RENDICONTI DEL SEMINARIO MATEMATICO

Università e Politecnico di Torino

Polynomial Interpolation and Projective Embeddings

Lecture Notes of the School

CONTENTS

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Preface

Two important research fields in Algebraic Geometry are the study of linear systems of curves with prescribed base points and the classification of defective varieties. These topics go back to the work of classical Italian authors, as F. Enriques, G. Scorza, B. Segre, C. Segre, F. Severi, A. Terracini, E. Togliatti and many others. These two research fields can be viewed in the more general setting of polynomial interpolation (the first one) and of projective embeddings (the second one).

On September 15–20, 2003 the meeting "School (and Workshop) on Polynomial Interpolation and Projective Embeddings" was held at the Department of Mathematics of the Politecnico di Torino.

The School was articulated in two series lectures delivered by L. Chiantini and R. Miranda. C. Bocci also gave exercises sessions. During the Workshop many researchers gave contributions concerning their results in the field. This special issue contains the expanded versions of the lectures given during the School.

The organizers would like to thank all the partecipants to the School/ Workshop and the Departement of Mathematics for the warm hospitality. Special thanks go to the main speakers for their work before, during and after the School/ Workshop.

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TOPICS ON INTERPOLATION PROBLEMS IN ALGEBRAIC GEOMETRY

Abstract. These are notes of the lectures given by the authors during the school/workshop "Polynomial Interpolation and Projective Embeddings". We mainly focus our attention on the planar case and on the Segre and Harbourne-Hirschowitz Conjectures. We discuss the state of the art introducing several results and different techniques.

1. Introduction

These are the expanded and detailed notes of the lectures given by the authors during the school and workshop entitled "Polynomial Interpolation and Projective Embeddings", held at the Politecnico di Torino during the period September 15 - 20, 2003.

The second author gave five lectures of length one hour each. He attempted to cover the basic facts in interpolation problems in algebraic geometry. Given the extensiveness of the subject, it was not possible to go into great detail in every proof. The first author gave two exercise sessions where he made examples and exercises and he introduced some topics that were complementary to the standard lectures.

We believe that these notes can be a valuable addition to the literature and give new details and new points of view of the subject that cannot be found in other expository work.

In the expository Section 2 we introduce the origin of the subject. We first focus our attention on the planar case and on the Segre and Harbourne-Hirschowitz Conjectures. We discuss the state of the art introducing several results and different techniques.

In Sections 3 and 4 we focus on one of these techniques based on the results of Lorentz and Lorentz [42] and others, which is related to a detailed study of the interpolation matrix.

Although the technique can be used more broadly, we will present the main ideas by concentrating on the study of linear systems in two variables with prescribed multiple base points, i.e., Hermite interpolation in two variables.

In Section 5 we will explain the essential features of a particular specialization technique introduced by Ciliberto and the second author in [22].

Although related closely to other specializations, the new feature is that the degeneration is not of sets of points, but, instead, we degenerate the surface where these points live. The idea is based on a degeneration method used by Z. Ran ([48]) to study enumerative problems on singular curves and consists in degenerating the plane to a reducible surface. The restriction of the limit linear system to the components of the surface are hopefully easier to understand than the system that one begins with.

In Section 6 we will explain some interesting applications of the previous degeneration technique. In particular, we will present some Lemmas that permit one to obtain information on the system by simply working with the degenerated system. At the end of the Section there will be some examples to illustrate and better understand these results.

In Section 7 we introduce a new topic for the interpolation problems: the concept of special effect varieties. These varieties are the main subject of the Ph.D. thesis of the first author. During the workshop he gave a communication about his recent results and we present this summary in the notes as a sixth lecture.

Both authors want to thank the main organizers of this School/Workshop, Gianfranco Casnati, Silvio Greco, Nadia Chiarli, Roberto Notari and Maria Luisa Spreafico. We are also most grateful to the participants with their mathematical discussions and communications that made for a very interesting and productive week in the lovely city of Torino.

2. Lecture one: overview

2.1. Interpolation problems

This section is dedicated to an overview on linear systems with base points and their relationship with polynomial interpolation.

Let us start with the following naive problem: fix points $\{P_i\}$ and values $\{c_i\}$; find f such that $f(P_i) = c_i$ for each i.

The first question we can pose is "from where do we take the function f?". Let us consider the case when f is a polynomial; to be specific, let us take $f \in V_d$, with $V_d = \{\text{polynomials of degree } \leq d\}$. Even in this case the nature of the solution depends on the **number of variables**.

In one variable, the classical polynomial interpolation theory of functions in numerical analysis and statistics gives that a single-variable polynomial $f \in K[x]$ of degree d over a field K is uniquely determined by $d+1$ distinct points $P_0, \dots, P_d$ on the affine line $\mathbb{A}^1_K$ and a set of values $c_i \in K, i = 0, \dots, d$ such that $f(P_i) = c_i$ for each $i = 0, \dots, d$. This is essentially due to the nonsingularity of the Vandermonde matrix.

We can generalize this problem slightly by asking not only for values of the function, but also for values of derivatives. Specifically, we can fix distinct points $z_0, \dots z_n$ and positive integers $m_i, i = 0, \dots, n$ such that $m_1 + \dots + m_n = d + 1$ and set the values of the derivatives:

$$f^{(j)}(z_i) = w_{i,j}, \quad i = 1, \dots, n, \quad j = 0, \dots, m_i - 1.$$

Again it is a standard exercise to show that one finds a unique polynomial $f(x)$ satisfying the previous conditions, for any desired values $\{w_{i,j}\}$. This is a linear problem in the vector space of polynomials $\{f\}$ of degree d, and if we set all values $w_{i,j}$ equal to zero we obtain the corresponding homogeneous linear problem, where we are seeking polynomials with values and derivatives equal to zero. In one variable, this linear problem has full rank, and the only solution to the homogeneous problem is the identically zero polynomial. We note that the only requirement is that the points $\{z_i\}$ be distinct; in particular it is not necessary that they be general in any way.

The situation for $r \geq 2$ variables is quite different. A polynomial of degree at most d in r variables $f \in K[x_1, \dots, x_r]$ depends on $N_{r,d} + 1 = \binom{d+r}{r}$ parameters. Suppose that we fix n points P_i in the r-dimensional affine space $\mathbb{A}^r_K$ and integers $m_1, \dots, m_n$ such that

$$\sum_{i=1}^{n} \binom{m_i+r-1}{r} = N_{r,d} + 1.$$

We can then impose that $D^{(j)} f(P_i) = 0, i = 1, \dots, n, j = 0, \dots, m_i - 1$, where $D^{(k)}$ is any derivative of order k. (This is the homogeneous linear problem.) In analogy with the one-variable case, we can ask if the only polynomial satisfying these conditions is identically zero; in several variables, there is as yet no answer to this problem in this generality.

Going back to our starting problem, it is possible to incorporate derivatives in a more general way. Define the set

$$D_d = \{\text{constant coefficient differentiable operators of order} \leq d\}.$$

If P is any point, then the mapping

$$\begin{array}{ccc} D_d \times V_d & \to & k \\ (L, f) & \mapsto & L(f)(P) \end{array}$$

is a perfect pairing. If we fix distinct points P_i and, for each i, fix a subspace $A_i \subseteq D_d$, we can pose the following problem: determine all $f \in V_d$ such that for each i, $L(f)(P_i) = 0$ for all $L \in A_i$. We denote by $\mathcal{L}_d(-\sum A_i P_i)$ the (projectived) subspace of polynomials verifying the previous condition, i.e.

$$\mathcal{L}_d(-\sum A_i P_i) = \{f \in V_d \text{ such that } L(f)(P_i) = 0, \forall L \in A_i, \forall i\}$$

minus zero, modulo scalars.

EXAMPLE 1. If $A_i =< I >$ for each i, then we are not asking for any derivatives; we are asking only for values. This case is called *Lagrange Interpolation*.

EXAMPLE 2. If $A_i = D_{m_i-1}$ for each i, then we are asking that all derivatives of order at most $m_i - 1$ are zero at P_i. Thus the coefficient of the Taylor expansions are zero up through order $m_i - 1$. This is a condition on the multiplicity of the polynomial at the point P_i; in particular it means that $\text{mult}_{P_i}(f) \geq m_i$. The corresponding interpolation problem is called *Hermite Interpolation*. The corresponding linear system is denoted by $\mathcal{L}_d(-\sum m_i P_i)$. This kind of interpolation is very important because it does not depend on the choice of coordinates.

EXAMPLE 3. Assume A_i is spanned by "monomials" $\frac{\partial^{a_1+a_2+\dots+a_r}}{\partial x_1^{a_1} \partial x_2^{a_2} \dots \partial x_r^{a_r}}$. The corresponding interpolation problem is called *Birkhoff Interpolation*.

EXAMPLE 4. Let C be a smooth curve through a point P and consider a polynomial f. We can ask $f_{|C}$ vanishes to order $\geq k$ at P on C. It is not difficult to see that this can be expressed with a particular k-dimensional subspace A_i as above. We refer to this type of problem as *Curvilinear Interpolation*.

EXAMPLE 5. The theory of splines can in some ways be considered as a generalization of interpolation problems. For splines, one considers collections of polynomials $(f_1, f_2, \dots, f_k) \in V \times \cdots \times V$, and imposes interpolation-type conditions on these collections. For example we can ask that

$$f_1(P_1) = f_2(P_1) \qquad \text{and} \qquad f_2(P_2) = f_3(P_3) \qquad \dots$$

2.2. The dimension problem

The main question we can pose in all previous examples is: what is the dimension of $\mathcal{L}_d(-\sum A_i P_i)$? As a first step we can define the **virtual dimension** of the system $\mathcal{L}_d(-\sum A_i P_i)$ as

$$\nu(\mathcal{L}_d(-\sum A_i P_i)) := \dim(\mathcal{L}_d) - \sum \dim A_i.$$

Note that this is the projective dimension; in particular we have that $\dim(\mathcal{L}_d) = N_{r,d} = \binom{d+r}{r} - 1$. This formula simply represents the expectation that each additional condition imposed by the space A_i will drop the dimension of the space by one. In other words, this formula will be true if all of the conditions imposed by the A_i are independent.

This number can be negative: in this case we expect that $\mathcal{L}_d(-\sum A_i P_i)$ is empty. We can then define the **expected dimension** of $\mathcal{L}_d(-\sum A_i P_i)$ as

$$\epsilon(\mathcal{L}_d(-\sum A_i P_i)) := \max\{\nu(\mathcal{L}_d(-\sum A_i P_i)), -1\};$$

here we take the convention that the empty projective space has dimension equal to -1.

REMARK 1. It is important to observe that the dimension (and all other phenomena) of the previous system depends in a critical way on the position of points. Consider, for example, a two-variable polynomial f of degree 5 vanishing at 8 points on a line l. The dimension of the space of quintics in two variables is 20, so that expected dimension of this linear system is $20 - 8 = 12$. However if f vanishes at the first 6 of the points, then by Bezout's Theorem f vanishes all along l, and therefore vanishes at all 8 of the points. Hence the conditions imposed by the vanishing at the final 2 points are not independent of the first 6 conditions; indeed, the dimension of this space is $20 - 6 = 14$, which is exactly the dimension of the space of residual quartics.

The "reason" this phenomenon has occured is that the points are related geometrically in an obvious way. To avoid this, we assume that the points P_i are in *general position*. This notion of general position means something different for every interpolation problem.

2.3. The interpolation matrix

To be more precise, we introduce a matrix M associated to the interpolation problem, called the *interpolation matrix*. This matrix has columns indexed by a basis $\{f_k\}$ for the vector space V from which we are drawing our polynomials and has rows doubly-indexed by the points P_i and a basis $D_{i,j}$ for the interpolation conditions A_i for each i. We form the entries of the interpolation matrix M by applying the $D_{i,j}$'s to the f_k's, and evaluating at P_i; specifically, the entry in row $(i, D_{i,j})$ and column f_k is $D_{i,j}(f_k)(P_i)$.

It is clear that the subspace of polynomials satisfying the interpolation problem may be identified with the kernel of multiplication by M. Therefore the dimension problem is equivalent to the computation of the rank of the interpolation matrix.

Now if we take the points P_i to have undetermined coordinates, then the various minors of M become polynomials in these coordinates. The largest size (say $s \times s$) minor of M which is not identically zero determines the rank of M, for values of the coordinates of P_i which makes at least one $s \times s$ minor nonzero. This condition (that at least one of these minors be nonzero) gives a Zariski open subset of the set parametrizing n points in r-space, and determines the precise notion of "general position" for this particular interpolation problem.

REMARK 2. Hermite interpolation, expressed via imposing multiplicities to polynomials at given points, lends itself also to working with homogeneous polynomials and points in projective space. In this way the multiplicity conditions give a homogeneous ideal in the homogeneous coordinate ring of projective space; the graded pieces (as we vary the degree d) represent the $\mathcal{L}_d$ interpolation problems. In this way the interpolation problem is equivalent to the study of the Hilbert function of the given ideal. Other commutative algebra tools now may come into play, and more complicated problems related to this ideal (such as determining generators, syzygies, ranks of multiplication maps, etc.) are of great interest also.

The Hermite interpolation problem for polynomials of degree d having multiplicities m_i at n points in general position will give a space which we will denote by $\mathcal{L}_d(m_1, \dots, m_n)$. This notation is convenient when we do not want or need to refer to the particular positions of the n points. If there are repetitions in the multiplicities, these might be denoted using superscripts; for example, $\mathcal{L}_d(m^n)$ means the linear system of polynomials of degree d having multiplicity m at n general points.

2.4. Special linear systems

As explained at the beginning of the section, the Hermite interpolation problem has full rank in one variable. Thus the above-mentioned questions are all relatively easy in $\mathbb{P}^1$. However when we consider $\mathbb{P}^2$, there is still much unknown. Here, as we will see later, we have some precise conjectures. From now on, we will assume that we are working with Hermite interpolation in two variables.

A naive conjecture would be that, for points in general position, every Hermite interpolation problem leads to a linear system which always has the expected dimension:

all Hermite conditions are linearly independent at distinct points. This turns out to be false, as the following example shows.

EXAMPLE 6. Consider the system of conics $\mathcal{L}_2$, and impose two double points P_1, P_2. The notation for this system would be: $\mathcal{L}_2(-2P_1 - 2P_2)$ or $\mathcal{L}_2(2^2)$. Since any double point imposes three conditions to a curve in $\mathbb{P}^2$, we obtain

$$\epsilon(\mathcal{L}_2(2^2)) = \nu(\mathcal{L}_2(2^2)) = 5 - 3 - 3 = -1$$

and we expect that the system is empty. But if $f(x, y, z) = 0$ is the homogeneous linear polynomial defining the line through P_1 and P_2, then $f(x, y, z)^2$ is a nonzero conic double at P_1 and P_2. This conic exists for any two distinct points P_1 and P_2, and in particular for general points; therefore $\dim(\mathcal{L}_2(2^2)) = 0 > -1 = \epsilon(\mathcal{L}_2(2^2))$. This system does *not* have the expected dimension. Moreover note that f^2 is singular (has multiplicity two) at every point of the line.

EXAMPLE 7. We have the same phenomenon with $\mathcal{L}_4(-\sum_{i=1}^{5} 2P_i) = \mathcal{L}_4(2^5)$ which is the linear system of quartics with five general double points. This system has expected dimension -1, but, if $q(x, y, z)$ is the polynomial of the conic through the points P_i's, then q^2 is a quartic double at every point of the conic, in particular at the five (general) points $P_1, \ldots, P_5$. This system does *not* have the expected dimension.

Note that in the first example, if we blow up the plane at the two points, the line in question becomes a (-1)-curve on the blowup, and the corresponding linear system on the blowup consists of this (-1)-curve, with multiplicity two. Similarly, in the second example, if we blow up the five points, the conic becomes a (-1)-curve, and the corresponding linear system becomes this curve, with multiplicity two.

These two examples show that the naive conjecture, that every such linear system in the plane has the expected dimension, is false.

DEFINITION 1. *A system* $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_n)$ *is* **special** *if its dimension is larger than the expected dimension:*

$$\dim(\mathcal{L}) > \epsilon(\mathcal{L});$$

otherwise $\mathcal{L}$ *is said to be* **non-special**.

Consider the blow-up $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of the plane $\mathbb{P}^2$ at $P_1, \ldots, P_n$ and let E_i, $i = 1, \ldots, n$ be the exceptional divisors corresponding to the blow-up of the points $P_i, i = 1, \ldots, n$. If we denote by H the pull-back of a general line of $\mathbb{P}^2$ via π, then we can write the strict transform of the system $\mathcal{L} := \mathcal{L}_d(\sum_{i=1}^{n} m_i P_i)$ as the complete linear system $\tilde{\mathcal{L}} = |dH - \sum_{i=1}^{n} m_i E_i|$. In the future, if confusion cannot arise, we will indicate both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\mathcal{L}$.

Note that on the blowup, $\nu(\mathcal{L}) = \frac{\mathcal{L}\cdot(\mathcal{L}-K)}{2}$.

By Riemann–Roch, remembering that $h^2(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) = 0$, we obtain

$$\dim(\mathcal{L}) = h^0(\tilde{\mathbb{P}}^2, \mathcal{L}) - 1 = \nu(\mathcal{L}) + h^1(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}). \tag{1}$$

Hence

(2) $\mathcal{L}$ is non-special if and only if $h^0(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) \cdot h^1(\tilde{\mathbb{P}}^2, \tilde{\mathcal{L}}) = 0$.

Using this cohomological reformulation, it is not hard to compute that if such a linear system $\mathcal{L}$ has a (-1)-curve E on the corresponding blowup with $\mathcal{L} \cdot E \leq -2$, then E occurs as (at least) a double fixed component of $\mathcal{L}$, and if $\mathcal{L}$ is not empty, it must be special. If this happens, we will call the linear system (-1)-*special*.

2.5. Conjectures in two variables

Going back to the conjectures for special systems in $\mathbb{P}^2$, B. Segre was the first author who claimed that speciality is related to the non-reducedness of the general curve of the given linear system with general multiple base points.

CONJECTURE 1 ((SC) B. SEGRE, 1961). If a linear system of plane curves with general multiple base points $\mathcal{L}_{2,d}(-\sum_{i=1}^n m_i P_i)$ is special, then its general member is non-reduced, i.e. the linear system has, according to Bertini's theorem, some multiple fixed component.

In 1987, Gimigliano [34] studied several examples of special linear systems on $\mathbb{P}^2$ and made the previous Conjecture more precise.

CONJECTURE 2 ((GC) A. GIMIGLIANO, 1987). Consider a linear system of plane curves with general multiple base points $\mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$. Then one has the following possibilities:

(i) the system is non-special and its general member is irreducible;

(ii) the system is non-special, its general member is non-reduced, reducible, its fixed components are all rational curves, except for at most one (this may occur only if the system has dimension 0), and the general member of its movable part is either irreducible or composed of rational curves in a pencil;

(iii) the system is non-special of dimension 0 and consists of a unique multiple elliptic curve;

(iv) the system is special and it has some multiple rational curve as a fixed component.

This conjecture, in the case of special systems, was made more precise by the following conjecture given separately by B. Harbourne and A. Hirschowitz.

CONJECTURE 3 ((HHC) HARBOURNE–HIRSCHOWITZ, 1989). A linear system of plane curves $\mathcal{L} := \mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ with general multiple base points is special if and only if it is $(-1)-$special, i.e. it contains some multiple rational curve of self-intersection -1 in the base locus.

The last conjecture we want to mention is related to the homogeneous case, i.e. when $m_1 = m_2 = \cdots = m_n = m$.

CONJECTURE 4 ((NC) NAGATA, 1960). $\mathcal{L}_d(m^n)$ is empty as soon as $n \geq 10$ and $d \leq \sqrt{n} \cdot m$

Recently, Ciliberto and Miranda, in [24], proved the following implications

$$\begin{array}{ccc} SC & \Longleftrightarrow & HHC \\ & \Downarrow & \\ & NC & \end{array}$$

Although the conjectures are still unproved it is important to note that, in more than a century of research, all known special systems are consistent with them.

2.6. Results to date

We now mention some results on these conjectures, in particular on the conjecture of Harbourne and Hirschowitz.

The first case we treat is $\mathcal{L}_d(1^n)$, that is, when all points have multiplicity one. In this case we are asking for polynomials of degree d that simply vanish at the points. It is easy to see that this always has the expected dimension. One argues by induction on the number of points n; the statement is clearly true for $n = 0$. Assume it is true for $n - 1$, and consider the system $\mathcal{L}_d(1^n)$, which is a subsystem of the system $\mathcal{L}_d(1^{n-1})$. These two systems, unless they are both empty, have expected dimensions which are different by one, and we must show that indeed $\mathcal{L}_d(1^n)$ has dimension one less. This is equivalent to showing that it is a proper subsystem of $\mathcal{L}_d(1^{n-1})$. It will be if we choose the n-th point so that it is not a base point of the system. This is true if the points are in general position. This proves the following:

THEOREM 1 (MULTIPLICITY ONE THEOREM). *If the points $P_i, \ldots, P_n$ are in general position, then the dimension of $\mathcal{L}(-\sum_{i=1}^n P_i)$ is equal to the expected dimension.*

Consider now the linear systems $\mathcal{L}_{2,d}(-\sum_{i=1}^n m_i P_i)$. When the number of points n is less than or equal to 9, the anticanonical class $-\tilde{K}$ is effective on the blowup $\tilde{\mathbb{P}}^2$ and we can use vanishing theorems (Kodaira's and Kawamata-Vieweg's or Mumford's and Franchetta-Ramanujam's on the specific case of surfaces) to establish that $h^1(\tilde{\mathcal{L}}, \tilde{\mathbb{P}}^2) = 0$ and use (2). In this way one proves the following result already known to Castelnuovo and later rediscovered by several authors:

THEOREM 2 (CASTELNUOVO, 1891; NAGATA, 1960; GIMIGLIANO 1986; HARBOURNE, 1986). *The Harbourne–Hirschowitz Conjecture holds for all linear systems with $n \leq 9$ general multiple base points.*

The second simple case is the one with only double points, i.e $m_1 = \cdots = m_n = 2$.

This case was examined by several authors, e.g. Campbell, Palatini, Terracini. More recently, Arbarello and Cornalba used an approach based on an infinitesimal deformation technique consisting in moving the base points of the system and computing the first order deformation of a curve which moves keeping its singularities. In general, in these deformation techniques, one tries to show that if there exists $C \in \mathcal{L}$ with isolated singularities, then $H^1(\mathcal{L}) = 0$ (and conclude that $\mathcal{L}$ is non-special). In order to do this one tries to interpret this H^1 as an obstruction space to deforming $C \in \mathcal{L}$ and to prove that every element of H^1 occurs as an obstruction. In essence one tries to construct a map

$$\{\text{ Deformations of } P_i\} \xrightarrow{\text{obstruction to moving } C} H^1(\mathcal{L})$$

and show that it is onto; the existence of C for points in general position allows one to claim that it is also zero, and one deduces that the $H^1 = 0$. Using this general idea, Arbarello and Cornalba proved the following:

THEOREM 3 (ARBARELLO–CORNALBA, 1981). *Consider* $\mathcal{L} = \mathcal{L}_d(2^n)$. *Assume:*

(i) $\frac{d(d+3)}{2} \geq 3n$, *i.e.* $v(\mathcal{L}) \geq 0$*;*

(ii) $\binom{d-1}{2} \geq n$, *i.e.* $g_{\mathcal{L}} \geq 0$.

Then $\mathcal{L}$ is non-special, and a general $C \in \mathcal{L}$ is irreducible, with nodes at the imposed general double points $P_1, \ldots P_n$, and smooth elsewhere, except for $\mathcal{L}_6(2^9)$ which is a double cubic.

Another result by (slightly different) deformation techniques is the following

THEOREM 4 (A. BRUNO, 1998). $\mathcal{L} = \mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ *is non-special if* $v(\mathcal{L}) \geq 0$ *and* $g_{\mathcal{L}} \geq 0$ *and the general curve has ordinary* m_i*-tuple points at* P_i*,* $i = 1, \ldots, n$.

Although the hypothesis is rather strong, the main tool in Bruno's proof is the use of moduli space of curves, of stable reduction, and of the theory of limit linear system that is a really new idea in this setting.

A different way to attack the problem is to argue by *degeneration*. In this technique, we specialize the base points of the linear system so as to make it easier to compute the dimension of the system. Since the dimension of $\mathcal{L}(-\sum_{i=1}^n m_i P_i)$ is upper semi-continuous in the position of the points, it is enough to find a particular set of points $Q_1, \ldots Q_h$ such that $\mathcal{L}(-\sum_{i=1}^n m_i Q_i)$ is non-special to conclude that also the general system $\mathcal{L}(-\sum_{i=1}^n m_i P_i)$ is non-special. In general, we try to put the points P_i in a special enough position that we can compute the dimension, but not so special that the dimension will rise.

EXAMPLE 8. Consider, for example, the system $\mathcal{L}_5(2^7)$. Its expected dimension is $20 - 7 \cdot 3 = -1$; in other words, we expect that this system (of quintics with seven general double points) is empty. Put three of the seven points on a line l. In this case, by Bezout's Theorem, every element in $\mathcal{L}$ contains the line. Then one has

$$\mathcal{L} = l + \mathcal{L}_4(2^4, 1^3)$$

and so the dimension is equal to the dimension of the system $\mathcal{L}_4(2^4, 1^3)$ (where the three simple points are collinear). Now if C is a conic through the four points that appear with multiplicity 2 in $\mathcal{L}_4(2^4, 1^3)$ and through one of the three points with multiplicity 1, one has

$$C \cdot \mathcal{L}_4(2^4, 1^3) = 8 - 8 - 1 = -1$$

and therefore C must be a base curve of the system. Since there are three such curves, we see that this system has a sextic in its base locus; but it only has degree four. We conclude that the system must be empty.

Unfortunately, very often, convenient particular positions of the points do increase the dimension of the system. Therefore this technique has its limitations.

In [35], Hirschowitz was able to improve this degeneration technique, introducing the Horace Method (*la méthode d'Horace*). This technique is not only applicable to the planar case, but can be used on every projective variety.

Using a refined version of the Horace Method, (the so-called *differential Horace Method*, see [6]), Alexander and Hirschowitz were able to prove the following asymptotic result:

THEOREM 5 (ALEXANDER–HIRSCHOWITZ, 1998). *Given any projective, reduced variety X and an ample line bundle $\mathcal{L}$ on it, there is a function $d(m)$ such that if $m_i < m$, $i = 1, \dots, n$, and $d > d(m)$ then $\mathcal{L}^{\otimes d}(-\sum_{i=1}^n m_i P_i)$ is non-special.*

The prototype for results of this sort is the following theorem of Hirschowitz ([36]):

THEOREM 6. *The system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ in $\mathbb{P}^2$ is non-special as soon as*

$$\left\lceil \frac{(d+3)^2}{4} \right\rceil > \sum_{i=1}^{n} \binom{m_i + 1}{2}.$$

Returning to the general Harbourne–Hirschowitz Conjecture, we mention some other recent results.

If we pass to the quasi-homogeneous case, i.e. all m_i's equal to m except one, the Harbourne–Hirschowitz Conjecture is proved for $m = 2, 3$ by Ciliberto and Miranda [22] and for $m = 4$ by Siebert and (independently) Laface (see [38]).

The following theorem is due to T. Mignon in his thesis ([46], [47]) and it is based on the use of the Horace method:

THEOREM 7 (T. MIGNON, 1998). *Let $\mathcal{L} = \mathcal{L}_{2,d}(-\sum_{i=1}^n m_i P_i)$. Then:*

(i) if $m_i \leq 4$ then the Harbourne–Hirschowitz Conjecture 3 holds;

(ii) if $g_{\mathcal{L}} \leq 4$ and $\nu(\mathcal{L}) \geq 0$ then the Harbourne–Hirschowitz Conjecture 3 holds;

(iii) if $m_i \leq 3$, $d \geq 33$, $\nu(\mathcal{L}) \geq 0$ and $g_{\mathcal{L}} \geq 0$ then the Harbourne–Hirschowitz Conjecture 3 holds.

Recently, S. Yang was able to generalize part (i) of the previous result to $m_i \leq 6$. She uses a combination of the Ciliberto–Miranda degeneration with a particular specialization of the points on a fixed line with a fixed point (see [52]).

Another result to mention is the following

THEOREM 8 (L. EVAIN, 1998). *$\mathcal{L}_d(m^n)$ is never special if n is of the form $n = 4^k$.*

The same result was obtained in [11] by A. Buckley and M. Zompatori using a degeneration technique; moreover they proved the same statement for the case $n = 9^k$, and for products of powers of 4 and 9.

Recently, Ciliberto and the second author, using a particular degenerations technique (which we will describe in sections 5 and 6) were able to prove the following (see [23]):

THEOREM 9 (CILIBERTO–MIRANDA,1998). *The Harbourne–Hirschowitz Conjecture holds in the quasi-homogeneous cases $\mathcal{L}_d(m_0, m^n)$, $m \leq 3$ and in the homogeneous cases $\mathcal{L}_d(m^n)$, $m \leq 12$.*

Another result of Ciliberto and Miranda in [22] is the full classification of homogeneous $(-1)-$special systems.

THEOREM 10 (CLASSIFICATION OF THE HOMOGENEOUS $(-1)-$SPECIAL SYSTEMS). *The only homogeneous linear systems $\mathcal{L}_d(m^n)$ which are $(-1)-$special are:*

$$\mathcal{L}_d(m^2) \text{ with } m \leq d \leq 2m-2$$

$$\mathcal{L}_d(m^3) \text{ with } \tfrac{3m}{2} \leq d \leq 2m-2$$

$$\mathcal{L}_d(m^5) \text{ with } 2m \leq d \leq \tfrac{5m-2}{2}$$

$$\mathcal{L}_d(m^6) \text{ with } \tfrac{12m}{5} \leq d \leq \tfrac{5m-2}{2}$$

$$\mathcal{L}_d(m^7) \text{ with } \tfrac{21m}{8} \leq d \leq \tfrac{8m-2}{3}$$

$$\mathcal{L}_d(m^8) \text{ with } \tfrac{48m}{17} \leq d \leq \tfrac{17m-2}{6}$$

For homogeneous systems, with all multiplicities equal, the Harbourne – Hirschowitz conjecture is then equivalent to stating that the only such systems that are special are on the above list. In particular, the conjecture implies that for nine or more points, there are no special homogeneous systems.

REMARK 3. The way Ciliberto and Miranda degenerate the systems can be seen as a way to degenerate the set of points $P_1, \dots, P_n$ by putting b of them on a line, and letting the line split from the curves of the linear system k times. This approach seems to be very systematic and, in [26] C. Ciliberto, F. Cioffi, R. Miranda and F. Orecchia applied a more refined computational algebra approach to improve the bound $m \leq 12$. In particular they have been able to work out a computer program to test the Harbourne–Hirschowitz Conjecture for $\mathcal{L}_d(m^n)$ and to prove it for $m \leq 20$.

2.7. Higher dimensions and Waring's problem

As we said in the first section, the general problem of computing the dimension of a system with imposed multiple points can be formulated in any dimension and for any ambient variety X, not only in the plane. But, unfortunately, just in the simplest case of $X = \mathbb{P}^r$, $r \geq 3$ very little is known. In this setting we fix notation and define $\mathcal{L}_{r,d} := |\mathcal{O}_{\mathbb{P}^r}(dH)|$.

The most important result is a theorem due to Alexander and Hirschowitz which classifies the special linear systems with imposed double points $\mathcal{L}_{r,d}(2^n)$.

THEOREM 11 (ALEXANDER–HIRSCHOWITZ, 1996). *The linear system $\mathcal{L}_{r,d}(2^n)$ is non-special unless r, d, and n are in one of the columns of the following table:*

r	*any*	2	3	4	4
d	2	4	4	4	3
n	$2, \dots, r$	5	9	14	7

The original proof of this theorem requires the Horace method, and occupies a whole series of papers [1], [2], [3], [4], [5]. Another proof, somewhat shorter, was recently given by K. Chandler in [19]. She still used the Horace method but in a particularly efficient way, specializing part of the points to a hyperplane.

Let us analyze the systems in Theorem 11. It is very easy to show that linear systems $\mathcal{L}_{r,2}(2^n)$ with $2 \leq n \leq r$ are special. We know that every quadric hypersurface is defined by a quadratic polynomial, which, if we homogenize, can be considered as a quadratic form in $r + 1$ variables. This in turn can be considered as a symmetric matrix Q of size $r + 1$. We can choose coordinates so that the first $r + 1$ points (if there are that many) occur at the "coordinates points" whose homogeneous coordinates correspond to the standard basis vectors, i.e. the points $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, etc. For the quadric hypersurface to have multiplicity at least two at $(1, 0, 0, \dots, 0)$, the first row (and column) of the matrix Q must be zero. This clearly imposes $r + 1$ conditions, as we expect. However, for the quadric to have multiplicity at least two at the second point $(0, 1, 0, \dots, 0)$, the second row and column of Q must be zero. If the first row and column are already zero, the first entries of the second row and column are automatically zero, so there are only r additional entries that must be zero. Hence the second point imposes only r conditions and the dimension of $\mathcal{L}_{r,2}(2^n)$ is one larger than the expected. This phenomenon continues until there are $r + 1$ points, in which case the matrix Q is all zero and there are no nontrivial quadratic polynomials satisfying the conditions: if $n \geq r + 1$ then $\mathcal{L}_{r,2}(2^n)$ is empty as we expect.

The cases $\mathcal{L}_{2,4}(2^5)$, $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$ are similar. The first one is already treated in example 7. For $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$ we observe that both have expected dimension -1, but there is an element given by the double of the quadric respectively in $\mathbb{P}^3$ and $\mathbb{P}^4$ through the 9 and the 14 points.

Finally consider the system $\mathcal{L}_{4,3}(2^7)$. Its virtual dimension is -1 whereas it is not empty. In fact there is an unique rational normal quartic curve C_4 through 7 general points $P_1, \dots, P_7$ in $\mathbb{P}^4$. Let X be the first secant variety of C, i.e. $X := \mathrm{Sec}_1(C)$. Then X is a cubic hypersurface and it is singular along C; therefore it is singular at $P_1, \dots, P_7$. Thus X sits in $\mathcal{L}_{4,3}(2^7)$.

More recently, some results on the higher dimension case are given by Bocci ($\mathbb{P}^r$ and $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_t}$), Catalisano, Geramita and Gimigliano ($\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_t}$) and De Volder, Laface and Ugaglia ($\mathbb{P}^3$).

Let $Q = [q_o : \cdots : q_r] \in \mathbb{P}^r$. We define the differential operator $\Delta_Q = \sum q_i \frac{\partial}{\partial x_i}$. Moreover, given a set of n points $\{Q_i\}$, we define

$$A_d(\sum Q_i) = \{\sum M_i \Delta_{Q_i}^{d-1}, \deg(M_i) = 1\}.$$

Thus we have a pairing

$$\begin{array}{ccc} \{\text{differential operators of degree} \le d\} & \longleftrightarrow & \{\text{polynomials of degree } \le d\} \\ \cup & & \cup \\ A_d(\sum Q_i) & & \mathcal{L}_d(-\sum 2Q_i) \end{array}$$

By Terracini's Lemma, with this pairing, $A_d(\sum Q_i)$ and $\mathcal{L}_d(-\sum 2Q_i)$ annihilate each other.

If we let W be the d-Veronese variety of $\mathbb{P}^r$, the space $A_d(\sum Q_i)$ can be identified with the tangent space to the n-secant variety to W (at the point corresponding to the n points Q_i). Therefore information about the dimension of $\mathcal{L}_d(\sum 2Q_i)$ will give information about the dimension of this secant variety. In particular, when this secant variety is the whole space, then the general form of degree d can be written as a sum of n pure d-th powers of linear forms. This is a version of Waring's Problem for Forms, and is a beautiful application of the Alexander–Hirschowitz theorem.
Recently, in [13], Carlini proposed an interesting generalization of Waring's problem.

3. Lecture two: the matrix approach I

3.1. Visualizing Birkhoff interpolation

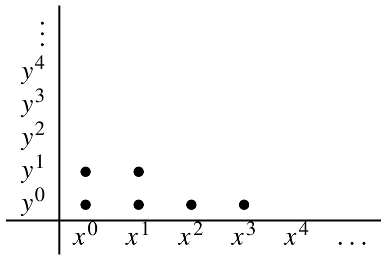
In the previous lecture we did not mention the results of Lorentz and Lorentz [42] and others, which use a different technique based on a detailed study of the interpolation matrix. We will now present the essential features of this approach.

Although the technique can be used more broadly, we will present the main ideas by concentrating on the study of linear systems in two variables with prescribed multiple base points, i.e., Hermite interpolation in two variables. We recall that a polynomial f

has multiplicity at least m at a point P if all derivatives of f, up through order $m-1$, are zero at P.

The technique for Hermite interpolation however immediately leads to considerations of the more general Birkhoff interpolation, where one considers a general set of derivatives (which may be considered as "monomial" differential operators) to be zero at a given point. Using suitable coordinates centered at the point in question, this means that the polynomial is contained in an ideal generated by monomials, which defines a zero-dimensional scheme. A graphical representation of such an ideal is often useful, where one uses the lattice of all monomials (in the first quadrant of the plane) and indicates those monomials which are not in the ideal.

