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RANK 2 REFLEXIVE SHEAVES ON A SMOOTH THREEFOLD

Abstract. We show that some properties of rank 2 reflexive sheaves true on $\mathbb{P}^3$ can be extended to a wide class of smooth projective threefolds, including smooth 3-dimensional complete intersections and some Fano threefolds. In particular, we extend the Hartshorne-Serre correspondence between rank 2 reflexive sheaves and curves lying on the threefold. Also, we establish the non negativity of the third Chern class of a rank 2 reflexive sheaf.

1. Introduction

Since the seventies rank 2 vector bundles (i.e. locally free sheaves) on projective three–space have been studied by many authors achieving several interesting results in some different research directions. One of these areas of research is the correspondence between rank 2 vector bundles on $\mathbb{P}^3$ on the one hand and algebraic subcanonical curves embedded in $\mathbb{P}^3$ on the other hand, where a curve is called subcanonical if its dualizing sheaf is isomorphic to a suitable twist of its structure sheaf. This correspondence is due, at least implicitly, to Serre (see [12]) in the affine case and to Horrocks (see [6]) in the projective case; but it was Hartshorne in [4], in the late seventies, to give a precise and explicit statement about the "rank 2 vector bundles–subcanonical curves" correspondence in $\mathbb{P}^3$. As reflexive sheaves are a natural generalization of vector bundles (in fact we can think of them as bundles with "singularities"), Hartshorne in [5] extends the above correspondence to rank 2 reflexive sheaves on $\mathbb{P}^3$, obtaining as counterpart curves in $\mathbb{P}^3$ which are equidimensional, locally Cohen-Macaulay and generically local complete intersection. This was the starting point to investigate the properties of rank 2 reflexive sheaves, also in order to look for applications to space curves. In fact the correspondence assures the existence of a short exact sequence linking a rank 2 reflexive sheaf $\mathcal{F}$ on $\mathbb{P}^3$ and a curve C zero locus of a global section of $\mathcal{F}$, namely

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{F} \to \mathcal{I}_C(c_1) \to 0$$

where c_1 is the first Chern class of $\mathcal{F}$ (considered as an integer). Thanks to this sequence we deduce that cohomological properties of the reflexive sheaf and of the curve are in very close connection. This interaction between reflexive sheaves (or vector bundles) and curves in $\mathbb{P}^3$ allows us to work on the ones or on the others to get as a consequence results on both. But the techniques used in the two fields, sheaves and curves, may be very different, so it's more convenient in some cases to work with reflexive sheaves and then translate the results in the language of curves or to do the opposite. Hartshorne's paper [5] was followed, between the seventies and the nineties, by several new results due to many authors (see the survey paper [11] for a partial overview on the matter). In the present paper we study rank 2 reflexive sheaves on a

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smooth algebraic projective polarized threefold $(X, \mathcal{O}_X(1))$ verifying some technical conditions, where $\mathcal{O}_X(1)$ is a fixed very ample invertible sheaf on X, with the aim of extending some results true for rank 2 reflexive sheaves on $\mathbb{P}^3$. The first goal is to give a complete proof of the correspondence between rank 2 reflexive sheaves and equidimensional, locally Cohen-Macaulay and generically local complete intersection curves on such a threefold (the so called "Hartshorne-Serre correspondence"). We follow, essentially, the construction given by Hartshorne in [5, Theorem 4.1]. If we consider only locally free sheaves we get the so called "Serre correspondence" between rank 2 vector bundles and curves in X which are locally complete intersection with dualizing sheaf isomorphic to the restriction to the curve of some invertible sheaf on the threefold. These two correspondences are already known to the experts of the area, but there are no adequate references for them in the literature (although in [4] there is a remark that the proof of the correspondence for vector bundles on $\mathbb{P}^3$ applies also on $\mathbb{P}^n$, $n \geq 3$, or more generally on any nonsingular projective variety, and moreover there is a concise proof of the correspondence for vector bundles on a smooth threefold in [2]). So we have decided to include a full proof of both of them. In section 3 we assume that the threefold X satisfies two technical conditions: the Picard group of X is isomorphic to $\mathbb{Z}$ (generated by $\mathcal{O}_X(1)$) and the 1–cohomology of $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ vanishes for any $n \in \mathbb{Z}$. Notice that the first condition is essential to regard the first Chern class of a reflexive sheaf as an integer, like on $\mathbb{P}^3$. In this way we restrict our attention to a wide class of smooth threefolds, which includes, besides the obvious case of the projective three–space, all nonsingular 3-dimensional complete intersections, in particular the smooth hypersurfaces of $\mathbb{P}^4$, but also some Fano threefolds, and others. We want to point out that the threefolds which we are interested in, although from a certain point of view look very much like $\mathbb{P}^3$, really give rise to new phenomena. For example on a smooth hypersurface of $\mathbb{P}^4$, of degree at least 2, Horrocks' splitting criterion for a rank 2 vector bundle does not hold (see [7, Theorem 1]). Under the above technical conditions on the threefold X, we extend some properties of rank 2 reflexive sheaves already known on $\mathbb{P}^3$ (see [5], [1], and [8]). In particular, on such a threefold, it holds that for every rank 2 reflexive sheaf $\mathcal{F}$ the third Chern class $c_3(\mathcal{F})$ is always non negative, in fact $c_3(\mathcal{F}) = \lambda(\mathcal{F}) \cdot \delta$, where $\lambda(\mathcal{F})$ is the length of the singular locus of $\mathcal{F}$ and δ is the degree of X with respect to the fixed very ample invertible sheaf $\mathcal{O}_X(1)$. For all general facts not explicitly mentioned we refer to Hartshorne's book [3].

2. Generalities

We work over an algebraically closed field k of characteristic zero. We always consider a smooth polarized threefold $(X, \mathcal{O}_X(1))$, where X is a nonsingular irreducible projective algebraic variety of dimension 3 and $\mathcal{O}_X(1)$ is a fixed very ample invertible sheaf on X. Actually, the polarization on the threefold X is not needed except in Proposition 2 and Remark 1, and from Definition 1 to the end of the section. The other results, like Serre duality, Riemann-Roch formula and Hartshorne-Serre correspondence, do not depend on a choice of a polarization. Given a sheaf $\mathcal{F}$ on X we use the following notations: $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$, and $h^i(\mathcal{F}) = \dim_k H^i(X, \mathcal{F})$, also

$\mathcal{F}^\vee$ denote the dual of $\mathcal{F}$, while V^* is the dual of the k-vector space V. We denote with $A(X) = \oplus_{i=0}^3 A^i(X)$ the Chow ring of X, where obviously $A^1(X) = \mathrm{Pic}(X)$ and $A^0(X) \cong \mathbb{Z}$. For every rank r coherent sheaf $\mathcal{F}$ on X it is defined the i-th Chern class $c_i(\mathcal{F}) \in A^i(X)$ for $i = 0, 1, \dots, r$, and the Chern polynomial of $\mathcal{F}$ is $c_t(\mathcal{F}) = c_0(\mathcal{F}) + c_1(\mathcal{F})t + \cdots + c_r(\mathcal{F})t^r$. We denote with $h = c_1(\mathcal{O}_X(1))$ the class of the “hyperplane” divisor in $A^1(X)$. Given a cycle Z on X of codimension i, that is $Z \in A^i(X)$, we define the degree of Z with respect to $\mathcal{O}_X(1)$ as

$$\deg(Z; \mathcal{O}_X(1)) = Z \cdot h^{3-i}$$

having identified codimension 3 cycles with integers through the degree map. So we denote with $\delta = h^3$ the degree of X with respect to $\mathcal{O}_X(1)$, and with $\bar{c}_i(\mathcal{F}) = \deg(c_i(\mathcal{F}); \mathcal{O}_X(1))$ the degree of the i-th Chern class of a coherent sheaf $\mathcal{F}$ on X. We are interested in rank 2 reflexive sheaves on a smooth threefold $(X, \mathcal{O}_X(1))$. The main references for reflexive sheaves are [5] and [8].

Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. The *singular locus* of $\mathcal{F}$ is the closed set

$$\mathrm{S}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\}$$

and it has codimension ≥ 3 (see [8, Lemma 1.1.10]), thus $\mathrm{S}(\mathcal{F})$ is a finite number of points or is empty, and in the latter case the sheaf $\mathcal{F}$ is actually locally free. Since $\mathcal{F}$ is reflexive we have

$$\mathrm{S}(\mathcal{F}) = \mathrm{Supp}\,(\mathcal{E}xt_X^1(\mathcal{F}, \mathcal{L})) \qquad \text{with} \quad \mathcal{L} \in \mathrm{Pic}(X).$$

We denote with $\lambda(\mathcal{F})$ the length of the 0-dimensional scheme $\mathrm{S}(\mathcal{F})$, therefore we have

$$\lambda(\mathcal{F}) = h^0(\mathcal{E}xt_X^1(\mathcal{F}, \mathcal{L}))$$

and it holds trivially the following

LEMMA 1. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then $\mathcal{F}$ is locally free if and only if $\lambda(\mathcal{F}) = 0$.*

PROPOSITION 1. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then $\mathcal{F}^\vee \simeq \mathcal{F} \otimes (\det \mathcal{F})^{-1}$.*

Proof. See [5, Proposition 1.10]. □

PROPOSITION 2. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and $l \in \mathbb{Z}$. Then the Chern classes of $\mathcal{F}(l) = \mathcal{F} \otimes \mathcal{O}_X(l)$ are*

$$\begin{aligned} c_1(\mathcal{F}(l)) &= c_1(\mathcal{F}) + 2lh \in A^1(X) \\ c_2(\mathcal{F}(l)) &= c_2(\mathcal{F}) + c_1(\mathcal{F}) \cdot lh + l^2h^2 \in A^2(X) \\ c_3(\mathcal{F}(l)) &= c_3(\mathcal{F}) \in A^3(X). \end{aligned}$$

Note in particular that the third Chern class does not change by twisting.

Proof. It is a straightforward computation. □

REMARK 1. Note that the degrees of the first two Chern classes of $\mathcal{F}(l)$ are

$$\bar{c}_1(\mathcal{F}(l)) = \bar{c}_1(\mathcal{F}) + 2\delta l$$
$$\bar{c}_2(\mathcal{F}(l)) = \bar{c}_2(\mathcal{F}) + \bar{c}_1(\mathcal{F})l + \delta l^2$$

while the third Chern class $c_3(\mathcal{F}(l)) = c_3(\mathcal{F})$ can be identified with an integer through the degree map.

PROPOSITION 3 (SERRE DUALITY). *Let $\mathcal{F}$ be a reflexive sheaf on X. Then there are isomorphisms*

$$H^0(\mathcal{F}^\vee \otimes \omega) \cong H^3(\mathcal{F})^*$$
$$H^3(\mathcal{F}^\vee \otimes \omega) \cong H^0(\mathcal{F})^*$$

and an exact sequence

$$0 \to H^1(\mathcal{F}^\vee \otimes \omega) \to H^2(\mathcal{F})^* \to H^0(\mathcal{E}xt^1_X(\mathcal{F}, \omega)) \to H^2(\mathcal{F}^\vee \otimes \omega) \to H^1(\mathcal{F})^* \to 0$$

where $\omega = \omega_X$ is the canonical bundle of X.

Proof. (See [5, Theorem 2.5]) We consider the spectral sequence of local and global Ext functors:

$$E_2^{pq} = H^p(\mathcal{E}xt^q_X(\mathcal{F}, \omega_X)) \ \Rightarrow \ E^{p+q} = \mathrm{Ext}^{p+q}(\mathcal{F}, \omega_X).$$

We have $\mathcal{E}xt^0_X(\mathcal{F}, \omega_X) = \mathcal{H}om_X(\mathcal{F}, \omega_X) \simeq \mathcal{F}^\vee \otimes \omega_X$ and, since $\mathcal{F}$ is reflexive, $\mathcal{E}xt^i_X(\mathcal{F}, \omega_X) = 0$ for $i > 1$. Thus the E_2^{pq} terms are zero except for $q = 0, 1$, furthermore $\mathcal{E}xt^1_X(\mathcal{F}, \omega_X)$ is a coherent sheaf supported at the points in which $\mathcal{F}$ is not locally free, it follows that $E_2^{p1} = 0$ for all $p \neq 0$. The spectral sequence degenerates into two isomorphisms and one 5-term exact sequence. The usual Serre duality theorem says that $\mathrm{Ext}^i_X(\mathcal{F}, \omega_X)$ and $H^{3-i}(\mathcal{F})$ are dual vector spaces (see [3, III Theorem 7.6]), so we obtain the thesis. □

PROPOSITION 4 (RIEMANN-ROCH). *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with Chern classes $c_1(\mathcal{F})$, $c_2(\mathcal{F})$ and $c_3(\mathcal{F})$. Then the Euler-Poincaré characteristic of $\mathcal{F}$ is*

$$\chi(\mathcal{F}) = \frac{1}{6}\Big(c_1(\mathcal{F})^3 - 3c_1(\mathcal{F}) \cdot c_2(\mathcal{F}) + 3c_3(\mathcal{F})\Big) + \frac{1}{4}\Big(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})\Big) \cdot c_1(X) + \\ + \frac{1}{12}c_1(\mathcal{F}) \cdot \Big(c_1(X)^2 + c_2(X)\Big) + \frac{1}{12}c_1(X) \cdot c_2(X)$$

where $c_1(X)$ and $c_2(X)$ are the Chern classes of X, that is the Chern classes of the tangent bundle $\mathcal{T}_X$ of X.

Proof. It is enough to apply the Hirzebruch-Riemann-Roch Theorem ([3, III Theorem 4.1]), which holds for every coherent sheaf on X, not only for locally free sheaves, so we get

$$\chi(\mathcal{F}) = \deg(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(\mathcal{T}_X))_3$$

where $\mathrm{ch}(\mathcal{F})$ is the exponential Chern character of $\mathcal{F}$, $\mathrm{td}(\mathcal{T}_X)$ is the Todd class of the tangent bundle $\mathcal{T}_X$, and $(\ )_3$ denotes the component of degree 3 in $A(X) \otimes \mathbb{Q}$. $\square$

Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and $s \in H^0(\mathcal{F})$ be a non-zero global section of $\mathcal{F}$. The section s defines a map $\varphi\colon \mathcal{O}_X \to \mathcal{F}$ which sends 1 to s. Dualizing such map we get a morphism $\varphi^\vee\colon \mathcal{F}^\vee \to \mathcal{O}_X$ whose image is an ideal sheaf in $\mathcal{O}_X$. This ideal sheaf defines a subscheme Y of X, which we call the *zero locus* of s and we denote with $(s)_0$. Let $\mathcal{L}$ be the kernel of the map $\varphi^\vee$, then we have the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{F}^\vee \to \mathcal{I}_Y \to 0$$

from which we deduce that $\mathcal{L}$ is an invertible sheaf. It may happen that $Y = (s)_0$ is the empty set. In that case the map $\varphi^\vee$ is surjective and we have the exact sequence

$$0 \to \mathcal{L} \to \mathcal{F}^\vee \to \mathcal{O}_X \to 0$$

so $\mathcal{F}$ is an extension of line bundles (in general not the trivial one). It may happen also that $Y = (s)_0$ has a component D of codimension 1. In that case D is a divisor on X and we can associate to the section s another section $s' \in H^0(\mathcal{F} \otimes \mathcal{O}_X(-D))$ with zero locus of codimension ≥ 2.

Now assume that $Y = (s)_0$ is non empty and of codimension ≥ 2. $\mathcal{F}$ is locally free of rank 2 except on the singular locus $\mathrm{S}(\mathcal{F})$ which is a closed set of codimension ≥ 3, so $\mathcal{F}$ is locally free except at finitely many points, therefore the ideal sheaf $\mathcal{I}_Y$ is locally generated by two elements on $X - \mathrm{S}(\mathcal{F})$. Then Y has exactly codimension 2.

On every open set $U \subseteq X - \mathrm{S}(\mathcal{F})$ on which $\mathcal{F}$ is locally free the ideal sheaf $\mathcal{I}_Y$ has a resolution given by the Koszul complex, and therefore on X we get the exact sequence

$$0 \to \det \mathcal{F}^\vee \to \mathcal{F}^\vee \to \mathcal{I}_Y \to 0.$$

Thus given a rank 2 reflexive sheaf $\mathcal{F}$ on X, and given a global section $s \in H^0(\mathcal{F})$, whose zero locus has codimension 2, we obtain a curve $Y = (s)_0$ in X, which may be reducible, disconnected, and also non reduced (here curve means closed scheme of dimension 1, really without 0-dimensional components, as explained in the following theorem).

