severi, which indeed in its full strength reads:

THEOREM 12. *Surfaces covered by a* 2*-dimensional family of plane curves are contained in* $\mathbb{P}^3$*, except for the Veronese surface and its projections.*

*Proof.* It remains only to prove that the curves are conics, when $r > 3$. This follows from the classical 3-secant lemma: if through a general pair of points of $X$ one finds a plane curve of degree bigger than 2, then the secant line through 2 general points of $X$ meets $X$ elsewhere. This is forbidden in characteristic 0: a general hyperplane section would be a curve in $\mathbb{P}^{r-1}$, $r-1 > 2$, whose general secant line is (at least) trisecant. □

EXAMPLE 14. Assume that $X$ is covered by a $(m+1)$-filling family of subvarieties of $\mathbb{P}^m$. Then $X$ lies in $\mathbb{P}^{m+1}$.

We refer to [60] or [53] for an account of results on varieties which contain big families of linear spaces.

An account of the method of focal loci in the theory of degenerate subvarieties is given in [52]. Let us just mention the following result by C. Segre (see [70], [52]):

THEOREM 13. *Let $X$ be a surface in $\mathbb{P}^5$ containing a 2-dimensional family of curves of $\mathbb{P}^3$. Then either:*
*(a) $X$ is contained in a 3-dimensional rational normal scroll of degree 3, or:*
*(b) $X$ is contained in the cone over a Veronese surface; or:*
*(c) the curves in the family have degree $\leq 5$.*

## 5. The Theorem session

We have now enough methods to start with some results on effective varieties. We follow here the chronological output of results, starting with Severi's classical theorem on 1-defective surfaces, up to recent results on the classification of defective threefolds and products.

We use as our main tool, an extensive inspection of the properties of the contact loci $\Sigma(H)$. We know that they are degenerate subvarieties. They are also uniquely determined by the points $P_0, \dots, P_k$. As these points move, the contact loci $\Sigma(H)$ describe a family of positive dimensional degenerate subvarieties of $X$.

PROPOSITION 11. *Assume $\nu_k(X) > 0$. Then the contact loci $\Sigma(H)$ determine a flat family $\mathcal{S}$ of subvarieties of $X$ with the following property: for a general choice of the points $P_0, \dots, P_k$, the set of elements in $\mathcal{S}$ passing through $P_0, \dots, P_k$ is a projective space.*

*Proof.* Let $V$ be the system of hyperplanes in $\mathbb{P}^r$. Then the family $\mathcal{S}$ is dominated by an open subset of the singular incidence variety

$$J = \{(P_0, \dots, P_k, H) \in X^{k+1} \times V : H \in V(-2P_0 - \cdots - 2P_k)\}$$

and if $(P_0, \dots, P_k, H)$, $(P_0, \dots, P_k, H')$ have the same locus, then the same holds for any $H''$ general in the pencil determined by $H, H'$. □

THEOREM 14. **(Severi)** *Let $X \subset \mathbb{P}^r$ be a 1-defective surface. Then $r \geq 5$ and $X$ is either a cone or the Veronese surface $V(2,2)$.*

*Proof.* Notice that surfaces in $\mathbb{P}^4$ cannot be defective, by exercise 26. On the other hand, cones in $\mathbb{P}^5$ and $V(2,2)$ are indeed 1-defective, by exercise 12 and section 4.2. So let $X \subset \mathbb{P}^r$, $r \geq 5$, be 1-defective. The defect $\delta_1$ is forced to be 1: it is positive and smaller than 2, because it is smaller than the projection defect, which cannot be 2 by Corollary (4).

For a general hyperplane $H$ which is tangent to $X$ at two points, call $\Sigma(H)$ the contact variety. We know that $\Sigma(H)$ has positive dimension. It cannot coincide with $X$, so $\nu_1(X) = 1 = \delta_1$.
Formula 5 thus says that $\Sigma(H)$ imposes at most 3 conditions to the hyperplane system. It follows that $\Sigma(H)$ is either a line or a plane curve.
Remind that for a choice of two general points of $X$ there exists an element in the family described by $\Sigma(H)$ which contains $P_0, P_1$. Since $\Sigma(H)$ is a plane curve, it has degree 2, by proposition 10.
We have two cases: either $\Sigma(H)$ is an irreducible conic, or it is a pair of incident lines, one for each point $P_0, P_1$.
Assume first that $\Sigma(H)$ is reducible. Then $X$ is ruled. Furthermore if $L_0, L_1$ are the lines passing through two general points, they meet somewhere. Just as in the proof of lemma 1, this is possible if and only if the meeting point is fixed. So $X$ is a cone.
Assume that $\Sigma(H)$ is irreducible. The classical theorem 10 of Severi proves that the only surface in $\mathbb{P}^5$ with a 2-dimensional family of generically irreducible conics is the Veronese surface $V(2, 2)$. The claim follows. □

A refinement of Severi's theorem, which works for $k$-defective surfaces, any $k$, was obtained by Terracini in [75]. Probably because it was published in a not–so–widely–distributed journal, most people was not aware of this result. Terracini's classification was rediscovered in recent times by Adlansvik and Dale ([1], [32]).

THEOREM 15. *Let $X$ be a surface with $\delta_k > 0$. Then either:*
*(i) the contact curve of a general $(k+1)$-tangent hyperplane is irreducible, and then $r = 3k+2$, $\delta_k = 1$ and $X$ is the 2-Veronese embedding of a rational normal surface $Y$ of degree $k$ in $\mathbb{P}^{k+1}$; or,*
*(ii) the contact curve of a general $(k+1)$-tangent hyperplane is reducible, and then $X$ sits in a $(s+2)$-dimensional cone over a curve, with vertex a linear space of dimension $s \leq k-1$ and $r \geq 2k+s+3$. The minimal such $s$ is characterized by the property that $X$ is $s$-defective but not $(s-1)$-defective and one has $\delta_k \geq k-s$.*

*Proof.* The proof relies on Severi's result, but of course it requires more technicalities, mainly on the contact variety $\Sigma(H)$. Details are omitted here.
The starting point is the observation that, under our assumptions, a general $m$-th tangential projection, for some $m \leq k-1$, maps $X$ either to a Veronese surface or to a cone. The former case happens when $\Sigma(H)$ is an irreducible curve, which thus maps, under a general $m$-th tangential projection, to a conic. So $\Sigma(H)$ is rational and determines a $(k+1)$-dimensional linear system. The corresponding map $X \to \mathbb{P}^{k+1}$ realizes $X$ as a surface of degree $k$.
If a general $m$-th tangential projection maps $X$ to a cone, then clearly $X$ itself sits naturally in a cone over a curve. We just need some projective argument to minimize the dimension of the vertex. We refer the reader to the computations in the last section of [16]. □

EXERCISE 50. Prove that the surfaces of type (i) and (ii) in the previous theorem are indeed $k$-defective.

