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E. Ballico - C. Fontanari

ON A LEMMA OF BOMPIANI

Abstract. We reprove in modern terms and extend to arbitrary dimension a classical result of Enrico Bompiani on algebraic surfaces $X \subset \mathbb{P}^r$ having very degenerate osculating spaces.

Let $X \subset \mathbb{P}^r$ be an integral nondegenerate projective variety of dimension n defined over the field $\mathbb{C}$. In order to introduce the osculating space of order m to a point $p \in X$, fix a lifting

$$\begin{array}{rcl} U \subseteq \mathbb{C}^n & \longrightarrow & \mathbb{C}^{r+1} \setminus \{0\} \\ t & \longmapsto & p(t) \end{array}$$

of a local parametrization of X centered in p and define $T(m, p, X)$ to be the linear span of the points $[p_I(0)] \in \mathbb{P}^r$, where I is a multi-index such that $|I| \leq m$. The starting point of our research was the following result, which we read in a recent paper by Luca Chiantini and Ciro Ciliberto (see [3], Proposition 2.3):

PROPOSITION 1. *Let X be a smooth variety and let $p \in X$ be a general point. Assume that* $\dim T(m, p, X) = \dim T(m+1, p, X) = r$. *Then $X \subseteq \mathbb{P}^r$.*

Our natural question was: can one go a little bit further? Namely, if the dimension of the osculating space at a general point does not increase too much while passing from order m to order $m+1$, what can one say about the projective geometry of X? Following a suggestion of Ciro Ciliberto, to whom we are grateful, we looked for an answer among the works of Bompiani (see [2]). Enrico Bompiani (1889–1975) was a student of Guido Castelnuovo and in more than three hundreds papers intensively studied the differential geometry of projective varieties; in particular, he deeply investigated the relationship between partial differential equations and algebraic geometry. In the beautiful paper [1] we found an explicit answer to our question. Unluckily (or fortunately, according to the points of view), Bompiani's explanation turns out to have a couple of drawbacks: first, he treats only the case of surfaces; and next, his arguments are a little bit involved and it is not so easy to understand them properly. Therefore we decided to work them out again in a hopefully more clear and rigorous form; as a result, we obtained the following generalization of Bompiani's Lemma (see [1], pp. 614–615):

THEOREM 1. *Let $X \subset \mathbb{P}^r$ be a smooth variety and let $p \in X$ be a general point. Assume that* $\dim T(m, p, X) = h$ *and* $\dim T(m+1, p, X) = h+k$ *with* $1 \leq k \leq n-1$. *Then either $X \subset \mathbb{P}^{h+k}$ or X is covered by infinitely many subvarieties Y of dimension at least $n-k$ such that $Y \subset \mathbb{P}^{h-m}$.*

Theorem 1 is a direct consequence of the following

CLAIM 1. Under the assumptions of Theorem 1, either $X \subset \mathbb{P}^{h+k}$, or there exists a subvariety Y of X such that $p \in Y$, $\dim Y \geq n - k$, and $T(m, q, X) = T(m, p, X)$ for a general $q \in Y$.

Indeed, assume for a moment that Claim 1 holds. Since $Y \subset T(m, p, X)$, we have $< (m+1)Y > \subseteq T(m, p, X)$. Moreover, from [3], Remark 2.4, it follows that either $< (m+1)Y >= \mathbb{P}^r$ or $\dim < (m+1)Y > \geq \dim < Y > +m$. Hence we deduce that either

$$X \subset \mathbb{P}^r =< (m+1)Y > \subseteq \mathbb{P}^h,$$

or

$$\dim < Y > \leq \dim < (m+1)Y > -m \leq h - m.$$

Therefore we are reduced to establish the claim. In order to do that, we recall a standard definition:

DEFINITION 1. *The order m osculating variety of X is*

$$T(m, X) := \overline{\bigcup_{p \in X} T(m, p, X)}.$$

We also recall a natural description of the tangent space to $T(m, X)$ at a general point $P \in T(m, p, X)$:

$$T_P(T(m, X)) = \left\{ \frac{d\gamma}{ds}(0) \in \mathbb{P}^r \right\}$$

where

$$\begin{array}{rcl} \gamma : \Delta & \longrightarrow & T(m, X) \subset \mathbb{P}^r \\ s & \longmapsto & \sum_{|I| \leq m} \alpha_I(s)[p_I(t(s))] \end{array}$$

is a holomorhic map defined on the unit disc $\Delta \subset \mathbb{C}$. Since $\dim T(m, X) = \dim T_P(T(m, X))$, Claim 1 is a direct consequence of the following Lemma, which we believe to be quite interesting also in its own:

LEMMA 1. *Let X be a smooth variety and let $P \in T(m, p, X)$ be a general point of $T(m, X)$. Then*

$$T_P(T(m, X)) \subseteq T(m+1, p, X).$$

Proof. Indeed, we have

$$\begin{aligned} \frac{d\gamma}{ds}(0) &= \sum_{|I| \leq m} \dot{\alpha}_I(0)[p_I(0)] + \sum_{|I| \leq m} \alpha_I(0) \sum_{j=1}^{n} [\frac{\partial}{\partial t_j} p_I(0)]\dot{t}(0) = \\ &= \sum_{|I| \leq m+1} \beta_I[p_I(0)] \in T(m+1, p, X) \end{aligned}$$

and the proof is over. □

References

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Edoardo BALLICO, Claudio FONTANARI, Università degli Studi di Trento, Dipartimento di Matematica, Via Sommarive 14, 38050 Povo (Trento), ITALIA
e-mail: `ballico@science.unitn.it`
e-mail: `fontanar@science.unitn.it`

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