If, for example, the ideal is given by $< y^2, x^2y, x^4 >$, one may visualize it as follows:



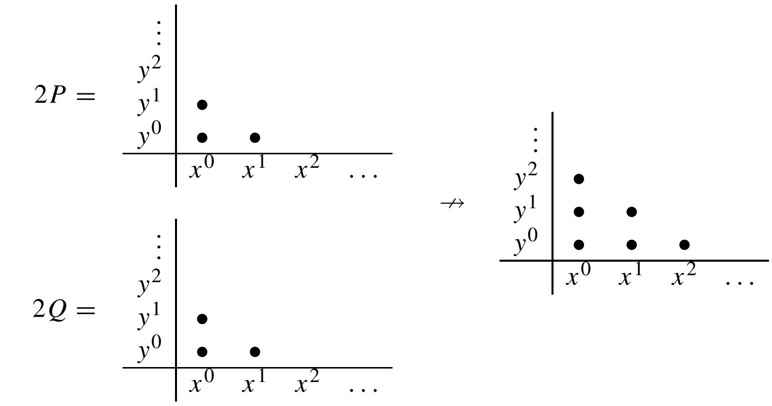
One may ask two natural questions about these zero-dimensional schemes:

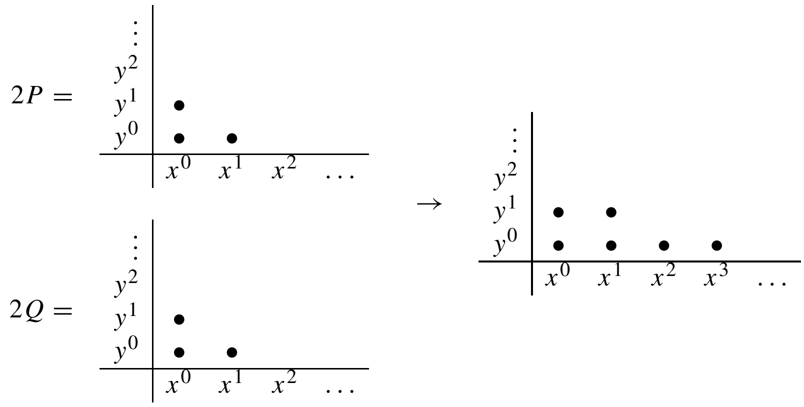
1) How do these zero-dimensional schemes degenerate ?

2) How do these zero-dimensional schemes collide?

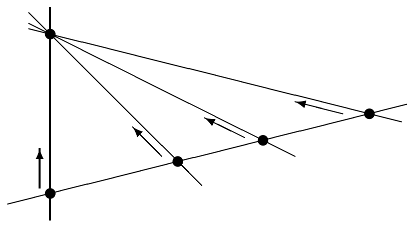
If the movement is along an axis, the answer is obtained by just "stacking" the monomials. In other words, if an ideal I_1 has the monomials $\{x^i y^j \mid 0 \le j < r_i\}$ not in it, and a second ideal I_2 has the monomials $\{x^i y^j \mid 0 \le j < s_i\}$ not in it, then the flat limit of these two zero-dimensional schemes, if they approach each other along the y-axis, has the monomials $\{x^i y^j \mid 0 \le j < r_i + s_i\}$ not in it. (This is a relatively easy exercise which we encourage the reader to attempt.)

This stacking algorithm however is not the only type of collision that can occur with such monomial ideals, and the question of what actually is possible to obtain as a flat limit is quite delicate. For example, if one stacks two ordinary double points, one obtains the tacnode ideal generated by $\{y^2, x^2y, x^4\}$ drawn above, which has the six monomials $\{1, y, x, xy, x^2, x^3\}$ not in it. Are there other possibilities for the collision of two double points? Flatness requires that the codimension of the ideals must be preserved; since a double point ideal has codimension three, the collision of two double points must have codimension six. Is a triple point (which also has codimension six) possible? We will see later that the collision of two double points can not be a triple point, even though both have codimension six: a triple point scheme is not the (flat) limit of two double point schemes.





Another limit that is not so obvious is the following one, of four multiple points on a line approaching a single point:



We can ask for example what is the limit here as the points come together.

3.2. The interpolation matrix (revisited)

Going back to our interpolation problem, fix a vector space V of bivariate polynomials which are spanned by monomials $x^i y^j$ indexed by a set of lattice points S. This means that a typical polynomial in V has the form

$$f(x, y) = \sum_{(i,j)\in S} a_{i,j} \cdot x^i y^j$$

where the numbers $a_{i,j}$ are the coefficients of the term $x^i y^j$.

Birkhoff interpolation (at a point $P = (x_P, y_P)$) would impose some (monomial) differential operators A at P, i.e. $L(f(P)) = 0$, for $L \in A$. Therefore A is spanned by monomial differential operators, e.g.

$$\Delta = \frac{\partial^{a+b}}{\partial x^a \partial y^b}.$$

Define

$$T_m = \{(r, s) | r \geq 0, s \geq 0, r + s < m\}$$

to be the "triangle" of derivatives orders at most $m - 1$. Thus, with Hermite interpolation, for any i, $A_i = T_{m_i}$ for some integer m_i. We can write $f \in V$ as a row-column product:

$$f = \sum a_{i,j} x^i y^j = \begin{pmatrix} \dots & x^i y^j & \dots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix}.$$

To analyze the condition that $\Delta(f)(P) = 0$, one takes the row vector of monomials

$$\begin{pmatrix} \dots & x^i y^j & \dots \end{pmatrix},$$

applies Δ, and evaluates at P:

$$\begin{pmatrix} \dots & \Delta(x_P^i y_P^j) & \dots \end{pmatrix}.$$

This gives a new row, and finally we ask the product

$$\begin{pmatrix} \dots & \Delta(x_P^i y_P^j) & \dots \end{pmatrix} \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix}$$

has to be zero.

The generalization of this process lead us to the definition of the *Matrix of the Interpolation Problem*. For that, consider

1. an integer n, the number of points;
2. for each i, $1 \leq i \leq n$, a point $P_i := (x_i, y_i)$;
3. For each i, $1 \leq i \leq n$, a finite set A_i of (monomial) differential operators.

From now on we assume that $\dim(V) = \sum_{i=1}^{n} \dim(A_i)$ so that we have a square linear problem. Denote by $\mathfrak{A} := \{A_i\}_{i=1}^{n}$. Let us denote the matrix corresponding to the linear system by $M_S(A_1, \ldots, A_n)$ or $M_S(\mathfrak{A})$ and its determinant by $D_S(\mathfrak{A}) = \det(M_S(\mathfrak{A}))$. The columns of $M_S(\mathfrak{A})$ are indexed by the monomials $x^j y^k$ in V. The rows of $M(\mathfrak{A})$ are doubly indexed by i and then by single partial differential operators in A_i. Thus the $(i, r, s) - (j, k)$ entry of $M_S(\mathfrak{A})$ is given by

$$\frac{\partial^{r+s} x^j y^k}{\partial x^r \partial y^s}(P_i) = r!s!\binom{j}{r}\binom{k}{s} x_i^{j-r} y_i^{k-s}.$$

The interpolation matrix itself is then

$$M_S(\mathfrak{A}) = \left|\begin{array}{c} \hline \\ P_1 \\ \\ \hline \\ P_2 \\ \\ \hline \\ P_3 \\ \\ \hline \\ \vdots \\ \\ \hline \end{array}\right| \begin{array}{l} A_1\{ \\ A_2\{ \\ A_3\{ \\ \vdots \end{array}.$$

Note that if the coordinates of the P_i's are undetermined, then $D_S(\mathfrak{A})$ is a polynomial in 2n variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. For consistency we will order the monomials and the derivatives in degree lexicographic order. Every vector $(\ldots, a_{i,j}, \ldots)$ such that

$$M_S(\mathfrak{A}) \begin{pmatrix} \vdots \\ a_{i,j} \\ \vdots \end{pmatrix} = 0$$

is the vector of coefficients of a polynomial satisfying the condition of the given interpolation problem, i.e.

$$\ker M_S(\mathfrak{A}) = \{f \in V | L(f)(P_i) = 0 \quad \forall L \in A_i, \forall i\}.$$

EXAMPLE 9. Consider the system $\mathcal{L}_2(2^2)$. We obtain

$$\begin{array}{c} \\ (x_1, y_1) \\ \\ \\ (x_2, y_2) \\ \\ \end{array} \begin{array}{|cccccc|} \hline 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 \\ \hline 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 0 & 1 & 0 & 2x_2 & y_2 & 0 \\ 0 & 0 & 1 & 0 & x_2 & 2y_2 \\ \hline \end{array} \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ a_{0,1} \\ a_{2,0} \\ a_{1,1} \\ a_{0,2} \end{pmatrix} = 0.$$

DEFINITION 2. *The interpolation problem is called*

1. regular *if the determinant $D_S(\mathfrak{A})$ is a non-zero constant, i.e., for any set of points $\{P_i\}$ there is no nonzero polynomial in V satisfying the interpolation conditions;*
2. almost regular *or* generically non-special *if the determinant $D_S(\mathfrak{A})$ is a non-constant polynomial in the x_i's and y_i's, i.e., for a general set of points, there is no nonzero polynomial in V satisfying the interpolation conditions;*
3. singular *if the determinant $D_S(\mathfrak{A})$ is identically zero, i.e., there is always a nonzero polynomial $P \in V$ satisfying the interpolation conditions.*

The determinant $D_S(\mathfrak{A})$ of an interpolation problem is in general a polynomial in the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. To prove that the interpolation problem is non-singular one must show that this determinant is not identically zero.

3.3. Derivatives of D and shifts

It suffices to show that a *derivative* of the determinant is not identically zero; this leads us to analyze such derivatives in more detail. Let us introduce the following notation, for any matrix M whose entries are functions of a variable x:

$$\frac{\partial(p)}{\partial x} M := \text{matrix obtained by applying } \frac{\partial}{\partial x} \text{to the } p\text{th row in the matrix } M.$$

REMARK 4. The product rule for derivatives implies that, if M is a square matrix,

$$\frac{\partial}{\partial x}(\det M) = \sum_p \det(\frac{\partial(p)}{\partial x} M)$$

where the sum is taken over all rows p of M.

From now we denote $D_S(\mathfrak{A})$ simply by D. Applying the previous remark to our interpolation matrices, we obtain

$$\frac{\partial}{\partial x_i} D_S(\mathfrak{A}) = \sum_{(r,s)\in A_i} det\left(\frac{\partial(i,r,s)}{\partial x_i} M_S(\mathfrak{A})\right), \tag{3}$$

recalling that the rows are indexed by triples (i, r, s). We note that the sum is taken over only those rows in the A_i part because the other rows do not involve the variable x_i, and hence the derivatives all vanish as does the determinant. The similar equation holds for taking y_i derivatives also.

Next we note that apply $\frac{\partial}{\partial x_i}$ to the rows is the same thing as replacing partial derivatives in A_i by partial derivatives with one additional x-derivative.

$$\begin{array}{ccc} \text{row} & \longrightarrow & \frac{\partial^{a+b}}{\partial^a x \partial^b y} \\ \downarrow & & \downarrow \\ \frac{\partial}{\partial x} - \text{row} & \longrightarrow & \frac{\partial^{a+b+1}}{\partial^{a+1} x \partial^b y} \end{array}$$

Thus we can write

$$\frac{\partial(i,r,s)}{\partial x_i} M_S(A_1,\ldots,A_i,\ldots,A_n) = M_S(A_1,\ldots,A_i^*,\ldots,A_n)$$

where A_i^* is the result of replacing the (r,s) lattice point with the $(r+1,s)$ lattice point: we are simply taking one more derivative with respect to x_i. The equation above is actually only true up to a possible re-ordering of the rows in the i-th part, since we may have to re-order to put the A_i^* rows in degree lex ordering. If $(r+1,s)$ is already in A_i, this leads to a matrix with two identical rows, whose determinant is therefore zero. In addition, if $(r+1,s)$ is no longer in the *lower closure* of S, (that is, the set of lattice points to the left and below some lattice point of S), we will have a matrix with an identically zero row whose determinant is therefore zero. Similarly, the derivatives with respect to y_i leads to replacing the (r,s) point in A_i with $(r,s+1)$. This leads us to the following concept; let A be the set of lattice points, such as one of the sets A_i of derivative orders as above.

DEFINITION 3. *A* ***right shift*** *of A moves a point $(r,s)\in A$ to the position $(r+1,s)$. An* ***up shift*** *moves a point $(r,s)\in A$ to the position $(r,s+1)$. A right or up shift gives a* collision *if the resulting lattice point $(r+1,s)$ or $(r,s+1)$ is already in A. A right or up shift gives an* exit *if the resulting lattice point leaves the lower closure of the set S.*

EXAMPLE 10. Consider shifts of the set $A_1 = T_2 = \{(0,0),(1,0),(0,1)\}$. Let us index these by $1=(0,0)$, $2=(1,0)$, and $3=(0,1)$. Consider $\frac{\partial}{\partial x_1}$ and apply it respectively to the elements $2, 3 \in A_1$. These right shifts can be visualized as follows:

$$\begin{array}{c|cccc} y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & \boxed{2} & & \\ \hline & x^0 & x^1 & x^2 & \ldots \end{array} \quad \xrightarrow{\frac{\partial}{\partial x_1}} \quad \begin{array}{c|cccc} y^2 & & & & \\ y^1 & 3 & & & \\ y^0 & 1 & & \boxed{2} & \\ \hline & x^0 & x^1 & x^2 & \ldots \end{array}$$

$$\begin{array}{c|cccc} y^2 & & & & \\ y^1 & \boxed{3} & & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \ldots \end{array} \quad \xrightarrow{\frac{\partial}{\partial x_1}} \quad \begin{array}{c|cccc} y^2 & & & & \\ y^1 & & \boxed{3} & & \\ y^0 & 1 & 2 & & \\ \hline & x^0 & x^1 & x^2 & \ldots \end{array}$$

We note that the right shift of the element 1 of A_1 gives a collision (with the element 2) and need not be considered in the determinant formula.

Using this notation, (3) becomes

$$\frac{\partial}{\partial x_i} D_S(A_1,\ldots,A_i,\ldots,A_n) = \sum_{\substack{\text{right shifts} \\ A_i^* \text{ of } A_i \\ \text{without collision} \\ \text{or exit}}} \mu D_S(A_1,\ldots,A_i^*,\ldots,A_n) \tag{4}$$

where $\mu = \pm 1$ depends on the particular right shift and comes from the possible re-ordering of the rows as noted above.

3.4. Higher-order derivatives and iterated shifts

We want to understand higher-order derivatives of the determinant, which leads to iterated shifts. Number each lattice point of the set A_i, using as the index set the integers $1, \ldots, |A_i|$. For any given index ℓ, denote by $(r(\ell), s(\ell))$ that element of A_i. A right (respectively up) shift of the ℓ-th element of A_i simply increments $r(\ell)$ (respectively $s(\ell)$) by one, and will be denoted by R_ℓ (respectively U_ℓ). Applying a right shifts (of the elements indexed by $\ell_1, \ldots, \ell_a$, with duplication allowed) followed by b up shifts (of the elements indexed $m_1, \ldots, m_b$, duplicates allowed) to A_i will be denoted by

$$U_{m_b} \cdots U_{m_1} R_{\ell_a} \cdots R_{\ell_1} A_i$$

and will be called an *iterated* (a, b)*-shift* of A_i. Such an operation has a *collision* if at any stage of the process, one of the separate $a + b$ shifts do.

With this notation, (4) applied $a + b$ times gives

$$\text{(5)} \quad \frac{\partial^b}{\partial y_i^b} \frac{\partial^a}{\partial x_i^a} D_S(A_1, \ldots, A_i, \ldots, A_n) = \sum_{\substack{(\underline{m},\underline{\ell}) \in A_i^b \times A_i^a \\ \text{giving iterated } (a,b)\text{-shifts} \\ \text{without collisions} \\ \text{or exits}}} \mu_{(\underline{m},\underline{\ell})} D_S(A_1, \ldots, U_{m_b} \cdots U_{m_1} R_{\ell_a} \cdots R_{\ell_1} A_i, \ldots, A_n)$$

where $\mu_{(\underline{m},\underline{\ell})} = \pm 1$ comes from possible re-orderings of the rows.

Re-organize the sum above based on the final set resulting from the various iterated shifts gives:

$$\text{(6)} \quad \frac{\partial^b}{\partial y_i^b} \frac{\partial^a}{\partial x_i^a} D_S(A_1, \ldots, A_i, \ldots, A_n) = \sum_{\substack{\text{final positions} \\ A_i^*}} \sum_{\substack{(\underline{m},\underline{\ell}) \in A_i^b \times A_i^a \\ \text{giving iterated } (a,b)\text{-shifts} \\ \text{without collisions or exits} \\ \text{ending with} A_i^*}} \mu_{(\underline{m},\underline{\ell})} D_S(A_1, \ldots, A_i^*, \ldots, A_n)$$

$$= \sum_{\substack{\text{final positions} \\ A_i^*}} \epsilon(A_i^*) D_S(A_1, \ldots, A_i^*, \ldots, A_n)$$

where

$$\epsilon(A_i^*) = \sum_{\substack{(\underline{m},\underline{\ell}) \in A_i^b \times A_i^a \\ \text{giving iterated } (a,\, b)\text{-shifts} \\ \text{without collisions or exits} \\ \text{ending with} A_i^*}} \mu_{(\underline{m},\underline{\ell})}$$

is an integer.

We say that the shifts resulting in a given A_i^* are *non-cancelling* if $\epsilon(A_i^*) \neq 0$.

EXAMPLE 11. Consider (2, 2)-shifts of the set $A = T_2 = \{(0, 0), (1, 0), (0, 1)\}$. Let us index these by $1 = (0, 0)$, $2 = (1, 0)$, and $3 = (0, 1)$.

y^2				
y^1	3			
y^0	1	2		
	x^0	x^1	x^2	...

One possible final position is the set $A^* = \{(2, 0), (1, 1), (0, 2)\}$. There are only four iterated shifts without collisions leading to this final position: $U_3U_1R_1R_2$, $U_1U_3R_1R_2$, $U_1U_1R_3R_2$, and $U_1U_1R_2R_3$. All four end up with 2 in the position (2, 0). The first two have 3 in position (0, 2) and 1 in position (1, 1), while the last two have these reversed.

y^2				
y^1	3			
y^0	1	2		
	x^0	x^1	x^2	...

$\xrightarrow{U_3U_1R_1R_2}$

y^2	3			
y^1		1		
y^0			2	
	x^0	x^1	x^2	...

y^2				
y^1	3			
y^0	1	2		
	x^0	x^1	x^2	...

$\xrightarrow{U_1U_3R_1R_2}$

y^2	3			
y^1		1		
y^0			2	
	x^0	x^1	x^2	...

$\vdots$				
y^2				
y^1	3			
y^0	1	2		
	x^0	x^1	x^2	...

$\xrightarrow{U_1U_1R_3R_2}$

$\vdots$				
y^2	1			
y^1		3		
y^0			2	
	x^0	x^1	x^2	...

$\vdots$				
y^2				
y^1	3			
y^0	1	2		
	x^0	x^1	x^2	...

$\xrightarrow{U_1U_1R_2R_3}$

$\vdots$				
y^2	1			
y^1		3		
y^0			2	
	x^0	x^1	x^2	...

Therefore two of the μ's are equal to 1 and two are equal to -1, and the resulting ϵ is zero. Therefore this is a *cancelling* set of shifts.

4. Lecture three: the matrix approach II

4.1. Coalescence

Suppose $D = D(x_1, \dots, x_n, y_1, \dots, y_n)$ is a polynomial in these $2n$ variables. Denote by coal(D) the polynomial obtained by coalescing the variables (x_1, y_1) with the variables (x_2, y_2): this is essentially setting $x_2 = x_1$ and $y_2 = y_1$:

$$\mathrm{coal}(D) = D(x_1, x_1, x_3, \dots, x_n, y_1, y_1, y_3, \dots, y_n).$$

Note that coal is a linear operation.

Let us apply this to the determinants of the interpolation matrices that we are considering. Coalescing the first two variables in $D_S(\mathfrak{A})$ exactly means that we are requiring the second set of derivatives A_2 to vanish at the first point $P_1 = (x_1, y_1)$. If there is overlap between A_1 and A_2, then the interpolation matrix will have two identical rows after this coalescence. Otherwise, we simply have the union of A_1 and A_2 at the first point. This proves the following:

LEMMA 1.

$$\mathrm{coal}\left(D_S(\mathfrak{A})\right) = \begin{cases} \pm D_S\big(A_1 \cup A_2, A_3, \dots A_n\big), & \text{if } A_1 \cap A_2 = \emptyset; \\ 0, & \text{if } A_1 \cap A_2 \neq \emptyset. \end{cases}$$

Applying coalescence to both sides of (6) gives the following, using Lemma 1.

$$\text{(7)} \quad \mathrm{coal}\, \frac{\partial^b}{\partial y_2^b} \frac{\partial^a}{\partial x_2^a} D_S(A_1, A_2, \dots, A_n) = \\ = \sum_{\substack{\text{final positions} \\ A_2^* \text{ such that} \\ A_1 \cap A_2^* = \emptyset}} \pm \epsilon(A_2^*) D_S(A_1 \cup A_2^*, \dots, A_n)$$

4.2. Minimal shifts

Our goal is to reduce the sum in the above formula to a single determinant. We observe that a and b are determined by the final position A_2^*. In particular we can give the following

DEFINITION 4. (a, b) *is a* minimal shift with final position A_2^* *for the pair* (A_1, A_2) *if* A_2^* *is the unique final position for an* (a, b)*-shift of* A_2 *for which* $A_1 \cap A_2^* = \emptyset$ *and* $\epsilon(A_2^*) \neq 0$.

COROLLARY 1. *Suppose* (a, b) *is a minimal shift for* (A_1, A_2) *with final position*

A_2^*. *Then*

$$coal\left(\frac{\partial^{a+b}}{\partial x_2^a \partial y_2^b} D_S(\mathfrak{A})\right) = \epsilon D_S(A_1 \cup A_2^*, A_3 \ldots, A_n).$$

for some nonzero constant ϵ.

The main application we have for this is the following.

PROPOSITION 1. *Suppose that* (a, b) *is a minimal shift for* (A_1, A_2) *with final position* A_2^*. *Suppose that the interpolation problem for* S *and* $\mathfrak{A}' = \{A_1 \cup A_2^*, \ldots, A_n\}$ *is non-singular. Then the original interpolation problem for* S *and* $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ *is non-singular.*

The goal is ultimately to reduce to the following situation, using the above Proposition.

PROPOSITION 2. *Suppose that* S *is lower closed. Then the determinant* $D_S(A_1)$ *with* $A_1 = S$ *is nonzero.*

EXAMPLE 12. Consider A_1 and A_2 both equal to $\{1, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. If we apply a $(5, 0)$-shift and coalesce P_1 and P_2, then the only final position for A_2^* which is disjoint from A_1 is $A_2^* = \{\frac{\partial^2}{\partial x^2}, \frac{\partial^3}{\partial x^3}, \frac{\partial^2}{\partial x \partial y}\}$. Thus

$$\text{coal}\left(\frac{\partial^5}{\partial x_1^5} D_S(A_1, A_2, \ldots, A_n)\right) = \epsilon D_S(A_1 \cup A_2^*, A_3 \ldots, A_n).$$

for some nonzero constant ϵ, where $A_1 \cup A_2^*$ is $\{1, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^3}{\partial x^3}\}$.

4.3. Shifts of T_1 and T_2

In this section we first suppose that $A_2 = T_1 = \{(0, 0)\}$. With only one element in A_2, the only condition for (a, b) to be a minimal shift is that the resulting A_2^* (which is $\{(a, b)\}$ of course) is disjoint from A_1. Hence it is a trivial matter, if $A_2 = T_1$, to apply Proposition 1 and simply add the lattice point (a, b) to A_1.

Next we suppose that $A_2 = T_2 = \{(0, 0), (1, 0), (0, 1)\}$. This case is already quite a bit more complicated. The following lemma, proved in [30], at least gives us good information about when an (a, b) shift of T_2 is non-cancelling:

LEMMA 2. *The final position* A_2^* *for initial position* $A_2 = T_2$ *is cancelling if and only if the three elements of* A_2^* *are collinear.*

The proof of this is a rather involved computation of the contributions to the ϵ factor.

We have already seen an example of this in Example 11 in the previous Lecture.

We note that if the final position A_2^* has two lattice points on one row and one on another, then they cannot be collinear, and the shift will be non-cancelling. This is the primary type of shift that is necessary to consider in most applications.

When is a shift of T_2 minimal? This question is equivalent to asking: when is a final position A_2^* (which is disjoint from A_1) unique, given the numbers a and b of right and up shifts, respectively? This is an easier question to address in most circumstances, and we simply note the following.

Suppose that S is lower closed, and that $A_1 \subset S$ is also lower closed. Suppose that the first k lowest rows of A_1 are equal to the first k lowest rows of S, and that the $k+1$-st row of A_1 is not equal to the $k+1$-st row of S.

LEMMA 3. *Suppose that there are at least two elements of S in the $k+1$-st row which are not in A_1, and at least one element of S in a higher row that is not in A_1. Then the shift of T_2 placing* $(0,0)$ *and* $(1,0)$ *into the first two elements of the $k+1$-st row of S which are not in A_1, and which places* $(0,1)$ *in the first element of the next higher row of S which has an element not in A_1, is a minimal shift.*

LEMMA 4. *Suppose that there is exactly one element of S in the $k+1$-st row which is not in A_1, and in the next higher row of S that has elements not in A_1, there are at least two elements of S that are not in A_1. Then the shift of T_2 placing* $(0,0)$ *into the final element of the $k+1$-st row of S which is not in A_1, and which places* $(0,1)$ *and* $(1,0)$ *in the first two elements of the next higher row of S which has the two elements not in A_1, is a minimal shift.*

To prove the above two lemmas, the readers need only convince themselves that these final positions are the unique ones disjoint from A_1 for $A_2 = T_2$; for this we use up shifts first, then right shifts.

EXAMPLE 13. We give here a visualization of the final position A_2^*.

Suppose A_1 fills completely the lattice indexing monomials in V untill the row k and it has some element at the row $k+1$. In the next figures we indicate the elements in A_1 with a bullet $\bullet$ while the elements in $A_2 = T_2$ are marked with $\circ_j$.

As a first case, we suppose there are at least two free boxes in the $(k+1)^{st}$−row.

$\vdots$											
$k+1$	$\bullet$	$\dots$	$\bullet$								
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\circ_3$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\circ_1$	$\circ_2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Then we start with all the up shifts and we reach the position

$\vdots$											
$k+2$	$\circ_3$										
$k+1$	$\circ_1$	$\circ_2$	$\bullet$								
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

After that, we move to right, performing all the right shifts:

$\vdots$											
$k+2$	$\circ_3$										
$k+1$	$\bullet$	$\dots$	$\bullet$	$\circ_1$	$\circ_2$						
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

and this is the final position for A_2^*.
We suppose now there is only one free box in the $(k+1)^{st}-$row.

$\vdots$											
$k+1$	$\bullet$	$\bullet$	$\dots$	$\bullet$							
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\circ_3$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\circ_1$	$\circ_2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Again we start with all the up shifts and we reach the position

$\vdots$											
$k+2$	$\circ_3$	$\circ_2$									
$k+1$	$\circ_1$	$\bullet$	$\dots$	$\bullet$							
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

After that, we move to right, performing all the right shifts, and we get the following as final position for A_2^*.

$\vdots$											
$k+2$	$\circ_3$	$\circ_2$									
$k+1$	$\bullet$	$\bullet$	$\dots$	$\bullet$	$\circ_1$						
k	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$					
$\vdots$											
3	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$			
2	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$		
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$	
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dots$	$\bullet$
	0	1	2	3	4	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

4.4. The paving strategy

Let us now specialize the interpolation problem and assume the following: S is a lower-closed set, A_1 is a lower-closed set, and all A_i sets for $i \geq 2$ are either T_1 or T_2. A strategy for proving that such an interpolation problem is nonsingular is to perform a minimal shift to A_2 and coalesce it with A_1, thereby reducing the number of A_i sets. If we can reduce to the case where there is only one such set, A_1, then $A_1 = S$ and the problem is nonsingular by Proposition 2.

The minimal shifts of T_1 and of T_2 will be all of the type introduced above. In particular, those of T_2 will be to place two elements on one row and one on another, systematically filling up the rows of S from the bottom up.

We refer to this strategy as "paving" the set S by the sets A_i.

EXAMPLE 14. Consider the system of conics with two double points, namely the linear system $\mathcal{L}_2(2^2)$. In this case we have $S = T_3$, and $A_1 = A_2 = T_2$; S has six elements, and both of the A_i's have three elements. We see that in this case, the paving strategy introduced above fails, since the three elements in $S - A_1$ (namely $(2, 0)$, $(1, 1)$, and $(0, 2)$) are collinear, and the shift placing A_2 into these elements as a final possible position A_2^* is cancelling, by Lemma 2. We are not particularly surprised that the paving strategy has failed in this case, since we know that the interpolation problem is indeed singular!

To continue the discussion, it is useful to illustrate the paving strategy by visualizing the set S (representing the basis of the underlying vector space V) and the successive increasingly larger sets A_1 (obtained by coalescing the next A_i set in its turn) as the algorithm using the paving strategy proceeds.

For example, Consider now the system of cubics with three double points and one simple point; we have $\dim(V) = \sum \dim(A_i) = 10$, with $S = T_4$, $A_1 = A_2 = A_3 = T_2$ and $A_4 = T_1$:

$$S = \begin{array}{c|cccc} 3 & \bullet & & & \\ 2 & \bullet & \bullet & & \\ 1 & \bullet & \bullet & \bullet & \\ 0 & \bullet & \bullet & \bullet & \bullet \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

$$\text{first } A_1 = \begin{array}{c|cccc} 3 & & & & \\ 2 & & & & \\ 1 & \circ & & & \\ 0 & \circ & \circ & & \\ \hline & 0 & 1 & 2 & 3 \end{array} \qquad \text{second } A_1 = \begin{array}{c|cccc} 3 & & & & \\ 2 & & & & \\ 1 & \circ & \circ & & \\ 0 & \circ & \circ & \circ & \circ \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

$$\text{third } A_1 = \begin{array}{c|cccc} 3 & & & & \\ 2 & \circ & \circ & & \\ 1 & \circ & \circ & \circ & \\ 0 & \circ & \circ & \circ & \circ \\ \hline & 0 & 1 & 2 & 3 \end{array} \qquad \text{fourth } A_1 = S = \begin{array}{c|cccc} 3 & \circ & & & \\ 2 & \circ & \circ & & \\ 1 & \circ & \circ & \circ & \\ 0 & \circ & \circ & \circ & \circ \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

In this case the paving strategy has succeeded. We note that in creating the second A_1, we have executed a $(5, 0)$ minimal shift of the T_2, moving two elements into the first row (filling it up) and one element into the second row. In creating the third A_1, we have executed a $(2, 4)$ minimal shift of the T_2, placing one element into the second row (filling it up) and two elements into the third row (filling it up also). Finally in creating the fourth A_1, we have executed a $(0, 3)$ minimal shift (namely three up shifts) of the T_1, placing the element into the fourth row at the top, and paving the entire set S.

It is more efficient to encode all of this a bit more simply as follows:

$$
S = \begin{array}{c|cccc}
3 & 4 & & & \\
2 & 3 & 3 & & \\
1 & 1 & 2 & 3 & \\
0 & 1 & 1 & 2 & 2 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

Here the numbers filling in the elements of S represent which A_i contribute to the final paving of that particular element. Another way of saying this is that the n-th A_1 in the paving strategy algorithm consists of the union of all of the elements labeled with integers between 1 and n.

EXAMPLE 15. In a similar way, if we consider the system of quartics with five double points, we again get stuck, in a similar way to the case of $\mathcal{L}_2(2^2)$. Indeed, applying the paving strategy as above, the fourth A_1 is

$$
\text{fourth } A_1 = \begin{array}{c|ccccc}
4 & \bullet & & & & \\
3 & 4 & \bullet & & & \\
2 & 4 & 4 & \bullet & & \\
1 & 1 & 2 & 3 & 3 & \\
0 & 1 & 1 & 2 & 2 & 3 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

and in order to finish paving S, we would need to make a minimal shift into the final three collinear elements, which is not possible by Lemma 2.

EXAMPLE 16. The paving strategy works very well for the system $\mathcal{L}_5(2^7)$ of quintics with seven double points:

$$
\begin{array}{c|cccccc}
5 & 7 & & & & & \\
4 & 7 & 7 & & & & \\
3 & 5 & 6 & 6 & & & \\
2 & 4 & 5 & 5 & 6 & & \\
1 & 1 & 2 & 3 & 4 & 4 & \\
0 & 1 & 1 & 2 & 2 & 3 & 3 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

This proves that $\mathcal{L}_5(2^7)$ is non-special.

4.5. Proof of the Double Points Theorem in dimension two

The matrix approach that we are presenting here is powerful enough to prove that, except for conics and quartics with two and five double points respectively, all linear systems with simple and double points are non-special.

THEOREM 12 (ALEXANDER–HIRSCHOWITZ FOR $\mathbb{P}^2$). *Suppose that $d \geq 5$ and $S = \{(i, j) | i + j \leq d\}$ (i.e. S represents all monomials of degree $\leq d$) and all A_i are*

T_2 or T_1. Then $D_S(\mathfrak{A}) \not\equiv 0$ and the linear system $\mathcal{L}_d(1^r, 2^s)$ is non-special whenever $r + 3s \le (d+2)(d+1)/2$.

The proof is simply an analysis of the paving strategy, proving that it does work and in this case, with $d \ge 5$, it never requires the three collinear elements for a minimal shift.

One uses the strategy of always moving two elements on the lowest unfilled row, and one on the next, whenever possible; if there is only one element left on the lowest unfilled row, one shows that there are at least two elements on the next unfilled row. This is relatively simple as long as there are more than four unfilled rows left, since using this algorithm, the lowest unfilled row fills up twice as fast as the next unfilled row, and hence when it does get near the end (with only zero or one element left) there are at least two unfilled elements in that next row.

Thus the first remark to make is that one get "near the top" using this algorithm without any problems. The next remark is that, when we have exactly filled the sixth-to-last row, there is at least one element filled in the fifth-to-last row. This again follows from the considerations above: one cannot simultaneously exactly fill the seventh-to-last and the sixth-to-last rows.

Finally one simply checks by hand that if there is at least one element filled in the fifth-to-last row, one can finish the paving of S from that point on. Since this fifth-to-last row has only five elements in it total, there are really only four cases to check:

One element in fifth-to-last row:

d	6				
$d-1$	4	5			
$d-2$	3	4	4		
$d-3$	1	2	3	3	
$d-4$	$\bullet$	1	1	2	2
	$d-4$	$d-3$	$d-2$	$d-1$	d

Two elements in fifth-to-last row:

d	5				
$d-1$	4	4			
$d-2$	3	3	4		
$d-3$	1	2	2	3	
$d-4$	$\bullet$	$\bullet$	1	1	2
	$d-4$	$d-3$	$d-2$	$d-1$	d

Three elements in fifth-to-last row:

d	4				
$d-1$	4	4			
$d-2$	2	3	3		
$d-3$	1	2	2	3	
$d-4$	$\bullet$	$\bullet$	$\bullet$	1	1
	$d-4$	$d-3$	$d-2$	$d-1$	d

Four elements in fifth-to-last row:

d	5				
$d-1$	3	4			
$d-2$	2	3	3		
$d-3$	1	1	2	2	
$d-4$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	1
	$d-4$	$d-3$	$d-2$	$d-1$	d

These four simple computations finish the proof of the Theorem.

EXAMPLE 17. One can use this method to study the problem in $\mathbb{P}^1 \times \mathbb{P}^1$ instead $\mathbb{P}^2$. This time S is given by $d_1 \times d_2$ boxes, arranged in a rectangle; these are the monomials in the complete linear systems on $\mathbb{P}^1 \times \mathbb{P}^1$.

$S =$

d_1				
$\vdots$				
1				
0				
	0	1	$\dots$	d_2

Assume $d_1 = 2$. If $A_1 = A_2 = T_2$ we have

2	2	2			
1	1	2			
0	1	1			
	0	1	2	$\dots$	d_2

Thus two double points fill exactly two columns. If d_2 is even, we have an odd number of columns (i.e. $d_2 + 1$) and the system is special. In Dent's thesis the case of rectangular S is analyzed more completely; see also [30]

The matrix approach presented here has several variations, and has been used to prove that several classes of interpolation problems are non-special. See [43] for a recent survey.

Dent, in her thesis ([29]), has used the method to prove the Alexander-Hirschowitz Theorem for three variables.

It would be a wonderful project to systematically relate these matrix approach methods for studying interpolation problems, to degeneration techniques coming from a more standard algebro-geometric approach, using upper semicontinuity.

5. Lecture four: degenerations of the plane

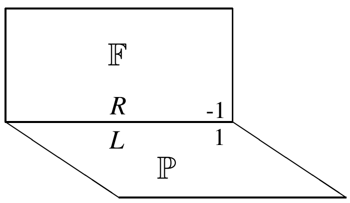
5.1. Blowing up the trivial family of planes

We already spoke, in Lecture One, about various specialization techniques to attack the interpolation problem. Since the dimension of a system with imposed multiple points is upper-semicontinuous in the position of the fat points $\{Z_i\}$, we can consider a degeneration of Z_1 and Z_2 to a suitable $Z_1 \cup Z_2$. In this way one reduces the study of $H^0(X_S, \mathcal{L} \otimes \mathcal{I}_{Z_1} \otimes \mathcal{I}_{Z_2} \otimes \cdots \otimes \mathcal{I}_{Z_n})$ to the study of $H^0(X_S, \mathcal{L} \otimes \mathcal{I}_{Z_1 \cup Z_2} \otimes \mathcal{I}_{Z_3} \otimes \cdots \otimes \mathcal{I}_{Z_n})$.

In the next two lectures we will explain the essential features of a particular specialization technique introduced by Ciliberto and Miranda in [22].

Although related closely to other specializations, the new feature is that the degeneration is not of sets of points, but, instead, we degenerate the surface where these points live. The idea is based on a degeneration method used by Z. Ran ([48]) to study enumerative problems on singular curves and consists in degenerating the plane to a reducible surface. The restriction of the limit linear system to the components of the surface are hopefully easier to understand than the system that we began with.

In more detail, let Δ be a complex disc around the origin. We consider the trivial family of planes which is the product $V = \mathbb{P}^2 \times \Delta$, with its two projections $p_1 : V \to \mathbb{P}^2$ and $p_2 : V \to \Delta$. We denote the fiber over $t \in \Delta$ by $V_t = p_2^{-1}(t) = \mathbb{P}^2 \times \{t\}$. Consider a line $L \subset V_0$ and blow it up to obtain a three-fold X with maps $f : X \to V$, $\pi_1 = p_1 \circ f : X \to \mathbb{P}^2$ and $\pi_2 = p_2 \circ f : X \to \Delta$. The map π_2 is a flat family of surfaces over Δ: for $t \neq 0$, $X_t = V_t = \mathbb{P}^2$, while, for $t = 0$, X_0 is the union of the proper transform $\mathbb{P}$ of V_0 (which is again isomorphic to $\mathbb{P}^2$) and of the exceptional divisor $\mathbb{F}$ of the blow-up (which is isomorphic to a Hirzebruch surface $\mathbb{F}_1$). They are joined transversally along a curve R which is a line L in $\mathbb{P}$ and is the exceptional curve on $\mathbb{F}$.