Turning to higher dimensional cases, of course everything becomes much more complicate. First of all, the contact variety $\Sigma(H)$ now needs not being a divisor: it may be a curve. Even worse: we have now two essentially different definitions for the contact locus.

The situation was classically explored and solved, in the case $k = 1$, by Scorza, who found in [63] the following classification:

THEOREM 16. *(**G. Scorza**, [63]) An irreducible, non-degenerate, projective* 3*-fold* $X \subset \mathbb{P}^r$ *is* 1*-defective if and only if* $r \geq 6$ *and* $X$ *is of one of the following types:*
*(i)* $X$ *is a cone;*
*(ii)* $X$ *sits in a* 4*-dimensional cone over a curve;*
*(iii)* $r = 7$ *and* $X$ *is contained in a* 4*-dimensional cone over the Veronese surface* $V(2,2)$ *in* $\mathbb{P}^5$*;*
*(iv)* $X$ *is the Veronese variety* $V(2,3) \subset \mathbb{P}^9$ *of quadrics in* $\mathbb{P}^3$ *or a projection of it in* $\mathbb{P}^r$, $r = 7, 8$*;*
*(v)* $r = 7$ *and* $X$ *is a hyperplane section of the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$ *in* $\mathbb{P}^8$.

Scorza's result is based on an accurate analysis of surfaces obtained from $X$ in a tangential projection (see [18]).

Assuming that $X$ is a 1-defective threefold, we know that a general tangential projection $\pi_k : X \to X_1$ has fibers of dimension 1 or 2. With respect to the singular defect, the following cases may occur:
(1) $X_1$ is a curve, $\pi$ has fibers of dimension 2 and the singular defect is 2;
(2) $X_1$ is a surface, $\pi$ has fibers of dimension 1 and the singular defect is 2;
(3) $X_1$ is a surface, $\pi$ has fibers of dimension 1 and the singular defect is 1.

Case (2) happens when the surface $X_1$ has singular defect 1: for instance it may be defective itself, or simply it may be 1-weakly defective.

To produce a result for threefold which works for every $k$, we need thus a deeper analysis on surfaces whose general $k$-contact locus has positive dimension, i.e. $k$-weakly defective surfaces. A classification for these surfaces is the main result of [16]:

THEOREM 17. *Let* $X \subset \mathbb{P}^r$ *be a reduced, irreducible, non degenerate, projective surface which is* $k$*-weakly defective, but not* $k$*-defective.*
*Then* $k \geq 1$ *and either:*
*(i) the contact curve of a general* $(k+1)$*-tangent hyperplane is irreducible, and then either* $k = 0$ *and* $X$ *is the tangent developable to a curve, or;*
*(ii)* $r = 9$, $k = 2$ *and* $X$ *is the* 2*-Veronese embedding of a surface of degree* $d \geq 3$ *in* $\mathbb{P}^3$*, or;*
*(iii)* $r = 3k+3$ *and* $X$ *sits in the cone with vertex a point over a* $k$*-defective surface, or;*
*(iv)* $r = 3k+3$ *and* $X$ *is the* 2*-Veronese embedding in* $\mathbb{P}^r$ *of a surface* $Y$ *of degree* $k+1$ *in* $\mathbb{P}^{k+1}$ *with curve sections of arithmetic genus* 1*, or;*

*(v) the contact curve of a general $(k+1)$-tangent hyperplane is reducible, and then $X$ sits in a $(s+2)$-dimensional cone over a curve, with vertex a linear space of dimension $s \le k$ and $r \ge 2k+s+3$. The minimal such $s$ is characterized by the property that $X$ is $s$-weakly defective but not $(s-1)$-defective.*

Assuming that $X$ is a $k$-defective threefold, and taking the minimal $k$ for which this happens, then we know that the $(k-1)$-st tangential projection $X_{k-1}$ is a 1-defective threefold. Mixing the classifi cation of Scorza with the previous classifi cation of defective surfaces, one obtains:

THEOREM 18. *(**Chiantini-Ciliberto**, [19]) Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \ge 2$. Then $X$ is in the following list:*
*(1) $X$ is contained in a cone over the 2-uple embedding of a threefold $Y$ of minimal degree in $\mathbb{P}^{k+1}$, with vertex either a point (hence $r = 4k+2$) or a line (hence $r = 4k+3$);*
*(2) $k = 3$ and either $r = 14 = 4k+2$ and $X$ is the 2-uple embedding of a hypersurface $Y$ in $\mathbb{P}^4$ of $\deg(Y) \ge 3$ or $r = 15 = 4k+3$ and $X$ is contained in the cone with vertex a point over the 2-uple embedding of a hypersurface $Y$ as above;*
*(3) either $r = 4k+2$ and $X$ is the 2-uple embedding of a threefold $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1 or $r = 4k+3$ and $X$ is contained in the cone with vertex a point over the 2-uple embedding of a threefold $Y$ as above ;*
*(4) $r = 4k+3$ and $X$ is the 2-uple embedding of a threefold $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of genus 0 which is either a cone with vertex a line over a smooth rational curve of degree $k$ in $\mathbb{P}^{k-1}$ or it has a double line;*
*(5) $k = 4$, $r = 4k+3 = 19$ and $X$ is the 2-uple embedding of a threefold $Y$ in $\mathbb{P}^5$ with $\deg(Y) \ge 5$, contained in a quadric;*
*(6) $r = 4k+3$ and $X$ is the 2-uple embedding of a threefold $Y$ of degree $k+1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 2;*
*(7) $r = 4k+3$ and $X$ is contained in a cone with vertex a space of dimension $k$ over the 2–uple embedding of a surface $Y$ of minimal degree in $\mathbb{P}^{k+1}$;*
*(8) $k = 2$, $r = 4k+3 = 11$ and $X$ sits in a cone with vertex a line over the 2–uple embedding of a surface $Y$ of $\mathbb{P}^3$ with $\deg(Y) \ge 3$;*
*(9) $r = 4k+3$ and $X$ sits in a cone with vertex of dimension $k-1$ over the 2–uple embedding of a surface $Y$ of degree $k+1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1;*
*(10) $X$ is contained in a cone with vertex of dimension $k-1$ over a surface which is not $k$–weakly defective;*
*(11) $X$ is contained in a cone with vertex of dimension $2k$ over a curve;*
*(12) $k = 2$, $r = 4k+2 = 10$, and $X$ is contained in a cone with vertex of dimension 2 over a curve; (13) $r = 4k+5$ and $X$ is the 2-uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$;*
*(14) $r = 4k+4$ and $X$ is the projection of the 2-uple embedding $Y' \subset \mathbb{P}^{4k+5}$ of a threefold $Y$ of minimal degree in $\mathbb{P}^{k+2}$ from a point $P \in \mathbb{P}^{4k+5}$;*
*(15) $r = 4k+3$ and either $X$ is the projection of the 2-uple embedding $Y' \subset \mathbb{P}^{4k+5}$ of a*

*threefold* $Y$ *of minimal degree in* $\mathbb{P}^{k+2}$ *from a line* $\ell \subset \mathbb{P}^{4k+5}$*, or* $X$ *is contained in the intersection of a space of dimension* $4k+3$ *with the Segre embedding of* $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ *in* $\mathbb{P}^{k^2+4k+3}$.

*Cases (1)–(9) correspond to the case in which the tangential contact locus is an irreducible divisor, cases (10)–(12) correspond to the case in which the tangential contact locus is reducible, cases (13)–(15) correspond to the case the tangential contact locus is an irreducible curve.*

*All threefolds in this list are actually k–defective.*

One should notice that all the listed types refer to few constructions: mainly varieties defined by linear systems of quadrics, starting with very special varieties (typically of low degree); or varieties contained in cones with small vertex, or varieties contained in some product.

Going to the next step (fourfolds), very few results are known. We have no advances since a result of Scorza ([65]) were a partial classification of 1-defective fourfolds is shown.

THEOREM 19. *(***G. Scorza***) Let* $X$ *be a smooth, non-degenerate, projective* 4*-fold* $X \subset \mathbb{P}^r$ *which is* 1*-defective with defect* $\delta_1 = 1$*. Then* $r \geq 6$ *and:*
*(i) if* $X$ *has singular defect* 3*, then it lies in a* 5*-dimensional cone over a curve or in a* 6*-dimensional cone over a Veronese surface;*
*(ii) if* $X$ *has singular defect* 1 *then it is the projection of some hyperplane section of* $\mathbb{P}^2 \times \mathbb{P}^3$.

No particular characterization has been found for defective 4-folds with $\nu_1 = 2$.

For specific varieties, as Grassmannians or Segre products, we have wider results on the dimension of the secant varieties, as we yet observed in section 3.5.

Finally let me skip to the other end of our theory.

Assuming that $X$ is a defective variety of large dimension, can we limit the defect?

EXAMPLE 15. We yet observed that the defect of any $n$-dimensional variety is at most $n-1$. On the other hand, cones over curve have defect exactly equal to $n-1$. So one cannot hope to improve the limit on the defect, in the category of irreducible varieties.

Imposing new conditions on $X$, however, it turns out that the defect is bounded. The main stream explored so far relates the maximal defect with the dimension of the singular locus of $X$. It is always a surprise to find out that the local structure of $X$ reflects the geometry of the embedding: the absence of singularities forces tangent spaces to be not–too–wildly glued together. This is the kernel of the celebrated Zak's result on linear normality:

THEOREM 20. *(**Zak**, [82]) Assume that $X$ is smooth. Then*

$$\dim(S_1(X)) \geq \frac{3}{2}n + 1 \qquad i.e. \quad \delta_1(X) \leq \frac{n}{2}.$$

EXERCISE 51. Prove that Zak's results imply the following statement:
Any smooth variety of dimension $n$ in $\mathbb{P}^r$, $r \leq (3/2)n + 1$, is linearly normal.

This theorem which bounds the defect is indeed a consequence of Fulton–Hansen connectedness principle:

THEOREM 21. *(**Fulton-Hansen**, see [41] or [42]) If $Z \to \mathbb{P}^m \times \mathbb{P}^m$ is a morphism whose image has dimension at least $m$, then the inverse image of the diagonal is connected.*

Let us see how the connectedness theorem implies the bound on the defect.

*Proof.* (proof of Zak's theorem) Assume that the first defect of $X$ is bigger than $3n/2$. Then taking a general tangential projection from a point $P \in X$, we get a variety $X_1$ of dimension $\dim(X_1) < n/2$. In other words, the projection $X \to X_1$ has fibers of dimension bigger than $n/2$. It follows that for $Q \in X$ general, the space $T = T_{P,Q}(X)$, which maps to the tangent space to $X_1$ at the image of $Q$, is tangent along a fiber $Y$ over $Q$. Write $t$ for the dimension of $T$. We have, by assumptions, $t \leq 3n/2$. Fix a general linear space $L$ of dimension $r - t - 1$, such that $L \cap T = \emptyset$. The projection $X \to \mathbb{P}^t$ from $L$ is finite. Consider now the associated map $Z = X \times Y \to \mathbb{P}^t \times \mathbb{P}^t$. It is finite, so its image has dimension at least $\dim(Z) \geq n + \dim(Y)$. Since $\dim(Y) > n/2$, we may apply the connectedness theorem. It turns out that the inverse image of the diagonal is connected. This inverse image contains any pair $(y, y)$, $y \in Y$. Assume it contains also a pair $(y, x)$, $y \neq x$. This means that $y, x$ have the same image in $\mathbb{P}^t$, hence the line $< x, y >$ meets $L$. We can move $y$ to the general point $P$ of $Y$ and $x$ to $y$, so that the line $< x, y >$ tends to a tangent line to $X$ at $P$, a contradiction since $T_P(X)$ does not meet $L$. Hence no line joining points of $Y$ to points of $X - Y$ can meet $L$. It follows that all these lines lie in $T$, whence $X$ itself lies in $T$: a clear contradiction. □

In fact Zak's result is much more precise: it says that:

THEOREM 22. *If the singular locus of $X$ has dimension $b$, then for any subvariety $Y \subset X$ of dimension $m$, if a linear space $T$ is tangent to $X$ at every point of $Y - X$, then*

$$\dim(L) \geq m + n - b - 1.$$

EXERCISE 52. Use the previous result and the procedure above to prove Zak's theorem 8 on the image of the Gauss map.