THEOREM 1 (HARTSHORNE-SERRE CORRESPONDENCE). *Let X be a nonsingular threedimensional projective algebraic variety. Fix an invertible sheaf $\mathcal{L}$ on X such that $H^1(\mathcal{L}^{-1}) = H^2(\mathcal{L}^{-1}) = 0$. Then there is a bijective correspondence between*

(i) *the set of triples $(\mathcal{F}, s, \varphi)$, where $\mathcal{F}$ is a rank 2 reflexive sheaf on X, $s \in H^0(\mathcal{F})$ is a global section whose zero locus $(s)_0$ has codimension 2 and $\varphi\colon \det \mathcal{F} \to \mathcal{L}$ is an isomorphism of invertible sheaves, modulo the equivalence relation $\sim$, where $(\mathcal{F}, s, \varphi) \sim (\mathcal{F}', s', \varphi')$ if there exists an isomorphism $\psi\colon \mathcal{F} \to \mathcal{F}'$ and a non-zero* *element $a \in k$ such that $s' = a\psi(s)$ and $\varphi' = a^2\varphi(\det\psi)^{-1}$, and*

(ii) *the set of pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally Cohen-Macaulay and generically local complete intersection and $\xi \in H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1})$ is a global section which generates the sheaf $\omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}$ except at finitely many points.*

If the triple $(\mathcal{F}, s, \varphi)$ and the pair (Y, ξ) are in correspondence, then there is the exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{F}^\vee \to \mathcal{I}_Y \to 0$$

or equivalently

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_Y \otimes \mathcal{L} \to 0$$

where $\mathcal{L} \simeq \det\mathcal{F}$.

Proof. (See [5, Theorem 4.1] for the case $X = \mathbb{P}^3$). Given $(\mathcal{F}, s, \varphi)$ as in (i) we get a curve Y in X zero locus of the section s, as observed above, moreover by hypothesis $\det\mathcal{F} \simeq \mathcal{L}$ through the morphism φ, so we have the exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{F}^\vee \to \mathcal{I}_Y \to 0. \tag{1}$$

Since $\mathcal{F}$ is locally free except at finitely many points, we see that $\mathcal{I}_Y$ is locally generated by two elements except at those points, so Y is a generically local complete intersection. Sequence (1) also shows that $\mathrm{depth}\mathcal{I}_{Y,x} = 2$ at the points where $\mathcal{F}$ is not locally free and this implies that $\mathrm{depth}\mathcal{O}_{Y,x} = 1$ for all $x \in S(\mathcal{F})$, so Y has no embedded or isolated points, that is Y is a locally Cohen-Macaulay curve of pure dimension. Now the exact sequence (1) defines an element $\xi \in \mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$. Using the exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, the hypothesis $H^1(\mathcal{L}^{-1}) = H^2(\mathcal{L}^{-1}) = 0$ and a result by Horrocks ([6]) we obtain the isomorphism

$$\mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) \cong H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1})$$

which allows to consider ξ as a non-zero element of $H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1})$. Applying the functor $\mathcal{H}om_X(-, \mathcal{L}^{-1})$ to sequence (1) we get the exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{F} \otimes \mathcal{L}^{-1} \to \mathcal{O}_X \xrightarrow{\sigma} \mathcal{E}xt^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) \to \mathcal{E}xt^1_X(\mathcal{F}^\vee, \mathcal{L}^{-1}) \to 0$$

where σ is the map which sends 1 to ξ, as $\mathcal{E}xt^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$ can be identified with $\omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}$. Since the sheaf $\mathcal{E}xt^1_X(\mathcal{F}^\vee, \mathcal{L}^{-1})$ is supported at the points where $\mathcal{F}$ is not locally free, it follows that ξ generates the sheaf $\omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}$ except at those points. This shows how the data (i) determines the data (ii).

Note that starting with a triple $(\mathcal{F}', s', \varphi')$ equivalent to the used one, we get $Y = (s)_0 = (s')_0$ as $s' = a\psi(s)$ (with $a \in k$) and also the extension

$$0 \to \mathcal{L}^{-1} \to \mathcal{F}'^\vee \to \mathcal{I}_Y \to 0$$

which is easy to see being equivalent to the extension (1), so giving the same element ξ in $\mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$. Therefore we have constructed a map between the sets (i) and (ii).

Conversely, suppose given (Y, ξ) as in (ii). Since $H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}) \cong \operatorname{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$, we may think the global section ξ of $\omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}$ as an element of $\operatorname{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$, which in turn determines an extension

$$0 \to \mathcal{L}^{-1} \to \mathcal{G} \to \mathcal{I}_Y \to 0 \tag{2}$$

where $\mathcal{G}$ is a coherent sheaf on X. Since Y is locally Cohen-Macaulay, $\mathcal{G}$ has depth ≥ 2 at every point. Therefore from the sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{G}^\vee \otimes \mathcal{L}^{-1} \to \mathcal{O}_X \to \mathcal{E}xt^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) \to \mathcal{E}xt^1_X(\mathcal{G}, \mathcal{L}^{-1}) \to 0$$

obtained applying the functor $\mathcal{H}om_X(-, \mathcal{L}^{-1})$ to sequence (2), it follows that $\mathcal{G}$ is locally free of rank 2 except at the points where ξ does not generate the sheaf $\mathcal{E}xt^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) \simeq \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}$. Then we define $\mathcal{F} = \mathcal{G}^\vee$, which is a reflexive sheaf (see [5, Corallary 1.2]). Dualizing the map $\mathcal{G} \to \mathcal{I}_Y$ of sequence (2) we get a map $\mathcal{O}_X \to \mathcal{F}$, so the section s is obtained as the image of $1 \in H^0(\mathcal{O}_X)$ through this map. As we saw above, given $\mathcal{F}$ and $s \in H^0(\mathcal{F})$ we get the exact sequence

$$0 \to \det \mathcal{F}^\vee \to \mathcal{F}^\vee \xrightarrow{s^\vee} \mathcal{I}_Y \to 0$$

so on each open set U on which $\mathcal{G}$ and $\mathcal{F}$ are locally free it holds $\det \mathcal{F}_U^\vee \simeq \mathcal{L}_U^{-1}$ and therefore $\det \mathcal{F} \simeq \mathcal{L}$. We remark that ξ determines the extension (2) only up to equivalence of extensions, so $\mathcal{F}$ is determined up to isomorphism, while s is determined up to a non-zero element $a \in k$ and φ up to the square of that a. This shows how the data (ii) determines the data (i).
Now we have maps both ways between the sets (i) and (ii), they are clearly inverse to each other, so we have established the desired correspondence between rank 2 reflexive sheaves and curves. $\square$

If we restrict our attention to rank 2 locally free sheaves on X we obtain the following (see [7, Theorem 3], and also [2] and [4, Theorem 1.1] for the case $X = \mathbb{P}^3$)

THEOREM 2 (SERRE CORRESPONDENCE). *Let X be a nonsingular threedimensional projective algebraic variety. Fix an invertible sheaf $\mathcal{L}$ on X such that $H^1(\mathcal{L}^{-1}) = H^2(\mathcal{L}^{-1}) = 0$. Then there is a bijective correspondence between*

(i) *the set of triples $(\mathcal{E}, s, \varphi)$, where $\mathcal{E}$ is a rank 2 locally free sheaf on X, $s \in H^0(\mathcal{E})$ is a global section whose zero locus $(s)_0$ has codimension 2 and $\varphi\colon \det \mathcal{E} = \wedge^2 \mathcal{E} \to \mathcal{L}$ is an isomorphism of invertible sheaves, modulo the equivalence relation $\sim$, where $(\mathcal{E}, s, \varphi) \sim (\mathcal{E}', s', \varphi')$ if there exists an isomorphism $\psi : \mathcal{E} \to \mathcal{E}'$ and a non-zero element $a \in k$ such that $s' = a\psi(s)$ and $\varphi' = a^2\varphi(\wedge^2\psi)^{-1}$, and*

(ii) *the set of pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally complete intersection and $\xi\colon \mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Y \to \omega_Y$ is an isomorphism of invertible sheaves.*

