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CONNECTED ZERO-DIMENSIONAL SUBSCHEMES OF A PROJECTIVE SPACE AND THEIR MINIMAL FREE RESOLUTION

Abstract. Here we study the existence of zero-dimensional curvilinear schemes $Z \subset \mathbb{P}^n$, $n = 2, 3$, with the expected minimal free resolution and with $\mathrm{card}(Z_{red})$ very small. In the plane we prove the existence of arbitrary degree connected curvilinear zero-dimensional schemes with the expected minimal free resolution.

1. Introduction

Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme. A classical problem is the study of the postulation of Z. When Z is a general union of d points a very important problem is to see if the minimal free resolution is " the expected one " (see [11] and [8]). Of course, this is interesting even for more general zero-dimensional subschemes of a projective space. Now assume that Z is a zero-dimensional subscheme of the integral projective variety X. In this paper we study the case in which $\mathrm{card}(Z_{red})$ is small. Fix $L \in \mathrm{Pic}(X)$ and a linear subspace $V \subset H^0(X, L)$. What is the rank of the restriction map $\rho_{Z,V} : V \to H^0(Z, L|Z)$? Obviously, $\rho_{Z,V}$ has maximal rank (i.e. it is either injective or surjective) if Z is a general union of d points. For the minimal free resolution it is easy to reduce the result to the computation of the rank of a restriction map with $V = H^0(\mathbb{P}^n, E)$, E not a line bundle, but a higher rank vector bundle: $E = \Omega^i_{\mathbb{P}^n}(t)$ for suitable t and all i, with $0 < i < n$ (see [11], [3] or the introduction of [8]). For any closed subscheme $Z \subset \mathbb{P}^n$ let $\rho_{Z,t,n} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ be the restriction map. We will say that Z has *maximal rank* if for every integer $t \geq 1$ the linear map $\rho_{Z,t,n}$ has maximal rank, i.e. it is either injective or surjective. Here we are interested in the case of curvilinear subschemes, i.e. zero-dimensional schemes, Z, such that for every $P \in Z_{red}$ the Zariski tangent space to Z at P has dimension at most one; equivalently a zero-dimensional subscheme Z of a smooth variety W is curvilinear if and only if it is locally contained in a smooth germ of curve in W. For the postulation of general curvilinear connected subschemes see Remark 5. In section 2 we will prove the following result.

THEOREM 1. *For any integer $d \geq 4$ and any $P \in \mathbb{P}^2$ there is a zero-dimensional curvilinear scheme $Z \subset \mathbb{P}^2$ such that $Z_{red} = \{P\}$, length$(Z) = d$, Z has maximal rank and the expected minimal free resolution.*

The upper bounds $k + 19$ and $k + 15$ in the next two theorems are very rough; the

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part $k+2$ for Theorem 2 (resp. $k+3$ for Theorem 3) comes from the inductive proof and the use of Propositions 3 and 4; the additional term 17 (resp. 12) comes from the initial cases respectively used in [2] and [3], where the initial cases were done with the use of a computer and hence only for a set of distinct points. Are the next two theorems true with $\mathrm{card}(Z) = 1$? Are the next two theorems true with $\mathrm{card}(Z) = 1$ and Z curvilinear ?

THEOREM 2. *Fix an integer $d \geq 5$ and let k be the only integer such that $k \geq 2$ and $\binom{k+2}{3} < d \leq \binom{k+3}{3}$. There is a degree d zero-dimensional scheme $Z \subset \mathbb{P}^3$ such that Z has maximal rank, it is curvilinear, the homogeneous ideal of Z has the expected number of generators and card$(Z_{red}) \leq k+19$.*

Take d, k and Z as in the statement of Theorem 2. By Castelnuovo - Mumford's lemma Z has maximal rank if and only if $h^1(\mathbb{P}^3, \mathcal{I}_Z(k)) = 0$ and $h^0(\mathbb{P}^3, \mathcal{I}_Z(k-1)) = 0$. If Z has maximal rank, the homogeneous ideal of Z has the expected number of generators if and only if it is generated by forms of degree d and by $\max\{0, \binom{k+4}{3} - d - 4(\binom{k+3}{3} - d)\}$ forms of degree $k+1$ (see e.g. the introduction of [2]).