Zak's results also determine a classification of smooth varieties for which the maximal first defect $\delta_1(X) = n/2$ occurs. They are called **Severi varieties** in honor of Severi's result on the classification of defective surfaces.

Even a sketch of the classification arguments are beyond the scopes of these notes and the reader is referred to Zak's book [82]. The starting point, however, easily derives from the previous analysis of the contact locus:

EXERCISE 53. The contact locus $\Sigma$ of a Severi variety is a hypersurface of degree at most 2.
Indeed we know from theorem 6 that $\Sigma$ imposes at most 2+(n/2) conditions to hyperplanes, so it lies in a projective space of dimension at most $1 + (n/2)$. As it has dimension at least $n/2$, it is a hypersurface. Through any pair of points $P, Q \in X$ there is a contact variety $\Sigma$. If $\deg(\Sigma) > 2$, then it meets the general secant line $< P, Q >$ in a third point, contradicting the trisecant lemma.

Indeed one shows that since $X$ is smooth, then $\deg(\Sigma) = 2$.

THEOREM 23. *The Severi varieties are:*
*(1) the Veronese surfaces in* $\mathbb{P}^5$ *(dimension* $2$*);*
*(2) the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$ *in* $\mathbb{P}^8$ *(dimension* $4$*);*
*(3) the Plücker embedding of the Grassmannian* $G(1, 5)$ *in* $\mathbb{P}^{14}$ *(dimension* $8$*);*
*(4) a Spinor variety of dimension* $16$ *in* $\mathbb{P}^{26}$.

EXERCISE 54. Show that the varieties of the first three types in the previous classification are indeed Severi varieties. (Also varieties of the fourth type are Severi varieties, but proving this fact is *not* just an exercise!).

Zak's classification also suggests that varieties of given dimension and defect cannot be realized in arbitrarily large projective space. Indeed in the last chapter of its book [82], he finds:

THEOREM 24. *Let $X$ be a smooth, non-degenerate variety of dimension $n$ and defect* $\delta_1 > 0$ *in* $\mathbb{P}^r$. *Then*

$$r \leq \frac{n(n + \delta_1 + 2) + e(\delta_1 - e - 2)}{2\delta_1}$$

*where $e$ is the remainder of* $n : \delta_1$.
*The maximum is attained only for:*
*(1) the* $2$*-Veronese embedding of* $\mathbb{P}^n$ $(\delta_1 = 1)$*;*
*(2) the Segre embedding of* $\mathbb{P}^m \times \mathbb{P}^m$ *or* $\mathbb{P}^m \times \mathbb{P}^{m+1}$ $(\delta_1 = 2)$*;*
*(3) the Plücker embedding of the Grassmannian* $G(1, 1 + (n/2))$ $(\delta_1 = 4)$*;*
*(4) the Severi variety of dimension* $16$.

Observe that the structure of these examples are always the same. In fact one obtains an infinite hyerarchy of examples, except for the Spinor variety, which stops with dimension 16.

EXERCISE 55. The maximal defect $\delta_2(X)$ of a smooth variety $X$ satisfies $\delta_2(X) \leq (3n/2) - 1$.

Indeed we know that two general tangent spaces to $X$ span at least a space of dimension $1 + (3n/2)$, moreover the image of the second tangential map $X \to X_2$ is at least a curve. It turns out that the tangent spaces at three general points of $X$ span a space of dimension at least $3 + (3n/2)$.

EXAMPLE 16. Catalano Johnson ([10]) found examples of rational scrolls on curves which are smooth, but whose image in a general tangential projection is a cone. Thus they reach the bound $\delta_2(X) = 3 + (3n/2)$.
An easy example of this type is the scroll $X = S(2, q)$ of dimension 2, obtained by joining corresponding points of a conic and a normal rational curve of degree $q \gg 0$. The plane spanned by the conic is contracted to a point under a general tangential projection, which thus sends $X$ to a cone over a curve.

Using rational scrolls, Catalano Johnson in fact proves that essetially all defects can be obtained, in the obvious range.

## 6. Under construction

Let me list, in this final section, whose boundary is supposed to move (fast?) onward, some open problem and some related topic where the theory has its natural developments.

### 6.1. The main stream

As observed above, our theory stops with the classification of defective threefolds, even for the first defect. Scorza's result on fourfolds is by no means complete, for varieties with singular defect 2.

Indeed one may hope to find general results when the singular defect is maximal or minimal. For very big defects, one gets:

THEOREM 25. *Assume that the first tangential projection sends $X$ to a curve $X_1$. Then $X$ is either a cone over a curve or a cone over a Veronese surface in $\mathbb{P}^5$.*

More general results may be obtained with the procedure used by Scorza in his investigation of defective fourfolds:

EXERCISE 56. Find a classification of varieties $X$ which are 1-defec- tive, with $\delta_1(X) = 1$ and $\nu_1(X) = n - 1$ (the divisorial case).

EXERCISE 57. Find a classification of varieties $X$ which are 1-defec- tive, with $\delta_1(X) = 1$ and $\nu_1(X) = 1$ (the curvilinear case).

In this kind of results, one has to work with the projective extensions of (usually rational) varieties in some class, in order to perform induction on the dimension of $X$.