Proof. Given $(\mathcal{E}, s, \varphi)$ as in (i) we get a curve Y in X zero locus of s and the exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{E}^\vee \to \mathcal{I}_Y \to 0.$$

Since $\mathcal{E}$ is locally free, we see that $\mathcal{I}_Y$ is locally generated by two elements, so Y is locally complete intersection and of pure dimension 1. The above sequence defines an element $\xi \in \mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$. Now it holds

$$\begin{aligned}\mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) &\cong H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1})\\ &\cong \mathrm{Hom}_Y(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Y, \omega_Y)\end{aligned}$$

so we can consider ξ as a morphism from $\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Y$ to ω_Y. In fact it is an isomorphism, because locally the extension ξ defines a Koszul complex and so ξ cannot vanish in each point. This shows how the data (i) determines the data (ii).
Notice that, like in the proof of Theorem 1, all is defined up to the equivalence relation $\sim$ between triples. So we have a map between the sets (i) and (ii).
Conversely, given (Y, ξ) as in (ii), we can identify the isomorphism ξ with an element of $\mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1})$, so we get an extension

$$(*) \qquad 0 \to \mathcal{L}^{-1} \to \mathcal{G} \to \mathcal{I}_Y \to 0.$$

where $\mathcal{G}$ is a coherent sheaf on X. Since ξ is an isomorphism, then ξ is locally a generator of the corresponding Ext module, and this implies that $\mathcal{G}$ is locally free of rank 2. Then we define $\mathcal{E} = \mathcal{G}^\vee$. Dualizing the map $\mathcal{G} \to \mathcal{I}_Y$ of the extension $(*)$ we get a map $\mathcal{O}_X \to \mathcal{E}$, so the section s is obtained as the image of $1 \in H^0(\mathcal{O}_X)$ through this map. Since we have also the exact sequence

$$0 \to \wedge^2(\mathcal{E}^\vee) \to \mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{I}_Y \to 0$$

we deduce an isomorphism $\varphi\colon \wedge^2 \mathcal{E} \to \mathcal{L}$. We note that ξ determines the extension $(*)$ only up to equivalence of extensions, so $\mathcal{E}$ is determined up to isomorphism, while s is determined up to a non-zero element $a \in k$ and φ up to the square of that a. This shows how the data (ii) determines the data (i).
The two maps between the sets (i) and (ii) are clearly inverse to each other, therefore we have established the desired correspondence between rank 2 locally free sheaves and curves. $\square$

COROLLARY 1. *Let the locally free sheaf $\mathcal{E}$ on X and the locally complete intersection curve Y in X be in correspondence. If Y is such that $h^0(\mathcal{O}_Y) = 1$, in particular if it is reduced and connected, then the locally free sheaf $\mathcal{E}$ is unique.*

Proof. By Serre correspondence the invertible sheaves $\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_Y$ and ω_Y on Y are isomorphic, so $\omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_Y$. So we have

$$\mathrm{Ext}^1_X(\mathcal{I}_Y, \mathcal{L}^{-1}) \cong H^0(Y, \omega_Y \otimes \omega_X^{-1} \otimes \mathcal{L}^{-1}) \cong H^0(\mathcal{O}_Y).$$

Therefore, if $H^0(\mathcal{O}_Y) \cong k$, then $\mathcal{E}$ is uniquely determined. $\square$

DEFINITION 1. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. We define the* first relevant level *of $\mathcal{F}$ as the integer*

$$\alpha = \alpha(\mathcal{F}) = \min\{l \in \mathbb{Z} \mid h^0(\mathcal{F}(l)) \neq 0\},$$

it is the minimum twist for which $\mathcal{F}$ has a non-zero global section.

LEMMA 2. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and $s \in H^0(\mathcal{F}(\alpha))$ a non-zero global section.*

1. *The subspace $sH^0(\mathcal{O}_X(l-\alpha))$ of $H^0(\mathcal{F}(l))$ is isomorphic to $H^0(\mathcal{O}_X(l-\alpha))$; in particular $h^0(\mathcal{F}(l)) \geq h^0(\mathcal{O}_X(l-\alpha))$.*
2. *If $s' \in H^0(\mathcal{F}(t))$, for some t, is a global section not belonging to $sH^0(\mathcal{O}_X(t-\alpha))$, then the sum of the subspaces $sH^0(\mathcal{O}_X(l-\alpha))$ and $s'H^0(\mathcal{O}_X(l-t))$ of $H^0(\mathcal{F}(l))$ is direct; in particular $h^0(\mathcal{F}(l)) \geq h^0(\mathcal{O}_X(l-\alpha)) + h^0(\mathcal{O}_X(l-t))$.*
3. *$\mathcal{F}$ splits if and only if for some t equality holds in (2) for all l.*

Proof. The proof of Lemma 1.1 in [10] works well also in our situation. □

DEFINITION 2. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. We define the* second relevant level *of $\mathcal{F}$ as the integer*

$$\beta = \beta(\mathcal{F}) = \min\{l \in \mathbb{Z} \mid h^0(\mathcal{F}(l)) > h^0(\mathcal{O}_X(l-\alpha))\},$$

it is the minimum twist for which $\mathcal{F}$ has a "new" section after the first one.
If $\mathcal{F}$ is non split we define also the third relevant level *of $\mathcal{F}$ as the integer*

$$\gamma = \gamma(\mathcal{F}) = \min\{l \in \mathbb{Z} \mid h^0(\mathcal{F}(l)) > h^0(\mathcal{O}_X(l-\alpha)) + h^0(\mathcal{O}_X(l-\beta))\}.$$

Obviously it holds $\alpha \leq \beta \leq \gamma$.

THEOREM 3. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then*

1. *for $l \gg 0$ the general section of $\mathcal{F}(l)$ gives rise to an integral curve;*
2. *if $\mathcal{F}$ is locally free, for $l \gg 0$ the general section of $\mathcal{F}(l)$ defines a smooth irreducible curve.*

Proof. 1. See [9, Teorema 3].
2. Let $\mathcal{F}$ be locally free, then $S(\mathcal{F}) = \emptyset$, so again by [9, Teorema 3] we get the thesis. □

3. Reflexive sheaves on a class of smooth threefolds

From now on we consider a smooth polarized threefold $(X, \mathcal{O}_X(1))$ which verifies the following conditions:

1. $\mathrm{Pic}(X) \cong \mathbb{Z}$ (generated by $\mathcal{O}_X(1)$),
2. $H^1_*\mathcal{O}_X = \oplus_{n\in\mathbb{Z}} H^1(\mathcal{O}_X(n)) = 0$.

By the first hypothesis every invertible sheaf on X is of type $\mathcal{O}_X(a)$ with $a \in \mathbb{Z}$ (up to isomorphism), so we pose $\omega_X = \mathcal{O}_X(\varepsilon)$. Furthermore by Serre duality the second hypothesis implies that $H^2_*\mathcal{O}_X = 0$. We set, as above, $\delta = h^3$, that is δ represents the degree of X with respect to $\mathcal{O}_X(1)$. As by assumption $A^1(X) = \mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$ through the map $h \mapsto 1$, where $h = c_1(\mathcal{O}_X(1))$, we identify the first Chern class of a reflexive sheaf with a whole number.
Some smooth threefolds which verify the above conditions are:

- the projective space $\mathbb{P}^3$;
- the smooth hypersurfaces of $\mathbb{P}^4$;
- the smooth complete intersections of dimension 3;
- some Fano threefolds, like the intersection of the Grassmannian of the lines of $\mathbb{P}^5$, in its Plücker embedding, with five general hyperplanes of $\mathbb{P}^{14}$, or the intersection of the Grassmannian of the lines of $\mathbb{P}^4$, in its Plücker embedding, with two general hyperplanes and a general quadric of $\mathbb{P}^9$.