5.2. The triple point formula

Note that we have $(R^2)_{\mathbb{F}} + (R^2)_{\mathbb{P}} = 0$. This is a special case of the so-called Triple Point Formula for double curves of the special fiber of a degeneration of surfaces. In fact, let X be a smooth 3-fold with a map $\pi : X \to \Delta$, whose general fiber is a smooth surface and whose central fiber X_0 is the union $\cup A_i$ of smooth A_i meeting transversally along smooth curves $R_{ij} \subset A_i$. The Triple Point Formula states that with this situation, one has:

$$(R_{ij}^2)_{A_i} + (R_{ij}^2)_{A_j} = -(\text{numbers of triple points on } R_{ij}). \tag{8}$$

Proof. This is rather elementary intersection theory on the threefold X. First note that $A_i^2 \cdot A_j$ can be viewed as taking the self-intersection of the surface A_i, then restricting to A_j. Restriction is a homomorphism of the intersection product, and so we can restrict A_i to A_j first (obtaining R_{ji}) and then take the self-intersection. Hence

$$(A_i^2 \cdot A_j)_X = (R_{ji}^2)_{A_j}.$$

Now consider $A_i \cdot A_j \cdot X_0$. On the one hand this is zero, since $X_0 \equiv X_t$ and, for $t \neq 0$, X_t is disjoint from the A_i's. Using that $X_0 = \sum_k A_k$, we have

$$0 = A_i \cdot A_j \cdot X_0 = A_i \cdot A_j \cdot \sum_k A_k = A_i^2 \cdot A_j + A_j^2 \cdot A_i + \sum_{k \neq i,j} A_i \cdot A_j \cdot A_k.$$

The sum on the right is the number of triple points on R_{ij}. This and the identification above of $A_i^2 \cdot A_j$ proves the Triple Point Formula (8). □

5.3. The degeneration of the linear system

We pass now to analyzing the invertible sheaf on X_0. The Picard group of X_0 is the fibered product of Pic($\mathbb{P}$) (generated by $\mathcal{O}(1)$) and Pic($\mathbb{F}$) (generated by the class H of a line and the class R of the exceptional divisor) over Pic(R). Since $H \cdot R = 0$ and $R \cdot R = -1$, we have

$$\mathcal{O}_{\mathbb{F}}(H)_{|R} \cong \mathcal{O}_R$$

and

$$\mathcal{O}_{\mathbb{F}}(R)_{|R} \cong \mathcal{O}_R(-1).$$

Hence if χ is a line bundle on X_0 given by $\chi_{\mathbb{P}}$ and $\chi_{\mathbb{F}}$, in order that the restrictions to R agree, one must have $\chi_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(d)$ and $\chi_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(cH - dR)$ for some c and d; we denote this line bundle on X_0 by $\chi(c, c-d)$. In particular for any d and k, $\chi(d,k)_{|\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(d-k)$ and $\chi(d,k)_{|\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(dH - (d-k)R)$.

Let $\mathcal{O}_X(d)$ be the line bundle $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(d))$. If $t \neq 0$ then the restriction to X_t is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$ and the restriction to X_0 is the line bundle $\chi(d, 0)$. Since the normal bundles of $\mathbb{P}$ and $\mathbb{F}$ on X are respectively $-L$ and $-R$ we have $\mathcal{O}_X(\mathbb{P})_{|\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-1)$ and $\mathcal{O}_X(\mathbb{P})_{|\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(R)$.

Consider now the line bundle $\mathcal{O}_X(d,k) := \mathcal{O}_X(d) \otimes \mathcal{O}_X(k\mathbb{P})$; from the previous discussion we have

$$\begin{aligned} \mathcal{O}_X(d,k)_{|X_t} &\cong \mathcal{O}_{\mathbb{P}^2}(d) \quad t \neq 0 \\ \mathcal{O}_X(d,k)_{|X_0} &\cong \chi(d,k) \end{aligned}$$

and we therefore have that all line bundles $\chi(d,k)$ on X_0 are flat limits of line bundles $\mathcal{O}_{\mathbb{P}^2}(d)$ on the general fiber X_t in this degeneration.

Fix a positive integer n and a non-negative integer $b \leq n$. We now consider $n-b$ general points $P_1, \ldots P_{n-b}$ in $\mathbb{P}$ and b general points $P_{n-b+1}, \ldots P_n$ in $\mathbb{F}$. These points are viewed as limits of n general points $P_{1,t}, \ldots, P_{n,t}$ on X_t. Our goal is to understand, on X_t, the linear system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_{i,t})$.

To this end we now consider the system $\mathcal{L}(k,b)$ on X_0 which is formed by divisors on $|\chi(d,k)|$ with the prescribed multiplicities at the points P_i, $i = 0, \ldots, n$. Then $\mathcal{L}(k,b)$ can be regarded as a flat limit on X_0 of the desired system $\mathcal{L}_d(-\sum_{i=1}^n m_i P_{i,t})$ and we call this a $(k,b)-$*degeneration* of the linear system. We can observe that $\mathcal{L}(k,b)$ restricts to $\mathbb{P}$ as $\mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i)$ and to $\mathbb{F}$ as a system of the form $\mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^n m_i P_i)$ where Q_0 is a point in $\mathbb{P}^2$ at which we blow up to obtain $\mathbb{F}$. (Here we are viewing the surface $\mathbb{F}$ as a blowup of the plane, and the corresponding line bundle on $\mathbb{F}$ as a linear system of the same form as the others we are considering.)

We note that the restricted system on R in which they must agree is given by $\mathcal{O}_R(d-k)$.

5.4. The computation of the limit linear system

A global section of $\mathcal{L}(k,b)$ is a section on $\mathbb{P}$ of $\mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i)$ and a section on $\mathbb{F}$ of $\mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^n m_i P_i)$ which agree on the intersection curve R. In other words, $H^0(X_0, \mathcal{L}(k,b))$ is the fiber product of

$$H^0(\mathbb{P}, \mathcal{L}_{d-k}(-\sum_{i=1}^{n-b} m_i P_i))$$

and

$$H^0(\mathbb{F}, \mathcal{L}_d(-(d-k)Q_0 - \sum_{i=n-b+1}^{n} m_i P_i))$$

over the restriction to R, which is $H^0(R, \mathcal{O}_R(d-k))$.

If we denote by l_0 the dimension of $\mathcal{L}(k,b)$, by semicontinuity we have

$$l_0 \geq \dim(\mathcal{L}_d(-\textstyle\sum_{i=1}^n m_i P_i)) \geq \epsilon(\mathcal{L}_d(-\textstyle\sum_{i=1}^n m_i P_i)).$$

Therefore we have the following

LEMMA 5. *If*

$$l_0 = \epsilon(\mathcal{L}_d(-\sum_{i=1}^n m_i P_i))$$

then $\mathcal{L}_d(-\sum_{i=1}^n m_i P_i)$ *is non-special.*

This is the basis of the approach: given the degree d and the multiplicities m_i, choose an appropriate k and b and make a computation of the limit dimension l_0. If that limit dimension is equal to the expected dimension of the system, then it is non-special.

This method has the capability of reducing the problem for the original linear system into two easier linear systems on $\mathbb{P}$ and $\mathbb{F}$, of the same general sort. On $\mathbb{P}$ the degree has gone down (if $k > 0$) and the number of points is less (if $b > 0$); on $\mathbb{F}$ the number of points is less (if $b < n$). We can hope to determine their dimensions by a process of induction, and complete the analysis of the limit dimension by a transversality argument which allows the easy computation of the relevant fibered product of systems.

5.5. Results from this method

This was the approach taken by Ciliberto and Miranda in [22] and [23], which resulted in the proof of the following theorem:

THEOREM 13. *For any m_0 and any $m \leq 3$ the Harbourne–Hirschowitz Conjecture holds in the quasi-homogeneous cases $\mathcal{L}_d(m_0, m^n)$. For any $m \leq 12$ this Conjecture holds in the homogeneous cases $\mathcal{L}_d(m^n)$.*

Later this has been extended via a more efficient computer algebra component, by Ciliberto, Cioffi, Miranda, and Orrechia [26], and we now have:

THEOREM 14. *The Harbourne–Hirschowitz Conjecture holds in the homogeneous cases $\mathcal{L}_d(m^n)$ for $m \leq 20$.*

One reason that a computer algebra technique needed to be developed for this is that unfortunately the degeneration procedure as described above, by computing the space of global sections of the limiting linear system, does not work in all cases, even for these low multiplicities. For example, to study the Dixmier example $\mathcal{L}_{19}(6^{10})$ worked out by Hirschowitz with the Horace method, there are no integers k and b which have the limit bundle having no global sections, as expected. For these cases, which are finite in number for any fixed m, the result above relied on a separate computer algebra computation.

In the next Lecture we will describe a recently developed technique which seems to offer some promise to avoid the computer algebra methods and to give a more systematic approach.

6. Lecture five: refined matching conditions

6.1. The fiber product condition

As noted in the previous lecture, a section of the relevant limit line bundle $\mathcal{L}_0$ on the reducible surface $X_0 = \mathbb{P} \cup \mathbb{F}$ is a section over $\mathbb{P}$ and a section over $\mathbb{F}$ which agree on the double curve R. In other words, if we denote by $\mathcal{L}_{\mathbb{P}}$ the line bundle on $\mathbb{P}$ and by

$\mathcal{L}_{\mathbb{F}}$ the line bundle on $\mathbb{F}$, one has natural restriction maps

$$\rho_{\mathbb{P}} : H^0(\mathbb{P}, \mathcal{L}_{\mathbb{P}}) \longrightarrow H^0(R, \mathcal{O}_R(d-k))$$

and

$$\rho_{\mathbb{F}} : H^0(\mathbb{F}, \mathcal{L}_{\mathbb{F}}) \longrightarrow H^0(R, \mathcal{O}_R(d-k))$$

and the global sections of the limit line bundle may be identified with the fiber product

$$H^0(X_0, \mathcal{L}_0) := \{(\alpha, \beta) \in H^0(\mathbb{P}, \mathcal{L}_P P) \times H^0(\mathbb{F}, \mathcal{L}_F) \\ \mid \rho_{\mathbb{P}}(\alpha) = \rho_{\mathbb{F}}(\beta) \text{ in } H^0(R, \mathcal{O}_R(d-k))\}.$$

Thinking in terms of divisors on the two surfaces, this condition means that if we have a divisor $A \in |\mathcal{L}_{\mathbb{P}}|$ on $\mathbb{P}$ and $B \in |\mathcal{L}_{\mathbb{F}}|$ on $\mathbb{F}$, in order that they patch together to give a divisor on X_0, we must have that $A|_R = B|_R$ as divisors on the curve R. For example, if A is tangent to R at a point $r \in R$, so that $A|_R$ contains r with some multiplicity, then $B|_R$ must also contain r with that multiplicity, which implies B must have some tangency (at least) with R at r.

However it could be the case that A has a singularity at r, which is not distinguishable from a tangency, when one only looks at the restriction to R. For example, if A has a triple point at r, with no tangent in the direction of R, then $A|_R$ will contain the divisor $3r$, and this will then force B to have a flexed tangent at r along R. It will not force B to have a triple point though.

Should the possible "extra" singularity of the divisor A have an effect on the divisor B, if we assume that the union $A \cup B$ is a limit of curves in the general surface? This is a relevant hypothesis, for the following reason.

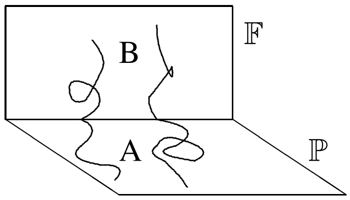
All of the dimension problems in interpolation theory that we have been considering can be reduced to proving that a certain linear system is in fact empty. Indeed, if the expected dimension is $e > -1$, then adding $e + 1$ simple base points to the linear system will result in a linear system which is expected to be empty; if it is, then the original linear system will have the correct (expected) dimension also.

6.2. Refined matching conditions

Now suppose that we are trying to prove that a certain linear system $\mathcal{L}$ is empty, using the degeneration method. We fix the geometric part of the degeneration (namely the number of points b that go to the $\mathbb{F}$ surface) and assume on the contrary that it is not. Thus there will exist a family of curves $\mathcal{C} \to \Delta$ such that C_t is an "unexpected" curve in X_t and C_0 is the curve in the central fiber, i.e., C_0 is the union of a curve A in $\mathbb{P}$ and a curve B in $\mathbb{F}$. Then C_0 must be a divisor for one of the limit bundles, i.e. there must exist an integer k for which A and B are divisors in the corresponding bundles on $\mathbb{P}$ and $\mathbb{F}$, and which agree on R (as sections of $\mathcal{O}_R(d-k)$).

Thus it is enough to show that for every k there are no sections in the fiber product which are different from zero in both factors, *and which could be limits of curves C_t in the general fiber.*

From now, assume that C_0 is given by $A + B$, where $A \subset \mathbb{P}$ and $B \subset \mathbb{F}$.



What we have been able to show is that if B has a *multiple component* which is a (-1)-curve in $\mathbb{F}$, and which meets R at a point r, then A must have a *singularity* at the point r. Precisely, we have been able to show the following:

LEMMA 6. *Suppose $E \subset \mathbb{F}$ is a (-1)-curve meeting R at a point $r \in R$, and $E \cdot B = -\sigma < 0$. Then B contains E as a component with multiplicity σ, and* $\text{mult}_r(A) \geq \sigma$.

The idea of the proof is to blow up the threefold X along E and analyze the proper transform $\mathcal{C}'$ of the family $\mathcal{C}$. For details see [25].

6.3. Cremona transforms and the Three-Point Lemma

In studying linear systems of plane curves with general multiple base points, the opportunity of applying a Cremona transformation of the plane is available at any time, and may indeed be useful in certain situations to reduce the degree or otherwise make the system more amenable to analysis. Let $\mathcal{L}$ be the system of plane curves of degree d with base points of multiplicity $m_1, \ldots, m_n$. Compute $s = m_1+m_2+m_3$; performing a Cremona transform based at these three points results in a linear system of curves of degree $2d-s$ and with base points of multiplicity m_1-s+d, m_2-s+d, m_3-s+d, $m_4, \ldots, m_n$. In particular if $s > d$ then both the degree and the first three multiplicities will drop under the Cremona transform.

Instead of applying a Cremona transformation to the system, let us degenerate it, putting the three points on the $\mathbb{F}$ surface. If we are interested in showing that the general system is empty, applying the technique explained above, we must show that for any integer k, there is no limit curve $A + B$ possible in the $(k, 3)$-degeneration. It is not hard to see that the only k that needs to be checked is $k = s - d$, so let us focus on this case.

Let P_1, P_2, P_3 be the three points on $\mathbb{F}$ with multiplicity m_1, m_2, m_3 such that $s = m_1 + m_2 + m_3 > d$. Let F_1, F_2, and F_3 be the corresponding fibers of the ruling of $\mathbb{F}$ through the three points. The linear system on $\mathbb{F}$ that any limit curve B must belong to is the system $|dH - (2d-s)R - m_1P_1 - m_2P_2 - m_3P_3|$. We note that if F is the class of the ruling, we have $H \cdot F = R \cdot F = 1$. Therefore $B \cdot F_i = s-d-m_i = m_j+m_k-d$. We may assume that this is negative, else the line joining P_j and P_k must split off the system anyway, and we would have reduced the degrees and multiplicities to consider already. Hence we can apply Lemma 6 and conclude that the curve A on the $\mathbb{P}$ surface must have a point of multiplicity $d - m_j - m_k$ at the point $R \cap F_i$.

In order to show that there are no such limits, we are therefore put into a position of showing that the system on $\mathbb{P}$, namely curves of degree $d - k = 2d - s$, with $n - 3$ points of multiplicity $m_4, \ldots, m_n$, has no divisors A *with three additional points of multiplicity* $d - m_1 - m_2$, $d - m_1 - m_3$, *and* $d - m_2 - m_3$, lying on the line R.

This is exactly the numerology of applying a Cremona transformation to the original system! The possible advantage to doing this instead of applying a Cremona transformation is that any geometric relationships between the other points would be preserved intact with this operation (e.g., if some subset of the others are not in fact general, but lie on a line), while applying a Cremona transformation would spoil this.

We call this the Three-Point Lemma.

LEMMA 7 (THREE-POINT LEMMA). *Suppose that the first three multiplicities of the original system* $\mathcal{L}$ *are* m_1, m_2, *and* m_3, *and set* $s = m_1 + m_2 + m_3$. *Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the system obtained by replacing the degree* d *by* $2d - s$, *and by replacing these first three multiplicities by* $d - m_1 - m_2$, $d - m_1 - m_3$, *and* $d - m_2 - m_3$, *and by enforcing that these three points are collinear, is empty.*

6.4. The Four-Point Lemma

In a similar way we can try to use four points on $\mathbb{F}$, and make a similar analysis.

Again set $s = m_1 + m_2 + m_3 + m_4$, assume that $s > d$, and write $s - d = 2t + e$, with $e = 0, 1$. Again it is not hard to see that the relevant k to analyze is $k = t + e$ (i.e. we drop the degree on $\mathbb{P}$ by $t + e$). The four fibers through the four points on $\mathbb{F}$ have intersection number with B equal to $-(m_i - t - e)$, and there is a fifth (-1)-curve (namely the conic through the four points and the point blown up to R) which has intersection number with B equal to $-t$. Using Lemma 6, we therefore have the following.

LEMMA 8 (FOUR-POINT LEMMA). *Suppose that the first four multiplicities of the original system* $\mathcal{L}$ *are* m_1, m_2, m_3, *and* m_4, *and set* $s = m_1 + m_2 + m_3 + m_4$. *Write* $s - d = 2t + e$ *as above, with* $e = 0$ *or* 1. *Assume that* $t + e \leq d$ *and* $t + e \leq m_i$ *for* $i = 1, \ldots, 4$. *Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the system obtained by replacing the degree* d *by* $d - t - e$, *by replacing these first four multiplicities* m_i *by* $m_i - t - e$, *by adding one additional point of multiplicity* t, *and by enforcing that these five points are collinear, is empty.*

It turns out that the virtual dimension of this reduced system is exactly the same as the virtual dimension of the original system.

6.5. The Five-Point Lemma

One can continue in this vein; let us present one more case, that of putting five points on the surface $\mathbb{F}$.

Again set $s = m_1+m_2+m_3+m_4+m_5$, assume that $s > d$, and write $s-d = 2t+e$, with $e = 0, 1$. Again it is not hard to see that the relevant k to analyze is $k = t + e$ (i.e. we drop the degree on $\mathbb{P}$ by $t + e$). The five fibers through the five points on $\mathbb{F}$ have intersection number with B equal to $-(m_i - t - e)$. Using Lemma 6, we therefore have the following.

LEMMA 9 (FIVE-POINT LEMMA). *Suppose that the first five multiplicities of the original system $\mathcal{L}$ are m_1, m_2, m_3, m_4, and m_5, and set $s = m_1 + m_2 + m_3 + m_4 + m_5$. Write $s - d = 2t + e$ as above, with $e = 0$ or 1. Assume that $t + e \leq d$ and $t + e \leq m_i$ for $i = 1, \dots, 5$. Suppose that the virtual dimension of this system is negative. In order to show that it is in fact empty, it suffices to show that the system obtained by replacing the degree d by $d - t - e$, by replacing these first five multiplicities m_i by $m_i - t - e$, and by enforcing that these five points are collinear, has the expected dimension.*

In the Five-Point Lemma, the virtual dimension of the system may go up; if it becomes non-negative, it is necessary to show that it has the expected dimension (not of course that it is empty).

6.6. Examples

Using the Three-, Four-, and Five-Point Lemmas, which incorporate the more refined matching conditions as explained above, we can handle several cases which were only possible using the computer algebra packages.

We have focused our attention on homogeneous systems with ten points, which is the first case of interest. Those with virtual dimension equal to -1 should be the hardest to prove are actually empty. The number theory to determine which degrees and multiplicities give virtual dimension -1 for ten points was worked out by A. Malone in her Master's degree paper [44]. The smallest one has $d = 19$ and $m = 6$, and the next smallest has $d = 38$ and $m = 12$. The third smallest has $d = 174$ and $m = 55$. The methods presented here are sufficient to handle the first two, but not the latter one.

The first such example was the one that Dixmier proposed, and that we referred to before; Hirschowitz was successful in using the Horace Method with this system, but we could not provide a proof using the original version of (k, b)-degenerations of the plane, with the naive matching conditions.

EXAMPLE 18. Consider the linear system $\mathcal{L}_{19}(6^{10})$ of curves of degree 19 with ten general points of multiplicity six. The virtual dimension of this system is -1, and so we expect the system to be empty.

Start by applying a four-point lemma with four of the $m = 6$ points. Here $s = 24$, $d = 19$, so that $s - d = 5$, and hence $t = 2$ and $e = 1$. Therefore we reduce to the system of curves of degree 16 with six general points of multiplicity 6, and five other collinear points, four of multiplicity 3 and one of multiplicity 2.

We then apply another four-point lemma, to one of the $m = 3$ points and three of the $m = 6$ points. Now $s = 21$, $d = 16$, so again $s - d = 5$, $t = 2$, and $e = 1$.

We therefore reduce to the system of curves of degree 13, with three general points of multiplicity six, and eight other base fat points, lying on two lines. Each of the two lines has a point with $m = 2$ and three points with $m = 3$. The intersection point between the two lines started as a point with $m = 3$, but the multiplicity was reduced by three, and therefore eliminated.

Apply another four-point lemma, to two of the $m = 6$ points, and two of the $m = 3$ points (one on each of the lines). Again since $s = 18$ and $d = 13$, we have $s - d = 5$ and $t = 2$, $e = 1$ as before. We reduce to the system of curves of degree 10, with one general $m = 6$ point, and nine other base fat points, lying on three lines; each of the lines has two $m = 3$ points and one $m = 2$ point. (Again the original two $m = 3$ points used in the four-point lemma application are removed by this process.)

Finally do one more four-point lemma, with the final $m = 6$ point, and one $m = 3$ point from each of the three lines. Again $s - d = 15 - 10 = 5$, $t = 2$, and $e = 1$. We reduce to the system of curves of degree 7, with eight general base fat points, four $m = 3$ points and four $m = 2$ points, lying on four general lines, one $m = 3$ and one $m = 2$ point on each line. (Again the three $m = 3$ points used in the four-point application are removed by this process.) At this point the eight points are in fact general points! We have reduced the problem to showing that the linear system $\mathcal{L}_7(2^4, 3^4)$ of septic curves with four general triple points and four general double points is empty.

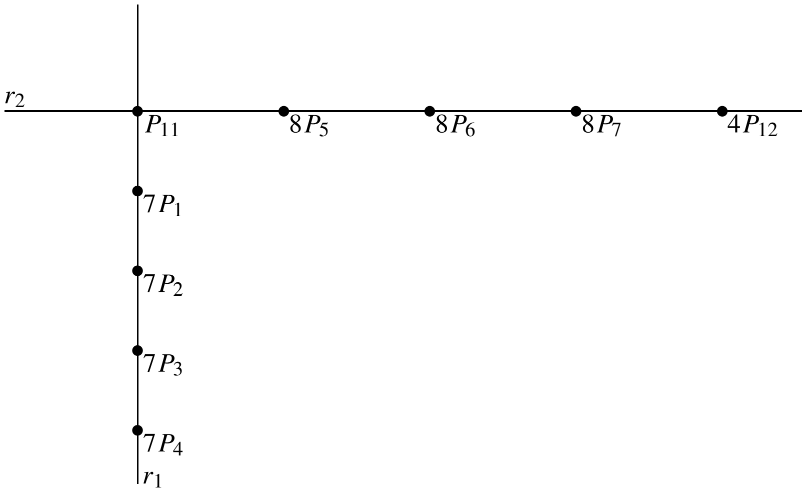
Performing a Cremona transformation at three of the four triple points gives the system $\mathcal{L}_5(1^3, 2^4, 3)$. Performing a second Cremona transformation at the triple point and two of the double points gives the system $\mathcal{L}_3(1^4, 2^2)$. At this point one notices that the line joining the two double points must split off this system, and the residual system is the system $\mathcal{L}_2(1^6)$ of conics through six general points, which is indeed empty.

This series of relatively simple applications of the Four-Point Lemma, followed by some Cremona transformations, suffices to prove that the original system $\mathcal{L}_{19}(6^{10})$ is empty.

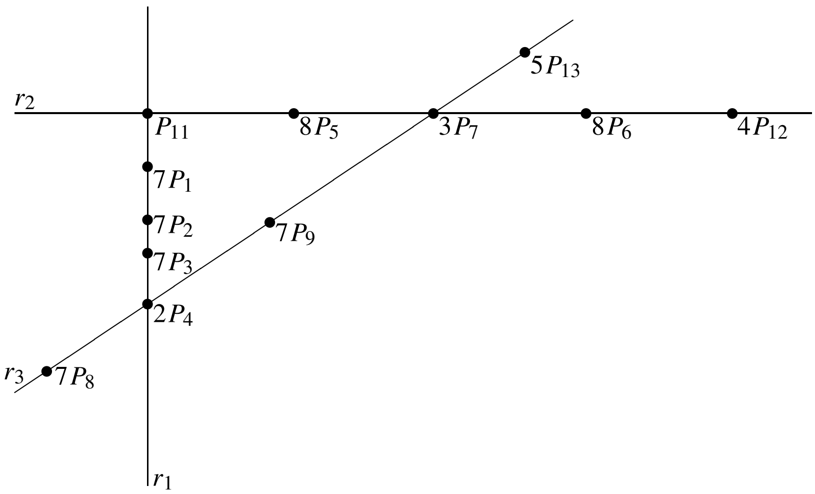
EXAMPLE 19. Consider the linear system $\mathcal{L}_{38}(12^{10})$ of curves of degree 38 with ten general points of multiplicity twelve. The virtual dimension of this system is -1, and so we expect the system to be empty. This system was analyzed in Gimigliano's thesis.

Call the points $P_1, \dots P_{10}$. Start by applying a Four-Point lemma with P_1, P_2, P_3 and P_4. Here $s = 48$, $d = 38$, so that $s - d = 10$, and hence $t = 5$ and $e = 0$. Thus we reduce to the system of curves of degree 33 with six general points of multiplicity twelve, and five other collinear points, four of multiplicity 7 and a point P_{11} of multiplicity 5, lying on a line r_1.

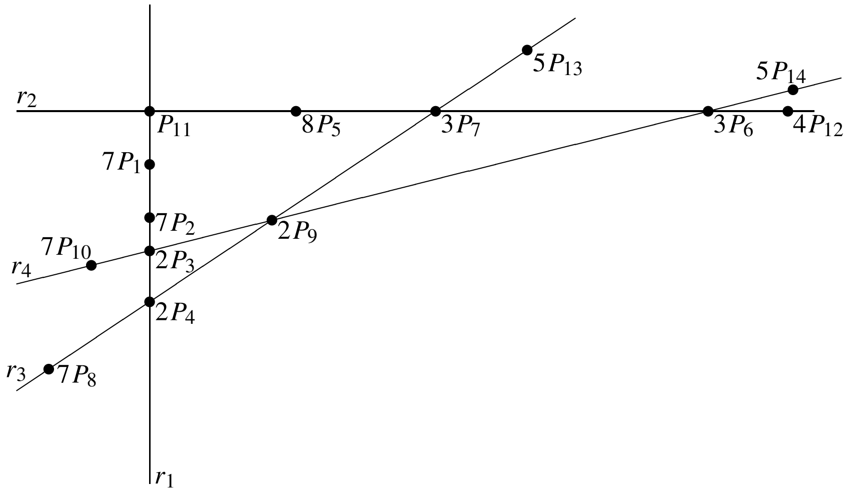
Apply a Four-Point Lemma with P_5, P_6, P_7 and P_{11}. We reduce to the system of curves of degree 29 with three points of multiplicity twelve, four points of multiplicity seven on the line r_1, three points of multiplicity eight (i.e. P_5, P_6 and P_7) and a new point of multiplicty four (i.e. P_{12}) on a line r_2. The intersection between r_1 and r_2 is the simple point P_{11}, having $m = 1$.



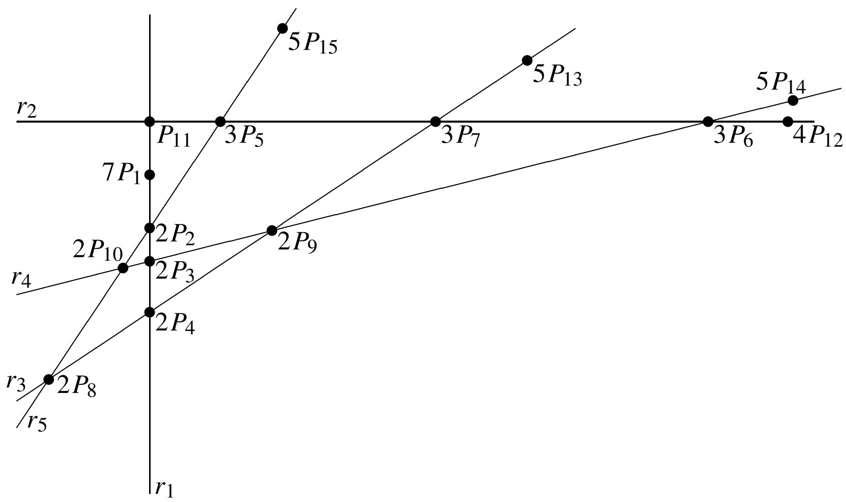
Apply a Four-Point Lemma with P_4, P_7, P_8 and P_9. Here $s = 39$, and we reduce to a system of curves of degree 24. We have a new point P_{13} of multiplicity $m_{13} = 5$. Moreover P_4 becomes a double point and P_7 a triple point, while P_8 and P_9 drop their multiplicity to $m_8 = m_9 = 7$. The points P_4, P_7, P_8, P_9 and P_{13} lie on the line r_3.



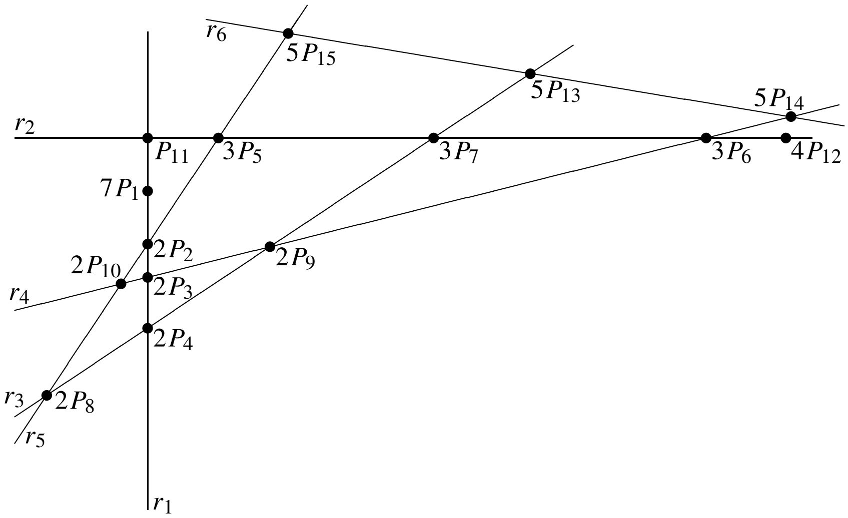
Apply a Four-Point Lemma with P_3, P_6, P_9 and P_{10}. Here $s = 34$; thus the degree of the system drops to 19. We have a new point P_{14} of multiplicity $m_{14} = 5$. The points P_3 and P_9 become double points, while P_6 drops its multiplicity to $m_6 = 3$, and P_{10} drops to $m_{10} = 7$. The points P_3, P_6, P_9 P_{10} and P_{14} lie on the line r_4.



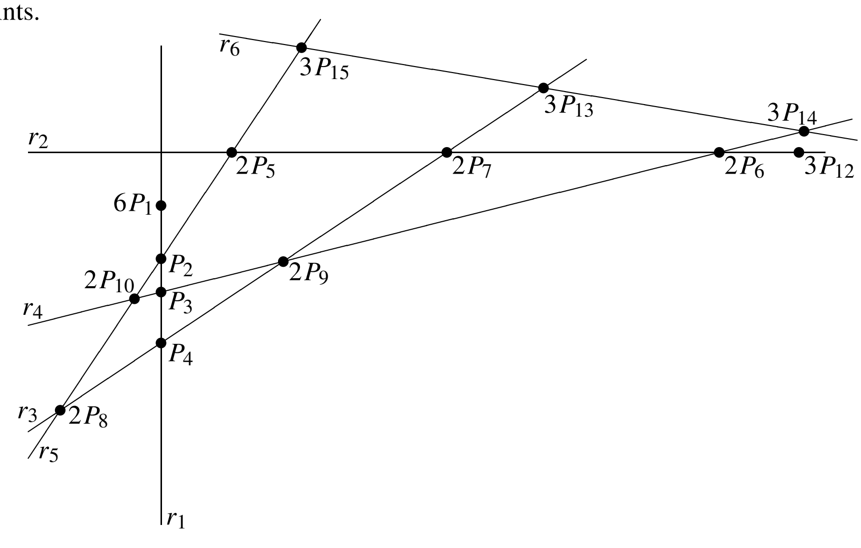
Finally apply another Four-Point Lemma with P_2, P_5, P_8 and P_{10}. In this way we reduce to a system of curves of degree 14. We have a new point P_{15} of multiplicity $m_{15} = 5$. The points P_2, P_8 and P_{10} become double points, while P_5 drops its multiplicity to $m_5 = 3$. The points P_2, P_5, P_8 P_{10} and P_{15} lie on the line r_5.



From this moment we can try to move the points in such a way that we only put three of them on a line at any one step. We start by putting P_{13}, P_{14} and P_{15} on a line r_6.



Since $m_{13} + m_{14} + m_{15} = 15 > 14$, r_6 splits off. We then reduce to a system of curves of degree 13. The points P_{13}, P_{14} and P_{15} now have multiplidity $m_{13} = m_{14} = m_{15} = 4$.
Now r_1 splits off once from the system (since $m_1 + m_2 + m_3 + m_4 + m_{11} = 7 + 2 + 2 + 2 + 1 = 14$). The point P_{11} drops its multiplicty to 0. The points P_2, P_3 and P_4 become simple points while P_1 has multiplicity $m_1 = 6$.
The degree of the system is now 12. Consider now the line r_2; we have $m_5 + m_6 + m_7 + m_{12} = 13$. Thus also r_2 splits off, leaving curves of degree 11. Moreover, the points P_5, P_6 and P_7 now have multiplicity $m_5 = m_6 = m_7 = 2$ and P_{12} has $m_{12} = 3$.
Now, since $m_{13} + m_{14} + m_{15} = 12$, the line r_6 splits once again from the system. Thus we reduce to a system of curves of degree 10 with the following configuration of points.



Now move the points such that P_1, P_8 and P_{14} lie a line r_7 and P_1, P_9 and P_{15} lie on a line r_8. Since $m_1 + m_8 + m_{14} = 6 + 2 + 3 = 11$ we can split the line r_7 from the system leaving curves of degree 9. On r_8 we now have $m_1+m_9+m_{15} = 5+2+3 = 10$; hence also r_8 splits. Thus the residual system has degree 8. Moreover we have P_1 with $m_1 = 4$, P_8, P_9 with $m_8 = m_9 = 1$ and P_{14}, P_{15} with $m_{14} = m_{15} = 2$.

Now the line r_2 splits ($m_5 + m_6 + m_7 + m_{12} = 9$) and we reduce to a system of curves of degree 7. The points P_5, P_6 and P_7 have multiplicity $m = 1$ while P_{12} has $m_{12} = 2$.

Now move P_1, P_5 and P_{13} such that they lie on a line r_9. Since $m_1+m_5+m_{13} = 8$ the line r_9 splits. We pass to a system of curves of degree 6 in which P_1 has $m_1 = 3$, P_{13} has $m_{13} = 2$ and P_5 has $m_5 = 0$. Now, the line r_4 with points P_3, P_6, P_9, P_{10} and P_{14} splits ($m_3 + m_6 + m_9 + m_{10} + m_{14} = 1 + 1 + 1 + 2 + 2 = 7$). Thus we can cancel the points P_3, P_6 and P_9 (since they have $m = 0$), while P_{10} and P_{14} drop their multiplicity to $m_{10} = m_{14} = 1$. The degree of the curves of the system drops to 5.

Move P_1, P_{10} and P_{12} in such a way they lie on a line r_{10}. Since $m_1+m_{10}+m_{12} = 3 + 1 + 2 = 6$, the line r_{10} splits and we reduce to a system of curves of degree 4. The point P_1 has $m_1 = 2$, P_{12} has $m_{12} = 1$, while P_{10} can be canceled since $m_{10} = 0$.

Now, since $m_4+m_7+m_8+m_{13} = 1+1+1+2 = 5 > 4$, the line r_3, with points P_4, P_7, P_8 and P_{13}, splits. Thus we can cancel the points P_4, P_7, P_8. The point P_{13} now has multiplicity $m_{13} = 1$. We reduce to a system of curves of degree 3.

Finally, we split the line r_6 with points P_{13}, P_{14} P_{15} ($m_{13}+m_{14}+m_{15} = 1+1+2 = 4$) and we reduce to the system of conics double at P_1 and passing through P_2, P_{12}, P_{15}. These give three distinct lines which must split off a system of degree two. We conclude that the system must be empty.

EXAMPLE 20. In a similar way to the previous examples we can treat the linear system $\mathcal{L}_{158}(50^{10})$ of curves of degree 158 with ten general points of multiplicity fifty. This was posed by J. Roe during the workshop as an interesting unknown case for the Nagata problem. The virtual dimension of this system is -31, and so we expect the system to be empty. We prove that, in fact, $\mathcal{L}_{158}(50^{10})$ is empty. We just give the essential steps and we leave the details to the reader.