THEOREM 3. *Fix an integer $d \geq 5$ and let k be the only integer such that $k \geq 2$ and $\binom{k+2}{3} < d \leq \binom{k+3}{3}$. There is a degree d zero-dimensional scheme $Z \subset \mathbb{P}^3$ such that Z has maximal rank, it is curvilinear, the homogeneous ideal of Z has the expected Cohen - Macaulay type and card$(Z_{red}) \leq k+15$.*

For an explanation of the Cohen - Macaulay type of a zero-dimensional scheme $Z \subset \mathbb{P}^3$ and its connection to $T\mathbb{P}^3$ see [3]. We do not know if in the statements of Theorems 2 and 3 we may take the same Z and hence if we may obtain a zero-dimensional scheme Z with the expected minimal free resolution; the problem for our approach comes from the fact the proofs of the two theorems gives schemes such that the lengths of their connected components are in general different. As in [2] and [3] to prove Theorems 2 and 3 we need some results on a smooth cubic surface $S \subset \mathbb{P}^3$ (see Propositions 3 and 4 and Remark 4). At the same price we will also prove the corresponding results for a smooth quadric surface (see Propositions 1 and 2).

2. The proofs

Let X be a projective variety, $Y \subset X$ a closed subscheme, E a vector bundle on X and D an effective Cartier divisor of X. Let $\mathrm{Res}_D(Y)$ denote the residual scheme of Y with respect to D. Hence $\mathrm{Res}_D(Y) \subseteq Y$. There is an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Y)},X} \otimes E(-D) \to \mathcal{I}_Y \otimes E \to \mathcal{I}_{Y \cap D, D} \otimes (E|D) \to 0 \tag{1}$$

From (1) we immediately get the following elementary form of Horace Lemma (see [7] or [1] for much more).

LEMMA 1. *Let X be a projective variety, $Y \subset X$ a closed subscheme, E a vector bundle on X and D an effective Cartier divisor of X. Then:*

(a) $h^0(X, \mathcal{I}_Y \otimes E) \leq h^0(X, \mathcal{I}_{Res_{D(Y)},X} \otimes E(-D)) + h^0(D, \mathcal{I}_{Y\cap D,D} \otimes (E|D))$;

(b) $h^1(X, \mathcal{I}_Y \otimes E) \leq h^1(X, \mathcal{I}_{Res_{D(Y)},X} \otimes E(-D)) + h^1(D, \mathcal{I}_{Y\cap D,D} \otimes (E|D))$.

REMARK 1. Let $C \subset \mathbb{P}^n$ be a rational normal curve. It is well-known (see e.g. [2], Lemma 1.3) that $\Omega_{\mathbb{P}^n}|C$ is a direct sum of n line bundles of degree $-n-1$.

LEMMA 2. *Fix an integer $t \geq 2$, a smooth conic $D \subset \mathbb{P}^2$, $P \in D$ and a zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that $Z_{red} = \{P\}$. Set $Z_1 := Z$ and define inductively Z_i, $i \geq 2$, by the formula $Z_i := Res_D(Z_{i-1})$. Set $a_i := length(Z_i \cap D)$, $i \geq 1$. If $a_i \geq 2i-2$ for every $i \geq 1$, then $h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega_{\mathbb{P}^2}(t)) = 0$. If $a_i \leq 2i-2$ for every $i \geq 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega_{\mathbb{P}^2}(t)) = 0$.*

Proof. By Remark 1 the vector bundle $\Omega_{\mathbb{P}^2}(t)|D$ is the direct sum of two line bundles of degree $2t-3$. Use the cohomology of line bundles on $D \cong \mathbb{P}^1$ and apply several times Horace Lemma, i.e. Lemma 1. □