PROPOSITION 5. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and $l \in \mathbb{Z}$. Then the Chern classes of $\mathcal{F}(l)$ are*

$$\begin{aligned} c_1(\mathcal{F}(l)) &= c_1(\mathcal{F}) + 2l \in \mathbb{Z} \\ c_2(\mathcal{F}(l)) &= c_2(\mathcal{F}) + c_1(\mathcal{F})lh^2 + l^2h^2 \in A^2(X) \\ c_3(\mathcal{F}(l)) &= c_3(\mathcal{F}) \end{aligned}$$

and the degree of the second Chern class of $\mathcal{F}(l)$ is

$$\bar{c}_2(\mathcal{F}(l)) = \bar{c}_2(\mathcal{F}) + c_1(\mathcal{F})\delta l + \delta l^2 \in \mathbb{Z}.$$

Proof. Apply Proposition 2 taking into account the identification of the first Chern class with an integer. □

PROPOSITION 6. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1. Then $\mathcal{F}^\vee \simeq \mathcal{F}(-c_1)$.*

Proof. By assumption $c_1(\det \mathcal{F}) = c_1(\mathcal{F}) = c_1$, so $\det \mathcal{F} \simeq \mathcal{O}_X(c_1)$, then by Proposition 1 we have

$$\mathcal{F}^\vee \simeq \mathcal{F} \otimes (\det \mathcal{F})^{-1} \simeq \mathcal{F} \otimes \mathcal{O}_X(-c_1) = \mathcal{F}(-c_1).$$

□

PROPOSITION 7. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then the Chern polynomial of $\mathcal{F}^\vee$ is*

$$c_t(\mathcal{F}^\vee) = 1 - c_1(\mathcal{F})t + c_2(\mathcal{F})t^2 + c_3(\mathcal{F})t^3.$$

Proof. We have $\mathcal{F}^\vee \simeq \mathcal{F}(-c_1)$ with $c_1 = c_1(\mathcal{F})$, so applying Proposition 5 we obtain the above formula. □

PROPOSITION 8 (SERRE DUALITY). *Let $\mathcal{F}$ be a reflexive sheaf on X with first Chern class c_1. Then for every $l \in \mathbb{Z}$ there are isomorphisms*

$$H^0(\mathcal{F}(m)) \cong H^3(\mathcal{F}(l))^*$$
$$H^3(\mathcal{F}(m)) \cong H^0(\mathcal{F}(l))^*$$

and an exact sequence

$$0 \to H^1(\mathcal{F}(m)) \to H^2(\mathcal{F}(l))^* \to H^0(\mathcal{E}xt^1_X(\mathcal{F}(l), \omega_X)) \to$$
$$\to H^2(\mathcal{F}(m)) \to H^1(\mathcal{F}(l))^* \to 0$$

where $m = -l - c_1 + \varepsilon$.

Proof. Apply Proposition 3 with $\mathcal{F}(l)^\vee \otimes \omega_X \simeq \mathcal{F}(m)$. □

REMARK 2. By Serre's vanishing theorem ([3, III Theorem 5.2]) it holds that $H^i(\mathcal{F}(l)) = 0$ for $i > 0$ and $l \gg 0$. If $\mathcal{F}$ is locally free on X this implies that $H^i(\mathcal{F}(l)) = 0$ for $i < 3$ and $l \ll 0$. If $\mathcal{F}$ is reflexive, the above version of Serre duality shows that $H^i(\mathcal{F}(l)) = 0$ for $i = 0, 1$ and $l \ll 0$, and that $H^2(\mathcal{F}(l))$ is of constant dimension $\lambda(\mathcal{F}) = h^0(\mathcal{E}xt^1_X(\mathcal{F}, \omega_X))$ for $l \ll 0$.

PROPOSITION 9 (RIEMANN-ROCH). *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with Chern classes c_1, c_2 and c_3. Then the Euler-Poincaré characteristic of $\mathcal{F}$ is*

$$\chi(\mathcal{F}) = \frac{1}{6}\left(c_1^3\delta - 3c_1\bar{c}_2 + 3c_3\right) + \frac{1}{4}\left(2\bar{c}_2 - c_1^2\delta\right)\varepsilon + \frac{1}{12}c_1\left(\varepsilon^2\delta + \tau\right) - \frac{1}{12}\varepsilon\tau$$

where $\tau = \bar{c}_2(X)$.

Proof. It is the Riemann-Roch formula of Proposition 4 where $c_1(\mathcal{F}) = c_1 h$ with $c_1 \in \mathbb{Z}$, $c_1(X) = -\varepsilon h$ and $\bar{c}_2(X) = \tau$. □

COROLLARY 2. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and $l \in \mathbb{Z}$. Then the Euler-Poincaré characteristic of $\mathcal{F}(l)$ is*

$$\begin{aligned}\chi(\mathcal{F}(l)) =& \frac{1}{3}\delta l^3 + \frac{1}{2}(c_1 - \varepsilon)\delta l^2 + \left(\frac{1}{2}c_1^2\delta - \bar{c}_2 - \frac{1}{2}c_1\varepsilon\delta + \frac{1}{6}\varepsilon^2\delta + \frac{1}{6}\tau\right)l + \\ &+ \frac{1}{6}\left(c_1^3\delta - 3c_1\bar{c}_2 + 3c_3\right) + \frac{1}{4}\left(2\bar{c}_2 - c_1^2\delta\right)\varepsilon + \\ &+ \frac{1}{12}c_1\left(\varepsilon^2\delta + \tau\right) - \frac{1}{12}\varepsilon\tau\end{aligned}$$

REMARK 3. The Riemann-Roch formula for a rank 2 reflexive sheaf $\mathcal{F}$ of Proposition 9 becomes

$$\chi(\mathcal{F}) = \frac{1}{6}c_1^3 - \frac{1}{2}c_1c_2 - 2c_2 + c_1^2 + \frac{11}{6}c_1 + \frac{1}{2}c_3 + 2$$

if X is the projective space $\mathbb{P}^3$, since in this case $\varepsilon = -4$, $\tau = 6$ and $\delta = 1$, while we have

$$\chi(\mathcal{F}) = \frac{1}{6}c_1^3\delta - \frac{1}{2}c_1c_2 + \frac{1}{2}c_2(\delta-5) + \frac{1}{4}c_1^2\delta(5-\delta) + \frac{1}{2}c_3 + \\ + \frac{1}{12}c_1\delta(35-15\delta+2\delta^2) + \frac{1}{12}\delta(5-\delta)(10-5\delta+\delta^2)$$

if X is a smooth hypersurface of degree δ in $\mathbb{P}^4$, since in this case $\varepsilon = \delta - 5$ and $\tau = \delta(10-5\delta+\delta^2)$.

LEMMA 3. *Let P be a (closed) point of X. Then the skyscraper sheaf $\mathcal{O}_P$ has Chern polynomial $c_t(\mathcal{O}_P) = 1 + 2\delta t^3$.*

Proof. Given $P \in X$, in the local ring $\mathcal{O}_{X,P}$ the point P is defined by the maximal ideal, then there exists an open neighbourhood U of P in which P is a complete intersection, so the sheaf $\mathcal{O}_P$ has a resolution on U of type

$$0 \to \mathcal{O}_U(-3) \to \mathcal{O}_U(-2)^3 \to \mathcal{O}_U(-1)^3 \to \mathcal{O}_U \to \mathcal{O}_P \to 0.$$

The Chern polynomial is multiplicative, then we have in $A(U)[t]$

$$c_t(\mathcal{O}_P) = 1 \cdot (1-\hbar t)^{-3}(1-2\hbar t)^3(1-3\hbar t)^{-1}$$

where $\hbar = c_1(\mathcal{O}_U(1))$. Therefore

$$c_t(\mathcal{O}_P) = \frac{(1-2\hbar t)^3}{(1-\hbar t)^3(1-3\hbar t)} = 1 + 2\hbar^3 t^3.$$

Now if we denote with $Y = X - U$ the complementary closed set we have the exact sequence (see [3, page 429])

$$A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U) \to 0$$

where $i : Y \to X$ and $j : U \to X$ are the inclusion maps. As it holds

$$j^*(h) = j^*c_1(\mathcal{O}_X(1)) = c_1(j^*\mathcal{O}_X(1)) = c_1(\mathcal{O}_U(1)) = \hbar$$

it follows that in $A(X)[t]$ we have $c_t(\mathcal{O}_P) = 1 + 2h^3t^3 = 1 + 2\delta t^3$. □

LEMMA 4. *Let $\mathcal{Q}$ be a sheaf on X supported in a finite number of points of length λ. Then $c_t(\mathcal{Q}) = 1 + 2\lambda\delta t^3$.*