We represent the system $\mathcal{L}_{158}(50^{10})$ by the following table

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
158	50	50	50	50	50	50	50	50	50	50

We start applying a four-point lemma to P_1, P_2, P_3 and P_4. Since $s = 200$ and $s - d = 200 - 158 = 42$, we have $t = 21$ and $e = 0$. Thus we reduce to the system with the following data

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
137	29	29	29	29	50	50	50	50	50	50	21
1	1	1	1	1	0	0	0	0	0	0	1

Here we have indicated that the points are no longer all general in the third row, by indicating the curve (in this case a line, of degree one) which passes through P_1, P_2, P_3, P_4, and P_{11}. Apply now a Cremona transformation centered at P_5, P_6 and P_7.

Apply a second Cremona centered at P_8, P_9, and P_{10}. Apply a third Cremona centered at P_5, P_6 and P_7, and finally a fourth Cremona centered at P_1, P_2, and P_3. We obtain

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
83	27	27	27	29	24	24	24	24	24	24	21
7	3	3	3	1	2	2	2	2	2	2	1

so that the resulting system has degree 83 with the indicated multiplicities, and the points lie on a curve of degree 7 with the indicated multiplicities. (The curve of degree 7 is the image of the line under the four Cremona transformations.)

Now execute a Four Point Lemma with points P_4, P_5, P_6, and P_7, which results in:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
74	27	27	27	20	15	15	15	24	24	24	21	9
7	3	3	3	1	2	2	2	2	2	2	1	0
1	0	0	0	1	1	1	1	0	0	0	0	1

Apply two more Cremona transformations. The first one is centered at P_1, P_2 and P_3 and the second one is centered at P_8, P_9 and P_{10}. The result is:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
62	20	20	20	20	15	15	15	19	19	19	21	9
4	1	1	1	1	2	2	2	1	1	1	1	0
4	1	1	1	1	1	1	1	2	2	2	0	1

The two quartics indicated above are the Cremona images of the septic and the line in the previous table.

At this point, we can use the four-point lemma on P_1, P_2, P_3 and P_{11}. We reduce to the system represented by

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
52	10	10	10	20	15	15	15	19	19	19	11	9	9
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

At this point one notices that the second quartic splits off the system, three times; and then the line splits off once. This results in the system of degree 39 indicated by:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
39	6	6	6	17	12	12	12	13	13	13	10	6	8
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

Now apply three Cremona transformations respectively centered at $(P_4,\ P_8, P_9)$,

(P_4, P_5, P_{10}) and (P_4, P_6, P_7). This reduces the system to

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
30	6	6	6	8	9	10	10	9	9	10	10	6	8
4	1	1	1	1	2	1	1	2	2	1	1	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1

where now the constraints on the points are that they lie on the indicated three quartic curves.

Now execute a Four Point Lemma with P_{10}, P_{11}, P_{12}, and P_{13}, to obtain the system

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
28	6	6	6	8	9	10	10	9	9	8	8	4	6	2
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

At this point the first quartic splits off the system, then the second quartic splits off, then the line splits off; at this point the first quartic splits off again, then the third quartic splits, then finally the line splits again. This leaves us with the system of degree ten:

d	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
10	2	2	2	2	3	5	5	3	3	1	3	1	3	0
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

Now three Cremona transformations, centered respectively at (P_6, P_7, P_{11}), (P_5, P_8, P_9) and (P_3, P_6, P_{13}), give a system of cubics which are double at P_1, P_2, P_4, and P_7, and passing simply through P_5, P_8, P_9, P_{10}, P_{12}, and P_{13}. This system is clearly empty (the six lines passing pairwise through the four double points must split off; but the degree of the system is only three).

Since this system is empty, the claim follows for $\mathcal{L}_{158}(50^{10})$.

7. Lecture six: special effect varieties

This section is devoted to the definition and the study of two different kinds of varieties called *special effect varieties*. The $\alpha-$*special effect variety* is defined by requiring some numerical conditions, while the definition of h^1-*special effect variety* concerns cohomology groups. We will start with the case of special effect curves in $\mathbb{P}^2$. As we will see, the existence of these curves is related to the speciality of a given linear system. This suggests two new conjectures for special systems in the planar case. Whenever not otherwise specified, we work over the field $\mathbb{C}$.

7.1. Basic definitions

$\alpha-$Special effect curves

We start with some preliminary definitions.

DEFINITION 5. *Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and $\mathcal{L}' := \mathcal{L}_{2,d'}(-\sum_{i=1}^{s} c_{j_i}\ P_{j_i})$ be two linear systems in $\mathbb{P}^2$. We will write $\mathcal{L}' <_{LS} \mathcal{L}$ if*

1) $d' \leq d$;

2) $\{P_{j_1}, \ldots P_{j_s}\} \subseteq \{P_1, \ldots P_h\}$;

3) $c_{j_k} \leq m_{j_k}$ for all $k = 1, \ldots, s$.

Let $Y \subset \mathbb{P}^2$ be a curve. Then we write $Y <_{LS} \mathcal{L}$ if the degree of Y is less than or equal to d and $\operatorname{mult}_{P_i}(Y) \leq m_i$ for each i.

DEFINITION 6. *Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of curves of degree d passing through the points P_i with multiplicity at least m_i. Let Y be an irreducible curve passing through the points $P_{j_1}, \ldots, P_{j_s}$ with multiplicity at least $c_{j_1}, \ldots, c_{j_s}$, such that $Y <_{LS} \mathcal{L}$. Then Y has the* **weak special effect property for** *$\mathcal{L}$ if*

(iP) $\nu(|Y|) \geq 0$,

(iiW) $\nu(\mathcal{L} - Y) \geq \nu(\mathcal{L})$.

Moreover, we will say that Y has the **special effect property for** *$\mathcal{L}$ if the inequality in (iiW) is strict, i.e.*

(iiP) $\nu(\mathcal{L} - Y) > \nu(\mathcal{L})$.

EXAMPLE 21. Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and consider a $(-1)-$curve E such that $\mathcal{L} \cdot E = -N < 0$. Thus $\mathcal{L} = NE + \mathcal{M}$, where $E \cdot \mathcal{M} = 0$. Using Riemann-Roch it is easy to prove

$$\nu(\mathcal{M}) = \nu(\mathcal{L}) + \binom{N}{2}.$$

Hence E has the special effect property if $N \geq 2$ and the weak special effect property if $N = 1$.

EXAMPLE 22. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,6}(-\sum_{i=1}^{9} 2P_i)$. The only element in $\mathcal{L}$ is the double cubic through the nine points $C = 3H - \sum_{i=1}^{9} P_i$. Since

$$\nu(\mathcal{L}) = \frac{6 \cdot (3+6)}{2} - 9 \cdot 3 = 0$$

and

$$\nu(\mathcal{L} - C) = \nu(C) = \frac{3 \cdot (3+3)}{2} - 9 = 0$$

we conclude that the cubic C has the weak special effect property for $\mathcal{L}$.

EXAMPLE 23. Let $\mathcal{L}$ be the system $\mathcal{L}_{2,9}(-6P_1-6P_2-6P_3)$. The only element in $\mathcal{L}$ is $3Y$, where Y is the union of the lines passing through two of the three points, i.e. $Y = L_{12} + L_{13} + L_{23}$, where L_{ij} is the line through P_i and P_j. We claim that each of the lines L_{ij} has the special effect property. We prove this for L_{12}. Obviously one has $\nu(|L_{12}|) \geq 0$; indeed, it is a (-1)-curve. Moreover $\mathcal{L} - L_{12}$ is the system $\mathcal{L}' := \mathcal{L}_{2,8}(-5P_1-5P_2-6P_3)$ and its virtual dimension is

$$\nu(\mathcal{L}') := \frac{8\cdot 11}{2} - 2\frac{5\cdot 6}{2} - \frac{6\cdot 7}{2} = 44 - 30 - 21 = -7$$

while $\nu(\mathcal{L}) = -9$. So the claim follows.

It is clear, now, in which way we proceed. If Y has one of the special effect properties, we substitute the system $\mathcal{L}$ with $\mathcal{L} - Y$ and we investigate this new system.

DEFINITION 7. *Let $\mathcal{L}$ be a system as above. Fix a sequence of (not necessarily distinct) irreducible curves $Y_1, \ldots Y_\alpha$, such that any two distinct members are disjoint. Suppose further that*

(1) Y_j has the weak special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} Y_i$, for $j = 1, \ldots, \alpha$,

(2) there exists at least one index j such that Y_j has the special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} Y_i$,

(3) $\nu(\mathcal{L} - \sum_{i=1}^{\alpha} Y_i) \geq 0$.

Then $X := \sum_{i=1}^{\alpha} Y_i$ is called a **special effect configuration for** $\mathcal{L}$*. In particular we write $X := \sum_{i=1}^{r} \alpha_i Y_i$ if r is the number of distinct curves and Y_i occurs α_i times in the list. We call both X and $\{Y_1, \ldots, Y_r\}$ an $(\alpha_1, \ldots, \alpha_r)-$***special effect configuration***. Finally, when $Y_1 = Y_2 = \cdots = Y_\alpha = Y$ we write $X = \alpha Y$ and we call both X and Y an $\alpha-$***special effect curve***.

Let us analyze the three requirements. Since $\mathcal{L} - \sum_{i=1}^{\alpha} Y_i$ is nothing else than the (residual) system $\mathcal{L}' := |(d - \sum_{i=1}^{\alpha} \deg(Y_i))H - \sum_{i=1}^{h}(m_i - \sum_{k=1}^{s_i} c_{j_{i_k}})P_i|$, condition (3) says that $\mathcal{L}'$ is not empty. Conditions (1) and (2) are surely the most interesting. As a matter of fact they tell us that the number of conditions imposed on the system of curves of degree d by imposing the curves $Y_1, \ldots, Y_\alpha$ and the points P_i with multiplicity $m_i - \sum_{k=1}^{s_i} c_{j_{i_k}}$ (such that the final multiplicity at the point P_i is at least m_i, $i = 1, \ldots, n$) is less than the number of conditions imposed to the same system $|dH|$ only imposing each P_i with multiplicity at least m_i, $i = 1, \ldots, n$. This sounds like a crazy requirement because, in general, we expect that a positive dimensional variety imposes more conditions than a zero-dimensional variety. It is important to notice the similarity with the "strange" requirement in the case of $(-1)-$curves (see for example [21]). We asked there for a curve C whose double is not expected to exist.

We recall that the existence of a $(-1)-$curve C such that $\mathcal{L} := NC + \mathcal{M}$ leads us to the inequality

$$\dim(\mathcal{L}) = \dim(\mathcal{M}) \geq \nu(\mathcal{M}) = \nu(\mathcal{L}) + \binom{N}{2}$$

which, under the assumption $\nu(\mathcal{M}) \geq 0$ and $N \geq 2$, implies that $\mathcal{L}$ is special. Also the existence of $\alpha-$special effect variety or special effect configuration X for a system $\mathcal{L}$ forces the system itself to be special. In fact we have the following chain of inequalities:

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L} - X) \geq \nu(\mathcal{L} - X) > \nu(\mathcal{L})$$

and, together with condition (3), one has $\dim(\mathcal{L}) > \epsilon(\mathcal{L})$.

EXAMPLE 24. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e $Y = H - P_1 - P_2$. Since

$$\nu(\mathcal{L}) = -1$$

and

$$\nu(\mathcal{L} - Y) = \nu(Y) = 0$$

we conclude that Y has the special effect property for $\mathcal{L}$ and it has the weak special effect property for $\mathcal{L} - Y$. From $\nu(\mathcal{L} - 2Y) = 0$ it follows that the line through P_1 and P_2 is a $2-$special effect curve for $\mathcal{L}$ and so $\mathcal{L}$ is special.

EXAMPLE 25. Consider again the system $\mathcal{L} := \mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$. We prove that $X = 3L_{12} + 3L_{13} + 3L_{23}$ is a $(3, 3, 3)-$special effect configuration. Recall that $\nu(\mathcal{L}) = -9$. In Example 23 we proved that L_{12} has the special effect property for $\mathcal{L}$ because

$$\nu(\mathcal{L} - L_{12}) = \nu(|8H - 5P_1 - 5P_2 - 6P_3|) = -7.$$

Define now $\mathcal{L}' := \mathcal{L} - L_{12}$. We have

$$\nu(\mathcal{L}' - L_{12}) = \nu(\mathcal{L} - 2L_{12}) = \nu(|7H - 4P_1 - 4P_2 - 6P_3|) = -6$$

so that L_{12} has the special effect property for $\mathcal{L}' = \mathcal{L} - L_{12}$. Define $\mathcal{L}'' := \mathcal{L}' - L_{12} = \mathcal{L} - 2L_{12}$. If we compute the virtual dimension of $\mathcal{L}'' - L_{12}$ we discover that it is again -6. Thus L_{12} has the weak special effect property for $\mathcal{L}''$. We can go ahead and apply the previous procedure with L_{13} and L_{23}. We obtain:

$$\begin{array}{l}
\nu(\mathcal{L} - 3L_{12} - L_{13}) = \nu(|5H - 2P_1 - 3P_2 - 5P_3|) = -4 \\
\nu(\mathcal{L} - 3L_{12} - 2L_{13}) = \nu(|4H - P_1 - 3P_2 - 4P_3|) = -3 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13}) = \nu(|3H - 3P_2 - 3P_3|) = -3 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - L_{23}) = \nu(|2H - 2P_2 - 2P_3|) = -1 \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 2L_{23}) = \nu(|H - P_2 - P_3|) = 0. \\
\nu(\mathcal{L} - 3L_{12} - 3L_{13} - 3L_{23}) = 0.
\end{array}$$

Thus $X = 3L_{12} + 3L_{13} + 3L_{23}$ is a $(3, 3, 3)-$special effect configuration for $\mathcal{L}_{2,9}(-6P_1 - 6P_2 - 6P_3)$.

DEFINITION 8. *A special system arising from the existence of an $\alpha-$special effect curve (or an $(\alpha_1, \dots \alpha_r)-$special effect configuration) is called* **Numerically Special**.

Finally, we can state the following

CONJECTURE 5 ((NSEC) "NUMERICAL SPECIAL EFFECT" CONJECTURE). A linear system of plane curves $\mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is numerically special.

h^1-Special effect curves

The second class of curves we introduce are defined via some particular conditions on certain cohomology groups. The original idea for these curves comes from a detailed analysis of the base locus in the special systems listed in Theorem 11, that is, linear systems with imposed double points in $\mathbb{P}^n$, $n \geq 2$.

DEFINITION 9. *Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of plane curves with general multiple base points. An irreducible curve $Y \subset \mathbb{P}^2$, with $\mathcal{O}_{\mathbb{P}^2}(Y) \ncong \mathcal{L}$, is an h^1-***special effect curve** *for the system $\mathcal{L}$ if the following conditions are satisfied:*

(a) $h^0(\mathcal{L}_{|Y}) = 0$*;*

(b) $h^0(\mathcal{L} - Y) \neq 0$*;*

(c) $h^1(\mathcal{L}_{|Y}) > 0$.

EXAMPLE 26. Let $\mathcal{L} := \mathcal{L}_{2,2}(-2P_1 - 2P_2)$ be the linear system of conics with two double points. Let Y be a line through P_1 and P_2, i.e $Y = H - P_1 - P_2$. Since $\mathcal{L} \cdot Y = -2$ the restricted system $\mathcal{L}_{|Y}$ has no effective divisors and $h^0(\mathcal{L}_{|Y})$ is empty. By Riemann–Roch we easily compute $h^1(\mathcal{L}_{|Y}) = g_Y - 1 - \deg(\mathcal{L}_{|Y}) = 1 > 0$. Finally $\mathcal{L} - Y$ is $|H - P_1 - P_2|$, so that $h^0(\mathcal{L} - Y) \neq 0$. Hence the line Y through P_1 and P_2 is an h^1-special effect curve for $\mathcal{L}$.

Let $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ and consider, on the blow-up of $\mathbb{P}^2$ at the points P_i $i = 1, \dots, n$, the exact sequence

$$0 \to \mathcal{L} - Y \to \mathcal{L} \to \mathcal{L}_{|Y} \to 0$$

which gives the following long exact sequence in cohomology:

$$0 \to H^0(\mathcal{L} - Y) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|Y}) \to H^1(\mathcal{L} - Y) \to H^1(\mathcal{L}) \to H^1(\mathcal{L}_{|Y}) \to 0.$$

Conditions (a) and (b) assure us that $H^0(\mathcal{L}) \neq 0$, while condition (c) implies $H^1(\mathcal{L}) \neq 0$. Thus the existence of such Y forces the system $\mathcal{L}$ to have $h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) \neq 0$ so that, by (2), $\mathcal{L}$ is special. Again, we can give a particular name to this kind of system:

DEFINITION 10. *A special linear system arising from the existence of an h^1-special effect curve is called* **Cohomologically Special**.

And again we can state a conjecture:

CONJECTURE 6 ((CSEC) "COHOMOLOGICAL SPECIAL EFFECT" CONJECTURE). A linear system of plane curves $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is cohomologically special.

The four conjectures

In the previous sections we introduced two new conjectures for the characterization of special linear systems in the planar case. At this point it is natural to ask if these new conjectures are equivalent to the Segre and Harbourne–Hirschowitz conjectures. The answer is given in the following

THEOREM 15. *Conjectures (SC) [1], (HHC) [3], (NSEC) [5] and (CSEC) [6] are equivalent.*

The proof of the previous theorem can be found in [10], Chapter 3. Here one can find additional interesting evidence relating these ideas with other conjectures for special systems on surfaces, in particular on Hirzebruch and K3 surfaces.

7.2. Results in higher dimensions

As already mentioned in Section 2, very little is known for special linear systems on a variety X with $\dim(X) > 2$, even when $X = \mathbb{P}^n$. In this last case the most important result is the classification of the homogeneous special systems for double points given by Alexander and Hirschowitz in Theorem 11.

Continuing with $\mathbb{P}^n$, $n \geq 3$ we can notice that there is not a precise conjecture. Although the Segre Conjecture can be generalized in every ambient variety using the statement concerning $H^1 \neq 0$, there is nothing that characterizes the special systems from a geometric point of view as, for example, in the case of $(-1)-$curves in $\mathbb{P}^2$.

A worthy goal would be to find a conjecture (C) in $\mathbb{P}^n$, [or in a generic variety X] such that, when we read (C) in $\mathbb{P}^2$, (C) is equivalent to the Segre (1) and Harbourne–Hirschowitz (3) Conjectures.

This goal is one of the main topics in [10]. Here we can see how both *Numerical Special Effect Conjecture* and *Cohomological Special Effect Conjecture* are potential candidates for the above-mentioned goal. Unluckily, in both cases it could be difficult to work with a generic special effect variety Y of codimension $c > 1$; we do not have, for example, a precise definition of virtual dimension of $\mathcal{L} - Y$ or it could be hard to compute $h^2(\mathcal{L} - Y)$.

Thus it is not so easy to define the special effect varieties in $\mathbb{P}^n$ with $n \geq 3$. Obviously, when the $\alpha-$special effect variety Y is a divisor on $\mathbb{P}^n$ (but in general, on every variety) we can generalize the definitions given in section 7.1 most easily. In [10] there are some different approaches to avoid the previous problem and, although there is not yet a general theory for the higher dimension case, several examples of $\alpha-$special effect varieties are shown. In particular we have the following theorem

THEOREM 16 ([10], THEOREM 4.1.17). *There exists a* $2-$*special effect variety* Y *for each of the special systems listed in Theorem 11.*

EXAMPLE 27. Consider the system $\mathcal{L} := \mathcal{L}_{3,4}(2^9)$. One has $\nu(\mathcal{L}) = -2$. Let Q the quadric in $\mathbb{P}^3$ through the nine points. Since $\nu(\mathcal{L} - 2Q) = 0 > \nu(\mathcal{L})$, Q is a $2-$special effect variety for $\mathcal{L}$.

We turn now to analyzing h^1-special effect varieties in higher dimension. Let $\mathcal{L} := \mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of hypersurfaces with general multiple base points and let X be the blow-up of $\mathbb{P}^n$ at the points $\{P_i\}$. Let $\tilde{\mathcal{L}}$ be the strict transform of $\mathcal{L}$. In general, if confusion cannot arise, we will denote both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\mathcal{L}$. If we denote by $\tilde{Y}$ the strict transform of a variety $Y \subset \mathbb{P}^n$, then we define $\mathcal{L} - Y := \mathcal{L} \otimes \mathcal{I}_{\tilde{Y}}$. The definition of the h^1-special effect variety is slightly modified by respect to the planar case.

DEFINITION 11. *Let* $\mathcal{L}$ *and* Y *be as above with* Y *irreducible. Moreover, if codim*$(Y, \mathbb{P}^n) = 1$ *then we require* $\mathcal{O}_{\mathbb{P}^n}(Y) \not\cong \mathcal{L}$*. Then* $Y \subset \mathbb{P}^n$ *is an* h^1-**special effect variety** *for the system* $\mathcal{L}$ *if the following conditions are satisfied:*

(a) $h^0(\mathcal{L}_{|Y}) = 0$*;*

(b) $h^0(\mathcal{L} - Y) \neq 0$*;*

(c) $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$.

The h^1-special effect varieties seem easier to treat than the $\alpha-$special effect varieties. In fact, we do not need to define the virtual dimension, but we just work with elements in cohomology. However, in several situations, it is very difficult to compute some cohomology groups, in particular $h^2(\mathcal{L} - Y)$.

As in the case of $\alpha-$special effect varieties, we do not have problems when Y is a divisor since $h^2(\mathcal{L} - Y) = 0$ if $\mathcal{L} - Y$ is effective. To see this, write $\mathcal{L}$ as $\mathcal{L} := |dH - \sum_{i=1}^{h} m_i P_i|$ and $Y := eH - \sum_{i=1}^{h} c_i P_i$; then $\mathcal{L} - Y = |(d-e)H - \sum_{i=1}^{h}(m_i - c_i)P_i|$, with $d \geq e$ and $m_i \geq c_i$. Hence the system $\mathcal{L} - Y$ has the form $\mathcal{L} - Y = |aH - \sum_{i=1}^{h} s_i P_i|$, with $a \geq 0$. Define Z as the union of the fat points $s_i P_i$; then we have the following exact sequence

$$0 \to \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(a) \to \mathcal{O}_{\mathbb{P}^n}(a) \to \mathcal{O}_Z \to 0.$$

where $\mathcal{L} - Y$ is exactly $\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(a)$. When we consider the cohomology groups, we have $h^i(\mathcal{O}_Z) = 0$ for $i \geq 1$, since Z is a zero-dimensional scheme. Moreover, $h^i(\mathcal{O}_{\mathbb{P}^2}(a)) = 0$ for $i \geq 1$. Thus $h^i(\mathcal{L} - Y) = 0$ for $i \geq 2$ (this motivates also conditions (c) in the planar case).

Unluckily, when Y is a divisor, it can be difficult to study the behaviour of $\mathcal{L}_{|Y}$. Instead, when $\operatorname{codim}(Y, \mathbb{P}^n) \geq 2$, the groups $h^i(\mathcal{L} - Y)$, $i = 1, 2$ can be computed on the blow-up of $\mathbb{P}^n$ along Y, but we need a deep understanding of the geometry and cohomology of Y.

In any case there is interesting evidence of the relationship between special systems and h^1-special effect varieties; in particular, we have the following:

THEOREM 17 ([10], CH 4, THEOREM 4.2.2). *There exists an h^1-special effect variety Y for each of the special systems listed in Theorem 11.*

We conclude this section with some examples of $\alpha-$special effect varieties on $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$ with $t \geq 2$ and $n_i \geq 1$ for $i = 1, \ldots, t$. In [10] the case mainly explored concerns $m = \alpha = 2$ and Y is a divisor. Surely this does not exhaust all possible special effect varieties on X, but we can observe how our results fit with the ones by Catalisano, Geramita and Gimigliano on secant varieties of products of projective spaces ([15], [16], [17], [18]). We suppose that the reader knows the relationship between special systems and defective varieties; we suggest [20] as reference.

Also in the case of $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$ we have some interesting evidence. First of all we recall a result by Catalisano, Geramita and Gimigliano.

THEOREM 18 ([18], THEOREM 2.1). *Let $\mathcal{L} := \mathcal{L}_{a_1,a_2}(2^h)$ be the linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ of divisors of bidegree (d_1, d_2) with h imposed double points. Then $\mathcal{L}$ is non-special unless*

$$a_1 = 2d, a_2 = 2, d \geq 1, \text{ and } h = 2d + 1.$$

From the study of $\alpha-$special effect varieties on $\mathbb{P}^a \times \mathbb{P}^b$ we are able to prove the following

THEOREM 19. *There exists a $2-$special effect curve for each of the special systems listed in Theorem 18.*

The second result we mention is related to the study of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

THEOREM 20 ([18], THEOREM 2.5). *Let $a_1 \geq a_2 \geq a_3 \geq 1$, $\alpha \in \mathbb{N}$ and $V = V_{\mathbf{a}}$ be a Segre–Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $Sec_k(V)$ has the expected dimension, except for:*

$$(a_1, a_2, a_3) = (2, 2, 2) \text{ and } k = 6;$$

$$(a_1, a_2, a_3) = (2\alpha, 1, 1) \text{ and } k = 2\alpha.$$

In these cases $Sec_k(V)$ is defective, and its defectivity is 2 in the first case and 1 in the second.

Once again we can try to check if there are special effect varieties for the special systems corresponding to the defective varieties listed before. It is easy to observe that, by numerical reasons, the second case cannot be treated with a $2-$special effect variety. However, using special effect configurations we can state a result as Theorem 19.

THEOREM 21. *There exists a* $2-$*special effect variety or a* $(1,1)-$*special effect configuration for each of the special systems listed in Theorem 20.*

The Theorems 19 and 21 follow from a deep studying of the combinatorial behaviour of $\alpha-$special effect varieties on $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$. In particular we can prove the following results.

PROPOSITION 3. *Let* $Y \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ *be a divisor of bidegree* (e_1, e_2)*, with* $e_i \neq 0$ *for at least one* i*; then* Y *is a* $2-$*special effect variety for* $\mathcal{L}_{(d_1,d_2)}(2^h)$*, with* $d_1 \cdot d_2 \neq 0$*, in the following cases*

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$	(d_1, d_2)	(e_1, e_2)	h
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2e_2)$	$(1, e_2)$	$2e_2 + 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2e_1, 2)$	$(e_1, 1)$	$2e_1 + 1$
$\mathbb{P}^1 \times \mathbb{P}^{n_2}$	$(2e_1, 2)$	$(e_1, 1)$	$m_1(e_1, n_2) \leq h \leq M_1(e_1, n_2)$
$\mathbb{P}^2 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$m_2(n_2) \leq h \leq M_2(n_2)$
$\mathbb{P}^3 \times \mathbb{P}^3$	$(2, 2)$	$(1, 1)$	15
$\mathbb{P}^3 \times \mathbb{P}^4$	$(2, 2)$	$(1, 1)$	19

where

$$m_1(e_1, n_2) := \lfloor \tfrac{(2e_1+1)(n_2+1)}{2} \rfloor \quad m_2(n_2) := \lfloor \tfrac{3n_2^2+9n_2+5}{n_2+3} \rfloor$$

$$M_1(e_1, n_2) := e_1 n_2 + e_1 + n_2 \quad M_2(n_2) := 3n_2 + 2.$$

PROPOSITION 4. *Let* $t \geq 3$*. Let* $\mathcal{L} := \mathcal{L}_{(d_1,\ldots d_t)}(2^h)$ *be a linear system of multidegree* $(d_1, \ldots, d_t)$*, with* $d_i \neq 0$ *for* $i = 1, \ldots, t$*, on* $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ *passing through* h *double points in general position and let* Y *be a divisor of multidegree* $(e_1, \ldots, e_t)$ *on* X *with* $e_i \neq 0$ *for* $i = 1, \ldots, t$*. Then* Y *is a* $2-$*special effect variety on* X *for* $\mathcal{L}$ *only if* $t = 3$ *and for the following values:*

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$	(d_1, d_2, d_3)	(e_1, e_2, e_3)	h
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2, 2)$	$(1, 1, 1)$	7
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	$(2, 2, 2)$	$(1, 1, 1)$	11
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$(2, 2, 2)$	$(1, 1, 1)$	15

We conclude with a short list of interesting special effect varieties.

- A curve of type $(n, 1)$ [resp. of type $(1, n)$] on a quadric $Q \subset \mathbb{P}^3$ is a $2-$special effect variety on Q for the system $\mathcal{L}(2n, 2)(2^{2n+1})$ [resp. for $\mathcal{L}(2, 2n)(2^{2n+1})$];
- the line in $\mathbb{P}^3$ is a $2-$special effect curve for $\mathcal{L}_{3,2}(2^2)$;
- the conic in $\mathbb{P}^3$ is a $2-$special effect curve for $\mathcal{L}_{3,2}(2^3)$;
- the union of the $\binom{n+1}{2}$ lines passing through the coordinate points in $\mathbb{P}^n$, $n \geq 3$, is an $(n-1)-$special effect variety for $\mathcal{L}_{n,n+1}(n^{n+1})$;
- the quadric $Q \subset \mathbb{P}^3$ is both a $1-$special effect variety and an h^1-special effect variety for the Laface–Ugaglia example (see [39]).

References

[1] ALEXANDER J., *Singularités imposables en positions générale à une hypersurface projective*, Compositio Math. **68** (1998), 305–354.

[2] ALEXANDER J. AND HIRSCHOWITZ A., *La méthode d'Horace éclatée: application à l'interpolation en degré quatre*, Inventiones Mathematicae **107** (1992), 582–602.

[3] ALEXANDER J. AND HIRSCHOWITZ A., *Un lemme d'Horace différentiel: application aux singularités hyperquartiques de* $\mathbb{P}^5$, J. Algebraic Geometry **1** (1992), 411–426.

[4] ALEXANDER J. AND HIRSCHOWITZ A., *Polynomial interpolation in several variables*, J. Algebraic Geometry **4** (1995), 201–222.

[5] ALEXANDER J. AND HIRSCHOWITZ A., *Generic hypersurface singularities*, Proc. Indian Acad. Sci. Math. Sci. **107** 2 (1997), 139–154.

[6] ALEXANDER J. AND HIRSCHOWITZ A., *An asympotic vanishing theorem for generic unions of multiple points*, Inventiones Mathematicae **140** (2000), 303–325.

[7] ARBARELLO E. AND CORNALBA M., *Footnotes to a paper of Beniamino Segre*, Math. Ann. **256** (1981), 341–362.

[8] ARBARELLO E. AND M. CORNALBA M., *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Alg. Geom. 5 (1996), 705–749.

[9] BALLICO E., BOCCI C. AND FONTANARI C., *Zero-dimensional subschemes of ruled varieties*, to appear on CEJM (2003).

[10] BOCCI C., *Special linear systems and special effect varieties*, Ph.D. Thesis, University of Torino, Torino 2003.

[11] BUCKLEY A. AND ZOMPATORI M., *Linear systems of plane curves with a composite number of base points of equal multiplicity*, Trans. Amer. Math. Soc. **355** 2 (2003), 539–549.

[12] BOYER C.B., *A history of mathematics*, John Wiley & Sons, Inc., New York 1991.

[13] CARLINI E., *Geometric aspects of some polynomial decompositions*, Ph.D. Thesis, University of Pavia, Pavia 2003.

[14] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *On the secant varieties to the tangential varieties of a Veronesean*, Proc. A.M.S. **130** (2001), 975–985.

[15] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Ranks of tensors, secant varieties of Segre varieties and fat points*, Linear Algebra Appl. **355** (2002), 263–285.

[16] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Higher secant varieties of Segre embeddings of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, preprint (2003).

[17] CATALISANO M.V., GERAMITA A.V. AND A. GIMIGLIANO A., *Higher secant varieties of Segre varieties of* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ preprint (2003).

[18] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Higher secant varieties of Segre–Veronese varieties*, preprint (2003).

[19] CHANDLER K., *A brief proof of a maximal rank theorem for generic double points in projective space*, Trans. Amer. Math. Soc. **353** 5 (2001), 1907–1920.

[20] CHIANTINI L., *Lectures on the structure of projective embeddings*, Rend. Sem. Mat. Univ. e Polit. Torino **64** 4 (2004), 59–114.

[21] CILIBERTO C., *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, in: "Proceedings of the Third European Congress of Mathematics, I (Barcelona 2000)", Prog. Math. **201**, Birkhäuser, Basel 2001, 289–316.

[22] CILIBERTO C. AND MIRANDA R., *Degenerations of planar linear systems*, J. Reine Angew. Math. **501** (1998), 191–220.

[23] CILIBERTO C. AND MIRANDA R., *Linear systems of plane curves with base points of equal multiplicity*, Trans. Amer. Math. Soc. **352** (2000), 4037–4050.

[24] CILIBERTO C. AND MIRANDA R., *The Segre and Harbourne–Hirschowitz conjectures*, in: “Applications of algebraic geometry to coding theory, physics and computation (Eilat 2001)”, NATO Sci. Ser. II Math. Phys. Chem. **36**, Kluwer Acad. Publ., Dordrecht 2001, 37–51.

[25] CILIBERTO C. AND MIRANDA R., *Matching conditions for degenerating plane curves and applications*, in preparation.

[26] CILIBERTO C., CIOFFI F., MIRANDA R., AND ORECCHIA F., *Bivariate Hermite interpolation and linear systems of plane curves with base fat points*, Proceedings of The Asian Symposium on Computer Mathematics [ASCM] 2003.

[27] DE VOLDER C. AND LAFACE A., *On linear systems of* $\mathbb{P}^3$ *through multiple points*, math.AG/0311447, preprint (2003).

[28] DE VOLDER C. AND LAFACE A., *Base locus of linear systems on the blowing-up of* $\mathbb{P}^3$ *along a most eight general points*, math.AG/0401244, preprint (2004).

[29] DENT A., *Variations on methods of Lorentz and Lorentz for dimensions two and three*, Ph.D. Thesis, Colorado State University 2003.

[30] DENT A. AND MIRANDA R., *Hermite interpolation on toric surfaces*, in preparation.

[31] EVAIN L., *Une minoration du degré des courbes planes à singularités imposées*, Bull. S.M.F. **126** (1998), 525–543.

[32] FONTANARI C., *On Waring’s problem for many forms and Grassmann defective varieties*, J. Pure Appl. Algebra **74** 3 (2002), 243–247.

[33] GERAMITA A.V., *Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, in: “The Curves Seminar at Queen’s, Vol. X (Kingston, 1995)” Queen’s Papers in Pure and Appl. Math. **102**, Queen’s Univ., Kingston, ON 1996, 2–114.

[34] GIMIGLIANO A., *On linear systems of plane curves*, Thesis, Queen’s University, Kingston 1987.

[35] HIRSCHOWITZ A., *La méthode d’Horace pour l’interpolation a plusieurs variables*, Manuscripta Math. **50** (1985), 337–378.

[36] HIRSCHOWITZ A., *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, J. Reine Angew. Math. **397** (1989), 208–213.

[37] HORN R.A. AND JOHNSON C.R., *Matrix Analysis*, The Press Syndicate of the University of Cambridge, Cambridge 1990.

[38] LAFACE A., *On linear systems of curves on rational scrolls*, Geom. Dedicata **90** (2002), 127 - 144.

[39] LAFACE A. AND UGAGLIA L., *A counterexample to a conjecture on linear systems on* $\mathbb{P}^3$, preprint (2003).

[40] LAFACE A. AND UGAGLIA L., *On a class of special linear system of* $\mathbb{P}^3$, math.AG/0311445, preprint (2003).

[41] LAFACE A. AND UGAGLIA L., *Special linear system on toric varieties*, math.AG/0310043, preprint (2003).

[42] LORENTZ G.G. AND LORENTZ R.A., *Bivariate Hermite interpolation and applications to algebraic geometry*, Numer. Math. **57** (1990), 669–680.

[43] LORENTZ R.A., *Multivariate Hermite interpolation by algebraic polynomials: a survey*, Journal of Computational and Applied Mathematics **122** (2000), 167–201.

[44] MALONE A., *The ten-point problem*, Master’s Paper, Colorado State University 2003.

[45] MIRANDA R., *Linear systems of plane curves*, Notices AMS **46** (1999), 192–202.

[46] MIGNON T., *Systémes linéaires de courbes planes*, Thése, Univ. de Nice, Nice 1997.

[47] MIGNON T., *Systémes linéaires de courbes planes à singularités ordinaires imposées*, C. R. Acad. Sci. Paris Sér I Math. **327** (1998), 651–654.

[48] RAN Z., *Enumerative geometry of singular plane curves*, Inventiones Mathematicae **97** (1989), 447–465.

[49] SEGRE B., *Alcune questioni su insiemi finiti di punti in geometria algebrica*, Atti Convegno Intern. di Geom. Alg. di Torino, Torino 1961, 15–33.

[50] SHAFAREVICH I., *Basic algebraic geometry 1: varieties in projective space*, Springer - Verlag, Berlin 1994 (second edition).

[51] FULTON W., *Introduction to toric varieties*, Princeton University Press, Princeton 1993.

[52] YANG S., *Linear systems of plane curves through base points of bounded multiplicity*, in: "Polynomial Interpolation and Projective Embeddings. Proceedings of the Workshop", Rend. Sem. Mat. Univ. e Polit. Torino (2005), to appear.

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Polynom. Interp. Lecture Notes

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LECTURES ON THE STRUCTURE OF PROJECTIVE EMBEDDINGS

Abstract. In this we draw a picture of the "status of the art" in the theory of defective varieties, i.e. varieties whose secant spaces fill up a variety of dimension smaller than expected. Some links between the theory of defective varieties and other fields of algebraic geometry and mathematics are outlined. Several open problems and current researches on the subject are presented.

1. Introduction

These notes are concerned with projective algebraic geometry. They are not self contained: at least they do not start with the general definition of what a general projective variety is. For our scopes, it will be enough to say that a **projective variety** X is just a subset of a projective space $\mathbb{P}^r$ defined by the vanishing of a set of **algebraic equations** (= homogeneous polynomials). For deeper details, we refer the reader to the massive literature on the subject. Let us just mention some classical books [36], [72], [46], [45], [47], whose foundations and methods are used freely through these lectures.