Proof of Theorem 1. Let k be the only integer such that $k(k+1)/2 < d \leq (k+2)(k+1)/2$. Thus $k \geq 2$. Let $A \subset \mathbb{P}^2$ be any degree d zero-dimensional scheme. By Castelnuovo - Mumford's lemma A has maximal rank if and only if $h^1(\mathbb{P}^2, \mathcal{I}_A(k)) = h^0(\mathbb{P}^2, \mathcal{I}_A(k-1)) = 0$. If A has maximal rank, then it has the expected minimal free resolution if and only if the restriction map $\eta_{A,k+1} : H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(k+1)) \to H^0(A, \Omega^1_{\mathbb{P}^2}(k+1)|A)$ has maximal rank, i.e. it is injective if $d \geq k(k+2)/2$ and it is surjective if $d \leq k(k+2)/2$ ([2], Remark 3.1). To fix the notation we will assume $d = k(k+2)/2$ and check the bijectivity of $\eta_{A,k+1}$ for a suitable A, the proof of the equalities $h^1(\mathbb{P}^2, \mathcal{I}_A(k)) = h^0(\mathbb{P}^2, \mathcal{I}_A(k-1)) = 0$ being similar, but easier. Let $D \subset \mathbb{P}^2$ be a smooth conic such that $P \in A$. Fix an integer $s > 0$ and integers $a_1 \geq \cdots \geq a_s > 0$. Set $a_i := 0$ for every $i > s$. There is a zero-dimensional subscheme $A \subset \mathbb{P}^2$ with the following properties. Set $A_1 := A$. For all integers $i \geq 2$ such that A_{i-1} is defined let $A_i := \mathrm{Res}_D(A_{i-1})$ be the residual scheme of A_{i-1} with respect to the Cartier divisor D of $\mathbb{P}^2$. We require that $\mathrm{length}(A_i \cap D) = a_i$ for every i. The construction of A is easy taking formal coordinates x, y around P in the plane such that the formal germ of D at P has equation $x = 0$. We do this construction for $s := k/2 - 1$ taking $a_i = a_1 - 2i + 2$ for $2 \leq i \leq s$. We apply $k/2 - 1$ times Horace Lemma with respect to the Cartier divisor D taking at the i^{th} step the vector bundle $E := \Omega^1_{\mathbb{P}^2}(k+3-2i)$, obtaining the bijectivity of $\eta_{A,k+1}$. Since the local Hilbert scheme of $K[[x, y]]$ is irreducible and a non-empty open subset of it is formed by the curvilinear subschemes with a fixed length ([4] over $\mathbb{C}$ and [9] or [6] in arbitrary characteristic), we may deform A to a zero-dimensional curvilinear scheme supported by P. Hence we conclude by semicontinuity. The irreducibility of the local Hilbert scheme of $K[[x, y]]$ allows us to obtain Z simultaneously with maximal rank and such that the restriction map $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(k+1)) \to H^0(Z, \Omega^1_{\mathbb{P}^2}(k+1)|Z)$ has maximal rank.

LEMMA 3. *Fix integers $x > 0$, $y > 0$, a smooth surface W, $P \in W$ and two germs $A, B \subset W$ of curves around P which are smooth and transversal at P. Then there* *exists a zero-dimensional scheme* $Z \subset W$ *such that* $Z_{red} = \{P\}$, *length*$(Z) = x+y-1$, *length*$(Z \cap C) = x$, *length*$(Z \cap D) = y$, *Res*$_D(Z) \subset C$, *and Res*$_C(Z) \subset D$.

Proof. Up to a formal change of coordinates around P we may assume $W = \mathbb{A}^2$ with coordinates z_1, z_2, $P = 0$, $C = \{z_2 = 0\}$ and $D = \{z_1 = 0\}$. Take $Z = \{z_1^x = z_2^y = z_1 z_2 = 0\}$. □

DEFINITION 1. *Let* $Z \subset W$ *the zero-dimensional scheme considered in Lemma 3. We will say that* Z *is a two-crossing scheme with lengths* (x, y),*with* A, B *as germs of supporting smooth curves and with* P *as support.*

REMARK 2. Consider a twist of the Euler's sequence of $T\mathbb{P}^3$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(t) \to \mathcal{O}_{\mathbb{P}^3}(t+1)^{\oplus 4} \to T\mathbb{P}^3(t) \to 0 \tag{2}$$

and a twist of its dual

$$0 \to \Omega_{\mathbb{P}^3}(t) \to \mathcal{O}_{\mathbb{P}^3}(t-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3}(t) \to 0 \tag{3}$$