Proof. By induction on λ. If $\lambda = 1$ by the above Lemma the statement holds. Now assume that the thesis holds for sheaves of length $\lambda - 1$, with $\lambda > 1$. Let $\mathcal{Q}$ be a sheaf of length λ. Let P be a point in the support of $\mathcal{Q}$. Then the skyscraper sheaf $\mathcal{O}_P$ in P is a subsheaf of $\mathcal{Q}$ and we have the exact sequence

$$0 \to \mathcal{O}_P \to \mathcal{Q} \to \mathcal{P} \to 0$$

where $\mathcal{P}$ is a sheaf of length $\lambda - 1$. Since c_t is multiplicative we get

$$c_t(\mathcal{Q}) = c_t(\mathcal{O}_P)\, c_t(\mathcal{P})$$

but by induction it results

$$\begin{aligned} c_t(\mathcal{O}_P) &= 1 + 2\delta t^3 \\ c_t(\mathcal{P}) &= 1 + 2(\lambda - 1)\delta t^3 \end{aligned}$$

and so

$$c_t(\mathcal{Q}) = (1 + 2\delta t^3)(1 + 2(\lambda - 1)\delta t^3) = 1 + 2\lambda\delta t^3.$$

□

PROPOSITION 10. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then $c_3(\mathcal{F}) = \lambda\delta$, where λ is the length of the singular locus of $\mathcal{F}$.*

Proof. $\mathcal{F}$ is a reflexive sheaf, therefore it has homological dimension ≤ 1 in every point of X, hence there exists an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

with $\mathcal{E}_0$ and $\mathcal{E}_1$ locally free sheaves. Then

$$c_t(\mathcal{F}) = c_t(\mathcal{E}_0)\, c_t(\mathcal{E}_1)^{-1}.$$

By duality we get the exact sequence

$$0 \to \mathcal{F}^\vee \to \mathcal{E}_0^\vee \to \mathcal{E}_1^\vee \to \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X) \to 0$$

hence

$$c_t(\mathcal{F}^\vee) = c_t(\mathcal{E}_0^\vee)\, c_t(\mathcal{E}_1^\vee)^{-1} c_t(\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X)).$$

We remind that for a locally free sheaf $\mathcal{E}$ it holds $c_t(\mathcal{E}^\vee) = c_{-t}(\mathcal{E})$. Moreover the sheaf $\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X)$ is supported at the points where $\mathcal{F}$ is not locally free and it has length $\lambda = \lambda(\mathcal{F})$, therefore by Lemma 4

$$c_t(\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X)) = 1 + 2\lambda\delta t^3.$$

So we obtain

$$\begin{aligned} c_t(\mathcal{F}^\vee) &= c_t(\mathcal{E}_0^\vee)\, c_t(\mathcal{E}_1^\vee)^{-1} c_t(\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X)) \\ &= c_{-t}(\mathcal{E}_0)\, c_{-t}(\mathcal{E}_1)^{-1}\, (1 + 2\lambda\delta t^3) \\ &= c_{-t}(\mathcal{F})(1 + 2\lambda\delta t^3) \\ &= (1 - c_1(\mathcal{F})t + c_2(\mathcal{F})t^2 - c_3(\mathcal{F})t^3)(1 + 2\lambda\delta t^3) \\ &= 1 - c_1(\mathcal{F})t + c_2(\mathcal{F})t^2 + (2\lambda\delta - c_3(\mathcal{F}))t^3. \end{aligned}$$

Comparing now with the expression of $c_t(\mathcal{F}^\vee)$ of Proposition 7 we get

$$c_3(\mathcal{F}) = 2\lambda\delta - c_3(\mathcal{F})$$

from which

$$c_3(\mathcal{F}) = \lambda\delta.$$

$\square$

REMARK 4. Notice that for $X = \mathbb{P}^3$ we recover the well known fact $c_3(\mathcal{F}) = h^0(\mathcal{E}xt^1_{\mathbb{P}^3}(\mathcal{F}, \omega_{\mathbb{P}^3}))$ (see [5, Proposition 2.6]), i.e. $c_3(\mathcal{F}) = \lambda(\mathcal{F})$.

COROLLARY 3. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. Then $c_3(\mathcal{F}) \geq 0$ and it holds $c_3(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is locally free.*

Proof. By the above Proposition $c_3(\mathcal{F}) = \lambda\delta$, where $\lambda = h^0(\mathcal{E}xt^1_X(\mathcal{F}, \omega_X))$ and $\delta = h^3$. Since $\delta > 0$ and $\lambda \geq 0$, it follows $c_3(\mathcal{F}) \geq 0$ and we have equality if and only if $\lambda = 0$, that is if and only if $\mathcal{F}$ is locally free (cf. Lemma 1). $\square$

THEOREM 4 (HARTSHORNE-SERRE CORRESPONDENCE). *Fix an integer c_1. Then there is a bijective correspondence between*

(i) *pairs $(\mathcal{F}, s)$, where $\mathcal{F}$ is a rank 2 reflexive sheaf on X with $c_1(\mathcal{F}) = c_1$ and $s \in H^0(\mathcal{F})$ is a global section whose zero locus has codimension 2, and*

(ii) *pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally Cohen-Macaulay and generically local complete intersection and $\xi \in H^0(Y, \omega_Y(-\varepsilon - c_1))$ is a global section which generates the sheaf $\omega_Y(-\varepsilon - c_1)$ except at finitely many points.*

If the pairs $(\mathcal{F}, s)$ and (Y, ξ) are in correspondence, then there is the exact sequence

$$0 \to \mathcal{O}_X(-c_1) \to \mathcal{F}^\vee \to \mathcal{I}_Y \to 0$$

or equivalently

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_Y(c_1) \to 0.$$

Proof. It is Theorem 1 under the additional hypothesis on X. $\square$

REMARK 5. If the sheaf $\mathcal{F}$ corresponds to the curve Y we have the above exact sequence which gives a cohomological connection between the reflexive sheaf $\mathcal{F}$ and the curve Y zero locus of a section of $\mathcal{F}$. In fact we get in cohomology the exact sequence

$$0 \to H^0(\mathcal{O}_X(l)) \to H^0(\mathcal{F}(l)) \to H^0(\mathcal{I}_Y(l + c_1)) \to 0$$

and the isomorphism

$$H^1(\mathcal{F}(l)) \cong H^1(\mathcal{I}_Y(l + c_1))$$

for all integer l.

If we restrict our attention to rank 2 locally free sheaves on X we obtain the following

THEOREM 5 (SERRE CORRESPONDENCE). *Fix an integer c_1. Then there is a bijective correspondence between*

(i) *pairs $(\mathcal{E}, s)$, where $\mathcal{E}$ is a rank 2 locally free sheaf on X with $c_1(\mathcal{E}) = c_1$ and $s \in H^0(\mathcal{E})$ is a global section whose zero locus has codimension 2, and*

(ii) *pairs (Y, ξ), where Y is a closed subscheme of X of pure dimension 1, locally complete intersection with $\omega_Y \simeq \mathcal{O}_Y(\varepsilon + c_1)$ and $\xi \in H^0(Y, \omega_Y(-\varepsilon - c_1))$ is a global section which generates the sheaf $\omega_Y(-\varepsilon - c_1)$ everywhere.*

Proof. It is Theorem 2 under the additional hypothesis on X. □

COROLLARY 4. *Let $\mathcal{E}$ be a locally free sheaf on X corresponding to the curve Y. Then Y is a complete intersection if and only if $\mathcal{E}$ splits.*

Proof. If $\mathcal{E}$ splits, then $\mathcal{E} \simeq \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_2)$. Let $Y = (s)_0$, then $s = (s_1, s_2)$ with $s_i \in H^0(\mathcal{O}_X(d_i))$, $i = 1, 2$, so Y is a complete intersection, namely of the two "surfaces" (i.e. effective divisors) S_1 and S_2 zero locus of the sections s_1 and s_2 respectively.