So, we aim to present some properties of the variety $X \subset \mathbb{P}^r$ and we agree that, from now on, our X is **irreducible** (in the topological sense), **reduced** (a technical assumption, for which we refer to [46]) and furthermore we assume that X is **non-degenerate**, which means that it lies in no proper hyperplane. Also, in order to avoid fuzzy behaviour with non–standard base fields, let us agree that everything is defined over the complex field $\mathbb{C}$. We advise that the theory may be considerably different (but not meaningless!) in positive characteristic or over non algebraically closed fields, like $\mathbb{R}$.

At the very beginning of this notes, for a while, let us consider the variety X as an abstract object and its plongement in the projective space as a map $\iota : X \to \mathbb{P}^r$. When is ι sufficiently good? Of course it must be an embedding: roughly speaking, no pair of distinct points should be glued together. Generalizing this fact, sets of distinct points should go to independent sets, at least as soon as this is possible (this is the *secant* point of view). There is a differential version of this principle (the *tangent* point of view): sets of tangent vectors applied at distinct points should remain independent, when we replace X with its image in ι, as soon as this is compatible with the fact that we arrive into $\mathbb{P}^r$.

There are other characterization of what a *good plongement* is. Nevertheless all of them prove to be strictly connected each other. One of the main connections between the tangent and the secant point of view is the celebrated Terracini's lemma, the cornerstone of our theory.

It turns out that **defective** varieties, i.e. varieties whose embedding is not "as good as possible", are indeed special from many points of view. What catches our attention is the considerable number of remarks that prove how such defective varieties appear in several fields of Mathematics, even outside Geometry, in many unexpected manners.

The aim of these notes is to present the theory of defective varieties, starting from the geometric point of view, and its several connections with many geometrical problems, with interpolation, decomposition of products, and so on.

As the thory is nowadays rapidly developing in many directions, the account of the actual situation outlined here is necessarily partial and (hopefully!) immediately obsolete. On the other hand we believe that marking some fixed points in the theory could be useful as a reference for people who wants to approach the study of defective varieties, and also to suggest how one can broaden the range of their applications.

2. Secant varieties

2.1. The linearization problem

It is even difficult to determine a starting point for an introduction to the theory of secant varieties. At least because there are so many approaches, each one valid from some point of view, that the choice of a beginning for the tale seems rather arbitrary. Here we present the problem from the point of view of consecutive partial linearizations of a (non-linear) projective variety X.
So let us start with $X \subset \mathbb{P}^r$, which, once forever, is an irreducible, reduced, non–degenerate projective variety. Since X is not contained in any proper hyperplane, the linear span of X is $\mathbb{P}^r$ itself. So, except for the trivial case $X = \mathbb{P}^r$ (which we exclude hereafter), the variety X is not linear.

A **partial linearization** is obtained by adding to X all the points which are linearly spanned by points of X. We obtain in this way an increasing hierarchy of subsets of $\mathbb{P}^r$, which eventually end up with the whole projective space ($\mathbb{P}^r$ is obtained, at least, taking all points spanned by $r + 1$ points of X. In fact, we will see that it is obtained much earlier).

These subsets turn out to be quasi-projective, highly ruled varieties, whose structure gives important information on the projective embedding of X. For an application, let us just say by now that they are linked with the *hidden variables* of parameter spaces or with the decomposition of tensors. From our point of view, they represent natural intermediate steps between X and the ambient space, in the sense of linear algebra, which is the ancestor of all our geometric investigations.

Our first easy exercise points out that the hierarchy is effective, unless X is trivial.

EXERCISE 1. If $X \neq \mathbb{P}^r$, then X cannot contain all the lines spanned by pairs $A, B \in X$.
If X contains all the quoted lines, then it also contains any point P spanned by a triple $A, B, C \in X$: indeed if $P = aA + bB + cC$, then the point $Q = bB + cC$ belongs to the line spanned by $B, C \in X$, hence it lies in X. But then observe that P belongs to the line spanned by A, Q. Thus $P \in X$.

Going on by induction, it turns out that any point of $\mathbb{P}^r$ which is spanned by k points of X (any k) actually lies in X. But we yet observed that, since X is non–degenerate, then $r+1$ general points of X span $\mathbb{P}^r$.

Now we are ready to introduce the main objects of our investigation.

DEFINITION 1. *For any non–negative integer k define the k-***secant variety** $S_k(X)$ *of X to be the (reduced) closure of the set:*
$\{P \in \mathbb{P}^r : P$ *lies in the span of* $k+1$ *independent points of* $X\}$

We take the closure (in the Zariski topology) because we want $S_k(X)$ to be a projective variety itself. Indeed, even before taking the closure, the set we get is locally algebraic, because, roughly speaking, is obtained moving $k+1$ points of X (which is algebraic) and taking the linear span, which is an algebraic procedure.
A more precise argument is contained in the following exercise, which should not be avoided also from those who are satisfied with the previous heuristic procedure, because it introduces some methods extensively used in the sequel.

EXERCISE 2. The set $\{P \in \mathbb{P}^r : P$ lies in the span of $k+1$ independent points of $X\}$ is locally algebraic in $\mathbb{P}^r$.
We need to introduce the formalism of Grassmannians. Let us indicate with $G(k,r)$ the Grassmannian of linear subspaces of (projective) dimension k in $\mathbb{P}^r$. $G(k,r)$ is an algebraic variety, and we have an algebraic map

$$sp : U \to G(k,r)$$

(the *span map*) defined over the open subset U which parametrizes independent $(k+1)$-tuples of points in the cartesian product $\mathbb{P}^r \times \cdots \times \mathbb{P}^r$ ($k+1$ times). The image of sp restricted to $U \cap (X \times \cdots \times X)$ is thus a locally closed subset Y of $G(k,r)$. Now consider the *incidence variety*

$$I(k,r) = \{(P,H) \in \mathbb{P}^r \times G(k,r) : P \in H\}$$

with the two projections $p_1 : I(k,r) \to \mathbb{P}^r$, $p_2 : I(k,r) \to G(k,r)$. Then $p_2^{-1}(Y)$ is algebraic, and $\{P \in \mathbb{P}^r : P$ lies in the span of $k+1$ independent points of $X\}$ coincides set-theoretically with $p_1(p_2^{-1}(Y))$. Hence it is locally closed. (The unexpert reader is strongly recommended to follow accurately these easy steps).

Using the formalism of the previous exercise, we may add some notation.

We indicate with $G_k(X)$ the *closure* of the image of $U \cap (X \times \cdots \times X)$ in $G(k,r)$, under the span map. So:
$G_k(X) =$ closure of $\{H \in G(k,r) : H$ is spanned by $k+1$ points of $X\}$.
$G_k(X)$ has no standard official name. We refer to it as the *k-th Grassmann secant variety* of X.

The inverse image $p_2^{-1}(G_k(X))$ is usually called the Grassmann *abstract* k-secant variety of X.

Finally let us set:

$$s_k(X) := \dim(S_k(X))$$

Of course the consecutive secant varieties of X define a chain:

$$X \subset S_1(X) \subset \cdots \subset S_r(X) = \mathbb{P}^r \tag{1}$$

Exercise 1 essentially says that $X \neq S_1(X)$, unless $X = \mathbb{P}^r$. We may generalize it for higher secant spaces.

EXERCISE 3. The inclusions $S_k(X) \subset S_{k+1}(X)$ in (1) are proper, unless $S_k(X) = \mathbb{P}^r$.
Assuming that $S_k(X) = S_{k+1}(X)$, one shows $S_{k+1}(X) = S_{k+2}(X)$, thus by induction $S_k(X) = S_r(X) = \mathbb{P}^r$. Indeed if $P \in S_{k+2}(X)$, then P is (a limit of) some sum $P = a_0P_0 + \cdots + a_{k+2}P_{k+2}$, with $P_i \in X$. But the point $Q = a_0P_0 + \cdots + a_{k+1}P_{k+1}$ belongs to S_{k+1}, hence by assumption it is (the limit of) a sum $b_0Q_0 + \cdots + b_kQ_k$, $Q_i \in X$. Then P is a limit of a sum $b_0Q_0 + \cdots + b_kQ_k + a_{k+2}P_{k+2}$, hence it belongs to $S_{k+1}(X)$.

Consequently we have the **secant dimensional sequence** associated to X:

$$n = \dim(X) < s_1(X) < \cdots < s_k(X) < \cdots < s_K(X) = r \tag{2}$$

in which we implicitly define K as the minimal integer for which $S_K(X) = \mathbb{P}^r$. This integer is an interesting invariant of the embedded variety X, to which we will refer as the **linearization constant** of X.

2.2. First motivations and examples

Let us play now an interlude. We feel urged to present here some example of interaction between the machinery introduced so far and everyday's mathematics.

A good starting point are those projective varieties which naturally represent tensor products. The most famous are Veronese varieties, Grassmannians and Segre products.

EXAMPLE 1. The **Veronese variety** $V(n, m)$ is defined in $\mathbb{P}^N$, $N = \binom{m+n}{n}$, as the set of points with homogeneous coordinates $(a_0^m, a_0^{m-1}a_1, a_0^{m-1}a_2, \ldots, a_n^m)$ for any choice of $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$. It corresponds to an algebraic image of $\mathbb{P}^n$ under a proper map, so it is an algebraic irreducible (and also smooth) subvariety of $\mathbb{P}^N$.

If one identifies $\mathbb{P}^n$ with the set of all linear forms in $n+1$ variables (modulo scalar multiplication), then $\mathbb{P}^N$ can be identified with the set of forms of degree m and $V(n, m)$ represents the subset of forms which decompose in a product of linear pieces.

With this representation in mind, one understands that the constant K for $X = V(n, m)$ is the minimum such that a *general* form of degree m in $n+1$ variables is a linear combination of $K + 1$ completely decomposable forms.
One may appreciate this interpretation of K comparing with the classical constants

introduced in number theory by Fermat and Waring, which compute the number of factors needed to express any large number as a sum of m-th powers.
We mention at this stage that a complete list of the values of K for all pairs n, m, although intensively studied by classical geometers, has been obtained only (relatively) recently by Alexander and Hirschowitz (see [3]).

EXAMPLE 2. We do not introduce **Grassmannians** here: we just want to stress that $G(h, s)$, embedded via Plücker relations in $\mathbb{P}^N$, $N = \binom{s+1}{h+1} - 1$, is the analogue of the Veronese varieties, when the symmetric product is replaced by the wedge product.

Thus the constant K for $X = G(h, s)$ is the minimum such that a *general* vector in $\Lambda^{h+1}\mathbb{C}^{s+1}$ is a linear combination of $K + 1$ completely decomposable wedge products.

A complete list of the values of K for all pairs h, s is yet unknown! We have the list for $h = 1$, plus some uncomplete results for higher dimensional vector spaces. (see [12] for a survey).

EXAMPLE 3. Continuing in the path, consider the **Segre products** $S(a, b)$, defined in $\mathbb{P}^{ab+a+b}$, as the set of points with homogeneous coordinates $(x_0y_0, x_0y_1, \dots, x_ay_b)$ for any choice of $(x_0, \dots, x_a), (y_0, \dots, y_b) \neq (0, \dots, 0)$. It corresponds to an algebraic image of $\mathbb{P}^a \times \mathbb{P}^b$ under a proper map, so it is algebraic.

If one identifies $\mathbb{P}^{ab+a+b}$ with the set of all matrices $(a + 1) \times (b + 1)$ (modulo scalar multiplication), then $S(a, b)$ represents the subset of matrices which decompose in a product of two vectors, hence have rank 1. The constant K represents here the minimum such that a *general* matrix is a linear combination of $K + 1$ rank one matrices. Its value is well-known for any a, b.

The situation becomes more complicate if we iterate the product process, considering the generalized Segre variety $X = S(a_1, \dots, a_m)$. In this case, the ambient space where X lies represents the set of m-dimensional boxes (= tensors) of type $(a_1 + 1) \times \cdots \times (a_m + 1)$ and the Segre products represents the equivalent of "rank 1" boxes. So K turns out to be here the minimum such that a *general* m-box is a linear combination of $K + 1$ boxes of rank 1. Worthless to say that very few facts are known about the value of K for general $a_1, \dots, a_m$ (refer to [13]).

EXAMPLE 4. Now the reader can imagine several variations on the theme. We can mix the procedures obtaining tensors made with symmetric, alternating and general products. All these multilinear procedures give rise to huge projective spaces where decomposable objects are represented by algebraic subvarieties, for which the determination of the linearizing constant K is relevant.

One may go further, taking weights for the variables involved in the products. One gets weighted projective spaces, which are realized as subvarieties of higher dimensional "ordinary" spaces, and still the linearization problem has considerable application for them. Also, one may consider some combinatorial subvarieties of the objects introduced in our previous examples, like Schubert cycles in Grassmannians or sparse tensors and ladders.

Our knowledge of the values of K for all these varieties is faint.

EXAMPLE 5. Even for general algebraic varieties $X \subset \mathbb{P}^r$, the knowledge of the constant K may have interesting applications.
Just to mention one of them: some probabilistic problems become easier to work with if one adds some "hidden variables" to the random variables directly involved in the process. In geometric terms, one plugs the original parameter space in a wider ambient space, as a non–linear subvariety X. Then the new points obtained taking linear combinations of the original ones must enter in the description of the process. These new points realize our secant varieties.

Coming back to our geometric birthplace, we cannot escape mentioning the applications of secant varieties to the simplification of the ambient space of the variety X.

EXAMPLE 6. For a given $X \subset \mathbb{P}^r$, one can try to realize the variety in a tinier projective space, with the most natural procedure of consecutive projections. However, one cannot expect for free that the projection from a general point $P \in \mathbb{P}^r - X$ sends isomorphically X to $\mathbb{P}^{r-1}$. First of all, it is clear that if P belongs to the line spanned by $A, B \in X$, then A, B are patched together in the projection. Thus when $P \in S_1(X) - X$, the projection from P is not one–to–one. It turns out that, in fact, the projection from $P \in \mathbb{P}^r - X$ is isomorphic to X if and only if $P \notin S_1(X)$ (this is essentially due to the fact that tangent lines to X lie in $S_1(X)$. See [46], §2.7).
Thus X can be projected isomorphically to $\mathbb{P}^{r-1}$ from a general point of $\mathbb{P}^r$ exactly when $K > 1$.

EXAMPLE 7. Generalizing the previous procedure, if $P \notin S_2(X)$, then for any triple of independent points $A, B, C \in X$, the plane spanned by A, B, C in $\mathbb{P}^r$ is sent to a plane in $\mathbb{P}^{r-1}$ under the projection from P. Thus A, B, C remain independent after the projection.
It turns out that when $K > 2$, under the projection from a general point of $\mathbb{P}^r$, X does not acquire any new trisecant line (of course, "old" trisecant lines are stable under projection). And one may easily generalize this principle for all K.

Our last observation, although elementary, is worthy of an explicit statement:

REMARK 1. K is the maximum such that the projection from a general point $P \in \mathbb{P}^r$ preserves the independence of any set of $K + 1$ points of X.

2.3. A dimension contest

Trying to extimate the dimension of the objects introduced in section 2.1, one realizes immediately that it is easy to find, in any case, an upper bound.

Starting with $G_k(X)$, it is clear that it corresponds to (the closure of) an algebraic image of (an open dense subset of) $X \times \cdots \times X$ ($k + 1$ times). Hence:

$$\dim(G_k(X)) \leq \min\{(k+1)n, \dim(G(k,r))\} = \min\{kn + n, (k+1)(r-k)\}$$

where, as usual, we use n for the dimension of X. We prove that equality always holds.

Let us show first two exercises on the connections between projections and secant varieties.

EXERCISE 4. Let X be an irreducible, non–degenerate variety in $\mathbb{P}^r$, $X \neq \mathbb{P}^r$. Then the fibers of projection of X from a general point $P \in X$ (internal projections) are generically finite.
This is an immediate consequence of exercise 1: the contrary would mean that X contains a general secant line.

Observe that, indeed, in the exercise one could invoke the celebrated **trisecant lemma**, which says that the general internal projection is even birational. Since its validity is restricted to characteristic 0, and we do not need its full strength at this level, we preferred to give a direct argument.

EXERCISE 5. Let X be an irreducible, non–degenerate variety in $\mathbb{P}^r$, $X \neq \mathbb{P}^r$. Fix a proper subvariety $Y \subset X$, of dimension m and take $k < r - m$. Then a general subset of $k+1$ points on X spans a k-space disjoint from Y.
Indeed everything is obvious when $k = 0$. For $k = 1$, take the projection of X from a general point P. The image has dimension n (by exercise 4) and does not coincide with the image of Y. This means exactly that the general secant line to X, passing through P, does not meet Y. The general case works similarly, by induction (details are left to the reader).

PROPOSITION 1. *$G_k(X)$ is irreducible, of dimension* $\min\{kn+n,\ (k+1)(r-k)\}$. *In particular, as soon as $k \leq r-n$, the intersection of X with a general $(k+1)$-secant k-space is finite.*

Proof. The irreducibility is obvious, since X is irreducible, so X^{k+1} is, and the closure of an algebraic image of any dense open subset of X^{k+1} is irreducible as well.
For the dimension, first assume $n > r-k$; then any k-plane π meets X in (at least) a curve; moving generically $k+1$ points of this curve, we see that $\pi \in G_k(X)$. It follows that $G_k(X)$ is the whole Grassmannian, hence its dimension is $(k+1)(r-k)$.
Assume now $n+k \leq r$ and assume $\dim G_k(X) < kn+n$. Since $G_k(X)$ is generically the image of a map $\phi_k : X^{k+1} \to G(k,r)$, and $\dim(X^{k+1}) = kn+n$, then necessarily ϕ cannot have finite fibers. Notice that the points of the fiber over some $\pi \in G_k(X)$ are contained in the intersection $\pi \cap X$. This is clearly impossible for $k = 1$: here π is a line, hence it would coincide with the fiber, which means that any secant line to X lies in X, i.e. $S_1(X) \subset X$, which contradicts the conclusion of exercise 1.
For general k, a similar contradiction is obtained by induction. Consider indeed a general point $P_0 \in X$ and let X' be the image of X in $\mathbb{P}^{r-1}$, under the projection from P_0. As above, this projection has finite general fibers. In particular $\dim(X') = n$, hence $k-1 < (r-1) - \dim(X')$. Call $Y \subset X'$ the proper subset where the fibers are not finite. It follows by induction that the span π of k general points of X' meets it in a finite set. Furthermore, the previous exercise implies that π misses Y. Since π is the

image of a general k-space, $(k+1)$-secant to X, the conclusion follows. □

Let us now turn to the dimension of secant varieties.

The incidence variety over the Grassmannian $G(k,r)$ is a $\mathbb{P}^k$-bundle. Thus the abstract secant variety, which corresponds to the bundle restricted to $G_k(X)$, has always dimension bounded by $\min\{(k+1)(r-k)+k, kn+n+k\}$.

$S_k(X)$ is the closure of an algebraic image of the abstract secant variety. Furthermore it lies in $\mathbb{P}^r$.

We get immediately:

COROLLARY 1. *The secant varieties $S_k(X)$ are irreducible.*

When we consider the dimension $s_k(X)$ of $S_k(X)$, we get another story. $s_k(X)$ is bounded by:

$$s_k(X) \leq \min\{r, (k+1)n+k\}.$$

The main point of all our lectures relies on the fact that the previous inequality may be strict.

Examples will be discussed soon. The easiest one is the Veronese surface $V(2,2) \subset \mathbb{P}^5$ (in the notation of example 1), for which $s_1 = 4$. This has the amazing consequence that we need more squares of linear forms than expected (i.e. 3 instead of 2) to generate a general (homogeneous) form of degree 2 in 3 variables.

We arrive thus at the following basic:

DEFINITION 2. *We say that the variety X is k-**defective** when*

$$s_k(X) < \min\{r, (k+1)n+k\}.$$

*The difference $\delta_k = \min\{r, (k+1)n+k\} - s_k(X)$ is referred to as the k-th **defect** of X. We say that X is defective when it is k-defective for some k. It is **minimally** k-defective if it is k-defective but not $(k-1)$-defective.*

In conclusion, it is implicit in the previous terminology that we will consider as *good* any embedding of X in $\mathbb{P}^r$ for which the *secant varieties* have the *maximal dimension*, compatibly with the ambient space.

Let us remark an easy characterization of defectiveness, via the computation of secant spaces through points.

EXERCISE 6. X is k-defective if and only if $S_k(X) \neq \mathbb{P}^r$ and the general point $P \in S_k(X)$ lies in infinitely many $(k+1)$-secant k-spaces.
Indeed X is k-defective when the map from the k-th abstract secant variety to $\mathbb{P}^r$ is not surjective and its general fiber is not finite.

A proper map $Z \to Z'$ of projective varieties has **maximal rank** when it is either surjective, or it has finite general fibers. With this notation, X is k-defective exactly when the map from the Grassmann abstract k-th secant variety to $\mathbb{P}^r$ is not of maximal rank.

EXERCISE 7. Assume that X is k-defective and $S_{k+1}(X) \neq \mathbb{P}^r$. Then X is also $(k+1)$-defective. Furthermore $\delta_k \leq \delta_{k+1}$.
Indeed since X is k-defective, then it is contained in a positive-dimensional family.

We warn the reader that all the previous definition are standard in the recent literature, *except* for the definition of defect, for which we are going to see some variations.

2.4. Further exercises

EXERCISE 8. Prove that no variety is 0-defective.

EXERCISE 9. Prove that the invariant K introduced in (2) is always smaller than $r-1$.

EXERCISE 10. For any X and for any a, b, prove that $S_{a+b+1}(X) = S_a(S_b(X))$.

EXERCISE 11. Assuming that no Veronese variety is defective (which, by the way, is completely **false**!), extimate the minimal number of powers of linear forms necessary to generate a general form of degree d in $m+1$ variables.

EXERCISE 12. Use the properties of rational cubics to prove that a general cubic form in 2 variables is combination of two cubes of linear forms.

EXERCISE 13. Find a quadratic form in 3 variables which is not a combination of two squares of linear forms. (It does not contradict the previous exercise!)

EXERCISE 14. Use Castelnuovo's formula for the genus of curves, to prove that curves in $\mathbb{P}^3$ are never defective.

3. The infinitesimal approach

3.1. Terracini's lemma

We said in the previous section that our main task concerns the computation of the dimension of secant varieties. Up to now, except for very particular cases, we found very few examples where the computation was effective.

Since secant varieties are irreducible (by corollary 1) and reduced (by construction), a possible way to compute consists in determining the dimension of the tangent space to $S_k(X)$ at a general point.

This is a general reduction step in Geometry: try to reduce the original problem in a problem concerning linear objects (as tangent spaces are).

The starting point in our reduction thus relies in determining the tangent space to secant varieties.

The result has been obtained by Terracini, at the beginning of last century. It not only opens the path for the computation of the invariants $s_k(X)$, but also provides a beautiful link between our setting and an apparently different characterization of *good* embeddings of projective varieties, with connections with the interpolation problem, inverse systems, algebraic products, and more.

Roughly speaking, the idea behind Terracini's lemma is simple. If $u \in S_k(X)$ is a general point, belonging to the linear span of the points $P_0, \dots, P_k \in X$, then a tangent vector to $S_k(X)$ at u can be interpreted as a direction in which u can be moved infinitesimally, leaving it *inside* $S_k(X)$, i.e. in the span of some set of $k+1$ points of X, infinitesimally near the P_i's. It should correspond then to an infinitesimal movement of the P_i's inside X, hence to a set of tangent vectors to X at the P_i's.

Before stating the lemma, let us introduce some useful pieces of notation.

DEFINITION 3. *For $P \in X$, we indicate with $T_P(X)$ the tangent space to X at P. Brackets $<,>$ will be used to denote linear spans.*
For $P_0, \dots, P_k \in X$, we abbreviate $< T_{P_0}, \dots, T_{P_k} >$ with $T_{P_0,\dots,P_k}$.

THEOREM 1. *(**Terracini's lemma** see [74]) Let u be a general point of $S_k(X)$, belonging to the span $< P_0, \dots, P_k >$ of the points $P_0, \dots, P_k \in X$. Then the tangent space of $S_k(X)$ at u is given by:*

$$T_u(S_k(X)) = T_{P_0,\dots,P_k}. \tag{3}$$

Proof. We deeply use the assumption that we work in characteristic 0. Here we refer to the theorem of generic smoothness (see [46] III.10.5) to conclude that, for general points, the tangent space to $S_k(X)$ at u is the image of the tangent space to the abstract secant variety at some preimage (π, u) of u. Here π is a k-space which meets X at $P_0, \dots, P_k$. For the same reason, the tangent space to $G_k(X)$ at π is an image of the product $T_{P_0}(X) \times \cdots \times T_{P_k}(X)$.
The abstract secant variety is a $\mathbb{P}^k$ bundle over $G_k(X)$, which can be trivialized around π by taking an identification for points in different linear spaces. A geometric way to do that uses a (general) projection of π and the neighbouring k-spaces, from some fixed space Q to a fixed k-space and identifies matching points.
So the tangent space to the abstract secant variety at (π, u) decomposes in a sum $T \oplus T'$, where T is the tangent space to $G_k(X)$ at π and T', the vertical space, corresponds canonically to the tangent space to π and is mapped in $\mathbb{P}^r$ to π itself. Now to prove the statement, it is enough to show that the composition:

$$T_{P_i}(X) \to T \to T \oplus T' \to \mathbb{P}^r$$

maps $T_{P_i}(X)$ to a space T_i which, together with the line $L_i =< u, P_i >$, spans $< u, T_{P_i}(X) >$. Indeed it follows that $T_u(S_k(X))$ is spanned by $T_{P_0,\dots,P_k}(X)$ and $< P_0, \dots, P_k, u >= \pi$, but observe that, clearly, π is yet contained in $T_{P_0,\dots,P_k}(X)$.
In order to prove the claim, say for $i = 0$, fix the points $P_1, \dots, P_k$ and take a tangent vector τ to P_0. Then τ implies a movement of the k-space π which defines, via the

projection from Q, a movement of u which is contained, for elementary reasons, in the span of π and τ. □

The unexpert reader will be surprised by the amount of consequences that such a natural result has and the consequences it spreads among apparently different fields of geometry.

Just to begin with, observe that the computation of the span of tangent spaces to X is sometimes easy to perform, and allows us to show that some classical varieties are defective.

EXERCISE 15. A hyperplane H contains the tangent space to X at a smooth point P if and only if the intersection $X \cap H$ has a singular point at P.
Indeed by assumption the tangent space to $T_P(X)$ has dimension $n = \dim(X)$ and the tangent space to $X \cap H$ at P is $H \cap T_P(X)$. Moreover P is singular in $H \cap X$ if and only if the tangent space to $T_P(X \cap H)$ has dimension greater than $\dim(X \cap H) = n - 1$. This clearly happens if and only if H contains $T_P(X)$. □

EXAMPLE 8. Let us use Terracini's lemma to prove that the Veronese variety $V(2,2) \subset \mathbb{P}^5$ of example 1 is 1-defective.
We need to prove that $s_1(X) = \dim S_1(X)$ is smaller than 5. Take a general point $u \in S_1(X)$ and assume it lies a line L which meets X at two points P, Q. Then, by Terracini's lemma, the tangent space to $S_1(X)$ at u is the span $T_{P,Q}$ of the two tangent spaces to X at P, Q. Since the span of linear spaces is linear, $\dim T_{P,Q} < 5$ if and only if there exists a hyperplane H containing $T_{P,Q}$. By exercise 15 this happens exactly if and only if there exists a hyperplane H with $H \cap V(2,2)$ singular at P, Q.
Now, $V(2,2)$ is the image of $\mathbb{P}^2$ in the map defined by the complete system of quadratic forms, which means that the intersections of $V(2,2)$ with the hyperplanes of $\mathbb{P}^5$ are the images of plane conics. If $P_0, Q_0 \in \mathbb{P}^2$ are pre–images for P, Q, then the double line defined by P_0, Q_0 is a conic which is singular at P_0, Q_0 (in fact this is the unique conic with this property). Hence it corresponds to a hyperplane section of X which is singular at P, Q. The proof is established: $\dim(T_{P,Q}) = 4 < 5$.

EXAMPLE 9. A similar computation shows that the Veronese variety $V(3,2) \subset \mathbb{P}^9$ is 1-defective.
We need to prove that $s_1(X) = \dim S_1(X)$ is $2 \cdot 3 < 7$. Take a general point $u \in S_1(X)$ and fix points $P, Q \in X$ such that u lies in the line $< P, Q >$. The tangent space to $S_1(X)$ at u is the span $T_{P,Q}$ of the two tangent spaces to X at P, Q. In order to prove that it is a linear space of dimension 6, it is sufficient to compute the dimension of the linear system of hyperplanes which contain it. Taking pre-images $P_0, Q_0 \in \mathbb{P}^3$ of P, Q, then we need to extimate the system of quadrics in $\mathbb{P}^3$ singular at P_0, Q_0. This is not, in general, an elementary task, as we shall discuss in a while.
In our specific case, however, observe that quadrics singular at P_0, Q_0 are singular along the line $L =< P_0, Q_0 >$, hence they split in a couple of planes along L. So the system has (projective) dimension 2. It turns out that $T_{P,Q}$ is contained in a system of hyperplanes of projective dimension 2, so it has dimension $9 - 3 < 7$, as claimed.

There are some features in the previous example, which will pop up continuosly in our computations. The first one is stressed here only for reference convenience:

REMARK 2. A linear subspace $\pi \subset \mathbb{P}^r$ has dimension x if and only if it is contained in a linear system of hyperplanes of projective dimension $r - x - 1$.

The second one leads to a link between our theory of projective embeddings and interpolation on algebraic varieties. The principle is:
$T_{P_0,\dots,P_k}$ is small exactly when there are many divisors, in the linear system cut on X by the hyperplanes of $\mathbb{P}^r$, which are singular at the P_i's.

Next section explores the connection in more details.

3.2. Interpolation on varieties

Let D be any divisor of the irreducible variety X. We use the symbol $|D|$ to indicate the *complete* linear system defined by D, considered either as a projective space, or as a collection of divisors.

The **interpolation problem** for any linear system $V \subset |D|$ consists in the study of the subsystem of divisors $E \in V$ which pass through fixed points $P_0, \dots, P_k$ with pre–assigned multiplicities $m_0, \dots, m_k$. We use the notation:

$$V(-m_0P_0 - \cdots - m_kP_k)$$

for this subsystem. The first question we have concerns its dimension.

EXERCISE 16. When the points $P_0, \dots, P_k$ are general and all the multiplicities are 1, then the system $V(-P_0 \cdots - P_k)$ has dimension $\dim(V) - k - 1$.
Indeed notice that $V(-P_0 \cdots - P_k)$ has dimension one less than $V(-P_1 \cdots - P_k)$, except than when P_0 is a fixed point $V(-P_1 \cdots - P_k)$, which cannot happen when P_0 is general. □

A similar statement fails as soon as the m_i's increase. There are varieties for which the dimension of $V(-m_0P_0 \cdots - m_kP_k)$ is bigger than the expected one. This may happen even for the system V of hyperplane sections of a smooth variety, and even if we have only one point involved! (see [77]).

For our matter, we are concerned with the case of interpolation for general points of multiplicity two and for the linear system of hyperplane sections.

Indeed, the linear system V determines a map

$$\phi_V : X \to \mathbb{P}^r$$

where $r = \dim(V)$. Now, when ϕ_V is birational, after replacing X with its image (does not matter if it is not an *isomorphic* image, for we are concerned with general points), the system V becomes the system of hyperplane sections of $X \in \mathbb{P}^r$ and $V(-2P_0 \cdots - 2P_k)$ is exactly the subsystem of all hyperplane sections which are singular at the P_i's.

EXERCISE 17. Let V be a system of hyperplane sections for the reduced variety $X \subset \mathbb{P}^r$, of dimension n. Then the interpolation problem for one general point P_0 and multiplicity two, has always the same answer: $\dim(V(-2P_0)) = \dim(V) - (n+1)$. This is a direct consequence of exercise 15: the set of divisors in V which are singular at P_0 corresponds to the set of hyperplanes containing $T_{P_0}(X)$. Since P_0 is smooth in X, then $\dim(T_{P_0}(X)) = n$ and we conclude. □

On the other hand, imposing singularity at two general points P_0, P_1, then we cannot immediately conclude that $\dim(V - 2P_0 - 2P_1) = \dim(V) - 2(n+1)$, as one would expect. The answer is indeed related to the dimension of the span $T_{P_0,P_1}(X)$, as explained in exercise 15.

It turns out that, discussing Terracini's lemma, we yet met a situation where the interpolation problem has an unexpected answer.

EXAMPLE 10. Take the Veronese variety $X = V(2,2) \subset \mathbb{P}^5$ and call V the system of hyperplane sections. Fix two general points $P, Q \in X$. Then, by example 8, we know that the span of the two tangent spaces $T_{P,Q}(X)$ has dimension 4 and it determines one hyperplane. Thus $V(-2P_0 - 2P_1)$ has dimension 0, which is bigger than $\dim(V) - 2(n+1) = 5 - 2 \cdot 3 = -1$.

Let us state directly the following:

PROPOSITION 2. *On the variety $X \subset \mathbb{P}^r$ consider the linear system V of hyperplane sections. Then X is k-defective if and only if, for general points $P_0, \dots, P_k \in X$, one has:*

$$\dim(V(-2P_0 - \cdots - 2P_k)) > \min(-1, r - (n+1)(k+1)).$$

In this case, the difference $\dim(V(-2P_0 - \cdots - 2P_k)) - \min(-1, r - (n+1)(k+1))$ *is the k-th defect of X.*

In interpolation theory, the linear system $V(-2P_0 - \cdots - 2P_k)$ is *special* when the previous inequality holds. So X is k-defective if and only if $V(-2P_0 - \cdots - 2P_k)$ is special.

PROPOSITION 3. *Assume that X is k-defective, with k-defect δ_k. Assume $S_{k+1}(X) \neq \mathbb{P}^r$. Then X is also $(k+1)$-defective.*
If moreover $r \geq (k+2)n + k$, then the $(k+1)$-th defect δ_{k+1} is greater or equal than δ_k.

Proof. By assumptions and by Terracini's lemma, if $P_0, \dots, P_k \in X$ are general points, then the span $T_{P_0,\dots,P_k}$, which is the tangent space at a general point of $S_k(X)$, has dimension $\min(r, (k+1)n + k) - \delta_k$. Hence adding one general point P_{k+1}, the space $T_{P_0,\dots,P_k,P_{k+1}}$, which is the span of $T_{P_0,\dots,P_k}$ and $T_{P_{k+1}}$, has dimension at most $\min(r, (k+1)n + k) - \delta_k + n + 1$. This last number, by assumptions, is smaller than r, while it is clearly smaller than $(k+2)n + k + 1$. So X is $(k+1)$-defective.

For the second assertion, observe that in our case

$$\delta_k = (k+1)n + k - \dim(T_{P_0,\dots,P_k})$$

$$\delta_{k+1} = (k+2)n + k - \dim(T_{P_0,\dots,P_k,P_{k+1}})$$

and the conclusion follows. □

The proposition justifies the reduction we will make in the study of defective varieties. Namely we are focused only on minimally defective ones.

COROLLARY 2. *The smallest integer K such that $S_K(X) = \mathbb{P}^r$ is also the smallest K such that, for general points $P_0, \dots, P_K$, the hyperplane linear system satisfies*

$$V(-2P_0 - \cdots - 2P_K) = \emptyset.$$

3.3. Further examples and exercises

We collect in this section examples of defective varieties, discovered using Terracini's lemma. Some of them are trivial and their proofs are left to the reader as an exercise.

EXERCISE 18. Use the fact that no variety is 0-defective to prove that the tangent space at a general point has the same dimension as X.

EXERCISE 19. A cone X of dimension 2 over a curve is 1-defective, as soon as $r \geq 5$.
Indeed the tangent spaces at two general points meet at the vertex of the cone, so they span at most a $\mathbb{P}^4$. □

EXERCISE 20. Generalize the previous example:
Any n-dimensional cone X with vertex of dimension m is 1-defective, when $r > 2n - m$.

The converse of the previous statement depends on the cone we work with:

EXERCISE 21. Find a cone X as above, which is 1-defective even if $r = 2n - m$.

PROPOSITION 4. *Let X be the cone over a variety Y, with vertex W. Fix general points $P_0 \dots, P_k$ in X and let $Q_0, \dots, Q_k \in Y$ be their images in the projection from W. Then $T_{P_0,\dots,P_k}(X)$ is the span of W and $T_{Q_0,\dots,Q_k}(Y)$.*

Proof. Indeed, for one points P_0, by generic smoothness ([46] III.10.5) the tangent space $T_{P_0}(X)$ maps onto $T_{Q_0}(Y)$ in the projection from W, hence $T_{P_0}(X) =< W, T_{Q_0}(Y) >$ for reasons of dimension. The general statement follows from elementary projective arguments. □

EXERCISE 22. Let $X \in \mathbb{P}^r$ be the cone over the variety $Y \subset \mathbb{P}^s$, with vertex at the linear space W of dimension $r - s - 1$. Then $S_k(X) = \mathbb{P}^r$ if and only if $S_k(Y) = \mathbb{P}^s$.

EXERCISE 23. Let $Z \subset \mathbb{P}^r$ be the cone over a variety $Y \subset \mathbb{P}^s$, with vertex W of dimension $r - s - 1$. Assume that $X \subset \mathbb{P}^r$ is an irreducible, non–degenerate subvariety of Z, which surjects onto Y in the projection from W. Put $n = dim(X)$, $m = dim(Y)$. Pick a positive integer k with $(n - m)(k + 1) > r - s$ and assume that $S_k(Y) \neq \mathbb{P}^s$. Then X is k-defective.

Indeed if $P_0, \ldots P_k$ are general points of X, which thus project to general points $Q_0, \ldots, Q_k \in Y$ then $T_{P_0,\ldots,P_k}(X)$ is contained in $T_{P_0,\ldots,P_k}(Z)$. This last space does not cover $\mathbb{P}^r$, by assumption, and it has dimension at most $m(k+1)+k+(r-s)$ by the previous proposition. The conclusion follows since our numerical hypothesis implies $m(k + 1) + k - (r - s) < n(k + 1) + k$. □

This last exercise explains that many defective varieties are found as subvarieties of cones. It is not easy in general to determine all the subvarieties of the cones over a given Y. However, taking at least complete intersections, we find some non elementary class of defective objects.