From (3) we obtain $h^1(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(t)) = 0$ for every $t \neq 0$, $h^1(\mathbb{P}^3, \Omega_{\mathbb{P}^3}) = 1$, $h^2(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(t)) = 0$ for every t, $h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(t)) = 0$ for every $t \leq 1$, $h^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(t)) = 4\binom{t+2}{3} - \binom{t+3}{3} = (t+2)(t+1)(t-1)/2$ for every $t \geq 2$. From (2) and/or Serre duality we obtain $h^0(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ for every $t \leq -2$, $h^0(\mathbb{P}^3, T\mathbb{P}^3(t)) = 4\binom{t+4}{3} - \binom{t+3}{3} = (t+5)(t+3)(t+2)/2$ for every $t \geq -1$, $h^1(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ for every t, $h^2(\mathbb{P}^3, T\mathbb{P}^3(t)) = 0$ for every $t \neq -4$ and $h^2(\mathbb{P}^3, T\mathbb{P}^3(-4)) = 1$. Fix a smooth quadric surface $Q \subset \mathbb{P}^3$ and set $E := \Omega_{\mathbb{P}^3}|Q$. We have the following exact sequences

$$0 \to T\mathbb{P}^3(t-2) \to T\mathbb{P}^3(t) \to (E^*)(t) \to 0 \tag{4}$$

$$0 \to \Omega_{\mathbb{P}^3}(t-2) \to \Omega_{\mathbb{P}^3}(t) \to E(t) \to 0 \tag{5}$$

From (5) or restricting (3) to Q we obtain $h^1(Q, E(t)) = 0$ for every $t \neq 0$, $h^1(Q, E) = 1$, $h^0(Q, E(t)) = 0$ for every $t \leq 1$ and $h^0(Q, E(t)) = (t+2)(t+1)(t-1)/2 - t(t-1)(t-3)/2 = (t-1)(3t+1)$ for every $t \geq 2$. From (4) or restricting (2) to Q we obtain $h^1(Q, E^*(t)) = 0$ for every $t \neq -2$, $h^0(Q, E^*(t)) = 0$ for every $t \leq -2$ and $h^0(Q, E^*(t)) = (t+5)(t+3)(t+2)/2 - (t+3)(t+1)t/2 = (t+3)(3t+5)$ for every $t \geq -1$.

REMARK 3. We will follow the set-up of [2] §2, and [3] §3. Let $\pi : M \to \mathbb{P}^2$ be the blowing-up of the plane at six points $P_1, \ldots, P_6$ not on a conic and no three of them collinear. We have $\text{Pic}(M) \cong \mathbb{Z}^7$. We will take as a basis of $\text{Pic}(M)$ the classes of $e := \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and of the exceptional divisors $e_i := \pi^{-1}(P_i)$, $1 \leq i \leq 6$. The anti-canonical divisor $3e - \sum_{i=1}^{6} e_i$ defines an embedding $j : M \to \mathbb{P}^3$. Set $S := j(M)$. Thus $S \subset \mathbb{P}^3$ is a smooth cubic surface. The linear system $|3e - 2e_1 - e_3 - e_4 - e_5 - e_6|$ (resp. $|3e - 2e_2 - e_3 - e_4 - e_5 - e_6|$) contains a smooth element C' (resp. D')

corresponding to an irreducible plane cubic passing through P_3, P_4, P_5, P_6 and with a node at P_1 (resp. P_2). Set $C := j(C')$ and $D := j(D')$, Thus $C, D \subset \mathbb{P}^3$ are smooth rational normal curves and $C \cup D$ is the complete intersection of S with a quadric surface. From the intersection theory of M we obtain $D \cdot D = 1$ and $D \cdot C = 5$. Set $F := \Omega_{\mathbb{P}^3}|S$. Hence we have the exact sequences