Conversely, if Y is a complete intersection of two surfaces S and T of degrees $a\delta$ and $b\delta$ respectively, then we have the Koszul complex

$$0 \to \mathcal{O}_X(-a - b) \to \mathcal{O}_X(-a) \oplus \mathcal{O}_X(-b) \to \mathcal{I}_Y \to 0,$$

so one possibility for $\mathcal{E}$ is the direct sum $\mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$, but Y has $h^0(\mathcal{O}_Y) = 1$. In fact it holds $h^0(\mathcal{I}_Y) = 0$ and $h^1(\mathcal{I}_Y) = h^1(\mathcal{O}_X(-a) \oplus \mathcal{O}_X(-b)) = 0$, then it follows $h^0(\mathcal{O}_Y) = h^0(\mathcal{O}_X) = 1$. Therefore, by Corollary 1, $\mathcal{E}$ is uniquely determined, and hence $\mathcal{E}$ splits. □

PROPOSITION 11. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X and Y a curve in X of degree d and arithmetic genus p_a which corresponds to $\mathcal{F}$. Then it holds*

$$\bar{c}_2(\mathcal{F}) = d$$
$$\lambda(\mathcal{F}) = 2p_a - 2 - d(\varepsilon + c_1).$$

Proof. Let $U = X - \mathrm{S}(\mathcal{F})$ be the open set complementary of the singular locus of $\mathcal{F}$. On U, $\mathcal{F}$ is a locally free sheaf, hence by the properties of Chern classes it holds $c_2(\mathcal{F}_U) = Y \cap U$ in $A^2(U)$, where $Y \cap U$ represents the cycle associated to the curve Y restricted to U. Therefore we have $d = \bar{c}_2(\mathcal{F})$.

By the proof of the Hartshorne-Serre correspondence we have the exact sequence

$$\mathcal{O}_X \to \omega_Y(-\varepsilon - c_1) \to \mathcal{E}xt^1_X(\mathcal{F}^\vee, \mathcal{O}_X(-c_1)) \simeq \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X) \to 0$$

and $\lambda(\mathcal{F}) = h^0(\mathcal{E}xt^1_X(\mathcal{F}, \mathcal{O}_X))$. Therefore

$$\lambda(\mathcal{F}) = \deg \omega_Y(-\varepsilon - c_1) = 2p_a - 2 - d(\varepsilon + c_1).$$

□

PROPOSITION 12. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. If there exists a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{L}' \to 0$$

where $\mathcal{L}$ and $\mathcal{L}'$ are invertible sheaves, then $\mathcal{F}$ splits as the direct sum of $\mathcal{L}$ and $\mathcal{L}'$.

Proof. By hypothesis there are $a, b \in \mathbb{Z}$ such that $\mathcal{L} \simeq \mathcal{O}_X(a)$ and $\mathcal{L}' \simeq \mathcal{O}_X(b)$, hence the sequence of the statement is an extension of $\mathcal{O}_X(b)$ by $\mathcal{O}_X(a)$, but it holds

$$\mathrm{Ext}^1_X(\mathcal{O}_X(b), \mathcal{O}_X(a)) \cong \mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X(a-b)) \cong H^1(X, \mathcal{O}_X(a-b)) = 0$$

that is the only possible extension is the trivial one, so the sequence splits and therefore $\mathcal{F} \simeq \mathcal{L} \oplus \mathcal{L}'$. □

PROPOSITION 13. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. $\mathcal{F}$ splits if and only if there is a global section s of $\mathcal{F}(\alpha)$ such that $(s)_0 = \emptyset$.*

Proof. Assume there is $s \in H^0(\mathcal{F}(\alpha))$ such that $(s)_0 = \emptyset$, then we have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F}(\alpha) \to \mathcal{O}_X(c_1 + 2\alpha) \to 0$$

namely

$$0 \to \mathcal{O}_X(-\alpha) \to \mathcal{F} \to \mathcal{O}_X(c_1 + \alpha) \to 0$$

so, by the above Proposition, we obtain that $\mathcal{F}$ splits.
Conversely, let $\mathcal{F} = \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ with $a \geq b$, then $\alpha(\mathcal{F}) = -a$ and the section $s = (1, 0)$ of $\mathcal{F}(-a)$ is such that $(s)_0 = \emptyset$. □

PROPOSITION 14. *Let $\mathcal{F}$ be a non split rank 2 reflexive sheaf on X. Then*

1. *the zero locus of every non-zero section of $\mathcal{F}(\alpha)$ is a curve;*
2. *the zero locus of every section of $\mathcal{F}(l)$, with $\alpha < l < \beta$, is not a curve;*
3. *the zero locus of the general section of $\mathcal{F}(l)$, with $l \geq \beta$, is a curve.*

Proof. 1. Let s be a non-zero section of $\mathcal{F}(\alpha)$. If $\mathrm{codim}\,(s)_0 = 1$, then the zero locus of s contains a "surface" (i.e. an effective divisor) S in X, thus $h^0(\mathcal{F}(\alpha - d)) \neq 0$, where $S \in |\mathcal{O}_X(d)|$, in contradiction with the definition of α. So every non-zero global section at the first relevant level gives rise to a curve.
2. If $\alpha < \beta$, then $h^0(\mathcal{F}(\alpha)) = 1$, so there is only one relevant section s at the first level of $\mathcal{F}$, and obviously every non-zero section of $\mathcal{F}(l)$, with $\alpha < l < \beta$, is of type sx, where $x \in H^0(\mathcal{O}_X(l - \alpha))$, so it has codimension 1.
3. (see also [1, Theorem 0.1]) Let $l \geq \beta$. We choose, once and for all, a basis $\{h_1, \ldots, h_m\}$ for the vector space $H^0(\mathcal{O}_X(l - \alpha))$ and a basis $\{g_1, \ldots, g_n\}$ for the vector space $H^0(\mathcal{O}_X(l - \beta))$. Given the non-zero sections $s_0 \in H^0(\mathcal{F}(\alpha))$ and $s_1 \in H^0(\mathcal{F}(\beta))$, with $s_1 \notin s_0 H^0(\mathcal{O}_X(\beta - \alpha))$, that is s_1 not a multiple of s_0, we set

$$s_a = (\sum_{i=1}^{m} a_i h_i) s_0 + (\sum_{j=1}^{n} a_{m+j} g_j) s_1$$

where $a = (a_1, a_2, \ldots, a_{m+n}) \in k^{m+n}$. Then s_a is a section of $\mathcal{F}(l)$. Choose a non empty affine open subset U of X such that:
a) $\mathcal{F}_U = \mathcal{F} \otimes \mathcal{O}_U$ is free;
b) $s_{0|U}$ and $s_{1|U}$ are linearly independent.
Assume that, for general a, $(s_a)_0$ contains a component of codimension 1. If $(s_a)_0$ contains a "surface" F_a, varying with a, then for general a such a surface cannot be contained in the closed set $X - U$, and so it must be contained in U, but this is impossible since by [9, Lemma 1(i)], for general a, $(s_a)_0 \cap U$ is empty or has codimension 2 in U. Thus, for general a, hence for all a, $(s_a)_0$ contains a fixed surface F. Now choose two elements x and y of a basis of the vector space $H^0(\mathcal{O}_X(1))$. Then the sections $\sigma = s_0 x^{l-\alpha}$ and $\tau = s_0 y^{l-\alpha}$ of $\mathcal{F}(l)$ have zero loci containing no common surface, contradiction. Therefore for general a it must hold $\operatorname{codim}(s_a)_0 = 2$. □

COROLLARY 5. *If $h^0(\mathcal{F}(l)) \neq h^0(\mathcal{O}_X(t))$ for every t, then $\mathcal{F}(l)$ has a section whose zero locus is a curve.*

Proof. Assume that $\mathcal{F}(l)$ has no section whose zero locus is a curve. Thus, if $l < \alpha$, then $h^0(\mathcal{F}(l)) = 0 = h^0(\mathcal{O}_X(-1))$ (e.g.), while if $\alpha < l < \beta$, then $h^0(\mathcal{F}(l)) = h^0(\mathcal{O}_X(l - \alpha))$. □

DEFINITION 3. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X. We define the* slope *of $\mathcal{F}$ (with respect to $\mathcal{O}_X(1)$) as the rational number*

$$\mu(\mathcal{F}) = \frac{c_1}{2}$$

where c_1 is the first Chern class of $\mathcal{F}$, while the slope of an invertible sheaf $\mathcal{L}$ is the whole number $\mu(\mathcal{L}) = c_1(\mathcal{L})$.
$\mathcal{F}$ is stable *(in the sense of Mumford-Takemoto) if for every invertible subsheaf $\mathcal{L}$ of $\mathcal{F}$ it holds $\mu(\mathcal{L}) < \mu(\mathcal{F})$.*
$\mathcal{F}$ is semistable *if for every invertible subsheaf $\mathcal{L}$ of $\mathcal{F}$ it holds $\mu(\mathcal{L}) \leq \mu(\mathcal{F})$.*
$\mathcal{F}$ is nonstable *if it is not semistable.*

REMARK 6. If $\mathcal{F}$ has odd first Chern class, then we have that $\mathcal{F}$ is stable if and only if $\mathcal{F}$ is semistable.