EXERCISE 24. Let Q be a smooth quadric in $\mathbb{P}^3$. Let X be the image of Q in the Veronese map which takes $\mathbb{P}^3$ to the variety $V(3, 2) \subset \mathbb{P}^9$. Then X is not 1-defective but it is 2-defective.

Indeed first observe that the system V of hyperplane sections of X corresponds to the system cut by quadrics on Q. Hence X spans $\mathbb{P}^8$ and indeed it corresponds to a hyperplane section of $V(3, 2)$.

Given two general points $Q_0, Q_1 \in Q$, calling P_0, P_1 their images in X, the system $V(-2P_0 - 2P_1)$ corresponds to the system of quadric sections of Q which are singular at the Q_i's. These sections are cut by quadrics Q' which are tangent to Q at the P_i's. On the other hand, if Q, Q' are tangent at Q_0, Q_1, i.e. they have the same tangent plane at the two points, then after replacing Q' with a suitable linear combination $aQ + bQ'$, the equations of the tangent planes of Q' at the Q_i's vanish. Thus we may assume that our system is cut on Q by quadrics which are singular at the Q_i's. We yet computed the dimension of quadrics with two singular points: it is 2. Hence $V(-2P_0 - 2P_1)$ has dimension $2 = r - 2(n + 1)$, so X is not 1-defective.

On the other hand, adding one general point $Q_2 \in Q$, which maps to P_2, then we have a quadric section of Q which is singular at Q_0, Q_1, Q_2: it is the intersection of Q with the plane $< Q_0, Q_1, Q_2 >$ counted twice. So $\dim(V(-2P_0 - 2P_1 - 2P_2)) = 0 > r - 3(n + 1) = -1$ and X is 2-defective. □

EXERCISE 25. Generalize the previous example:

Let Q be a smooth quadric in $\mathbb{P}^m$, $m > 2$. Let X be the image of Q in the Veronese map which takes $\mathbb{P}^m$ to the variety $V(m, 2) \subset \mathbb{P}^M$, $M = (m^2 + 3m + 2)/2 - 1$. Then X is $(m - 1)$-defective.

The following lemma will be useful for many applications:

LEMMA 1. (**Linear lemma**) *Any set of m-planes in $\mathbb{P}^r$, such that any two of them meet in a $(m-1)$-plane, either is contained in some fixed $\mathbb{P}^{m+1}$ or has a $(m-1)$-plane for base locus.*

Proof. Call H the $(m+1)$-plane spanned by two elements L', L'' of the family. Assume that some L in the set does not lie in H. Then L meets H in a $(m-1)$-plane, since it meets L', L'' in a $(m-1)$-plane, and $L \cap H = L' \cap L''$. Any element of the set contained in H must then contain $L \cap H$. Hence $L' \cap L''$ is the base locus. □

EXERCISE 26. When $X \subset \mathbb{P}^r$ has codimension 2, it is not defective.
We show indeed that $S_1(X) = \mathbb{P}^r$. Assume that $S_1(X) \neq \mathbb{P}^r$. Then for two general points $P, Q \in X$ we have $T_{P,Q} \neq \mathbb{P}^r$. So $T_P(X)$ and $T_Q(X)$ must meet in dimension $n-1$. Consider now the set of all tangent spaces to X (at smooth points). By the Linear lemma, either they are contained in a fixed hyperplane (a contradiction, since X spans $\mathbb{P}^r$) or they contain a fixed linear space H of dimension $n-1$. But this last conclusion cannot hold in characteristic 0. Indeed otherwise, taking a general section of X with a linear space W of dimension 3, we get a curve $X \cap W$ which is irreducible ([46] III.7.9.1) and whose tangent lines all pass through the point $W \cap H$, which contradicts [46] IV.3.9. □

A generalization of the Linear lemma would be useful for our purposes. Unfortunately it seems rather hard and involved. See [55] for classical results on the subject.

Our last topic of this section concerns Grassmannians. In order to study secant varieties and tangent spaces to Grassmannians, let us recall the following characterization of the tangent spaces to a Grassmannian of lines at a point.

EXERCISE 27. In the Grassmannian of lines $G(1,s)$ fix a point L and fix a $s-2$-space π in $\mathbb{P}^s$ with $\pi \supset L$. The Schubert cycle S_π, of lines which meet π is the intersection of $G(1,s)$ (in its Plücker embedding in $\mathbb{P}^r$, $r = (s^2+s-2)/2$) with a hyperplane tangent to $G(1,s)$ at L.
S_π is clearly a hyperplane section of $G(1,s)$, so it is enough to prove that L is a singular point of S_π. Fixing L_ϵ infinitesimally close to L, one uses linear algebra to show that the conditions for L_ϵ to meet π are less than the expected value $(2s-3)-\dim(S_\pi)$. □

We refer to the hyperplane sections S_π as above as *Schubert tangent sections* and the corresponding hyperplanes as *Schubert tangent hyperplanes*.

EXERCISE 28. The Grassmannian $G(1,s)$ is k-defective for all k with $2k+1 \leq s-2$ and $(4s-2)k \geq s^2-s-2$.
Indeed fix $k+1$ general lines $L_0, \dots, L_k$ in $\mathbb{P}^s$. They span a linear subspace π of dimension $2k+1 \leq s-2$. Fix a $(s-2)$-space Π containing π. The Schubert cycle of lines meeting Π represents a hyperplane section of $G(1,s)$ which is singular at all L_i's. Hence $T_{L_0,\dots,L_k}(G(1,s)) \neq \mathbb{P}^r$. On the other hand, if $n = \dim(G(1,s)) = 2s-2$, then by assumptions $nk+n+k \geq r$. It follows $\dim(S_k(X)) < \min(r, nk+n+k) = r$ and the claim is proved. □

In particular, the Grassmannian of lines in $\mathbb{P}^5$ is 1-defective.

EXERCISE 29. It is highly unelementary to extend, with this method, similar result on Grassmannians of planes and higher dimensional varieties (but see [12]). The reader is invited to study the Grassmannians $G(2, s)$ of planes, for $s \leq 8$.

3.4. Tangential projections

Terracini's lemma suggests an interpretation of defective varieties using general tangential projections of varieties. It allows us to explain the hierarchical nature of defects and leads to an easy understanding of some results in the projective embedding of varieties.

We start stating explicitely another easy consequence of the theorem of generic smoothness:

LEMMA 2. *Let $X \subset \mathbb{P}^r$ be an irreducible variety and let L be a linear space of dimension s, not containing X. Let Y be the image of X (with the reduced structure) in the projection from L to some fixed general $\mathbb{P}^{r-s-1}$. Fix a general point $P \in X$ and call Q its image on Y. Then the tangent space to Y at Q is the projection of $T_P(X)$ from L.*
In particular, if P is general, then:

$$\dim(Y) = \dim(X) - \dim(T_P(X) \cap L) - 1.$$

Proof. Just see [46] III.10.5 and [46] III.10.4. □

We are going to use the result with L = general tangent space to X. Observe that since X is non–degenerate, it cannot contain X.

COROLLARY 3. *X is 1-defective if and only if a general tangential projection sends X to a variety $X_1 \subset \mathbb{P}^{r-n-1}$ which is different from $\mathbb{P}^{r-n-1}$ and has dimension smaller than $n = \dim(X)$.*

Proof. Fix general points $P_0, P_1 \in X$ and put $L = T_{P_0}(X)$, Q = image of P_1 in the projection from L ($Q \in X_1$). We know by Terracini's lemma that X is 1-defective if and only if the span $T_{P_0,P_1}(X)$ does not fill $\mathbb{P}^r$ and has dimension smaller than $2n + 1$. The latter condition means that $T_{P_1}(X)$ meets $L = T_{P_0}(X)$ at least at some point. Hence X is 1-defective if and only if $T_Q(X_1)$ does not fill $\mathbb{P}^{r-n-1}$ and has dimension smaller than n. Since Q is general in X_1, the claim follows. □

The previous result suggests a new way for a definition of defect:

DEFINITION 4. *Assume that X is 1-defective. Call 1-st* **projection defect** *of X the difference $p_1(X) = \dim(X) - \dim(X_1)$, X_1 being a general tangential projection of X.*

We warn the reader that the projection defect may be different from our former defect in some cases, namely when $r < 2n + 1$. In general we have:

EXERCISE 30. Prove that X_1 is non–degenerate in $\mathbb{P}^{r-n-1}$.

EXERCISE 31. Let X be 1-defective, with defect δ_1 and projection defect p_1. Then:

$$\delta_1 = p_1 - \max(0, 2n + 1 - r).$$

Indeed call m the dimension of $T_{P_0,P_1}(X)$ for a general choice of the points in X ($2n + 1 \geq m > n$). The number $2n - m$ computes the dimension of the intersection between $T_{P_0}(X)$ and $T_{P_1}(X)$, so that, as above, $\dim(X_1) = n - \dim(T_{P_0}(X) \cap T_{P_1}(X)) - 1 = m - n - 1$.
Now if X is defective, then $2n - m < r$ and $p_1 = 2n - m + 1$. On the other hand $\delta_1 = 2n + 1 - \min(r, 2n + 1) - m$. The conclusion follows.

Of course, using successive tangential projections, one has a pattern for a definition of successive projection defects. We give new pieces of notation.

Given the variety $X = X_0$, write $\pi_0 : X \to X_1$ for a general tangential projection. Taking a general tangential projection $X_1 \to X_2$, write $\pi_2 : X \to X_2$ for the composition of two general tangential projections. Continuing with the same procedure, define a sequence of tangential projections:

$$\pi_k : X = X_0 \to X_1 \to \cdots \to X_{k-1} \to X_k.$$

The process stops when X_k is a linear space.

EXERCISE 32. π_k is the projection of X from the space $T_{P_0,\dots,P_{k-1}}(X)$ for a general choice of the points $P_0, \dots, P_{k-1} \in X$.

EXERCISE 33. If we set $m = \dim(T_{P_0,\dots,P_{k-1}}(X))$, then X_k is non–degenerate in the space $\mathbb{P}^{r-m-1}$. Hence X_k is a linear space if and only if it coincides with $\mathbb{P}^{r-m-1}$.

EXERCISE 34. X_k is a linear space if and only if $T_{P_0,\dots,P_k}(X) = \mathbb{P}^r$.
X_k is non–degenerate. Hence $X_k = \mathbb{P}^s$ is a linear space if and only if $s = r - m - 1$, with $m = \dim(T_{P_0,\dots,P_{k-1}}(X))$. This happens if and only if the tangent space to X_k at a general point Q coincides with $\mathbb{P}^s$. This means that its pull–back in π_k coincides with $\mathbb{P}^r$. Now just notice that if P_k is a (general) pre-image of Q in X, then the pull–back coincides with $T_{P_0,\dots,P_k}(X)$.

COROLLARY 4. *X is k-defective if and only if π_k sends X to a variety X_k which is not linear and has dimension smaller than $n = \dim(X)$.*

Proof. Induction on k, the case $k = 1$ being corollary 3.
If $X_k = \mathbb{P}^s$ is a linear space, the previous example shows that it is not k-defective.
So assume that X_k is not linear, and use induction. $\dim(X_k) < \dim(X)$ is equivalent to say that $\dim(T_Q(X_k)) < \dim(X)$, for a general point $Q \in X_k$. Taking a general

pre–image $P_k \in X$ for Q, one obtains:

$$\dim(T_{P_0,\dots,P_k}(X)) < \dim(T_{P_0,\dots,P_{k-1}}(X)) + n + 1 \leq nk + n + k$$

Since moreover $T_{P_0,\dots,P_k}(X)$ is not $\mathbb{P}^r$, then X is k-defective in this case.

Assuming on the contrary $\dim(X_k) = \dim(X)$, then one gets $\dim(X_{k-1}) = n$ so that X_{k-1} is not defective, by induction. Since it is not linear, then $\dim(T_{P_0,\dots,P_{k-1}}(X)) < r$, hence $\dim(T_{P_0,\dots,P_{k-1}}(X)) = n(k-1) + n + k - 1$. Thus:

$$\dim(T_{P_0,\dots,P_k}(X)) = \dim(T_{P_0,\dots,P_{k-1}}(X)) + n + 1 = nk + n + k$$

and we are done. □

DEFINITION 5. *Assume that X is 1-defective. Call k-th* **projection defect** *of X the difference $p_k(X) = \dim(X) - \dim(X_k)$, X_k being a k-th general tangential projection of X.*

We have thus a non–decreasing chain of numbers:

$$p_0 = 0 \leq p_1 \leq \cdots \leq p_K$$

which starts being positive at the minimal k such that X is k-defective and stops at the invariant K introduced in (2), for which $T_{P_0,\dots,P_K}(X)$ reaches $\mathbb{P}^r$.

The relationship between projection defects and our original defects becomes less elementary for $k > 1$. We exploit them in a remark.

REMARK 3. The tangent space $T_{P_0,\dots,P_k}(X)$ projects to $T_Q(X_k)$ in π_k (here Q is the image of P_k). Thus:

$$\dim(T_{P_0,\dots,P_{k-1}}(X)) + \dim(X_k) + 1 = \dim(T_{P_0,\dots,P_k}(X))$$

or, in other words:

$$p_k = n + 1 - \dim(T_{P_0,\dots,P_k}(X)) + \dim(T_{P_0,\dots,P_{k-1}}(X))$$

now recalling the definition of defect:

$$p_k = n + 1 - (\min(r, nk + n + k) - \delta_k) + \min(r, nk + k - 1) - \delta_{k-1}.$$

Let us consider the various cases.
If $r \geq nk + n + k$, then:

$$p_k = n + 1 - (nk + n + k - \delta_k) + nk + k - 1 - \delta_{k-1} = \delta_k - \delta_{k-1}.$$

On the other hand, when $r \leq nk + k - 1$:

$$p_k = n + 1 + \delta_k - \delta_{k-1}.$$

Finally in the remaining case $nk + k - 1 \leq r \leq nk + n + k$

$$p_k = n + 1 - (r - \delta_k) + nk + k - 1 - \delta_{k-1} = nk + n + k + \delta_k - \delta_{k-1}.$$

In particular, the case $r \geq nk + n + k$ should be fixed in mind, since some author defines p_k as the natural defect. Also some authors (e.g. see [82] and [37]) defines the defect as $p_k - p_{k-1}$, which measures the drop of dimension introduced at the k-th tangential projection, i.e. the new defectivity introduced at level k.

Times has come to introduce some result. We are ready to use the previous machinery to prove some bounds on the defects.

THEOREM 2. *If X is a curve, then it is never defective.*

Proof. It is an immediate consequence of corollary 4. Indeed if X is k-defective, then X_k is not linear, hence cannot be a point, and $\dim(X_k) < \dim(X)$. This is clearly impossible when X is an irreducible curve. □

REMARK 4. Notice how the irreducibility condition plays in the previous claim. Indeed when X is a reducible curve, its first secant variety $S_1(X)$ may have dimension $2 < 2n+1$. Indeed for $X=$ union of 3 lines, meeting at a point, then $S_1(X)$ is the union of the three planes spanned by the pairs of lines.

It is clear that the tangent space $T_{P_0}(X)$ at a general point cannot contain X, which is non–degenerate. Hence $\dim(T_{P_0,P_1}(X)) \geq n+1$. We can say something more:

PROPOSITION 5. *If $r > n+1$, then* $\dim(T_{P_0,P_1}(X)) \geq n+2$.

Proof. Assume $\dim(T_{P_0,P_1}(X)) = n+1$. Then $\dim(T_{P_0}(X) \cap T_{P_1}(X)) = n-1$. By the Linear lemma 1 this means that all the tangent spaces are contained in the same $\mathbb{P}^{n+1}$ or they contain a fixed $\mathbb{P}^{n-1}$. The former case is impossible, since X is non degenerate and $r > n+1$. The latter case cannot hold in characteristic 0, for the reasons explained in exercise 26. □

EXERCISE 35. If $n + 2k \leq r$ then $\dim(T_{P_0,\dots,P_k}(X)) \geq n+2k$.
Induction on k, the case $k = 1$ being considered in the previous proposition. Assume then $\dim(T_{P_0,\dots,P_{k-1}}(X)) \geq n+2k-2$. If $T_{P_0,\dots,P_{k-1}}(X)$ coincides with $\mathbb{P}^r$, then our numerical assumption implies the claim. Otherwise, since X is non-degenerate, then it cannot be contained in $T_{P_0,\dots,P_{k-1}}(X)$, thus if the strict inequality holds above, the claim follows. It remains to consider the case $\dim(T_{P_0,\dots,P_{k-1}}(X)) = n+2k-2$. Take the $(k-1)$-th tangential projection X_{k-1}. In this projection $T_{P_0,\dots,P_{k-1}}(X)$ maps to $T_Q(X_{k-1})$ and $T_{P_0,\dots,P_k}(X)$ maps to $T_{Q,Q'}(X_{k-1})$, where Q, Q' are the images of P_{k-1}, P_k. Then the claim follows from the previous proposition.

EXERCISE 36. For any k, if X is k-defective, then $\delta_k \leq (n-1)k$.

In general, we cannot expect to improve the previous bound on the defects, even for the first defect. Indeed:

EXAMPLE 11. Let C be a non–degenerate curve in $\mathbb{P}^{r-n-1}$ and take the cone X over C, with vertex W of dimension $n-2$. X has dimension n and it is non–

degenerate in $\mathbb{P}^r$. Assume $r > n + 2k + 2$. Fix general points $P_0, \ldots, P_k \in X$ and call $Q_0, \ldots, Q_k$ their images in the projection from W. Then $T_{P_0,\ldots,P_k}(X)$ is the span of W and $T_{Q_0,\ldots,Q_k}(X)$. Since $r - n - 1 > 2k + 1$ and C is not k-defective by theorem 2, then $\dim(T_{Q_0,\ldots,Q_k}(X)) = 2k + 1$. Hence:

$$\dim(T_{P_0,\ldots,P_k}(X)) = 2k + n < r$$

and X is k-defective, with k-th defect $\delta_k = (n - 1)k$.

Clearly the variety X of the previous example is rather singular along W. It turns out indeed that one can improve the bound considerably if one assumes the smoothness of X. This is a consequence of a deep result proved by Fulton and Lazarsfeld on the connectedness of some subvarieties of X. We will be back on the argument later on.

EXERCISE 37. Rephrase all the previous bounds for the projection defects.

3.5. Inverse systems

Occasionally, it turns out that some ad hoc technique computes the dimension of some secant variety, when X has some well-defined structure. This is the case, for instance, of some varieties which parametrizes products.

In this section, we introduce an appropriate method for these objects and suggest how to extend it to arbitrary varieties.

We illustrate how the method works for Grassmannian. Consider the Grassmannian $X = G(h, s)$ of h-dimensional projective subspaces in $\mathbb{P}^s$, with its Plücker embedding in $\mathbb{P}^r$, $r = \binom{s+1}{h+1} - 1$. If $E = \mathbb{C}^{s+1}$ is the vector space which defines $\mathbb{P}^s$, then we may identify $\mathbb{P}^r$ with $\mathbb{P}(\Lambda^{h+1}E)$. Given $P \in G(h, s)$ and a basis $v_0, \ldots, v_h$ for P, the Plücker embedding maps P to the point $v_0 \wedge \cdots \wedge v_h \in \mathbb{P}(\Lambda^{h+1}E) = \mathbb{P}^r$. As remarked above, $G(h, s)$ corresponds, from this point of view, to the set of decomposable vectors (i.e. wedge-monomials), while the secant variety $S_k(G(h, s))$ is the (closure of the) set of vectors which can be written as a sum of $k + 1$ wedge-monomials.

Now consider the exterior algebra $\mathrm{Ext}(E) = \oplus_{i=0}^{\infty} \Lambda^i E$. Using the canonical pairing $\Lambda^i E \times \Lambda^{s+1-i} E \to \mathbb{C}$ one defines the perpendicular Y^* of any homogeneous subspace $Y \subset \Lambda^i E$.

With this notation, we have the following interpretation of the tangent space to $X = G(h, s)$ at the point $P = v_0 \wedge \cdots \wedge v_h$:

EXERCISE 38. The perpendicular space $Y = T_P(X)^*$ is generated by the elements of degree $s - h$ in the ideal $(v_0, \ldots, v_h)^2 \subset \mathrm{Ext}(E)$.

Thus we get the following characterization:

PROPOSITION 6. *If* $P_0 = v_{0,0} \wedge \cdots \wedge v_{0,h}, \ldots, P_k = v_{k,0} \wedge \cdots \wedge v_{k,h}$ *are general points of* $X = G(h, s)$, *then for* u *general in* $< P_0, \ldots, P_k >$, *the tangent space*

$T_u(S_k(X))$ is perpendicular to

$$W = [(v_{0,0}, \dots, v_{0,h})^2 \cap \dots \cap (v_{k,0}, \dots, v_{k,h})^2]_{s-h}.$$

Proof. The perpendicular to the span of $< T_{P_0}(X), \dots, T_{P_k}(X) >= T_u(S_k(X))$ is the intersection of the perpendicular spaces to each $T_{P_i}(X)^*$. □

Now we provide an example in which the dimension of the perpendicular space can be effectively computed:

THEOREM 3. *(***Catalisano, Geramita, Gimigliano***) When $(h+1)(k+1) \le s+1$ and $h \ge 2$, the k-th secant variety to $G(h,s)$ has the expected dimension*

$$\dim(S_k(G(h,s))) = \min(r, (k+1)(h+1)(s-h) + k).$$

Proof. One uses the reduction of proposition 6.
The condition $(h+1)(k+1) \le s+1$ implies that we can choose the points $P_0, \dots, P_k$ from a canonical basis $e_0, \dots, e_s$ of E, using separate variables:

$$P_0 = e_0 \wedge \dots \wedge e_h,\ P_1 = e_{h+1} \wedge \dots \wedge e_{2h+1}, \dots P_k = e_{kh+k} \wedge \dots \wedge e_{kh+k+h}.$$

Notice that the natural action of the linear group on $G(h,s)$ is transitive over the $(k+1)$-tuples of points, in our setting. So any $(k+1)$-tuple of points can be reduced to the previous one.
Now one finds a basis of

$$[(e_0, \dots, e_h)^2 \cap \dots \cap (e_{kh+k}, \dots, e_{kh+k+h})^2]_{s-h}$$

by taking all the products of $s-h$ elements of the basis in which two elements sits in $\{e_0, \dots, e_h\}$, two elements sits in $\{e_{h+1}, \dots, e_{2h+1}\}$ and so on.
The proof follows from an easy dimension count. □

A completely similar approach works for symmetric tensor products, which are the algebraic analogue of Veronese varieties, and general tensor products, which are the algebraic analogue of Segre products of projective spaces.

In the former case, however, the results one can easily obtain in this way are weaker than the classical results obtained using the techniques we will explain in the next chapters.

In the latter case, the use of perpendicular spaces is harder, because we need to work with several different vector spaces. We refer to the paper [13] for a reference on results in progress. We just cite:

THEOREM 4. *(see [13], Proposition 3.3) Let $X = \mathbb{P}^{n_0} \times \dots \times \mathbb{P}^{n_t}$, with its Segre embedding in $\mathbb{P}^r$, $r = (n_0+1)\cdots(n_t+1) - 1$. Fix an integer k such that*

$$\prod_{i=1}^{t}(n_i+1) - \sum_{i=1}^{t}(n_i) + 1 \le k \le \min(n_0, \prod_{i=1}^{t}(n_i+1) - 1).$$

Then X is k-defective.

REMARK 5. One of the main facilities in the proof of the two previous results relies in the number $k + 1$ of points, which is small with respect to the intrinsic coordinates of X. This implies that under the action of the linear group, one can assume that $P_0, \dots, P_k$ are independent points with very special coordinates, so that the span of their tangent spaces is easy.
Similar simplifications apply as soon as X is $(k + 1)$-homogeneous, i.e. there exists a group acting on X, so that any $(k+1)$-tuples of independent points of X lie in the same orbit.

For Grassmannians the natural pairing allows to translate our question on the dimension of secant varieties to a question on the intersection of suitable perpendicular spaces. When X is arbitrary, one may create a similar procedure, introducing the pairing by means of the theory of **inverse systems**.

Identify $\mathbb{P}^r$ with a set of forms of given degree d in a given number of variables j (modulo scalar multiplication). Then consider the polynomial ring

$$A = \mathbb{C}[d_0, \dots, d_j]$$

as a ring of differentials on the forms which represent points of $\mathbb{P}^r$.
In other words, any $P \in \mathbb{P}^r$ represents a form in j variables, over which the elements of A act as sequences of partial derivatives.

DEFINITION 6. *We say that $d \in A$ and $f \in \mathbb{P}^r$ are* **apolar** *when $d(f) = 0$.*
For any subset $X \subset \mathbb{P}^r$, the **apolar set** *$X^\perp$ is the set of all $d \in A$ such that $d(f) = 0$ for all $f \in X$.*
For any subset $D \subset A$, the **inverse system** *of D is the set $D^{-1} = \{f \in \mathbb{P}^r : d(f) = 0\}$ for all $d \in D$.*

Of course this definition is invariant when f is multiplied by some scalars, hence it is well–posed.

EXERCISE 39. If $f, g \in \mathbb{P}^r$ belong to D^{-1}, then all the points in $< f, g >$ also belong to D^{-1}, hence D^{-1} is always a linear space.

EXERCISE 40. The k-th secant variety $S_k(X)$ is then also the closure of:
$S_k(X) = \{f$: there are $f_0, \dots, f_k \in X$ with $f \in ((f_0, \dots, f_k)^\perp)^{-1}\}$.

One should wonder why it is interesting to give such a complicate characterization of secant varieties. Apart from the fact that the theory of inverse systems has several interesting applications, one should consider that apolarity is indeed an algebraic condition, hence one has a way to reproduce algebraically the construction of secant varieties.

See [48], [9], [24] for applications to the study of the dimension of some secant varieties.

4. Degenerate subvarieties

Probably some readers are wondering that up to now, we put down on the paper several introductory facts, but no really ultimate results (except for theorem 2, perhaps). We ask the readers to patient only few more lines. Indeed we must still introduce our last tool, then the investigation could start and unelementary results will come at hand.

4.1. The Infinitesimal Bertini's Theorem

The infinitesimal Bertini's Theorem links defectivity with set of points at which the general tangent space $T_{P_0,\dots,P_k}(X)$ is actually tangent to X.

THEOREM 5. **Infinitesimal Bertini's Theorem** *Let X be an irreducible variety and let Y be a reduced, irreducible algebraic subvariety of some linear system V of divisors in X. Let $y \in Y$ be a general point, let $S := S_y$ be a component of the singular locus of the divisor H parametrized by y, not contained in Sing(X). Then the projective tangent space to Y at y in V is contained in $V(-S)$.*

Before we sketch a proof for the theorem, let us justify its nickname.

Bertini's classical theorems say that, in characteristic 0, if a linear system has a moving singular point, then this point is contained in the base locus, off Sing(X). In the previous setting, the family Y of divisors we work with is not, in general, a linear system: it can be any subfamily of the linear system V. So we cannot conclude that a moving singularity lies in the base locus. On the other hand, any variety Y "becomes linear" when it is replaced with a tangent space at some of its points. The theorem says that the tangent space contains the moving singularities (off Sing(X)). So the infinitesimal movements of H in the family have indeed the moving singularities in the base locus.

Proof. Let v be any tangent vector to Y at y and let $s \in H^0(H, N_{H,X})$ be the section of the normal bundle $N_{H,X}$ of H in X corresponding to the first order deformation H_s of H determined by v. Since $y \in Y$ is general, if $x \notin \mathrm{Sing}(X)$ is any singular point of H, there is a first order deformation of x along which H_σ stays singular.
Fix coordinates $z_1, \dots, z_n$ on X centered at the smooth point x, let $f(z_1, \dots, z_n) = 0$ be the equation of H in these coordinates and let $f(z_1, \dots, z_n) + \epsilon g(z_1, \dots, z_n) = 0$ be the equation of H_σ. The Taylor expansions of f and g give us:

$$f(z_1, \dots, z_n) = \sum_{i=2}^{\infty} f_i(z_1, \dots, z_n)$$

$$g(z_1, \dots, z_n) = \sum_{i=0}^{\infty} g_i(z_1, \dots, z_n)$$

where the f_i, g_i's are homogeneous polynomials of degree i.
Any infinitesimal deformation ξ of x is given by $z_i = \epsilon a_i$, $i = 1, \dots, n$.

Notice that H_σ is singular at ξ if and only if the partial derivatives of $f + \epsilon g$ vanish at $(\epsilon a_1, \dots, \epsilon a_n)$. Since $f_1 \equiv 0$ by assumption, we can find ξ with H_σ is singular at ξ if and only if the derivatives of $\epsilon g(\epsilon a_1, \dots, \epsilon a_n)$ vanish, i.e. if and only if $g_0 \equiv 0$. This means precisely that the section of the normal bundle associated with σ vanishes at any point of S not in $\mathrm{Sing}(X)$. The claim follows. □

The classical analysis of the structure of defective varieties is based on the previous theorem and the following consequence, pointed out by Terracini:

COROLLARY 5. (Terracini) *Assume that X is k-defective. Then for a general choice of the points $P_0, \dots, P_k \in X$, a general hyperplane H of $\mathbb{P}^r$ which contains the space $T_{P_0,\dots,P_k}(X)$, is indeed tangent along a subvariety $\Sigma(H)$ of positive dimension.*

Proof. We follow the modern proof introduced by Ciliberto and Hirschowitz in [26]. Let X_0^{k+1} be the open subset of X^{k+1} described by $(k+1)$-tuples formed by independent points. Call V the linear system of hyperplane divisors in X. Consider now the closure $I \subset V \times X_0^{k+1}$ of the **singular incidence correspondence**:

$$\{(H \cap X, S) \in V \times X_0^{k+1} : S \subset \mathrm{Sing}(H \cap X)\}$$

and let $p_1 : I \to V$ and $p_2 : I \to X_0^{k+1}$ be the two projections.
Since X is k-defective, for a general choice of the points $T_{P_0,\dots,P_k}(X) \neq \mathbb{P}^r$, so it lies in some hyperplane. This means that p_2 dominates X_0^{k+1}. Its general fiber is a projective space of dimension

$$w = \dim(V(-2P_0 - \dots - 2P_k)) > r - (k+1)(n+1)$$

by proposition 2. Then there is only one irreducible component J of I which dominates X_0^{k+1} and one has:

$$\dim(J) > r - k - 1.$$

Consider now the family of divisors on X given by $Y = p_1(J) \subset V$. If $(H \cap X, S) \in J$ is general, with $S = (P_0, \dots, P_k)$, then there is a component Σ of the locus where H is tangent to X, which intersects the set of smooth points of X. Hence we may apply the infinitesimal Bertini's theorem and conclude that the projective tangent space to Y at $H \cap X$ is contained in $V(-\Sigma) \subset V(-S)$. Since S is formed by $k+1$ general points of X, then $\dim(V(-S)) = r - k - 1$, hence:

$$\dim(Y) \le \dim(V(-\Sigma)) \le \dim(V(-S)) = r - k - 1 \tag{4}$$

Hence the map $J \to Y$ has positive dimensional fibers. This means that, for fixed general H containing $T_{P_0,\dots,P_k}(X)$, there exists a positive dimensional subset $\Sigma \subset H \cap X$ such that $H \cap X$ is singular along Σ. □

Setting some new piece of notation, one can be much more precise on the structure of Σ.

DEFINITION 7. *For a hyperplane H of $\mathbb{P}^r$, denote with $Z(H)$ the set of points $\{Q \in X - Sing(X) : T_Q(X) \subset H\}$.*
If $P_0, \dots, P_k \in X$ are general and H is a general hyperplane which contains $T_{P_0,\dots,P_k}$, call **contact locus** *or entry locus of H the union $\Sigma(H) = \Sigma_{P_0,\dots,P_k}(X)$ of all the components of $Z(H)$ which contain some P_i.*
For a general choice of the points, the dimension of the contact locus $\Sigma(H)$ is an invariant which depends only on X. $\nu_k(X) = \dim(\Sigma(X))$ is called the k-th **singular defect** *of X.*

The previous corollary can be rephrased as follows:

If X is k-defective, then the singular defect $\nu_k(X)$ is positive.

With a simple refinement of the previous argument, one can find much more precise information on the contact locus of a defective variety. The main (classical) remark says that the contact locus tends to be considerably degenerate. This remark, together with Terracini's lemma, is the main tool which makes possible the investigation of the world of defective varieties.

DEFINITION 8. *If V is a linear system on X and $X' \subset X$ is a subvariety, then the number $h_V(X') = \dim(V) - \dim(V(-X'))$ is called the* **number of conditions** *imposed by X' to V.*

THEOREM 6. *Assume that X is k-defective, with $\dim(S_k(X)) = s_k$. Choose general points $P_0, \dots, P_k \in X$, a general hyperplane H of $\mathbb{P}^r$ which contains the space $T_{P_0,\dots,P_k}(X)$ and call $\Sigma(H)$ the contact locus of H. Then the number of conditions $h_V(\Sigma(H))$ imposed by $\Sigma(H)$ to the system V of hyperplane sections satisfies:*

$$k+1 \le h_V(\Sigma(H)) \le 1 + s_k - (n-\nu_k)(k+1) \tag{5}$$

(remind that $\nu_k(X) = \dim(\Sigma(H))$). In other words:

$$\dim(<\Sigma(H)>) \le s_k - (n-\nu_k)(k+1)$$

Proof. Just repeat the proof of the corollary. Taking the notation, by definition of defect one can write precisely the dimension w of a general fiber of p_2:

$$w = \dim(V(-2P_0 - \dots - 2P_k)) \ge r - s_k - 1$$

so that $\dim(J) = r - s_k - 1 + n(k+1)$. Now, by definition, the general fibre of the restriction of p_1 to J has dimension $(k+1)\nu_k$. In conclusion we have:

$$r - s_k - 1 + n(k+1) = \dim(J) = \dim(p_1(J)) + (k+1)\nu_k$$

while as above

$$\dim(p_1(J)) \le \dim(V(-\Sigma)) \le r - k - 1$$

which yields the assertion. □

In normal words, the results above say:
if X is k-defective, then a general hyperplane which is tangent at $k+1$ points of X is indeed tangent along a positive dimensional subvariety $\Sigma(H)$, which imposes not many conditions to the hyperplanes, hence it is degenerate.

EXERCISE 41. If V is the hyperplane linear system on X and X' is a subvariety, then $h_V(X') = w < r$ if and only if X' is degenerate, contained in some $\mathbb{P}^w$.
Indeed observe that $V(-X')$ has a base formed by $r+1-w$ elements and X' sits in the base locus of $V(-X')$, which is the intersection of the elements of a base.

EXAMPLE 12. Let X be the cone over a curve $C \subset \mathbb{P}^4$, with vertex at a point. Then $X \subset \mathbb{P}^5$ is 1-defective. As explained in exercise 19, the intersection of two general tangent spaces is the vertex, hence the defect δ_1 is 1. If $P_0, P_1 \in X$ are general points, then any hyperplane H which is tangent to X at P_0, P_1 is indeed tangent to X along the two lines L_0, L_1 of the ruling passing through P_0, P_1. Hence the contact locus $\Sigma(H)$ is the (plane conic) $L_0 \cup L_1$ and $\nu_1(X) = 1$, $h_V(\Sigma(H)) = 3$. Formula (5) reads:

$$2 \leq h_V(\Sigma(H)) \leq 2 \cdot (1+1) - 1$$

and works.

EXERCISE 42. Repeat the previous example for general cones.

On the other hand, the reader is invited to reflect immediately on the fact that when a variety X has a positive k-th singular defect, then it is *not necessarily* k-defective.

A first example is provided by cones X: imposing to a hyperplane the tangency at one point, we get tangency along a line. So $\nu_0(X) > 0$. But X is not 0-defective (which does not make any sense!).

For a more sophisticated example, we need two elementary results:

EXERCISE 43. Fix two disjoint linear spaces $L, R \subset \mathbb{P}^s$. Let X be a subvariety of R and call W the cone over X, with vertex L. Fix a linear space L' containing L. Then the projection of W from L' coincides with the projection of X from $L' \cap R$.

EXERCISE 44. Let L be a linear space and let $L' \subset L$ be a subspace. Then the projection of X from L' is contained in the cone over the projection of X from L, with vertex at the projection of L from L'.

EXAMPLE 13. Let $W \subset \mathbb{P}^6$ be the cone, with vertex at a point P, over the Veronese surface $V(2,2) \subset \mathbb{P}^5$. Let X be the intersection of W with a general hypersurface of degree bigger than 1. We will see later on that X is not 1-defective.
At a general point $P_0 \in X$, the tangent space $T_{P_0}(X)$ is a hyperplane in $T_{P_0}(W)$, which in turn is $< P, T_Q(V(2,2)) >$, Q be the point of $V(2,2)$ corresponding to P_0 in the projection from P. By exercise 43, the projection of W from $T_{P_0}(W)$ coincides with the projection of $V(2,2)$ from $T_Q(V(2,2))$, which is a plane curve, since $V(2,2)$ is defective. Hence by exercise 44 the projection of W from $T_{P_0}(X)$ is a cone Z over a

plane curve. Z contains the projection of X from $T_{P_0}(X)$. Since X is not 1-defective, then Z coincides with this projection.
Now fix another general point $P_1 \in X$ and call A its image in Z. The tangent space to the cone Z at A contains the tangent spaces along a positive dimensional subvariety $Z' \subset Z$. Hence $T_{P_0,P_1}(X)$ contains the tangent spaces to X along the counterimage of Z'. It follows that any hyperplane which contains T_{P_0,P_1} is indeed tangent along a positive dimensional subvariety, i.e. $\nu_1(X) > 0$.

4.2. An example: the Veronese surface

Let us visit again the Veronese surface $X = V(2,2)$, corresponding to the image of $\mathbb{P}^2$ in $\mathbb{P}^5$, via the linear system of conics.

We know that X is 1-defective, with defect $\delta_1(X) = 1$.