A flavour of these results is given by statements of the following type:

PROPOSITION 12. *(see [6]) Let $X \in \mathbb{P}^r$ be a reduced, irreducible, non–degenerate, 1-defective variety of dimension n which is a developable scroll. Then X is a cone over a developable scroll.*

Going back to Scorza's analysis of defective 4-folds, one may observe that he considers only the case $\delta_1(X) = 1$. One clearly has:

EXERCISE 58. If $\delta_1(X) > 1$, then a general hyperplane section of $X$ is 1-defective.

Still, even applying induction, it is not clear which varieties of dimension 4 have a general hyperplane section fitting in the list of defective threefolds.

This is indeed a piece of a well–known, difficult problem of determining whether or not a given variety is the hyperplane section of a variety of higher dimension (except for cones). The extension problem has been studied by several classical and modern mathematicians, and there is no evidence that in its specific applications to defective varieties is simpler with respect to the general theory.

As soon as one classifies $k$-defective varieties for a fixed $k$, one has a guess for determining $(k + 1)$-defective varieties. These are varieties whose general tangential projection fits in the previous list. Unfortunately there is no simple way to see directly when some variety is the (even birational!) projection of something living in some higher dimensional projective space. This a piece of the so–called **geometric linear normality** problem, which is not completely understood, even for (singular) complete intersection curves.

Let me point out something missing in Scorza's analysis of defective fourfolds.

If $\delta_1(X) = 1$ and the singular defect is 2, then a general tangential projection sends $X$ to a threefold $X_1$ with the following property: a general tangent hyperplane to $X$ at one point is in fact tangent to $X$ along a curve. These are 0-weakly defective threefolds. So in order to classify defective fourfolds, the situation of weakly defective threefold must be understood. We do not have, up to now, a classification of $k$-weakly defective threefolds. This seems one of the main task necessary to extend our knowledge of defective varieties of dimension 4.

Let me stress again, however, that many particular results are known when one looks at some specific class of projective varieties, even in dimension bigger than 4. In particular, many cases of Grassmannians and Segre products are understood, while the classification of defective Veronese varieties is complete.

### 6.2. Grassmann defective varieties

Going back to the classical point of view, one may generalize the study of secant varieties in several ways.

For instance: instead of asking the reconstruction of points $P$ in the ambient space as a linear combination of $k+1$ points of $X$, one may try to obtain any *pair* of points $P, Q \in \mathbb{P}^r$ with linear combinations of the *same* $k+1$ points of $X$. In fact, this is equivalent to ask that the line joining two general points $P, Q \in \mathbb{P}^r$ lies in some $(k+1)$-secant $k$-space. Of course this makes sense when $S_k(X) = \mathbb{P}^r$.

Let us formalize the consequent theory:

DEFINITION 13. *For non–negative integers $h \leq k$ define the $k$-***Grassmann** $(h,k)$**-secant variety** $G_{h,k}(X)$ *of $X$ to be the (reduced) closure of the set:*

$$\{L \in G(h,r) : L \text{ lies in the span of } k+1 \text{ independent points of } X\}$$

*inside the Grassmannian $G(h,r)$ of $h$-planes in $\mathbb{P}^r$ (remind: projective dimensions).*

We have an obvious diagram:

$$G_{h,k}(X) \leftarrow \mathcal{G} \rightarrow G_k(X)$$

where $G_k(X)$ is the $k$-th Grassmann secant variety of $X$ defined in §2.1 and $\mathcal{G}$ indicates the universal Grassmannian of $h$-spaces inside the elements of $G_k(X) \subset G(k,r)$.

The expected dimension of the Grassmann secant variety $G_{h,k}(X)$ comes from the case where the leftmost map in the previous diagram is (generically) finite.

EXERCISE 59. Prove that the expected dimension of $G_{h,k}(X)$ is

$$\min\{(k+1)(r-k), (k+1)n + (h+1)(k-h)\}$$

and it is equal to the effective dimension if and only if a general $h$-space which lies in some $(k+1)$-secant $k$-space is in fact contained only in a finite number of such spaces.

So one says, as usual, that $X$ is **Grassmann** $(h,k)$**-defective** as soon as $G_{h,k}(X)$ has not the expected dimension.

The dimension of $G_{h,k}(X)$ has something to do with the projection of $X$ from a general set of $h+1$ points (i.e. from a general $h$-space). Namely it concerns the "exceptional" secant spaces or the singularities which may arise in $h+1$ successive general projections of $X$. Indeed one has:

REMARK 6. Fix a general $h$-space $L$ and assume that $L$ does not intersects $X$ (i.e. assume that $n+h<r$). Call $X_L$ the projection of $X$ from $L$. Then $X_L$ acquires a new $(k+1)$-secant $(k-h-1)$-space if $L \in G_{h,k}(X)$.
In particular for $k = h+1$, $X$ acquires a new $(k+1)$-fold point when $L \in G_{h,k}(X)$.

The reader should be advised that, unfortunately, projecting from some $L \in G_{k-1,k}(X)$ is *not* likely to be the unique way in which the projection of $X$ may acquire a new $(k+1)$-fold point $P$. Indeed in principle such points may arise projecting from some highly tangent space $L$ which is *not* a limit of $(k+1)$-secant spaces.
The problem is not definitely settled. However there is a construction by Flenner which seems to obtain multiple points of the previous type in the projection of smooth, complete intersection varieties of high dimension. We refer the interested reader to the discussion in [15] and its bibliography.

In the Waring setting of example 1, the problem of finding when $G_{h,k}(X)$ coincides with the Grassmannian $G(k,r)$ for some Veronese embedding $X$ of $\mathbb{P}^n$ is equivalent to ask for a "simultaneous" decomposition of $h+1$ general forms as linear combination of the *same* $k+1$ powers of linear forms.
In this formulation, the problem was classically considered by Terracini (see [76]) and Bronowski (see [7]). In particular Terracini found:

THEOREM 26. *(**Terracini**) If $X = V(2,3)$ is the 3-Veronese embedding of the plane $\mathbb{P}^2$ in $\mathbb{P}^9$, then $X$ is Grassmann $(1,4)$-defective. No other Veronese embedding of $\mathbb{P}^2$ is Grassmann defective.*

Terracini's method is based on the following lemma, which reduces Grassmann defectivity to the "usual" defectivity of some product of $X$ (see [34] for a modern proof):

LEMMA 3. *$X$ is Grassmann $(h,k)$-defective if and only if $\mathbb{P}^h \times X$ is $k$-defective (in its Segre embedding).*

EXERCISE 60. Assume that $X$ is Grassmann $(h,k)$-defective. Then prove that either $X$ is also $k$-defective or $r < nk + n + k$.