$$0 \to F(n-2) \to F(D) \to F(D)|D \to 0 \tag{6}$$

$$0 \to F(n-2)(D) \to F(n) \to F(n)|C \to 0 \tag{7}$$

$$0 \to F^*(n-2) \to F^*(D) \to F^*(D)|D \to 0 \tag{8}$$

$$0 \to F^*(n-2)(D) \to F^*(n) \to F^*(n)|C \to 0 \tag{9}$$

From the exact sequences (6), (7), (8) and (9) and Remark 1 we easily obtain (or see [2], §2, and [3], §3) $h^0(S, F(n)) = 0$ for every $n \leq -2$, $h^0(S, F(n)) = (n-1)(9n-6)/2$ for every $n \geq -1$, $h^1(S, F(n)) = 0$ for every $n \neq 3$, $h^1(S, F(n-2)(D)) = 0$ for every $n \geq 3$, $h^0(S, F(n-2)(D)) = (9n^2-33n+24)/2$ for every $n \geq 3$, $h^0(S, F^*(n)) = 0$ for every $n \leq -2$, $h^0(S, F^*(n)) = 3(n+2)(3n+5)/2$ for every $n \geq -1$, $h^1(S, F^*(n)) = 0$ for every $n \neq -1$, $h^0(S, F^*(n-2)(D)) = 3n(3n-1)/2 + 3n$ for every $n \geq 2$ and $h^1(S, F^*(n-2)(D)) = 0$ for every $n \geq 1$.

PROPOSITION 1. *Fix a smooth quadric surface $Q \subset \mathbb{P}^3$ and integers $t > 0$, $d > 0$ such that $(t-2)(3t-2) < 3d \leq (t-1)(3t+1)$. Then there exists a zero-dimensional scheme $Z \subset Q$ such that Z is the disjoint union of at most $[(t+2)/3]$ two-crossing schemes such that:*

(a) the restriction map $\eta_{Z,Q,t} : H^0(Q, E(t)) \to H^0(Z, (E(t)|Z))$ is surjective;

(b) the restriction map $\eta_{Z,Q,t-1} : H^0(Q, E(t-1)) \to H^0(Z, (E(t-1)|Z))$ is injective;

(c) the restriction map $\rho_{Z,Q,t-1} : H^0(Q, \mathcal{O}_Q(t-1)) \to H^0(Z, \mathcal{O}_Z(t-1))$ is surjective;

(d) the restriction map $\rho_{Z,Q,t-2} : H^0(Q, \mathcal{O}_Q(t-2)) \to H^0(Z, \mathcal{O}_Z(t-2))$ is injective.

Proof. Fix $P \in Q$ and two smooth curves $A \subset Q$ and $B \subset Q$ respectively of type $(2, 1)$ and of type $(1, 2)$ such that $P \in A \cap B$ and A and B are transversal at P. Thus A and B are rational normal curves of $\mathbb{P}^3$, $A \cdot A = B \cdot B = 4$, $A \cdot B = 5$ (intersection numbers in Q) and $A \cup B$ is the complete intersection of Q with a cubic surface. Apply Lemma 3 for $W := Q$, $x := 3t - 9$ and $y = 3t - 5$. Call Z' the corresponding two-crossing scheme. We have the following exact sequences on Q:

$$0 \to E(t-3) \to E(t-3)(A) \to E(t-3)(A)|A \to 0 \tag{10}$$

(11) $$0 \to E(t-3)(A) \to E(t) \to E(t)|B \to 0$$

By Remark 1 the vector bundle $E(t-3)(A)|A$ is the direct sum of three line bundles of degree $3t-9$, while the vector bundle $E(t)|B$ is the direct sum of three line bundles of degree $3t-4$. Set $Z'' := Res_B(Z')$. Thus $Z'' \subset A$ and $\text{length}(Z'') = x-1 = 3t-9$. By Horace Lemma with respect to B we obtain $h^i(Q, \mathcal{I}_{Z'} \otimes E(t)) = h^i(Q, \mathcal{I}_{Z''} \otimes E(t-2)(A))$, $i = 0, 1$. By Horace Lemma with respect to A we have $h^i(Q, \mathcal{I}_{Z''} \otimes E(t-2)(A)) = h^i(Q, E(t-2))$. Hence we may take Z' as one of the connected components of Z and use induction on t. The restriction map $\rho' : H^0(Q, \mathcal{O}_Q(t-2)) \to H^0(B, \mathcal{O}_B(t-2))$ is surjective because $H^1(Q, \mathcal{O}_Q(t-3, t-4)) = 0$. The restriction map $H^0(B, \mathcal{O}_B(t-2)) \to H^0(Z' \cap B, \mathcal{O}_{Z' \cap B}(t-2))$ is surjective because $y = 3t-5 = 1 + \deg(\mathcal{O}_{Z' \cap B}(t-2))$ and $B \cong \mathbb{P}^1$. Similarly, the restriction maps $\rho_1 : H^0(A, \mathcal{O}_A(t-3, t-4)) \to H^0(Z'', \mathcal{O}_{Z''}(t-3, t-4))$ and $\rho_2 : H^0(Q, \mathcal{O}_Q(t-3, t-4)) \to H^0(A, \mathcal{O}_A(t-3, t-4))$. Hence we obtain the surjectivity of $\rho_{Z,Q,t-1}$ using again Horace Lemma. □