PROPOSITION 15.

1. *Every invertible sheaf is stable.*

2. *$\mathcal{F}$ is stable (respectively semistable) if and only if $\mathcal{F}^\vee$ is stable (respectively semistable).*

3. *$\mathcal{F}$ is stable (respectively semistable) if and only if $\mathcal{F}(l)$ is stable (respectively semistable), with $l \in \mathbb{Z}$.*

Proof. See [8, Chap. II Lemma 1.2.4]. □

PROPOSITION 16. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with first Chern class c_1 and first relevant level α. Then*

1. *$\mathcal{F}$ is stable if and only if $c_1 + 2\alpha > 0$;*
2. *$\mathcal{F}$ is semistable if and only if $c_1 + 2\alpha \geq 0$;*
3. *$\mathcal{F}$ is nonstable if and only if $c_1 + 2\alpha < 0$.*

Proof. 1. As $\mathrm{Pic}(X) \cong \mathbb{Z}$ every invertible sheaf on X is of type $\mathcal{O}_X(t)$ with $t \in \mathbb{Z}$. Then $\mathcal{F}$ is stable if and only if for each invertible subsheaf $\mathcal{O}_X(t)$ of $\mathcal{F}$ it holds $t < c_1/2$. By definition $h^0(\mathcal{F}(\alpha)) \neq 0$, then there is a non-zero global section of $\mathcal{F}(\alpha)$ which defines an injective map $\mathcal{O}_X(-\alpha) \to \mathcal{F}$, therefore $\mathcal{F}$ is stable if and only if $-\alpha < c_1/2$, that is $c_1 + 2\alpha > 0$.
2. It is analogous to 1.
3. Let $\mathcal{F}$ be nonstable, then there is an invertible subsheaf $\mathcal{O}_X(t)$ of $\mathcal{F}$ such that $t > c_1/2$. Hence there is an injective map $\mathcal{O}_X \to \mathcal{F}(-t)$, that is $h^0(\mathcal{F}(-t)) \neq 0$, therefore by definition $-t \geq \alpha$, i.e. $t \leq -\alpha$, then $-\alpha > c_1/2$, that is $c_1 + 2\alpha < 0$.
Conversely, assume that it holds $c_1 + 2\alpha < 0$. By definition $h^0(\mathcal{F}(\alpha)) \neq 0$, then there is an injective map $\mathcal{O}_X(-\alpha) \to \mathcal{F}$ with $-\alpha > c_1/2$, that is $\mathcal{F}$ is nonstable. ☐

REMARK 7. Note that the quantity $c_1 + 2\alpha$ is invariant by twisting, in fact for all $l \in \mathbb{Z}$ we have

$$c_1(\mathcal{F}(l)) + 2\alpha(\mathcal{F}(l)) = c_1(\mathcal{F}) + 2l + 2(\alpha(\mathcal{F}) - l) = c_1(\mathcal{F}) + 2\alpha(\mathcal{F}).$$

COROLLARY 6. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with $c_1 = 0$ or -1, then*

1. *$\mathcal{F}$ is stable if and only if $\alpha > 0$;*
2. *$\mathcal{F}$ is semistable if and only if $\alpha + c_1 \geq 0$;*
3. *$\mathcal{F}$ is nonstable if and only if $\alpha + c_1 < 0$.*

COROLLARY 7. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X with $c_1 = 0$ or -1. Then $\mathcal{F}$ is stable if and only if $H^0(\mathcal{F}) = 0$. If $c_1 = 0$, then $\mathcal{F}$ is semistable if and only if $H^0(\mathcal{F}(-1)) = 0$.*

Proof. (See [5, Lemma 3.1] for the case $X = \mathbb{P}^3$) By the above Corollary, $\mathcal{F}$ is stable if and only if $\alpha > 0$, that is if and only if $H^0(\mathcal{F}) = 0$. If $c_1 = 0$, then $\mathcal{F}$ is semistable if and only if $\alpha \geq 0$, that is if and only if $H^0(\mathcal{F}(-1)) = 0$. If $c_1 = -1$, then the two notions of stability and semistability coincide. ☐

PROPOSITION 17. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf on X which corresponds to a curve Y in X. Then $\mathcal{F}$ is stable (respectively semistable) if and only if*

1. *$c_1 > 0$ (respectively $c_1 \geq 0$), and*
2. *$h^0(\mathcal{I}_Y(l)) = 0$ for $l \leq \frac{1}{2}c_1$ (respectively for $l < \frac{1}{2}c_1$).*

Proof. (See [5, Proposition 4.2] for the case $X = \mathbb{P}^3$) By Hartshorne-Serre correspondence we have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{I}_Y(c_1) \to 0.$$

We assume c_1 even, the proof for c_1 odd being completely analogous. We consider the sequence

$$0 \to \mathcal{O}_X\left(-\frac{1}{2}c_1\right) \to \mathcal{F}\left(-\frac{1}{2}c_1\right) \to \mathcal{I}_Y\left(\frac{1}{2}c_1\right) \to 0.$$

Now $\mathcal{F}$ is stable if and only if $c_1 + 2\alpha > 0$, that is $\alpha > -\frac{1}{2}c_1$. This is equivalent to (1) $h^0\left(\mathcal{O}_X\left(-\frac{1}{2}c_1\right)\right) = 0$, which says that $c_1 > 0$, and (2) $h^0(\mathcal{I}_Y\left(\frac{1}{2}c_1\right)) = 0$.
Similarly, $\mathcal{F}$ is semistable if and only if $c_1 + 2\alpha \geq 0$, that is $\alpha > -\frac{1}{2}c_1 - 1$, and this is equivalent to (1) $h^0\left(\mathcal{O}_X\left(-\frac{1}{2}c_1 - 1\right)\right) = 0$, which says that $c_1 \geq 0$, and (2) $h^0(\mathcal{I}_Y\left(\frac{1}{2}c_1 - 1\right)) = 0$. □

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References

[1] GERAMITA A.V., ROGGERO M., VALABREGA P., *Subcanonical curves with the same postulation as Q skew complete intersections in projective 3-space*, Istituto Lombardo (Rend. Sc.) A **123** (1989), 111–121.

[2] GRIFFITHS F. AND HARRIS J., *Two proofs of a theorem concerning algebraic space curves*, in: Proceedings of the Eighth National Mathematics Conference (Arya-Mehr Univ. Tech., Tehran, 1977), Arya-Mehr Univ. Tech., Tehran, 1977, pp. 350–370.

[3] HARTSHORNE R., *Algebraic Geometry*, GTM 52, Springer-Verlag, New York 1977.

[4] HARTSHORNE R., *Stable vector bundles of rank 2 on* $\mathbb{P}^3$, Math. Ann. **238** (1978), 229–280.

[5] HARTSHORNE R., *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[6] HORROCKS G., *A construction for locally free sheaves*, Topology **7** (1968), 117–120.

[7] MADONNA C., *A splitting criterion for rank 2 vector bundles on hypersurfaces in* $\mathbb{P}^4$, Rend. Sem. Mat. Univ. Pol. Torino **56** (1998), 43–54.

[8] OKONEK C., SCHNEIDER M., SPINDLER H., *Vector Bundles on Complex Projective Spaces*, Progress in Mathematics 3, Birkhäuser, Boston 1980.

[9] ROGGERO M., *Sulle sezioni di un fascio riflessivo di rango 2 defi nenti sottovarietà integre*, Boll. U.M.I. **IV-D** (1985), 57–61.

[10] ROGGERO M. AND VALABREGA P., *Some vanishing properties of the intermediate cohomology of a reflexive sheaf on* $\mathbb{P}^n$, J. Algebra **170** (1994), 307–321.

[11] ROGGERO M., VALABREGA P., VALENZANO M., *Rank two bundles and reflexive sheaves on* $\mathbb{P}^3$ *and corresponding curves: an overview*, in: Geometric and Combinatorial Aspects of Commutative Algebra (Messina 1999), LNPAM 217, Marcel Dekker, 2001, pp. 327–343.

[12] SERRE J.-P., *Sur les modules projectifs*, Sèminaire Dubreil–Pisot 1960/61, Secr. Math. Paris, 1961.

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