In the interpretation suggested above, we get that a general tangential projection of X to $\mathbb{P}^2$ must be a curve. This is clear, indeed. Tangential projection means that we restrict our linear system to the system of conics with one fixed double point P. The linear system is then composed with the pencil of lines through P. Thus it contracts $\mathbb{P}^2$ to a curve, which is the image of a line not passing through P under the complete g^2_2. In other words, the image X_1 of X in a general tangential projection is a plane conic. In particular the projection defect p_1 is equal to δ_1. Compare with remark 3 (here $r = 5$ is equal to $nk + n + k$).

X has a singular defect. A general tangent line to X_1 meets it in a double point. Since this line is the image of a space T_{P_0,P_1} under the projection from T_{P_0} (with $P_0, P_1 \in X$ general points), it turns out that $T_{P_0,P_1} = H$ is a hyperplane, tangent to X exactly along a fibre of the projection $X \to X_1$. As explained above, these fibers are the image in $\mathbb{P}^5$ of lines through P, hence conics in $\mathbb{P}^5$. So the singular defect is $\nu_1 = 1$ and a general contact variety is a (plane) conic Σ.

The conditions imposed by Σ to the hyperplanes of $\mathbb{P}^5$ are $h_V(\Sigma) = 3$. Formula (5) reads:

$$2 \le h_V(\Sigma(H)) \le 2 \cdot (1+1) - 1$$

and works.

Notice that, as the points P_0, P_1 vary, the corresponding contact variety of the hyperplane T_{P_0,P_1} determine a family of conics in X. This family is indeed a linear system and this linear system determines an isomorphism between the Veronese surface X and $\mathbb{P}^2$.

4.3. The Gauss map

It turns out that the definition of contact locus has some clones which are relevant for the study of defective varieties. The first one starts from a remark concerning the intersection of X with the tangent space $T_{P_0,\dots,P_k}$.

DEFINITION 9. *For a general choice of the points $P_0, \ldots, P_k$, consider the set T of regular points $Q \in X$ such that $T_Q(X) \subset T_{P_0,\ldots,P_k}(X)$. Call $\Gamma_{P_0,\ldots,P_k}(X)$ the union of all the components of the closure of T which contains one of the points P_i's. $\Gamma_{P_0,\ldots,P_k}(X)$ will be referred to as the k-th* **tangential contact locus**. *We call* **tangential singular defect** *the dimension $\gamma_k(X)$ of $\Gamma_{P_0,\ldots,P_k}(X)$, for a general choice of the points.*

It is clear that all the hyperplanes H which are tangent to X at $P_0, \ldots, P_k$ contain $T_{P_0,\ldots,P_k}(X)$; hence for all such hyperplanes H one has

$$\Gamma_{P_0,\ldots,P_k}(X) \subset \Sigma(H).$$

More precisely one gets the following characterization:

EXERCISE 45. $\Gamma_{P_0,\ldots,P_k}(X)$ is the intersection of $\Sigma(H)$ for all hyperplanes H which are tangent to X at $P_0, \ldots, P_k$.
Enough to remind that $T_{P_0,\ldots,P_k}(X)$ is the intersection of all the hyperplanes which are tangent to X at $P_0, \ldots, P_k$.

Using the classical Bertini's theorem, one gets:

PROPOSITION 7. *For $P_0, \ldots, P_k$ general, if H is a hyperplane which contains $T_{P_0,\ldots,P_k}$, then $\Sigma(H) \subset X \cap T_{P_0,\ldots,P_k}(X)$.*

Proof. The set of hyperplanes containing $T_{P_0,\ldots,P_k}(X)$ cuts on X a linear system of hyperplane sections, whose moving singular part must be contained, by Bertini's theorem, in the base locus of the system. Since $\Sigma(H)$ lies in the singular locus of the divisor $H \cap X$ and the base locus is exactly $X \cap T_{P_0,\ldots,P_k}(X)$, the claim follows. □

Thus we have, for a general set of points $P_0, \ldots, P_k \in X$ and for a hyperplane H which is tangent to X at all the P_i's, a sequence of inclusions:

$$\Gamma_{P_0,\ldots,P_k}(X) \subset \Sigma(H) \subset T_{P_0,\ldots,P_k}(X) \cap X \tag{6}$$

and $\gamma_k(X) \leq \nu_k(X)$.

THEOREM 7. *Assume that X is k-defective. Then for a general choice of the points $P_0, \ldots, P_k \in X$, the tangential contact locus $\Gamma_{P_0,\ldots,P_k}(X)$ has positive dimension.*

Proof. Fix general points $P_0, \ldots, P_k$. Let X_k be the tangential projection of X from the space $T_{P_1,\ldots,P_k}(X)$. Then we know from corollary 3 that $X \to X_k$ has positive dimensional fibers. If Q is the image of P_0 in the projection, then $T_{P_0,\ldots,P_k}(X)$ maps to $T_Q(X_k)$, hence it is tangent to X along the fiber of the projection over Q. □

PROPOSITION 8. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, k–defective threefold. For a general choice of $P_0, \ldots, P_k \in X$ and a general choice* *of the hyperplane* $H \in \mathcal{H}(-2P_0 - \cdots - 2P_k)$*, the contact loci* $\Gamma = \Gamma_{P_0,\ldots,P_k}$ *and* $\Sigma = \Sigma(H) = \Sigma_{P_0,\ldots,P_k}(H)$ *are equidimensional. Furthermore either they are irreducible or they consist of* $k+1$ *irreducible components, one through each of the points* $P_0, \ldots, P_k$.

Proof. Let us give the proof only for $\Sigma(H)$, the other proof being similar.
First of all, let us move slightly the points P_i's on $\Sigma(H)$ to a new set of points $\{Q_0, \ldots, Q_k\}$. Then $Q_0, \ldots, Q_k$ are also general points on X. Furthermore $T_{X,P_0,\ldots,P_k}$ contains the tangent spaces to X at the points Q_i's, so for reasons of dimension, it coincides with $T_{X,Q_0,\ldots,Q_k}$. Then we may assume that $P_0, \ldots, P_k$ are smooth points for $\Sigma(H)$, hence there is only one irreducible component of $\Sigma(H)$ through each one of the points $P_0, \ldots, P_k$. By monodromy we may exchange the points P_i and therefore all components of $\Sigma(H)$ have the same dimension.
Assume now that there is a component of Σ which contains more than one of the P_i's, say P_0 and P_1. By monodromy, we can let P_0 stay fixed and we can move P_1 to any one of the points P_i, $i > 1$. Then we see that also P_0 and P_i, $i > 1$, stay on an irreducible component of Σ. Since P_0 sits on only one irreducible component of Σ, then this component has to contain all the points $P_0, \ldots, P_k$ and therefore it has to coincide with $\Sigma(H)$. □

EXERCISE 46. If X is a surface, whose general k-th contact locus has positive dimension, then the first inclusion in (6) is an equality and the second is a local, set-theoretical equality.
Indeed $T_{P_0,\ldots,P_k}(X) \cap X$ is a divisor in X. As H moves in the linear system of hyperplanes tangent to X at the P_i's, every $\Sigma_{P_0,\ldots,P_k}(H)$ is a divisor, so it is composed with some components of $T_{P_0,\ldots,P_k}(X) \cap X$. Finally observe that moving H, $\Sigma_{P_0,\ldots,P_k}(H)$ is fixed thus it coincides with $\Gamma_{P_0,\ldots,P_k}$, by the previous result.

EXERCISE 47. If X is a defective threefold, then the first inclusion in (6) is an equality.
Indeed $T_{P_0,\ldots,P_k}(X) \cap X$ is a divisor in X and $\Gamma_{P_0,\ldots,P_k}$ has dimension at least one. None of them depends on the hyperplane $H \supset T_{P_0,\ldots,P_k}$. Hence also $\Sigma(H)$ is fixed, for general H.

Let us go back to the exercise 46. We know that the general k-th contact loci Σ and Γ are positive-dimensional, but X is not k-defective. See example 13. So one is led to the following:

DEFINITION 10. *We say that* X *is* **k-weakly defective** *(resp.* **tangentially k-weakly defective***) when for a general choice of* $P_0, \ldots, P_k \in X$ *and the hyperplane* $H \supset T_{P_0,\ldots,P_k}$*, the contact locus* $\Sigma_{P_0,\ldots,P_k}(H)$ *(resp. tangential contact locus* $\Gamma_{P_0,\ldots,P_k}$*) are positive dimensional.*

It is clear now that:

defective → tangentially weakly defective → weakly defective.

On the other hand, these concepts are disjoint. There are examples of varieties which are weakly defective but not tangentially weakly defective.

The study of tangentially defective varieties has interest from his own point of view. Usually tangentially defective varieties are referred to as *varieties with degenerate Gauss map*.

DEFINITION 11. *The* **Gauss map** *of* X *is the rational map* $g(X) : X \to G(n, r)$ *which sends a general (smooth) point* $P \in X$ *to the point of the Grassmannian which parametrizes the tangent space* $T_P(X)$.
The **dual variety** *of* X *is the (closure of the) variety* $X^v \subset (\mathbb{P}^r)^v$ *of the dual projective space which parametrizes hyperplanes which contain some tangent space to* X *at some smooth point.*

EXERCISE 48. The image of the Gauss map $g(X)$ has dimension smaller than X if and only if X is tangentially 0-weakly defective.

EXERCISE 49. The dual variety of X has dimension at most $r-1$. It has dimension $r-1$ if and only if X is not 0-weakly defective.
Indeed consider the singular incidence variety:

$$I = \{(P, H) \in X \times (\mathbb{P}^r)^v : H \supset T_P(X)\}$$

Since the fiber of the first projection of this variety over $\mathrm{Reg}(X)$ has fixed dimension, then $\dim(I) = r-1$. X^v is the image of I in the second projection. So $\dim(X^v) < r-1$ if and only if a general hyperplane which is tangent to X at some point P, is indeed tangent to X in infinitely many points.

The theories of varieties for which the Gauss map has image of dimension $< n$ (varieties with *degenerate Gauss map*) and varieties with “small” dual variety, are indeed classically considered and also studied from a modern point of view.

PROPOSITION 9. *The set* Σ *of points where a general tangent hyperplane is tangent to* X *is a linear subvariety of* X.
The general fibers of the Gauss map are linear subspaces.

Proof. The first assertion is just an application of (5) for $k = 0$. Indeed ν_0 is exactly the dimension of Σ and $h(\Sigma) = \nu_0 + 1$, i.e. Σ is contained in a $\mathbb{P}^{\nu_0}$.
The second assertion follows now, since the general fiber of the Gauss map is just the intersection of several loci Σ. □

The main result about the dimension of Gauss image is due to Zak, as a consequence of Fulton–Hansen connectedness principle that will be discussed later, in section 5:

THEOREM 8. *The dimension of the image of the Gauss map is at least* $n - b - 1$, *where* $b \geq -1$ *is the dimension of the singular locus of* X.
In particular, when X *is smooth, then the Gauss map is birational.*

A classification of surfaces with degenerate Gauss map is classical (see e.g. [44]):

THEOREM 9. *Surfaces whose images in the Gauss map are curves are either cones or the union of tangent lines to a fixed curve.*

A similar result for higher dimensional varieties seems quite challenging. Several partial results are known. See [2] and [54] for an account of the theory and many extremal examples, essentially obtained taking the 1-secant variety to some extremal varieties.

For varieties with small dual variety, we refer to [83].

4.4. Varieties with many degenerate subvarieties

An important principle in Algebraic geometry, still to be fully explored and understood, suggests that a general projective variety X of small codimension contains few irreducible subvarieties of special type. This is the philosophy behind the Noether-Lefschetz theorem on general hypersurfaces and more recent results on the geometric genus of their subvarieties (see e.g. [29], [80], [81], [23], [30]).

For our scopes, the principle suggests that there are only "few" degenerate subvarieties in X, unless X has a very particular structure. This is more or less the base for our classification of defective varieties: since they have indeed a family of degenerate objects passing through general $(k+1)$-tuples of points, we may hope to use the family to derive complete information on X itself.

The study of varieties covered by degenerate subvarieties is however a subject interesting by itself and intensively studied in classical projective geometry. We give in this section some hint on our knowledge on the field, with particular focus on results which are relevant for the classification of defective varieties.

Let us start with two classical facts:

PROPOSITION 10. *Assume that X contains a family of plane curves, with the property that there exists an element of the family through any pair of general points of X. Then either X is a projective space, or the plane curves have degree* 2.

Proof. If the curves are lines, then there is a line in X which connects a general pair of points. This is a classical characterization of projective spaces. Assuming that the plane curves have degree 3 or bigger, we get a contradiction. Namely, by assumption, any secant line to X lies in the plane of some curve of the family, so it is at least trisecant. This is excluded by the trisecant lemma. □

THEOREM 10. (**Severi**) *The only surfaces in $\mathbb{P}^r$, $r \leq 5$ with a 2-dimensional linear system of generically irreducible conics are the Veronese surface $V(2,2) \subset \mathbb{P}^5$ and its projection to lower dimensional spaces.*

Proof. The first step is to reduce to the case in which the self intersection of the system

is 1. Clearly we have infinitely many divisors of the system through a general point, so the self intersection is positive. Since the general divisor of the system is irreducible, if the self intersection is at least 2, then two general planes defined by the conics meet in a line. By lemma 1, either $r = 3$ or these planes have a fixed line. The latter case is impossible, since we have a divisor of the system through two general points of X. The former implies that the system is cut by the family of planes through a point. Hence X has degree 2, and it is the projection of a Veronese surface from a secant line.
So, assume that two general elements of the system meet at one point. This implies that the map $\phi : X \to \mathbb{P}^2$ induced by the linear system is birational. It sends conics to lines. Thus, if ϕ' is its inverse, then ϕ' sends lines to conics. It follows that X is the image of $\mathbb{P}^2$ in a linear system of conics. □

The previous characterization of the Veronese surface can be strenghtened. Indeed, we need not to assume that the 2-dimensional family of conics is a linear system: it is a consequence of our setting. Let us explain in details.

DEFINITION 12. *We say that a family $\mathcal{F}$ of X is k-**filling** if for a general choice of k points of X, there exists an element of the family passing through the points.*
*We say that the family is an **involution** of dimension k if for a general choice of k points of X, there exists exactly one element of the family passing through them.*

Notice that k-filling implies that, writing m for the dimension of the elements of $\mathcal{F}$, then:

$$\dim(\mathcal{F}) \geq k(n-m) \tag{7}$$

and the map from the total space of k-th cartesian products of elements of $\mathcal{F}$ to X^k is dominant.

We have an involution when the previous map is birational, which implies that the equality holds in (7).

Clearly, any linear system of divisors is an involution. Conversely there are involutions which are not linear systems: just take any family of dimension 2 composed with a pencil. It turns out that, essentially, families composed with pencils are the only involutions which are not linear systems. This fact was classically observed by Castelnuovo and Humbert for divisors on curves, and extended to higher dimension in [16], §5.

THEOREM 11. *(**Chiantini-Ciliberto**) Let X be a reduced, irreducible variety of dimension $n > 1$. Let $\mathcal{F}$ be a k-dimensional involution of divisors on X, which has no fixed divisor and whose general divisor D is reduced. Then either $\mathcal{F}$ is a linear system or it is composite with a pencil.*

Proof. We just give a sketch of the proof, referring the interested reader to [16]. Since the problem is birational, we may assume X is smooth. We argue by induction on k, the case $k = 1$ being classically known.
Suppose $k = 2$ and the general divisor $D \in \mathcal{F}$ is irreducible. If P is a general point

of X, then $\mathcal{F}(-P)$ is an 1-dimensional involution with no fixed divisor. $\mathcal{F}(-P)$ is a linear system of dimension 1. Indeed one can prove that for a general choice of P, then the divisors in $F(-P)$ have no fixed tangent directions at the point. So $\mathcal{F}(-P)$ sends the blowing up of X at P to a curve which is dominated by the exceptional divisor: this curve is rational.
$\mathcal{F}(-P)$ is different from $\mathcal{F}(-Q)$, if P and Q are two general points of a general divisor in $\mathcal{F}$. This immediately implies that the natural map $Y \to Pic(X)$ (here Y is the space which parametrizes the family) is constant, i.e. $\mathcal{F}$ is contained in a linear system. We want to prove now that $\mathcal{F}$ itself is a linear system.

Take $D, D' \in \mathcal{F}$ general divisors. Let Z be the scheme-theoretic intersection of D and D'. First we notice that $Z \neq \emptyset$. Otherwise we would have $\dim|D| \leq 1$, contrary to the fact that $\dim|D| \geq \dim \mathcal{F} = m \geq 2$. Furthermore we claim that, as D' varies in $\mathcal{F}$, Z describes a dense Zariski subset of D. Otherwise, since D is irreducible, Z would stay fixed. By blowing up Z we would then reduce to the case $Z = \emptyset$ which we excluded already. This implies that we can choose a general point P on X in such a way that it lies on $D \cap D'$. Hence D and D' are connected by the $(m-1)$-dimensional linear system $\mathcal{F}(-P)$ inside $\mathcal{F}$. This proves that $\mathcal{F}$ itself is a linear system.
Suppose now the general divisor $D \in \mathcal{F}$ is reducible and $\dim(\mathcal{F}) > 1$. Let P be a general point on X. Suppose $\mathcal{F}(-P)$ has no fixed divisor. By induction there is a pencil $f : X \to C$ and an involution $\mathcal{E}$ on C such that $\mathcal{F}(-P) = f^*\mathcal{E}$. Since all divisors in $\mathcal{F}(-P)$ contain P, then all fibres of f contain P. Let $d > 1$ be the degree of divisors in $\mathcal{E}$. Then a general divisor D in $\mathcal{F}(-P)$ would consist of d fibres of f, all passing through P. Hence D would be singular at P, against the generality of P and D. In conclusion $\mathcal{F}(-P)$ has a fixed divisor. This implies the claim. □

The study of varieties X covered by degenerate subvarieties is a classical subject of investigation and the mess of results obtained in this theory is so big that we do not even try to give a short account here.

The starting case is given of course by ruled varieties, and this yet measures the task of classifying such objects.

Classically, the main tool was introduced by C. Segre in [68]: the spaces spanned by the degenerate subvarieties describe a family of linear spaces whose **foci** of any order determine linear systems, useful to reconstruct a canonical image of X.

For surfaces, we have the theorem 10 of Severi, which indeed in its full strength reads:

THEOREM 12. *Surfaces covered by a 2-dimensional family of plane curves are contained in $\mathbb{P}^3$, except for the Veronese surface and its projections.*

Proof. It remains only to prove that the curves are conics, when $r > 3$. This follows from the classical 3-secant lemma: if through a general pair of points of X one finds a plane curve of degree bigger than 2, then the secant line through 2 general points of X meets X elsewhere. This is forbidden in characteristic 0: a general hyperplane section would be a curve in $\mathbb{P}^{r-1}$, $r - 1 > 2$, whose general secant line is (at least) trisecant. □

EXAMPLE 14. Assume that X is covered by a $(m+1)$-filling family of subvarieties of $\mathbb{P}^m$. Then X lies in $\mathbb{P}^{m+1}$.

We refer to [60] or [53] for an account of results on varieties which contain big families of linear spaces.

An account of the method of focal loci in the theory of degenerate subvarieties is given in [52]. Let us just mention the following result by C. Segre (see [70], [52]):

THEOREM 13. *Let X be a surface in $\mathbb{P}^5$ containing a 2-dimensional family of curves of $\mathbb{P}^3$. Then either:*
(a) X is contained in a 3-dimensional rational normal scroll of degree 3, or:
(b) X is contained in the cone over a Veronese surface; or:
(c) the curves in the family have degree ≤ 5.

5. The Theorem session

We have now enough methods to start with some results on effective varieties. We follow here the chronological output of results, starting with Severi's classical theorem on 1-defective surfaces, up to recent results on the classification of defective threefolds and products.

We use as our main tool, an extensive inspection of the properties of the contact loci $\Sigma(H)$. We know that they are degenerate subvarieties. They are also uniquely determined by the points $P_0, \ldots, P_k$. As these points move, the contact loci $\Sigma(H)$ describe a family of positive dimensional degenerate subvarieties of X.

PROPOSITION 11. *Assume $\nu_k(X) > 0$. Then the contact loci $\Sigma(H)$ determine a flat family $\mathcal{S}$ of subvarieties of X with the following property: for a general choice of the points $P_0, \ldots, P_k$, the set of elements in $\mathcal{S}$ passing through $P_0, \ldots, P_k$ is a projective space.*

Proof. Let V be the system of hyperplanes in $\mathbb{P}^r$. Then the family $\mathcal{S}$ is dominated by an open subset of the singular incidence variety

$$J = \{(P_0, \ldots, P_k, H) \in X^{k+1} \times V : H \in V(-2P_0 - \cdots - 2P_k)\}$$

and if $(P_0, \ldots, P_k, H)$, $(P_0, \ldots, P_k, H')$ have the same locus, then the same holds for any H'' general in the pencil determined by H, H'. □

THEOREM 14. *(**Severi**) Let $X \subset \mathbb{P}^r$ be a 1-defective surface. Then $r \geq 5$ and X is either a cone or the Veronese surface $V(2, 2)$.*

Proof. Notice that surfaces in $\mathbb{P}^4$ cannot be defective, by exercise 26. On the other hand, cones in $\mathbb{P}^5$ and $V(2, 2)$ are indeed 1-defective, by exercise 12 and section 4.2. So let $X \subset \mathbb{P}^r$, $r \geq 5$, be 1-defective. The defect δ_1 is forced to be 1: it is positive and smaller than 2, because it is smaller than the projection defect, which cannot be 2 by Corollary (4).

For a general hyperplane H which is tangent to X at two points, call $\Sigma(H)$ the contact variety. We know that $\Sigma(H)$ has positive dimension. It cannot coincide with X, so $\nu_1(X) = 1 = \delta_1$.
Formula 5 thus says that $\Sigma(H)$ imposes at most 3 conditions to the hyperplane system. It follows that $\Sigma(H)$ is either a line or a plane curve.
Remind that for a choice of two general points of X there exists an element in the family described by $\Sigma(H)$ which contains P_0, P_1. Since $\Sigma(H)$ is a plane curve, it has degree 2, by proposition 10.
We have two cases: either $\Sigma(H)$ is an irreducible conic, or it is a pair of incident lines, one for each point P_0, P_1.
Assume first that $\Sigma(H)$ is reducible. Then X is ruled. Furthermore if L_0, L_1 are the lines passing through two general points, they meet somewhere. Just as in the proof of lemma 1, this is possible if and only if the meeting point is fixed. So X is a cone.
Assume that $\Sigma(H)$ is irreducible. The classical theorem 10 of Severi proves that the only surface in $\mathbb{P}^5$ with a 2-dimensional family of generically irreducible conics is the Veronese surface $V(2, 2)$. The claim follows. □

A refinement of Severi's theorem, which works for k-defective surfaces, any k, was obtained by Terracini in [75]. Probably because it was published in a not–so–widely–distributed journal, most people was not aware of this result. Terracini's classification was rediscovered in recent times by Adlansvik and Dale ([1], [32]).

THEOREM 15. *Let X be a surface with $\delta_k > 0$. Then either:*
(i) the contact curve of a general $(k+1)$-tangent hyperplane is irreducible, and then $r = 3k + 2$, $\delta_k = 1$ and X is the 2-Veronese embedding of a rational normal surface Y of degree k in $\mathbb{P}^{k+1}$; or,
(ii) the contact curve of a general $(k+1)$-tangent hyperplane is reducible, and then X sits in a $(s+2)$-dimensional cone over a curve, with vertex a linear space of dimension $s \leq k-1$ and $r \geq 2k + s + 3$. The minimal such s is characterized by the property that X is s-defective but not $(s-1)$-defective and one has $\delta_k \geq k - s$.

Proof. The proof relies on Severi's result, but of course it requires more technicalities, mainly on the contact variety $\Sigma(H)$. Details are omitted here.
The starting point is the observation that, under our assumptions, a general m-th tangential projection, for some $m \leq k-1$, maps X either to a Veronese surface or to a cone. The former case happens when $\Sigma(H)$ is an irreducible curve, which thus maps, under a general m-th tangential projection, to a conic. So $\Sigma(H)$ is rational and determines a $(k+1)$-dimensional linear system. The corresponding map $X \to \mathbb{P}^{k+1}$ realizes X as a surface of degree k.
If a general m-th tangential projection maps X to a cone, then clearly X itself sits naturally in a cone over a curve. We just need some projective argument to minimize the dimension of the vertex. We refer the reader to the computations in the last section of [16]. □

EXERCISE 50. Prove that the surfaces of type (i) and (ii) in the previous theorem are indeed k-defective.

Turning to higher dimensional cases, of course everything becomes much more complicate. First of all, the contact variety $\Sigma(H)$ now needs not being a divisor: it may be a curve. Even worse: we have now two essentially different definitions for the contact locus.

The situation was classically explored and solved, in the case $k = 1$, by Scorza, who found in [63] the following classification:

THEOREM 16. *(***G. Scorza***, [63]) An irreducible, non-degenerate, projective* 3*-fold* $X \subset \mathbb{P}^r$ *is* 1*-defective if and only if* $r \geq 6$ *and* X *is of one of the following types:*
(i) X *is a cone;*
(ii) X *sits in a* 4*-dimensional cone over a curve;*
(iii) $r = 7$ *and* X *is contained in a* 4*-dimensional cone over the Veronese surface* $V(2, 2)$ *in* $\mathbb{P}^5$*;*
(iv) X *is the Veronese variety* $V(2, 3) \subset \mathbb{P}^9$ *of quadrics in* $\mathbb{P}^3$ *or a projection of it in* $\mathbb{P}^r$*,* $r = 7, 8$*;*
(v) $r = 7$ *and* X *is a hyperplane section of the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$ *in* $\mathbb{P}^8$.

Scorza's result is based on an accurate analysis of surfaces obtained from X in a tangential projection (see [18]).

Assuming that X is a 1-defective threefold, we know that a general tangential projection $\pi_k : X \to X_1$ has fibers of dimension 1 or 2. With respect to the singular defect, the following cases may occur:
(1) X_1 is a curve, π has fibers of dimension 2 and the singular defect is 2;
(2) X_1 is a surface, π has fibers of dimension 1 and the singular defect is 2;
(3) X_1 is a surface, π has fibers of dimension 1 and the singular defect is 1.

Case (2) happens when the surface X_1 has singular defect 1: for instance it may be defective itself, or simply it may be 1-weakly defective.

To produce a result for threefold which works for every k, we need thus a deeper analysis on surfaces whose general k-contact locus has positive dimension, i.e. k-weakly defective surfaces. A classification for these surfaces is the main result of [16]:

THEOREM 17. *Let* $X \subset \mathbb{P}^r$ *be a reduced, irreducible, non degenerate, projective surface which is* k*-weakly defective, but not* k*-defective.*
Then $k \geq 1$ *and either:*
(i) the contact curve of a general $(k + 1)$*-tangent hyperplane is irreducible, and then either* $k = 0$ *and* X *is the tangent developable to a curve, or;*
(ii) $r = 9$*,* $k = 2$ *and* X *is the* 2*-Veronese embedding of a surface of degree* $d \geq 3$ *in* $\mathbb{P}^3$*, or;*
(iii) $r = 3k + 3$ *and* X *sits in the cone with vertex a point over a* k*-defective surface, or;*
(iv) $r = 3k+3$ *and* X *is the* 2*-Veronese embedding in* $\mathbb{P}^r$ *of a surface* Y *of degree* $k+1$ *in* $\mathbb{P}^{k+1}$ *with curve sections of arithmetic genus* 1*, or;*

(v) the contact curve of a general $(k+1)$*-tangent hyperplane is reducible, and then* X *sits in a* $(s+2)$*-dimensional cone over a curve, with vertex a linear space of dimension* $s \leq k$ *and* $r \geq 2k+s+3$*. The minimal such* s *is characterized by the property that* X *is* s*-weakly defective but not* $(s-1)$*-defective.*

Assuming that X is a k-defective threefold, and taking the minimal k for which this happens, then we know that the $(k-1)$-st tangential projection X_{k-1} is a 1-defective threefold. Mixing the classification of Scorza with the previous classification of defective surfaces, one obtains:

THEOREM 18. *(**Chiantini-Ciliberto**, [19]) Let* $X \subset \mathbb{P}^r$ *be an irreducible, non–degenerate, projective, minimally k–defective threefold with* $k \geq 2$*. Then* X *is in the following list:*
(1) X *is contained in a cone over the* 2*-uple embedding of a threefold* Y *of minimal degree in* $\mathbb{P}^{k+1}$*, with vertex either a point (hence* $r = 4k+2$*) or a line (hence* $r = 4k+3$*);*
(2) $k = 3$ *and either* $r = 14 = 4k+2$ *and* X *is the* 2*-uple embedding of a hypersurface* Y *in* $\mathbb{P}^4$ *of* $\deg(Y) \geq 3$ *or* $r = 15 = 4k+3$ *and* X *is contained in the cone with vertex a point over the* 2*-uple embedding of a hypersurface* Y *as above;*
(3) either $r = 4k+2$ *and* X *is the* 2*-uple embedding of a threefold* Y *of degree* k *in* $\mathbb{P}^{k+1}$ *with curve sections of arithmetic genus* 1 *or* $r = 4k+3$ *and* X *is contained in the cone with vertex a point over the* 2*-uple embedding of a threefold* Y *as above ;*
(4) $r = 4k+3$ *and* X *is the* 2*-uple embedding of a threefold* Y *of degree* k *in* $\mathbb{P}^{k+1}$ *with curve sections of genus* 0 *which is either a cone with vertex a line over a smooth rational curve of degree* k *in* $\mathbb{P}^{k-1}$ *or it has a double line;*
(5) $k = 4$*,* $r = 4k+3 = 19$ *and* X *is the* 2*-uple embedding of a threefold* Y *in* $\mathbb{P}^5$ *with* $\deg(Y) \geq 5$*, contained in a quadric;*
(6) $r = 4k+3$ *and* X *is the* 2*-uple embedding of a threefold* Y *of degree* $k+1$ *in* $\mathbb{P}^{k+1}$ *with curve sections of arithmetic genus 2;*
(7) $r = 4k+3$ *and* X *is contained in a cone with vertex a space of dimension* k *over the* 2*–uple embedding of a surface* Y *of minimal degree in* $\mathbb{P}^{k+1}$*;*
(8) $k = 2$*,* $r = 4k+3 = 11$ *and* X *sits in a cone with vertex a line over the* 2*–uple embedding of a surface* Y *of* $\mathbb{P}^3$ *with* $\deg(Y) \geq 3$*;*
(9) $r = 4k+3$ *and* X *sits in a cone with vertex of dimension* $k-1$ *over the* 2*–uple embedding of a surface* Y *of degree* $k+1$ *in* $\mathbb{P}^{k+1}$ *with curve sections of arithmetic genus* 1*;*
(10) X *is contained in a cone with vertex of dimension* $k-1$ *over a surface which is not k–weakly defective;*
(11) X *is contained in a cone with vertex of dimension* $2k$ *over a curve;*
(12) $k = 2$*,* $r = 4k+2 = 10$*, and* X *is contained in a cone with vertex of dimension* 2 *over a curve; (13)* $r = 4k+5$ *and* X *is the* 2*-uple embedding of a threefold of minimal degree in* $\mathbb{P}^{k+2}$*;*
(14) $r = 4k+4$ *and* X *is the projection of the* 2*-uple embedding* $Y' \subset \mathbb{P}^{4k+5}$ *of a threefold* Y *of minimal degree in* $\mathbb{P}^{k+2}$ *from a point* $P \in \mathbb{P}^{4k+5}$*;*
(15) $r = 4k+3$ *and either* X *is the projection of the* 2*-uple embedding* $Y' \subset \mathbb{P}^{4k+5}$ *of a*

threefold Y of minimal degree in $\mathbb{P}^{k+2}$ from a line $\ell \subset \mathbb{P}^{4k+5}$, or X is contained in the intersection of a space of dimension $4k+3$ with the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ in $\mathbb{P}^{k^2+4k+3}$.

Cases (1)–(9) correspond to the case in which the tangential contact locus is an irreducible divisor, cases (10)–(12) correspond to the case in which the tangential contact locus is reducible, cases (13)–(15) correspond to the case the tangential contact locus is an irreducible curve.

All threefolds in this list are actually k–defective.

One should notice that all the listed types refer to few constructions: mainly varieties defined by linear systems of quadrics, starting with very special varieties (typically of low degree); or varieties contained in cones with small vertex, or varieties contained in some product.

Going to the next step (fourfolds), very few results are known. We have no advances since a result of Scorza ([65]) were a partial classification of 1-defective fourfolds is shown.

THEOREM 19. *(**G. Scorza**) Let X be a smooth, non-degenerate, projective 4-fold $X \subset \mathbb{P}^r$ which is 1-defective with defect $\delta_1 = 1$. Then $r \geq 6$ and:*
(i) if X has singular defect 3, then it lies in a 5-dimensional cone over a curve or in a 6-dimensional cone over a Veronese surface;
(ii) if X has singular defect 1 then it is the projection of some hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^3$.

No particular characterization has been found for defective 4-folds with $\nu_1 = 2$.

For specific varieties, as Grassmannians or Segre products, we have wider results on the dimension of the secant varieties, as we yet observed in section 3.5.

Finally let me skip to the other end of our theory.

Assuming that X is a defective variety of large dimension, can we limit the defect?

EXAMPLE 15. We yet observed that the defect of any n-dimensional variety is at most $n-1$. On the other hand, cones over curve have defect exactly equal to $n-1$. So one cannot hope to improve the limit on the defect, in the category of irreducible varieties.

Imposing new conditions on X, however, it turns out that the defect is bounded. The main stream explored so far relates the maximal defect with the dimension of the singular locus of X. It is always a surprise to find out that the local structure of X reflects the geometry of the embedding: the absence of singularities forces tangent spaces to be not–too–wildly glued together. This is the kernel of the celebrated Zak's result on linear normality:

THEOREM 20. *(**Zak**, [82]) Assume that X is smooth. Then*

$$\dim(S_1(X)) \geq \frac{3}{2}n + 1 \qquad i.e. \quad \delta_1(X) \leq \frac{n}{2}.$$

EXERCISE 51. Prove that Zak's results imply the following statement:
Any smooth variety of dimension n in $\mathbb{P}^r$, $r \leq (3/2)n + 1$, is linearly normal.

This theorem which bounds the defect is indeed a consequence of Fulton–Hansen connectedness principle:

THEOREM 21. *(**Fulton-Hansen**, see [41] or [42]) If $Z \to \mathbb{P}^m \times \mathbb{P}^m$ is a morphism whose image has dimension at least m, then the inverse image of the diagonal is connected.*

Let us see how the connectedness theorem implies the bound on the defect.

Proof. (proof of Zak's theorem) Assume that the first defect of X is bigger than $3n/2$. Then taking a general tangential projection from a point $P \in X$, we get a variety X_1 of dimension $\dim(X_1) < n/2$. In other words, the projection $X \to X_1$ has fibers of dimension bigger than $n/2$. It follows that for $Q \in X$ general, the space $T = T_{P,Q}(X)$, which maps to the tangent space to X_1 at the image of Q, is tangent along a fiber Y over Q. Write t for the dimension of T. We have, by assumptions, $t \leq 3n/2$. Fix a general linear space L of dimension $r - t - 1$, such that $L \cap T = \emptyset$. The projection $X \to \mathbb{P}^t$ from L is finite. Consider now the associated map $Z = X \times Y \to \mathbb{P}^t \times \mathbb{P}^t$. It is finite, so its image has dimension at least $\dim(Z) \geq n + \dim(Y)$. Since $\dim(Y) > n/2$, we may apply the connectedness theorem. It turns out that the inverse image of the diagonal is connected. This inverse image contains any pair (y, y), $y \in Y$. Assume it contains also a pair (y, x), $y \neq x$. This means that y, x have the same image in $\mathbb{P}^t$, hence the line $< x, y >$ meets L. We can move y to the general point P of Y and x to y, so that the line $< x, y >$ tends to a tangent line to X at P, a contradiction since $T_P(X)$ does not meet L. Hence no line joining points of Y to points of $X - Y$ can meet L. It follows that all these lines lie in T, whence X itself lies in T: a clear contradiction. □

In fact Zak's result is much more precise: it says that:

THEOREM 22. *If the singular locus of X has dimension b, then for any subvariety $Y \subset X$ of dimension m, if a linear space T is tangent to X at every point of $Y - X$, then*

$$\dim(L) \geq m + n - b - 1.$$

EXERCISE 52. Use the previous result and the procedure above to prove Zak's theorem 8 on the image of the Gauss map.

Zak's results also determine a classification of smooth varieties for which the maximal first defect $\delta_1(X) = n/2$ occurs. They are called **Severi varieties** in honor of Severi's result on the classification of defective surfaces.

Even a sketch of the classification arguments are beyond the scopes of these notes and the reader is referred to Zak's book [82]. The starting point, however, easily derives from the previous analysis of the contact locus:

EXERCISE 53. The contact locus Σ of a Severi variety is a hypersurface of degree at most 2.
Indeed we know from theorem 6 that Σ imposes at most 2+(n/2) conditions to hyperplanes, so it lies in a projective space of dimension at most $1 + (n/2)$. As it has dimension at least $n/2$, it is a hypersurface. Through any pair of points $P, Q \in X$ there is a contact variety Σ. If $\deg(\Sigma) > 2$, then it meets the general secant line $< P, Q >$ in a third point, contradicting the trisecant lemma.

Indeed one shows that since X is smooth, then $\deg(\Sigma) = 2$.

THEOREM 23. *The Severi varieties are:*
(1) the Veronese surfaces in $\mathbb{P}^5$ *(dimension* 2*);*
(2) the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ *in* $\mathbb{P}^8$ *(dimension* 4*);*
(3) the Plücker embedding of the Grassmannian $G(1, 5)$ *in* $\mathbb{P}^{14}$ *(dimension* 8*);*
(4) a Spinor variety of dimension 16 *in* $\mathbb{P}^{26}$.

EXERCISE 54. Show that the varieties of the first three types in the previous classification are indeed Severi varieties. (Also varieties of the fourth type are Severi varieties, but proving this fact is *not* just an exercise!).