The same proof (just using the exact sequences)

(12) $$0 \to E^*(t-3) \to E^*(t-3)(A) \to E^*(t-3)(A)|A \to 0$$

(13) $$0 \to E^*(t-3)(A) \to E^*(t) \to E^*(t)|B \to 0$$

instead of the exact sequences (10) and (11)) gives the following result.

PROPOSITION 2. *Fix a smooth quadric surface $Q \subset \mathbb{P}^3$ and integers $t > 0$, $d > 0$ such that $(t+2)(3t+2) < 3d \le (t+3)(3t+5)$. Then there exists a zero-dimensional scheme $Z \subset Q$ such that Z is the disjoint union of at most $[(t+2)/3]$ two-crossing schemes such that:*

(a) the restriction map $\gamma_{Z,Q,t} : H^0(Q, E^*(t)) \to H^0(Z, (E^*(t)|Z))$ is surjective;

(b) the restriction map $\gamma_{Z,Q,t-1} : H^0(Q, E^*(t-1)) \to H^0(Z, (E^*(t-1)|Z))$ is injective;

(c) the restriction map $\rho_{Z,Q,t+1} : H^0(Q, \mathcal{O}_Q(t-1)) \to H^0(Z, \mathcal{O}_Z(t-1))$ is surjective;

(d) the restriction map $\rho_{Z,Q,t-2} : H^0(Q, \mathcal{O}_Q(t-2)) \to H^0(Z, \mathcal{O}_Z(t-2))$ is injective.

PROPOSITION 3. *Fix a smooth cubic surface $S \subset \mathbb{P}^3$ and integers $t > 0$, $d > 0$ such that $(t-2)(9t-15)/2 < 3d \le (t-1)(9t-6)/2$. Then there exists a zero-dimensional scheme $Z \subset S$ such that Z is the disjoint union of at most $[(t+2)/3]$ two-crossing schemes such that:*

(a) the restriction map $\delta_{Z,S,t} : H^0(S, F(t)) \to H^0(Z, (F(t)|Z))$ is surjective;

(b) *the restriction map* $\delta_{Z,S,t-1} : H^0(S, F(t-1)) \to H^0(Z, (F(t-1)|Z))$ *is injective;*

(c) *the restriction map* $\rho_{Z,S,t-1} : H^0(S, \mathcal{O}_S(t-1)) \to H^0(Z, \mathcal{O}_Z(t-1))$ *is surjective;*

(d) *the restriction map* $\rho_{Z,S,t-2} : H^0(S, \mathcal{O}_S(t-2)) \to H^0(Z, \mathcal{O}_Z(t-2))$ *is injective.*

Proof. Use that the rational normal curves $C, D \subset S$ have as union the complete intersection of S with a smooth quadric (Remark 3) and then modify the proof of Proposition 1. □

As in the case of a quadric surface the same proof of Proposition 2 gives the following result.

PROPOSITION 4. *Fix a smooth cubic surface* $S \subset \mathbb{P}^3$ *and integers* $t > 0$, $d > 0$ *such that* $(t+1)(3t+2)/2 < d \leq (t+2)(3t+5)/2$. *Then there exists a zero-dimensional scheme* $Z \subset S$ *such that* Z *is the disjoint union of at most* $[(t+2)/3]$ *two-crossing schemes such that:*

(a) *the restriction map* $\lambda_{Z,S,t} : H^0(S, F^*(t)) \to H^0(Z, (F^*(t)|Z))$ *is surjective;*

(b) *the restriction map* $\lambda_{Z,S,t-1} : H^0(S, F^*(t-1)) \to H^0(Z, (F^*(t-1)|Z))$ *is injective;*

(c) *the restriction map* $\rho_{Z,S,t-1} : H^0(S, \mathcal{O}_S(t-1)) \to H^0(Z, \mathcal{O}_Z(t-1))$ *is surjective;*

(d) *the restriction map* $\rho_{Z,S,t-2} : H^0(S, \mathcal{O}_S(t-2)) \to H^0(Z, \mathcal{O}_Z(t-2))$ *is injective.*

REMARK 4. Recall that the local Hilbert scheme of a smooth surface is irreducible and that a dense open subset of it is formed by the curvilinear connected subscheme ([2] over $\mathbb{C}$, [9] or [6] in arbitrary characteristic). Thus Propositions 1, 2, 3 and 4 are true if we take as Z a curvilinear subscheme with the same upper bound for card(Z_{red}).