The status of the theory can be actually resumed as follows:

There are no $(h,k)$-defective curves (see [17]).

For surfaces. A classification of smooth $(1,2)$-defective surface was achiev- ed in [22]. Fontanari gave a criterion for detecting the Grassmann $(1,k)$-defectivity of surfaces (see [40]). A complete classification of $(1,k)$-defective surfaces has been obtained in [20], using the classification of defective threefolds and lemma 3:

THEOREM 27. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective surface which is minimally $(1,k)$-defective. Then $k$ is even and $X$ is in the following list:*

*(1) $X$ is contained in a cone with vertex of dimension $\frac{k}{2}-1$ and not smaller over a curve $C$ with $\dim(< C >) \geq \frac{3}{2}k + 1$;*

*(2) $k = 4$ and $X$ is the 3–Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$;*

*(3) $X \subset \mathbb{P}^{2k+1}$ is a rational normal scroll $S(a_1, a_2)$ with $a_1 \geq \frac{k}{2}$.*

*All surfaces in the list are actually minimally $(1,k)$–defective. Surfaces of type (1) can be $(1,h)$–defective for $h > k$, whereas surfaces of types (2) and (3) are not $(1,h)$–defective for $h > k$.*

For higher dimensional varieties, very few things are known. Of course, for some Veronese embeddings $X$ of $\mathbb{P}^n$, as soon as we have a decription of the defectivity of some Segre product $\mathbb{P}^h \times \mathbb{P}^n$, then by lemma 3 we have information on the $(h,k)$-defectivity of $X$. Refer to [13] for details.

Let me cite the preprint of Coppens [31], where smooth Grassmann $(2, 3)$-defective threefolds are classified:

THEOREM 28. *A smooth Grassmann* $(2, 3)$*-defective threefold* $X \subset \mathbb{P}^r$ *is either:*
*(1) a threefold of minimal degree in* $\mathbb{P}^7$*;*
*(2) a threefold of minimal degree in* $\mathbb{P}^8$*;*
*(3) a projection in* $\mathbb{P}^7$ *of the previous threefold.*

Finally observe that, in the same setting, following some suggestions arisen from number theory, Voisin studied the general problem of determining linear spaces contained in some secant variety (see [79] and [73]). The main result is:

THEOREM 29. *Let* $X$ *be a smooth curve of genus* $g$ *and degree* $d$ *and assume* $d \geq 2g + 2k + 1$. *Then if* $L$ *is a linear space of dimension* $\geq k$ *contained in* $S_k(X)$, *we have* $\dim(L) = k$ *and* $L$ *is a space spanned by* $k + 1$ *independent points of* $X$ *or a limit of such spaces.*

### 6.3. The number of apparent secant spaces

If $P$ is a general point of the secant variety $S_k(X)$, then one may ask "how many" $(k + 1)$-secant $k$-spaces pass through $P$.
Clearly we have infinitely many such spaces as soon as $r < nk+n+k$. If $r \geq nk+n+k$, then as observed in the first section (see exercise 6) a general $P \in S_k(X)$ lies in finitely many secant $(k + 1)$-secant $k$-spaces, unless $X$ is $k$-defective.

DEFINITION 14. *Assume* $r \geq nk+n+k$ *and assume that* $X$ *is not* $k$*-defective. The (finite) number of* $(k + 1)$*-secant* $k$*-spaces passing through a general point* $P \in S_k(X)$ *is called the* **number of apparent** $k$**-secant spaces** *to* $X$ *and is indicated with* $App_k(X)$

When $r = nk + n + k$, that is when $S_k(X)$ is $\mathbb{P}^r$ (remind: we assume here that $X$ is not $k$-defective), then $App_k(X)$ is a powerful invariant of the projective embedding of $X$.
For instance, in the case $n = k = 1$, i.e. looking at secant lines to curves in $\mathbb{P}^3$, then $App_1(X)$ is exactly the number of nodes in a general projection of $X$ to $\mathbb{P}^2$. Hence it relates the degree (external invariant) with the genus (internal invariant). Furthermore Halphen's theory shows that this number is strictly influenced by the postulation of $X$.
For $k = 2$, $App_2(X)$ measures the number of "apparent trisecant lines" in a general projection of an $n$-fold $X$ to $\mathbb{P}^{3n+1}$. And so on.
These invariants seem to be not yet completely understood for $k > 1$, and their relation with the geometry of $X$ is quite unexplored. Just notice that even for surfaces $X \subset \mathbb{P}^5$, the relations between the number of apparent double points and the postulation of $X$ are almost completely obscure.

Remaining in the case $r = nk + n + k$, notice that $k$-defective varieties are just varieties such that one has 0 $(k + 1)$-secant $k$-spaces through a general point of $\mathbb{P}^r$. So varieties for which $App_k(X) = 1$ are eventually a generalization of defective varieties. Thus there is some hope that some methods introduced in the previous chapters lead to a classifi cation of such varieties.

Let me briefly recall the actual situation.

EXERCISE 61. Prove that the only irreducible curve $X \subset \mathbb{P}^3$ with $App_1(X) = 1$ is the rational normal curve.

EXERCISE 62. Prove that if $X \subset \mathbb{P}^3$ is a reducible, smooth curve with $App_1(X) = 1$, then $X$ is a disjoint union of two lines.