Zak's classification also suggests that varieties of given dimension and defect cannot be realized in arbitrarily large projective space. Indeed in the last chapter of its book [82], he finds:

THEOREM 24. *Let* X *be a smooth, non-degenerate variety of dimension* n *and defect* $\delta_1 > 0$ *in* $\mathbb{P}^r$. *Then*

$$r \leq \frac{n(n + \delta_1 + 2) + e(\delta_1 - e - 2)}{2\delta_1}$$

where e *is the remainder of* $n : \delta_1$.
The maximum is attained only for:
(1) the 2*-Veronese embedding of* $\mathbb{P}^n$ *(*$\delta_1 = 1$*);*
(2) the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^m$ *or* $\mathbb{P}^m \times \mathbb{P}^{m+1}$ *(*$\delta_1 = 2$*);*
(3) the Plücker embedding of the Grassmannian $G(1, 1 + (n/2))$ *(*$\delta_1 = 4$*);*
(4) the Severi variety of dimension 16.

Observe that the structure of these examples are always the same. In fact one obtains an infinite hyerarchy of examples, except for the Spinor variety, which stops with dimension 16.

EXERCISE 55. The maximal defect $\delta_2(X)$ of a smooth variety X satisfies $\delta_2(X) \leq (3n/2) - 1$.

Indeed we know that two general tangent spaces to X span at least a space of dimension $1 + (3n/2)$, moreover the image of the second tangential map $X \to X_2$ is at least a curve. It turns out that the tangent spaces at three general points of X span a space of dimension at least $3 + (3n/2)$.

EXAMPLE 16. Catalano Johnson ([10]) found examples of rational scrolls on curves which are smooth, but whose image in a general tangential projection is a cone. Thus they reach the bound $\delta_2(X) = 3 + (3n/2)$.
An easy example of this type is the scroll $X = S(2, q)$ of dimension 2, obtained by joining corresponding points of a conic and a normal rational curve of degree $q \gg 0$. The plane spanned by the conic is contracted to a point under a general tangential projection, which thus sends X to a cone over a curve.

Using rational scrolls, Catalano Johnson in fact proves that essetially all defects can be obtained, in the obvious range.

6. Under construction

Let me list, in this final section, whose boundary is supposed to move (fast?) onward, some open problem and some related topic where the theory has its natural developments.

6.1. The main stream

As observed above, our theory stops with the classification of defective threefolds, even for the first defect. Scorza's result on fourfolds is by no means complete, for varieties with singular defect 2.

Indeed one may hope to find general results when the singular defect is maximal or minimal. For very big defects, one gets:

THEOREM 25. *Assume that the first tangential projection sends X to a curve X_1. Then X is either a cone over a curve or a cone over a Veronese surface in $\mathbb{P}^5$.*

More general results may be obtained with the procedure used by Scorza in his investigation of defective fourfolds:

EXERCISE 56. Find a classification of varieties X which are 1-defec- tive, with $\delta_1(X) = 1$ and $\nu_1(X) = n - 1$ (the divisorial case).

EXERCISE 57. Find a classification of varieties X which are 1-defec- tive, with $\delta_1(X) = 1$ and $\nu_1(X) = 1$ (the curvilinear case).

In this kind of results, one has to work with the projective extensions of (usually rational) varieties in some class, in order to perform induction on the dimension of X.

A flavour of these results is given by statements of the following type:

PROPOSITION 12. *(see [6]) Let $X \in \mathbb{P}^r$ be a reduced, irreducible, non–degenerate, 1-defective variety of dimension n which is a developable scroll. Then X is a cone over a developable scroll.*

Going back to Scorza's analysis of defective 4-folds, one may observe that he considers only the case $\delta_1(X) = 1$. One clearly has:

EXERCISE 58. If $\delta_1(X) > 1$, then a general hyperplane section of X is 1-defective.

Still, even applying induction, it is not clear which varieties of dimension 4 have a general hyperplane section fitting in the list of defective threefolds.

This is indeed a piece of a well–known, difficult problem of determining whether or not a given variety is the hyperplane section of a variety of higher dimension (except for cones). The extension problem has been studied by several classical and modern mathematicians, and there is no evidence that in its specific applications to defective varieties is simpler with respect to the general theory.

As soon as one classifies k-defective varieties for a fixed k, one has a guess for determining $(k + 1)$-defective varieties. These are varieties whose general tangential projection fits in the previous list. Unfortunately there is no simple way to see directly when some variety is the (even birational!) projection of something living in some higher dimensional projective space. This a piece of the so–called **geometric linear normality** problem, which is not completely understood, even for (singular) complete intersection curves.

Let me point out something missing in Scorza's analysis of defective fourfolds.

If $\delta_1(X) = 1$ and the singular defect is 2, then a general tangential projection sends X to a threefold X_1 with the following property: a general tangent hyperplane to X at one point is in fact tangent to X along a curve. These are 0-weakly defective threefolds. So in order to classify defective fourfolds, the situation of weakly defective threefold must be understood. We do not have, up to now, a classification of k-weakly defective threefolds. This seems one of the main task necessary to extend our knowledge of defective varieties of dimension 4.

Let me stress again, however, that many particular results are known when one looks at some specific class of projective varieties, even in dimension bigger than 4. In particular, many cases of Grassmannians and Segre products are understood, while the classification of defective Veronese varieties is complete.

6.2. Grassmann defective varieties

Going back to the classical point of view, one may generalize the study of secant varieties in several ways.

For instance: instead of asking the reconstruction of points P in the ambient space as a linear combination of $k+1$ points of X, one may try to obtain any *pair* of points $P, Q \in \mathbb{P}^r$ with linear combinations of the *same* $k+1$ points of X. In fact, this is equivalent to ask that the line joining two general points $P, Q \in \mathbb{P}^r$ lies in some $(k+1)$-secant k-space. Of course this makes sense when $S_k(X) = \mathbb{P}^r$.

Let us formalize the consequent theory:

DEFINITION 13. *For non–negative integers $h \leq k$ define the k-***Grassmann** (h,k)**-secant variety** *$G_{h,k}(X)$ of X to be the (reduced) closure of the set:*
$\{L \in G(h,r) :$ *L lies in the span of $k+1$ independent points of X*$\}$
inside the Grassmannian $G(h,r)$ of h-planes in $\mathbb{P}^r$ (remind: projective dimensions).

We have an obvious diagram:

$$G_{h,k}(X) \leftarrow \mathcal{G} \rightarrow G_k(X)$$

where $G_k(X)$ is the k-th Grassmann secant variety of X defined in §2.1 and $\mathcal{G}$ indicates the universal Grassmannian of h-spaces inside the elements of $G_k(X) \subset G(k,r)$.

The expected dimension of the Grassmann secant variety $G_{h,k}(X)$ comes from the case where the leftmost map in the previous diagram is (generically) finite.

EXERCISE 59. Prove that the expected dimension of $G_{h,k}(X)$ is

$$\min\{(k+1)(r-k), (k+1)n + (h+1)(k-h)\}$$

and it is equal to the effective dimension if and only if a general h-space which lies in some $(k+1)$-secant k-space is in fact contained only in a finite number of such spaces.

So one says, as usual, that X is **Grassmann** (h,k)**-defective** as soon as $G_{h,k}(X)$ has not the expected dimension.

The dimension of $G_{h,k}(X)$ has something to do with the projection of X from a general set of $h+1$ points (i.e. from a general h-space). Namely it concerns the "exceptional" secant spaces or the singularities which may arise in $h+1$ successive general projections of X. Indeed one has:

REMARK 6. Fix a general h-space L and assume that L does not intersects X (i.e. assume that $n+h < r$). Call X_L the projection of X from L. Then X_L acquires a new $(k+1)$-secant $(k-h-1)$-space if $L \in G_{h,k}(X)$.
In particular for $k = h+1$, X acquires a new $(k+1)$-fold point when $L \in G_{h,k}(X)$.

The reader should be advised that, unfortunately, projecting from some $L \in G_{k-1,k}(X)$ is *not* likely to be the unique way in which the projection of X may acquire a new $(k+1)$-fold point P. Indeed in principle such points may arise projecting from some highly tangent space L which is *not* a limit of $(k+1)$-secant spaces.
The problem is not definitely settled. However there is a construction by Flenner which seems to obtain multiple points of the previous type in the projection of smooth, complete intersection varieties of high dimension. We refer the interested reader to the discussion in [15] and its bibliography.

In the Waring setting of example 1, the problem of finding when $G_{h,k}(X)$ coincides with the Grassmannian $G(k,r)$ for some Veronese embedding X of $\mathbb{P}^n$ is equivalent to ask for a "simultaneous" decomposition of $h+1$ general forms as linear combination of the *same* $k+1$ powers of linear forms.
In this formulation, the problem was classically considered by Terracini (see [76]) and Bronowski (see [7]). In particular Terracini found:

THEOREM 26. *(**Terracini**) If $X = V(2,3)$ is the 3-Veronese embedding of the plane $\mathbb{P}^2$ in $\mathbb{P}^9$, then X is Grassmann $(1,4)$-defective. No other Veronese embedding of $\mathbb{P}^2$ is Grassmann defective.*

Terracini's method is based on the following lemma, which reduces Grassmann defectivity to the "usual" defectivity of some product of X (see [34] for a modern proof):

LEMMA 3. *X is Grassmann (h,k)-defective if and only if $\mathbb{P}^h \times X$ is k-defective (in its Segre embedding).*

EXERCISE 60. Assume that X is Grassmann (h,k)-defective. Then prove that either X is also k-defective or $r < nk + n + k$.

The status of the theory can be actually resumed as follows:

There are no (h,k)-defective curves (see [17]).

For surfaces. A classification of smooth $(1,2)$-defective surface was achiev- ed in [22]. Fontanari gave a criterion for detecting the Grassmann $(1,k)$-defectivity of surfaces (see [40]). A complete classification of $(1,k)$-defective surfaces has been obtained in [20], using the classification of defective threefolds and lemma 3:

THEOREM 27. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective surface which is minimally $(1,k)$-defective. Then k is even and X is in the following list:*

(1) X is contained in a cone with vertex of dimension $\frac{k}{2} - 1$ and not smaller over a curve C with $\dim(< C >) \geq \frac{3}{2}k + 1$;

(2) $k = 4$ and X is the 3–Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$;

(3) $X \subset \mathbb{P}^{2k+1}$ is a rational normal scroll $S(a_1, a_2)$ with $a_1 \geq \frac{k}{2}$.

All surfaces in the list are actually minimally $(1,k)$–defective. Surfaces of type (1) can be $(1,h)$–defective for $h > k$, whereas surfaces of types (2) and (3) are not $(1,h)$–defective for $h > k$.

For higher dimensional varieties, very few things are known. Of course, for some Veronese embeddings X of $\mathbb{P}^n$, as soon as we have a decription of the defectivity of some Segre product $\mathbb{P}^h \times \mathbb{P}^n$, then by lemma 3 we have information on the (h,k)-defectivity of X. Refer to [13] for details.

Let me cite the preprint of Coppens [31], where smooth Grassmann $(2,3)$-defective threefolds are classified:

THEOREM 28. *A smooth Grassmann* $(2,3)$*-defective threefold* $X \subset \mathbb{P}^r$ *is either:*
(1) a threefold of minimal degree in $\mathbb{P}^7$*;*
(2) a threefold of minimal degree in $\mathbb{P}^8$*;*
(3) a projection in $\mathbb{P}^7$ *of the previous threefold.*

Finally observe that, in the same setting, following some suggestions arisen from number theory, Voisin studied the general problem of determining linear spaces contained in some secant variety (see [79] and [73]). The main result is:

THEOREM 29. *Let* X *be a smooth curve of genus* g *and degree* d *and assume* $d \geq 2g + 2k + 1$*. Then if* L *is a linear space of dimension* $\geq k$ *contained in* $S_k(X)$*, we have* $\dim(L) = k$ *and* L *is a space spanned by* $k + 1$ *independent points of* X *or a limit of such spaces.*

6.3. The number of apparent secant spaces

If P is a general point of the secant variety $S_k(X)$, then one may ask "how many" $(k + 1)$-secant k-spaces pass through P.
Clearly we have infinitely many such spaces as soon as $r < nk+n+k$. If $r \geq nk+n+k$, then as observed in the first section (see exercise 6) a general $P \in S_k(X)$ lies in finitely many secant $(k + 1)$-secant k-spaces, unless X is k-defective.

DEFINITION 14. *Assume* $r \geq nk+n+k$ *and assume that* X *is not* k*-defective. The (finite) number of* $(k + 1)$*-secant* k*-spaces passing through a general point* $P \in S_k(X)$ *is called the* **number of apparent k-secant spaces** *to* X *and is indicated with* $App_k(X)$

When $r = nk + n + k$, that is when $S_k(X)$ is $\mathbb{P}^r$ (remind: we assume here that X is not k-defective), then $App_k(X)$ is a powerful invariant of the projective embedding of X.
For instance, in the case $n = k = 1$, i.e. looking at secant lines to curves in $\mathbb{P}^3$, then $App_1(X)$ is exactly the number of nodes in a general projection of X to $\mathbb{P}^2$. Hence it relates the degree (external invariant) with the genus (internal invariant). Furthermore Halphen's theory shows that this number is strictly influenced by the postulation of X.
For $k = 2$, $App_2(X)$ measures the number of "apparent trisecant lines" in a general projection of an n-fold X to $\mathbb{P}^{3n+1}$. And so on.
These invariants seem to be not yet completely understood for $k > 1$, and their relation with the geometry of X is quite unexplored. Just notice that even for surfaces $X \subset \mathbb{P}^5$, the relations between the number of apparent double points and the postulation of X are almost completely obscure.

Remaining in the case $r = nk + n + k$, notice that k-defective varieties are just varieties such that one has 0 $(k + 1)$-secant k-spaces through a general point of $\mathbb{P}^r$. So varieties for which $App_k(X) = 1$ are eventually a generalization of defective varieties. Thus there is some hope that some methods introduced in the previous chapters lead to a classification of such varieties.

Let me briefly recall the actual situation.

EXERCISE 61. Prove that the only irreducible curve $X \subset \mathbb{P}^3$ with $App_1(X) = 1$ is the rational normal curve.

EXERCISE 62. Prove that if $X \subset \mathbb{P}^3$ is a reducible, smooth curve with $App_1(X) = 1$, then X is a disjoint union of two lines.

THEOREM 30. *Put $k = 1$, $r = 2n + 1$ and assume that X is a smooth, irreducible variety of dimension n, which is not 1-defective (hence $S_1(X) = \mathbb{P}^r$). Assume $App_1(X) = 1$.*
In the case of curves ($n = 1$) then X is a rational normal cubic in $\mathbb{P}^3$ (classical).
If $n = 2$ (surfaces in $\mathbb{P}^5$), then X is either a quartic normal scroll or a Del Pezzo quintic (Russo, [61]).
Smooth threefolds in $\mathbb{P}^7$ with $App_1(X) = 1$ are either $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or the residual intersection of $\mathbb{P}^1 \times \mathbb{P}^3$ and a quadric, with respect to one 3-dimensional ruling or scrolls in planes. (refer to [27] for details).

When k is bigger, then few things are known for dimension > 2. We have for curves the following result by Ciliberto and Russo ([28], but see also [11] for the smooth case):

THEOREM 31. *Assume that X is an irreducible curve in $\mathbb{P}^{2k+1}$ with $App_1(X) = 1$. Then X is a rational normal curve.*

While for surfaces:

THEOREM 32. *(**Ciliberto - Russo**, see [28] for details) Let $X \subset \mathbb{P}^{3k+2}$, $k \geq 2$ be an irreducible surface which is linearly normal and satisfies $App_k(X) = 1$. Then X is one of the following:*
(1) A rational normal scroll with sectional genus 0.
(2) A Castelnuovo surface of odd degree (Castelnuovo surfaces are surfaces of minimal sectional genus with respect to r) or an internal projection of such surfaces from three points of X.
(3) The 5-Veronese embedding $V(2, 5)$ of $\mathbb{P}^2$ or its general tangential projection from 0, 1, 2, 3 points.

The main tools for the previous results rely in the observation that the number $App_k(X)$ is bounded by, and sometimes it is equal to, the degree of a general tangential projection of X. This can be proved using some degeneration argument, but we are not going through the details here.

When $r > nk+n+k$ (and X is not k-defective) the situation becomes easier. Indeed as a consequence of the infinitesimal Bertini's theorem 5 for non–defective varieties X, we get a description of the intersection of X with a general $(k+1)$-tangent space:

EXERCISE 63. Let $X \subset \mathbb{P}^r$ be an irreducible, reduced, non-degenerate projective variety of dimension n and let k be a non-negative integer such that $k < r - n$. Let $P_0, \dots, P_k$ be general points of X. Then the schematic intersection of X with the subspace $< P_0, \dots, P_k >$ is the union of the points $P_0, \dots, P_k$.
By taking the section of X with a general subspace of codimension $n-1$, it suffices to prove the assertion only for curves. Then, by taking the projection of the curve X from a general point $P \in X$, it suffices to prove the assertion for $k = 1$, in which case it is a consequence of the trisecant lemma.

THEOREM 33. *Let $X \subset \mathbb{P}^r$ be a reduced, irreducible, non degenerate, projective variety. Then the general point of every irreducible component of the contact variety of a general $(k+1)$-tangent hyperplane H is a double point for H. If, in addition, X is not k-weakly defective for a given k such that $r \geq (n+1)(k+1)$, then given $P_0, \dots, P_k$ general points on X, the general $(k+1)$-tangent hyperplane $H \in |H(-2P_0 - \cdots - 2P_k)|$ is tangent to X only at $P_0, \dots, P_k$.*

Proof. We may assume that X is smooth, by passing, if necessary, to a resolution of the singularities.
Use the notation of theorem 5. Consider the variety $Y \subset |H|$ of hyperplanes which are tangent to X at $k+1$ general points $P_0, \dots, P_k$. Since X is not k-weakly defective, a general hyperplane $H \in Y$ has isolated singularities at $P_0, \dots, P_k$. Then the infinitesimal Bertini theorem tells us that the tangent space to Y at H is contained in $|H(-P_0 - \cdots - P_k)|$. Since the two spaces have the same dimension, they coincide. On the other hand $|H(-P_0 - \cdots - P_k)|$ is cut out on X by the hyperplanes through $P_0, \dots, P_k >$. Then exercise 63 and the infinitesimal Bertini theorem again forbid the presence of singularities for H other than $P_0, \dots, P_k$. $\square$

EXERCISE 64. Prove that indeed for a general hyperplane H which is tangent to X at general points $P_0, \dots, P_k$, the intersection $H \cap X$ has ordinary double points at the P_i's.

As a consequence we find:

THEOREM 34. *Assume that $X \subset \mathbb{P}^r$, $r > nk + n + k$, is not k-defective. Then $App_k(X) = 1$, unless X is k-weakly defective.
Conversely assume that X is k-weakly defective, but not k-defective. Call Σ a general k-contact locus of X. If Σ is irreducible, then $App_k(X) = App_k(\Sigma)$. If Σ is reducible, $\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_k$, then for general points $P_0 \in \Sigma_0, \dots, P_k \in \Sigma_k$ and for A general in the span $< P_0, \dots, P_k >$, there are exactly $App_k(X)$ $(k+1)$-tuples of points $\{P_{0i}, \dots, P_{ki}\}$ with $P_{ij} \in \Sigma_j$ and $A \in < P_{0i}, \dots, P_{ki} >$.*

Proof. Take $A \in S_k(X)$ general and assume that $A \in < P_0, \dots, P_k >$ and $A \in < Q_0, \dots, Q_k >$ with $P_i, Q_j \in X$ and $\{P_0, \dots, P_k\} \neq \{Q_0, \dots, Q_k\}$. Since A is general, then the tangent space of $S_k(X)$ at A coincides both with the span $T_{P_0,\dots,P_k}$ and

with $T_{Q_0,\dots,Q_k}$. Thus these spans are equal.
It follows that all the hyperplanes of $\mathbb{P}^r$ which are tangent to X at $P_0,\dots,P_k$ (they exist by the assumption $r > nk+n+k$), also are tangent to X at $Q_0,\dots,Q_k$. Since the two sets of points are different, this contradicts theorem 33.
To see the converse, assume Σ irreducible (the reducible case is left to the reader as an exercise).
For a general choice of $P_0,\dots,P_k \in X$, by our assumption, there exists a hyperplane H which is tangent to X at the P_i's. Call Σ its contact locus.
Then the inequality $App_k(\Sigma) \le App_k(X)$ follows from the generality of $P_0,\dots,P_k$.
To see the inverse inequality, let Σ is a general (irreducible) k-contact locus and fix a general point $A \in S_k(\Sigma)$. Then A is also a general point of $S_k(X)$, so there are exactly $m = App_k(X)$ choices of $(k+1)$-tuples of points $\{P_{01},\dots,P_{0k}\},\dots,\{P_{m1},\dots,P_{mk}\}$ such that $A \in < P_{0i},\dots,P_{ki} >$. It follows:

$$T_{P_{01},\dots,P_{0k}} = \cdots = T_{P_{m1},\dots,P_{mk}} = T_{S_k(X),A}$$

thus any hyperplane tangent to X at the points of one of these $(k+1)$-tuples, is also tangent at all P_{ij}'s. Hence our original contact variety Σ, which is completely determined by one of these $(k+1)$-tuples and one hyperplane tangent to it, also contains all P_{ij}'s. The inequality $App_k(\Sigma) \ge m$ follows. □

So in particular we know that varieties which are not k-weakly defective in $\mathbb{P}^r$, $r > nk+n+k$, must have $App_k(X) = 1$. This is a bit surprising, for the case $App_k(X) = 1$ is exceptional, when $r = nk+n+k$.
Notice that the converse is false:

EXAMPLE 17. There are examples of k-weakly defective varieties, with $App_k(\Sigma) = 1$, thus satisfying $App_k(X) = 1$.
One of them is obtained as follows: consider a cone $W \subset \mathbb{P}^6$ over a Veronese surface S. Let $Z \subset W$ be the cone over a conic of S. Call X the residual intersection of W with a quadric passing through Z. Then X is 1-weakly defective (by theorem 17) and one immediately sees that Σ is a rational normal cubic. Thus $App_1(X) = 1$.

A list of surfaces in $\mathbb{P}^r$, $r > 5$ with $App_k(X) > 1$ was obtained by Dale ([32]) when $k = 1$. For general k the classification has been obtained in [21], by mixing the previous result with the classification theorem 17 of weakly defective surfaces:

THEOREM 35. *Let X be an irreducible surface in* $\mathbb{P}^r$, $r > 3k+2$. *Then* $App_k(X) > 1$ *if and only if X is k-weakly defective, with the following exceptions:*
(1) X sits in the cone over a k-defective surface X', *2-uple embedding of a minimal surface* $Y \subset \mathbb{P}^{k+1}$, *and* $X \simeq 2h + f - 2e$ *in the Picard group of the desingularization of the cone, where h is the transform of a hyperplane section of Y, f is the transforms of a fiber of Y and e is the exceptional divisor.*
(2) There is an irreducible curve E and a non costant map $\phi : E \to \mathbb{P}^k$, *whose image spans* $\mathbb{P}^k$, *such that X is the ruled surface*

$$X = \cup_{P\in E} < P, \phi(P) > .$$

In higher dimensions, there is a recent result by Mella which computes $App_k(X)$ for some Veronese embeddings of $\mathbb{P}^n$:

THEOREM 36. *(see [51]) Let $X = V(n, d)$, $d > n$ be the Veronese embedding of degree d of $\mathbb{P}^n$. Then $App_d(X) = 1$ if and only if $n = 2$ and $d = 5$.*

EXERCISE 65. Define and set the first properties of $App_{h,k}(X)$, the number of apparent $(k+1)$-secant k-spaces through a general h-space in $\mathbb{P}^r$.

Many other variation on the theme of secant varieties (as secant varieties of scrolls, see [9] or the behaviour of osculating spaces, see [77], or the behaviour of successive defects, see [37]) are not listed here. Also I do not go further in the possible generalization of the definition of defective objects.

Let me just finish with two remarks, which link the end of these notes with the initial problem.

REMARK 7. Even if in some applications it is relevant to know the general geometric properties of the secant varieties to some X, nevertheless one often faces the "membership problem": for a given projective varieties X and a given point P, determine the minimum k such that $P \in S_k(X)$, i.e. the minimum such that P is linearly generated by $k+1$ points of X.
In practice, we would like to know the equations for $S_k(X)$. This is not an easy task. It can be solved for rational normal curves. But in general, even the degree of secant varieties is far from being easily calculated.
We refer to [11] and [59] for results on curves, to [49] and [78] for the case of Veronese re-embedding of some varieties.

REMARK 8. Assume we positively know that some P is a general point of $S_k(X)$. Then we know that there is a linear compbination $\sum_{i=0}^{k} a_i P_i$, $P_i \in X$, which gives P. How can one find the points P_i's starting with P?
This is relevant, for instance, when X is the variety of decomposable tensors of some sort and we know that $S_k(X)$ fills the entire space of tensors $\mathbb{P}^r$. An algorithm which produces a decomposition of a tensors in elementary products would be of valuable help in many computations.
The problem would become easier as soon as $App_k(X) = 1$, for in this case the decomposition is uniquely determined by the (general) point $P \in S_k(X)$. For instance, we know that a general form of degree 5 in 3 variable can be decomposed in the sum of 5 powers of linear forms in a unique way. Can someone compute such a decomposition? As far as I know, only very partial classical results by Sylvester are known in this setting.
When a general $P \in S_k(X)$ lies in infinitely many $(k+1)$-secant k-spaces, an initial study for the variety $D_k(P) = \{(P_0, \dots, P_k) \in X^{k+1} : P \in< P_0, \dots, P_k >\}$ (a sort of "Moduli space" for the decompositions of P) can be found in [24] and [9].

References

[1] ADLANSVIK B., *Joins and higher secant varieties*, Math. Scand. **61** (1987), 213–222.

[2] AKIVIS M., GOLDBERG V. AND LANDSBERG J., *Varieties with degenerate Gauss mapping*, math.AG/9908079, preprint (1999).

[3] ALEXANDER J. AND HIRSCHOWITZ A., *Polynomial interpolation in several variables*, J. Alg. Geom. **4** (1995), 201–222.

[4] ARBARELLO E., CORNALBA M., GRIFFITHS P. AND HARRIS J., *Geometry of algebraic curves*, Grundl. der Math. **267**, Springer-Verlag, Berlin 1985.

[5] ARBARELLO E. AND CORNALBA M., *Footnotes to a paper of B. Segre*, Math. Ann. **256** (1981), 341–362.

[6] BALLICO E. AND CILIBERTO C., *On gaussian maps for projective varieties*, in: "Geometry of Complex Varieties, Cetraro, June 1990", Mediterranean Press, Rende 1993, 35–54.

[7] BRONOWSKI J., *The sums of powers as simultaneous canonical expressions*, Proc. Camb. Phyl. Soc. **8** (1933), 465–469.

[8] BRONOWSKI J., *Surfaces whose prime sections are hyperelliptic*, J. London Math. Soc. **8** (1933), 308–312.

[9] CARLINI E., *Geometric aspects of some polynomial decompositions*, Ph.D. Thesis, Univ. Pavia, Pavia 2003.

[10] CATALANO-JOHNSON M., *The possible dimensions of the higher secant varieties*, Amer. J. Math. **118** (1996), 355–361.

[11] CATALANO-JOHNSON M., *The homogeneous ideal of higher secant varieties*, J. Pure Appl. Alg. **158** (2001), 123–129.

[12] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Rank of skew–symmetric tensors and secant varieties of Grassmannians*, preprint (2002).

[13] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Rank of tensors, secant varieties of Segre varieties and fat points*, Lin. Alg. and Applic. **355** (2002), 263–285.

[14] CATALISANO M.V., GERAMITA A.V. AND GIMIGLIANO A., *Higher secant varieties of the Segre variety* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, preprint (2003).

[15] CHIANTINI L., CHIARLI N. AND GRECO S., *Bounding Castelnuovo-Mumford regularity for varieties with good general projection*, J. Pure Appl. Alg. **52** (2000), 65–74.

[16] CHIANTINI L. AND CILIBERTO C., *Weakly defective varieties*, Trans. Amer. Math. Soc. **354** (2002), 151–178.

[17] CHIANTINI L. AND CILIBERTO C., *The grassmannians of secant varieties of curves are not defective*, Indag. Math. **13** (2002), 23–28.

[18] CHIANTINI L. AND CILIBERTO C., *Threefolds with degenerate secant variety: on a theorem of G. Scorza*, in: "Geometric and combinatorial aspects of commutative algebra (Messina, 1999)", Lect. Notes in Pure and Appl. Math. **217**, Marcel Dekker, New York 2001, 111–124.

[19] CHIANTINI L. AND CILIBERTO C., *The classification of defective threefolds*, mathAG.0312518, preprint (2003).

[20] CHIANTINI L. AND CILIBERTO C., *The classification of* $(1, k)$*-defective surfaces*, preprint.

[21] CHIANTINI L. AND CILIBERTO C., *On the concept of* k*-secant order of a variety*, preprint.

[22] CHIANTINI L. AND COPPENS M., *Grassmannians of secant varieties*, Forum Math. **13** (2001), 615–628.

[23] CHIANTINI L., LOPEZ A. AND RAN Z., *Subvarieties of generic hypersurfaces in any variety*, Math. Proc. Cambr. Phyl. Soc. **130** (2002), 259–268.

[24] CHIPALKATTI J., *Apolar schemes of algebraic forms*, Can. J. of Math., to appear.

[25] CILIBERTO C., *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, in: "European Congress of Mathematics, Vol. I (Barcelona, 2000)", Progr. Math. **201**, Binkenhäuser, Basel 2001, 289–316.

[26] CILIBERTO C. AND HIRSCHOWITZ A., *Hypercubiques de* $\mathbb{P}^4$ *avec sept pointes singulieres generiques*, C. R. Acad. Sci. Paris **313** (1991), 135–137

[27] CILIBERTO C., MELLA M. AND RUSSO F., *Varieties with one apparent double point*, mathAG/0210008, preprint (2002).

[28] CILIBERTO C. AND RUSSO ÊF., *Varieties with minimal secant degree and linear systems of maximal dimension on surfaces*, preprint (2004).

[29] CLEMENS H., *Curves in generic hypersurfaces*, Ann. Sci. Ec. Norm. Sup. **19** (1986), 629–636.

[30] CLEMENS H. AND RAN Z., *Twisted genus bounds for subvarieties of generic hypersurfaces*, Amer. J. Math, to appear.

[31] COPPENS M., *Smooth threefolds with* $G_{2,3}$*-defect*, preprint.

[32] DALE M., *Terracini's lemma and the secant variety of a curve*, Proc. London Math. Soc. **49** (1984), 329-339.

[33] DALE M., *On the secant variety of an algebrac surface*, University of Bergen, Dept. of Math. preprint n. 33 (1984).

[34] DIONISI C. AND FONTANARI C., *Grassmann defectivity á la Terracini*, Le Matematiche **56** (2001), 245–255.

[35] EISENBUD D. AND HARRIS J., *Curves in projective spaces*, Montreal University Press, Montreal 1982.

[36] ENRIQUES F. AND CHISINI O., *Teoria geometrica delle equazioni e delle funzioni algebriche, vol. III*, Zanichelli, Bologna 1982.

[37] FANTECHI B., *On the superadditivity of secants defects*, Bull. Soc. Math. France **118** (1990), 85–100.

[38] FUJITA T., *Projective threefolds with small secant varieties*, Sci. Papers College Gen. Ed. Univ. Tokyo **32** (1982), 33–46.

[39] FUJITA T. AND ROBERTS J., *Varieties with small secant varieties: the extremal case*, Amer. J. of Math. **103** 1 (1981), 953–976.

[40] FONTANARI C., *Grassmann defective surfaces*, to appear on Boll. UMI.

[41] FULTON W. AND HANSEN J., *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. **110** (1979), 159-166.

[42] FULTON W. AND LAZARSFELD R., *Connectivity and its application in algebraic geometry*, Lect. Notes in Math. **862**, Springer-Verlag, Berlin 1981, 26–92.

[43] GALLARATI D., *Alcune osservazioni sopra le varietà i cui spazi tangenti si appoggiano irregolarmente a spazi assegnati*, Rend. Accad. Naz. Lincei, **20** VIII (1956), 193–199.

[44] GRIFFITHS P. AND HARRIS J., *Algebraic geometry and local differential geometry*, Ann. Scient. Ec. Norm. Sup. **12** (1979), 335-432.

[45] GRIFFITHS P. AND HARRIS J., *Principles of algebraic geometry*, John Wiley and Sons, New York 1994.

[46] HARTSHORNE R., *Algebraic Geometry*, Graduate Texts in Math. **52**, Springer-Verlag, Berlin 1977.

[47] HARRIS J., *Algebraic Geometry: a first course*, Graduate Texts in Math. **133**, Springer-Verlag, Berlin 1992.

[48] IARROBINO A., KANEV ÊV., *Power sums, Gorenstein algebras and determinantal loci*, Lect. Notes in Math. **1723**, Springer-Verlag, Berlin 1999.

[49] KANEV V., *Chordal varieties of Veronese varieties and catalecticant matrices*, math.AG/9804141, preprint 1998.

[50] LANDSBERG J.M., *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. J. **85** (1996), 605–634.

[51] MELLA M., *Singularities of linear systems and the Waring problem*, preprint.

[52] MEZZETTI E., *Projective varieties with many degenerate subvarieties*, Boll. UMI **8B** (1994), 807–832.

[53] MEZZETTI E. AND TOMMASI O., *On projective varieties of dimension $n + k$ covered by k-spaces*, Illinois J. Math. **46** (2002), 443–465.

[54] MEZZETTI E. AND TOMMASI O., *Some remarks on varieties with degenerate Gauss image*, mathAG/0304127, preprint (2003).

[55] MORIN U., *Su sistemi di S_k a due a due incidenti e sulla generazione proiettiva di alcune varietà algebriche*, Atti R. Ist. Veneto di Scienze, Lettere ed Arti **101** (1941-42), 183–196.

[56] PALATINI F., *Sulle superficie algebriche i cui $S_h(h + 1)$-seganti non riempiono lo spazio ambiente*, Atti. Accad. delle Scienze di Torino **41** (1906), 634–640.

[57] PALATINI F., *Sulle varietà algebriche per le quali sono di dimensione minore dell'ordinario, senza riempire lo spazio ambiente, una o alcune delle varietà formate da spazi seganti*, Atti. Accad. delle Scienze di Torino **44** (1909), 362–374.

[58] RAN Z., *The (dimension+2)-secant lemma*, Inv. Math. **106** (1991), 65–71.

[59] RAVI M.S., *Determinantal equations for secant varieties of curves*, Comm. Alg. **22** (1994), 3103–3106.

[60] ROGORA E., *Varieties with many lines*, Manuscripta Math. **82** (1994), 207–226.

[61] RUSSO F., *On a theorem of Severi*, Math. Ann. **316** (2000), 1–17.

[62] SEVERI F., *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e ai suoi punti tripli apparenti*, Rend. Circ. Mat. Palermo **15** (1901), 33–51.

[63] SCORZA G., *Determinazione delle varietá a tre dimensioni di S_r, $r \geq 7$, i cui S_3 tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo **25** (1908), 193–204.

[64] SCORZA G., *Un problema sui sistemi lineari di curve appartenenti a una superficie algebrica*, Rend. R. Ist. Lombardo **41** 2 (1908), 913–920.

[65] SCORZA G., *Sulle varietà a quattro dimensioni di S_r ($r \geq 9$) i cui S_4 tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo **27** (1909), 148-178.

[66] SCORZA G., *Sopra una certa classe di varietà razionali*, Rend. Circ. Mat. Palermo **28** (1909), 400-401.

[67] SEGRE B., *Sulle V_n aventi più di ∞^{n-k} S_k*, Atti Accad. Naz. Lincei **5** (1948), 275–280.

[68] SEGRE C., *Preliminari di una teoria delle varietá luoghi di spazi*, Rend. Circ. Mat. Palermo **30** (1910), 87-121.

[69] SEGRE C., *Sulle varietà normali a tre dimensioni composte di serie semplici razionali di piani*, Atti. Accad. Sci. Torino **56** (1885), 95-115.

[70] SEGRE C., *Le superfici degli iperspazi con una doppia infinitá di curve piane o spaziali*, Atti. Accad. Sci. Torino **21** (1920), 75-89.

[71] SEVERI F., *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni e ai suoi punti tripli apparenti*, Rend. Circ. Mat. Palermo **15** (1901), 33-51.

[72] SHAFAREVICH I.R., *Basic algebraic geometry*, Grund. Math. Wissen. Einz. **213** Springer-Verlag, Berlin 1974.

[73] SOULÉ C., *Secant varieties and successive minima*, mathAG./0110254, preprint (2001).

[74] TERRACINI A., *Sulle V_k per cui la varietà degli S_h $(h + 1)$-seganti ha dimensione minore dell' ordinario*, Rend. Circ. Mat. Palermo **31** (1911), 392-396.

[75] TERRACINI A., *Su due problemi, concernenti la determinazione di alcune classi di superficie, considerati da G. Scorza e F. Palatini*, Atti Soc. Natur. e Matem. Modena **6** V (1921-22), 3-16.

[76] TERRACINI A., *Sulla rappresentazione di coppie di forme ternarie mediante somme di potenze di forme lineari*, Atti Matem. Pura Applic. **XXIV** (1915), 91-100.

[77] TOGLIATTI E., *Alcuni esempi di superficie algebriche degli iperspazi che rappresentano un'equazione di Laplace*, Comm. Math. Helv. **1** (1929), 255-272.

[78] VERMEIRE P., *Secant varieties and birational geometry*, mathAG/9911078, preprint (1999).

[79] VOISIN C., *On linear subspaces contained in the secant variety of projective curves*, AG/0110256, preprint (2003).

[80] VOISIN C., *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **44** (1996), 200–214.

[81] XU G., *Subvarieties of general hypersurfaces in projective space*, J. Diff. Geom. **39** (1994), 139-172.

[82] ZAK F., *Tangents and secants of algebraic varieties*, Transl. Math. Monogr. **127**, Amer. Math. Soc., Providence RI 1993.

[83] ZAK F., *Some properties of dual varieties and their applications in projective geometry*, Lect. Notes in Math. **1479**, Springer-Verlag, Berlin 1991, 273–280.

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