Proof of Theorem 2. Let k' be the only integer such that $2 \leq k' \leq 4$ and $k' \equiv k$ (mod 3). Fix a smooth cubic surface $S \subset \mathbb{P}^3$, $P \in S$ and a germ T at P of a smooth curve of S. There is a germ at P of a smooth surface $S' \subset \mathbb{P}^3$ which is transversal to S at P and such that $S \cap S' = T$. Fix an integer $s > 0$ and integers $a_1 \geq \cdots \geq a_s > 0$. Set $a_i := 0$ for every $i > s$. Thus T is the germ at P of a Cartier divisor of S'. There is a zero-dimensional subscheme $A \subset S'$ with the following properties. Set $A_1 := A$. For all integers $i \geq 2$ such that A_{i-1} is defined let $A_i := \mathrm{Res}_T(A_{i-1})$ be the residual scheme of A_{i-1} with respect to the Cartier divisor T of S'. We require that length$(A_i \cap D) = a_i$ for every i. The construction of A is easy taking formal coordinates x, y around P in the plane such that the formal germ of D at P has equation $x = 0$. We take $s := [(k+2)/3]$ and we apply several times Horace Lemma and the part of Remark 4 related to Proposition 2 first for the integer $t = k - 1$, then for the integer $t = k-4$, then for the integer $t = k-7$, and so on, untill we use it for the integer $t = k' - 1$. Then we use that for all integers x such that $5 \leq x \leq 17$ a general union of x points of $\mathbb{P}^3$ has the homogeneous ideal with the expected number of minimal generators. The local Hilbert scheme of $K[[x, y]]$ is irreducible and a non-empty open subset of it is formed by the curvilinear subschemes with a fixed length ([4] over $\mathbb{C}$ and [9] or [6] in arbitrary characteristic). Hence every connected zero-dimensional scheme contained in the germ Σ of a smooth surface may be flatly deformed inside Σ to a connected germ of curvilinear schemes of Σ with the same support. Hence we may deform any connected component of any zero-dimensional surfilinear scheme to a connected zero-dimensional curvilinear scheme with the same support. Hence we conclude by semicontinuity. □

Proof of Theorem 3. Just copy the proof of Theorem 2 quoting the part of Remark 4 related to Proposition 4 instead of the part related to Proposition 3. □