THEOREM 30. *Put $k = 1$, $r = 2n + 1$ and assume that $X$ is a smooth, irreducible variety of dimension $n$, which is not 1-defective (hence $S_1(X) = \mathbb{P}^r$). Assume $App_1(X) = 1$.*
*In the case of curves ($n = 1$) then $X$ is a rational normal cubic in $\mathbb{P}^3$ (classical).*
*If $n = 2$ (surfaces in $\mathbb{P}^5$), then $X$ is either a quartic normal scroll or a Del Pezzo quintic (Russo, [61]).*
*Smooth threefolds in $\mathbb{P}^7$ with $App_1(X) = 1$ are either $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or the residual intersection of $\mathbb{P}^1 \times \mathbb{P}^3$ and a quadric, with respect to one 3-dimensional ruling or scrolls in planes.* (refer to [27] for details).

When $k$ is bigger, then few things are known for dimension $> 2$. We have for curves the following result by Ciliberto and Russo ([28], but see also [11] for the smooth case):

THEOREM 31. *Assume that $X$ is an irreducible curve in $\mathbb{P}^{2k+1}$ with $App_1(X) = 1$. Then $X$ is a rational normal curve.*

While for surfaces:

THEOREM 32. *(**Ciliberto - Russo**, see [28] for details) Let $X \subset \mathbb{P}^{3k+2}$, $k \geq 2$ be an irreducible surface which is linearly normal and satisfies $App_k(X) = 1$. Then $X$ is one of the following:*
*(1) A rational normal scroll with sectional genus 0.*
*(2) A Castelnuovo surface of odd degree (Castelnuovo surfaces are surfaces of minimal sectional genus with respect to $r$) or an internal projection of such surfaces from three points of $X$.*
*(3) The 5-Veronese embedding $V(2, 5)$ of $\mathbb{P}^2$ or its general tangential projection from 0, 1, 2, 3 points.*

The main tools for the previous results rely in the observation that the number $App_k(X)$ is bounded by, and sometimes it is equal to, the degree of a general tangential projection of $X$. This can be proved using some degeneration argument, but we are not going through the details here.

When $r > nk+n+k$ (and $X$ is not $k$-defective) the situation becomes easier. Indeed as a consequence of the infi nitesimal Bertini's theorem 5 for non–defective varieties $X$, we get a description of the intersection of $X$ with a general $(k+1)$-tangent space:

EXERCISE 63. Let $X \subset \mathbb{P}^r$ be an irreducible, reduced, non-degenerate projective variety of dimension $n$ and let $k$ be a non-negative integer such that $k < r - n$. Let $P_0, \dots, P_k$ be general points of $X$. Then the schematic intersection of $X$ with the subspace $< P_0, \dots, P_k >$ is the union of the points $P_0, \dots, P_k$.
By taking the section of $X$ with a general subspace of codimension $n-1$, it suffices to prove the assertion only for curves. Then, by taking the projection of the curve $X$ from a general point $P \in X$, it suffices to prove the assertion for $k = 1$, in which case it is a consequence of the trisecant lemma.

THEOREM 33. *Let $X \subset \mathbb{P}^r$ be a reduced, irreducible, non degenerate, projective variety. Then the general point of every irreducible component of the contact variety of a general $(k+1)$-tangent hyperplane $H$ is a double point for $H$. If, in addition, $X$ is not $k$-weakly defective for a given $k$ such that $r \geq (n+1)(k+1)$, then given $P_0, \dots, P_k$ general points on $X$, the general $(k+1)$-tangent hyperplane $H \in |H(-2P_0 - \cdots - 2P_k)|$ is tangent to $X$ only at $P_0, \dots, P_k$.*

*Proof.* We may assume that $X$ is smooth, by passing, if necessary, to a resolution of the singularities.
Use the notation of theorem 5. Consider the variety $Y \subset |H|$ of hyperplanes which are tangent to $X$ at $k+1$ general points $P_0, \dots, P_k$. Since $X$ is not $k$-weakly defective, a general hyperplane $H \in Y$ has isolated singularities at $P_0, \dots, P_k$. Then the infinitesimal Bertini theorem tells us that the tangent space to $Y$ at $H$ is contained in $|H(-P_0 - \cdots - P_k)|$. Since the two spaces have the same dimension, they coincide. On the other hand $|H(-P_0 - \cdots - P_k)|$ is cut out on $X$ by the hyperplanes through $P_0, \dots, P_k >$. Then exercise 63 and the infinitesimal Bertini theorem again forbid the presence of singularities for $H$ other than $P_0, \dots, P_k$. □

EXERCISE 64. Prove that indeed for a general hyperplane $H$ which is tangent to $X$ at general points $P_0, \dots, P_k$, the intersection $H \cap X$ has ordinary double points at the $P_i$'s.

As a consequence we find:

THEOREM 34. *Assume that $X \subset \mathbb{P}^r$, $r > nk + n + k$, is not $k$-defective. Then $App_k(X) = 1$, unless $X$ is $k$-weakly defective.*
*Conversely assume that $X$ is $k$-weakly defective, but not $k$-defective. Call $\Sigma$ a general $k$-contact locus of $X$. If $\Sigma$ is irreducible, then $App_k(X) = App_k(\Sigma)$. If $\Sigma$ is reducible, $\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_k$, then for general points $P_0 \in \Sigma_0, \dots, P_k \in \Sigma_k$ and for $A$ general in the span $< P_0, \dots, P_k >$, there are exactly $App_k(X)$ $(k+1)$-tuples of points $\{P_{0i}, \dots, P_{ki}\}$ with $P_{ij} \in \Sigma_j$ and $A \in < P_{0i}, \dots, P_{ki} >$.*