REMARK 5. Let X be a positive-dimensional integral projective variety and V a non-empty linear system, i.e. a non-zero linear subspace of $H^0(X, L)$ for some $L \in \mathrm{Pic}(X)$. In characteristic zero for a general curvilinear scheme $Z \subset X$ supported by a general point of X the restriction map $V \to V|Z$ has maximal rank ([5]), but this is not true in positive characteristic, not even if X is a smooth curve (see the examples in [10]). Here we will check the well-known fact that this is true in arbitrary characteristic if $X = \mathbb{P}^n$ and $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ is a complete linear system. By the homogeneity of $\mathcal{O}_{\mathbb{P}^n}(k)$ the result does not depend on the choice of the supporting point. The result is obvious if $n = 1$ and hence we may assume $n > 1$ and that the result is true for the integer $n' := n - 1$. The result is also obvious if $k \leq 1$ and hence we may assume $k > 1$ and that the result is true in $\mathbb{P}^n$ for all integers $k' < k$. Fix a hyperplane $H \subset \mathbb{P}^n$ and $P \in H$. By Castelnuovo - Mumford's lemma and the fact that every degree z curvilinear scheme supported by P is contained in a curvilinear scheme of degree $z + 1$ supported by P and (if $z \geq 2$) it contains a degree $z - 1$ subscheme, to check the result for the pair (n, k) it is sufficient to check it for a general curvilinear degree d scheme supported by P just the integer $d := \binom{n+k}{n}$. Choose homogeneous coodinates $x_0, \ldots, x_n$ such that $H = \{x_n = 0\}$. By the inductive assumption on the integer n there is a curvilinear scheme $Z' \subset H$ such that $(Z')_{red} = \{P\}$, $\deg(Z') = \binom{n+k-1}{n-1}$ and $h^0(H, \mathcal{I}_{Z'}(k)) = h^1(H, \mathcal{I}_{Z'}(k)) = 0$. Take homogeneous equations $\{g_i(x_0, \ldots, x_{n-1})\}_{i \in I}$ of Z' in H. By the inductive assumption on the integer k there is a zero-dimensional curvilinear scheme $Z'' \subset \mathbb{P}^n$ such that $(Z'')_{red} = \{P\}$, $\deg(Z'') = \binom{n+k-1}{n}$ and $h^0(\mathbb{P}^n, \mathcal{I}_{Z''}(k-1)) = h^1(\mathbb{P}^n, \mathcal{I}_{Z''}(k-1)) = 0$. By the irreducibility of the local Hilbert scheme of curvilinear subschemes of H and of $\mathbb{P}^n$ we may also assume that either $Z' \subseteq Z''$ (case $k \leq n - 1$) or $Z'' \subset Z'$ (case $k \geq n$). Take homogeneous equations $\{f_j(x_0, \ldots, x_n)\}_{j \in J}$ of Z'' and let $Z \subset \mathbb{P}^n$ be the zero-dimensional scheme with g_i, $i \in I$, and $x_n f_j$, $j \in J$, as defining equations. We have $Z' = Z \cap H$ and $Z'' = \mathrm{Res}_H(Z)$. By Horace Lemma we have $h^0(\mathbb{P}^n, \mathcal{I}_Z(k)) = h^1(\mathbb{P}^n, \mathcal{I}_Z(k)) = 0$. Since $Z \cap H$ is curvilinear, the Zariski tangent space of Z has dimension at most two. Thus Z is contained in the germ of a smooth surface. By the irreducibility of the local Hilbert scheme of a smooth surface ([4] over $\mathbb{C}$, [9] or [6] in arbitrary characteristic) we may deform Z to a curvilinear subscheme supported by P and conclude by semicontinuity.

References

[1] ALEXANDER J. AND HIRSCHOWITZ A., *An asymptotic vanishing theorem for generic unions of multiple points*, Invent. Math. **140** 2 (2000), 303–325.

[2] BALLICO E., *Generators for the homogeneous ideal of s general points in* $\mathbb{P}_3$, J. Algebra **106** 1 (1987), 46–52.

[3] BALLICO E. AND GERAMITA A.V., *The minimal free resolution of the ideal of s general points in* $\mathbb{P}^3$, in: "Proceedings of the *1984 Vancouver Conference in Algebraic Geometry*", CMS Confrerence Proc. Canadian Math. Soc. and Amer. Math. Soc., Providence, RI, 1986,1–10.

[4] BRIANÇON J., *Description of* $Hilb^n\mathbb{C}\{x, y\}$, Invent. Math. **41** 1 (1977), 45–89.

[5] CILIBERTO C. AND MIRANDA R., *Interpolations on curvilinear schemes*, J. Algebra **203** 2 (1998), 677-678.

[6] GÖTTSCHE L., *Hilbert Schemes of Zero-dimensional Subschemes of Smooth Varieties*, Lect. Notes in Math. **1572**, Springer-Verlag, Berlin-Heidelberg 1994.

[7] HIRSCHOWITZ A., *La methode d'Horace pour l'interpolation a plusieurs variables*, Manuscripta Math. **50** (1985), 337–378.

[8] HIRSCHOWITZ A. AND SIMPSON C., *La résolution minimal de l'idéale d'un grand nombre de points dans* $\mathbf{P}^n$, Invent. Math. **126** (1996), 467–503.

[9] IARROBINO A., *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **188** (1977).

[10] LAKSOV D., *Wronkians and Plücker formulas for linear systems on curves*, Ann. scient. Éc. Norm. Sup. **17** (1984), 45–66.

[11] LORENZINI A., *The minimal free resolution conjecture*, J. Algebra **156** 1 (1993), 5–35.

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