*Proof.* Take $A \in S_k(X)$ general and assume that $A \in < P_0, \dots, P_k >$ and $A \in < Q_0, \dots, Q_k >$ with $P_i, Q_j \in X$ and $\{P_0, \dots, P_k\} \neq \{Q_0, \dots, Q_k\}$. Since $A$ is general, then the tangent space of $S_k(X)$ at $A$ coincides both with the span $T_{P_0,\dots,P_k}$ and

with $T_{Q_0,\dots,Q_k}$. Thus these spans are equal.
It follows that all the hyperplanes of $\mathbb{P}^r$ which are tangent to $X$ at $P_0,\dots,P_k$ (they exist by the assumption $r > nk+n+k$), also are tangent to $X$ at $Q_0,\dots,Q_k$. Since the two sets of points are different, this contradicts theorem 33.
To see the converse, assume $\Sigma$ irreducible (the reducible case is left to the reader as an exercise).
For a general choice of $P_0,\dots,P_k \in X$, by our assumption, there exists a hyperplane $H$ which is tangent to $X$ at the $P_i$'s. Call $\Sigma$ its contact locus.
Then the inequality $App_k(\Sigma) \leq App_k(X)$ follows from the generality of $P_0,\dots,P_k$.
To see the inverse inequality, let $\Sigma$ is a general (irreducible) $k$-contact locus and fix a general point $A \in S_k(\Sigma)$. Then $A$ is also a general point of $S_k(X)$, so there are exactly $m = App_k(X)$ choices of $(k+1)$-tuples of points $\{P_{01},\dots,P_{0k}\},\dots,\{P_{m1},\dots,P_{mk}\}$ such that $A \in < P_{0i},\dots,P_{ki} >$. It follows:

$$T_{P_{01},\dots,P_{0k}} = \cdots = T_{P_{m1},\dots,P_{mk}} = T_{S_k(X),A}$$

thus any hyperplane tangent to $X$ at the points of one of these $(k+1)$-tuples, is also tangent at all $P_{ij}$'s. Hence our original contact variety $\Sigma$, which is completely determined by one of these $(k+1)$-tuples and one hyperplane tangent to it, also contains all $P_{ij}$'s. The inequality $App_k(\Sigma) \geq m$ follows. □

So in particular we know that varieties which are not $k$-weakly defective in $\mathbb{P}^r$, $r > nk+n+k$, must have $App_k(X) = 1$. This is a bit surprising, for the case $App_k(X) = 1$ is exceptional, when $r = nk+n+k$.
Notice that the converse is false:

EXAMPLE 17. There are examples of $k$-weakly defective varieties, with $App_k(\Sigma) = 1$, thus satisfying $App_k(X) = 1$.
One of them is obtained as follows: consider a cone $W \subset \mathbb{P}^6$ over a Veronese surface $S$. Let $Z \subset W$ be the cone over a conic of $S$. Call $X$ the residual intersection of $W$ with a quadric passing through $Z$. Then $X$ is 1-weakly defective (by theorem 17) and one immediately sees that $\Sigma$ is a rational normal cubic. Thus $App_1(X) = 1$.

A list of surfaces in $\mathbb{P}^r$, $r > 5$ with $App_k(X) > 1$ was obtained by Dale ([32]) when $k = 1$. For general $k$ the classification has been obtained in [21], by mixing the previous result with the classification theorem 17 of weakly defective surfaces:

THEOREM 35. *Let $X$ be an irreducible surface in $\mathbb{P}^r$, $r > 3k+2$. Then $App_k(X) > 1$ if and only if $X$ is $k$-weakly defective, with the following exceptions:*
*(1) $X$ sits in the cone over a $k$-defective surface $X'$, 2-uple embedding of a minimal surface $Y \subset \mathbb{P}^{k+1}$, and $X \simeq 2h+f-2e$ in the Picard group of the desingularization of the cone, where $h$ is the transform of a hyperplane section of $Y$, $f$ is the transforms of a fiber of $Y$ and $e$ is the exceptional divisor.*
*(2) There is an irreducible curve $E$ and a non costant map $\phi : E \to \mathbb{P}^k$, whose image spans $\mathbb{P}^k$, such that $X$ is the ruled surface*

$$X = \cup_{P\in E} < P, \phi(P) > .$$

In higher dimensions, there is a recent result by Mella which computes $App_k(X)$ for some Veronese embeddings of $\mathbb{P}^n$:

THEOREM 36. *(see [51]) Let $X = V(n,d)$, $d > n$ be the Veronese embedding of degree $d$ of $\mathbb{P}^n$. Then $App_d(X) = 1$ if and only if $n = 2$ and $d = 5$.*

EXERCISE 65. Define and set the first properties of $App_{h,k}(X)$, the number of apparent $(k+1)$-secant $k$-spaces through a general $h$-space in $\mathbb{P}^r$.

Many other variation on the theme of secant varieties (as secant varieties of scrolls, see [9] or the behaviour of osculating spaces, see [77], or the behaviour of successive defects, see [37]) are not listed here. Also I do not go further in the possible generalization of the definition of defective objects.

Let me just finish with two remarks, which link the end of these notes with the initial problem.

REMARK 7. Even if in some applications it is relevant to know the general geometric properties of the secant varieties to some $X$, nevertheless one often faces the "membership problem": for a given projective varieties $X$ and a given point $P$, determine the minimum $k$ such that $P \in S_k(X)$, i.e. the minimum such that $P$ is linearly generated by $k+1$ points of $X$.
In practice, we would like to know the equations for $S_k(X)$. This is not an easy task. It can be solved for rational normal curves. But in general, even the degree of secant varieties is far from being easily calculated.
We refer to [11] and [59] for results on curves, to [49] and [78] for the case of Veronese re-embedding of some varieties.

REMARK 8. Assume we positively know that some $P$ is a general point of $S_k(X)$. Then we know that there is a linear compbination $\sum_{i=0}^{k} a_i P_i$, $P_i \in X$, which gives $P$. How can one find the points $P_i$'s starting with $P$?
This is relevant, for instance, when $X$ is the variety of decomposable tensors of some sort and we know that $S_k(X)$ fills the entire space of tensors $\mathbb{P}^r$. An algorithm which produces a decomposition of a tensors in elementary products would be of valuable help in many computations.
The problem would become easier as soon as $App_k(X) = 1$, for in this case the decomposition is uniquely determined by the (general) point $P \in S_k(X)$. For instance, we know that a general form of degree 5 in 3 variable can be decomposed in the sum of 5 powers of linear forms in a unique way. Can someone compute such a decomposition? As far as I know, only very partial classical results by Sylvester are known in this setting.
When a general $P \in S_k(X)$ lies in infinitely many $(k+1)$-secant $k$-spaces, an initial study for the variety $D_k(P) = \{(P_0, \ldots, P_k) \in X^{k+1} : P \in <P_0, \ldots, P_k>\}$ (a sort of "Moduli space" for the decompositions of $P$) can be found in [24] and [9].

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