Rend. Sem. Mat. Univ. Pol. Torino - Vol. 62, 3 (2004)

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LEVEL ALGEBRAS - SOME REMARKS

Prologue

These notes are an expanded version of a lecture I gave at the Politecnico di Torino in February, 2003, during a Workshop on Commutative Algebra and Algebraic Geometry. I want to thank the organizers of this Workshop for their invitation to speak and for the care they put into the organization. I also want to warmly thank Prof. Carla Massaza for her encouragement and patience during the preparation of these notes.

My goal in the lecture was to highlight a few of the results from the monograph on Level Algebras that I've written in collaboration with Tadahito Harima, Juan Migliore and Yong Su Shin [22]. I also wanted to discuss some work I've done with Anna Lorenzini [25] which is related to Level Algebras. The reader interested in finding proofs for any unsupported statements in this exposition can find them in one or the other of the two papers mentioned above. I approached the lecture in a very informal way and have tried to maintain that same "lightness" in these notes.

I would like to take this opportunity to thank all my co-authors (mentioned above) for giving me so many hours of fun working on the various projects connected to Level Algebras. I've enjoyed (almost!) every minute!

1. Level Algebras

The idea of a *level algebra* seems to have been first introduced by Richard Stanley in [55] in a combinatorial context. Indeed there are many connections between this notion and problems in combinatorics. However, once the idea was isolated with a definition, it was easy to see that level algebras are ubiquitous. For example, we'll see later, that for almost all positive integers s, a set of s general points in $\mathbb{P}^n$ has coordinate ring which is a level algebra.

But, before going any further, let's get down to establishing some notation and getting a formal definition of a level algebra in place.

Let $P = k[x_1, \ldots, x_n]$ be the usual polynomial ring over the field k. We will consider P with its usual grading by setting

$$P_t := \text{ the } k \text{ subvectorspace of } P \text{ generated by the monomials of degree } t.$$

Then $P = \oplus_{i\geq 0} P_i$ and $\dim_k P_t = \binom{t+n-1}{n-1}$.

It is worth noting that P is, what is now called, a *standard* graded algebra since $P = k[P_1]$, i.e. P is generated (as a k-algebra) by its piece of degree 1.

If I is a homogeneous ideal of P then $A = P/I$ is again a standard graded k-algebra. Moreover, if I has height n in P then $A = P/I$ is an *Artinian* k-algebra, i.e.

$\dim_k A < \infty$. Thus, we can write

$$A = k \oplus A_1 \oplus \cdots \oplus A_s, \text{ where } A_s \neq 0.$$

We call s the *socle degree* of A.

We associate to the graded Artinian algebra A two vectors of non-negative integers:

1) the first is an $(s+1)$-tuple, called the *h-vector* of A and denoted $h(A)$ (the "h" recalls *Hilbert*). It is defined as follows (where A is as above):

$$h(A) := (1, \dim_k A_1, \ldots, \dim_k A_s) = (1, h_1, \ldots, h_s) \text{ with } h_s \neq 0;$$

2) the second is an s-tuple, called the *socle vector* of A and denoted $s(A)$. It is defined as follows: the *socle* of A is $soc(A) := 0 : m$ ($m = \oplus_{i=1}^{s} A_i$). It is a graded ideal of A. Then

$$s(A) := (\dim_k (0:m)_1, \ldots, \dim_k (0:m)_s).$$

REMARK 1. a) $0 : m = 0 : A_1$; b) $A_s = (0 : m)_s$.

This observation implies that we always have A_s as part of the socle of A.

We are now ready for our first definition of a level algebra.

DEFINITION 1. *An Artinian standard graded algebra A, having socle degree s, is a level algebra if*

$$soc(A) = A_s\ .$$

Equivalently, if $s(A) = (0, \ldots, 0, h_s = \dim_k A_s)$. If A is a level algebra with socle degree s, we say that h_s is the type of A.

There is an alternate way to make the definition of a level algebra which will allow us to extend the definition to algebras which are not necessarily Artinian.

Let $\mathcal{F}_\bullet$ be the *minimal free resolution* (MFR) of A, considered as a P-module. $\mathcal{F}_\bullet$ has the form

$$0 \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_0 \to P \to A \to 0$$

where

$$\mathcal{F}_j = \oplus_{t=1}^{r_j} P(-(j+1+t))^{\beta_{j,j+1+t}}.$$

The integers $\beta_{j,j+1+t} \geq 0$ are called the *graded Betti numbers of A* (or of I).

The following is well known:

$$s(A) = (a_1, \ldots, a_s) \Leftrightarrow \beta_{n-1,n+i} = a_i.$$

This means that, for an Artinian algebra we can read off the socle vector from the last free module in the MFR of that algebra. In particular, if A is an Artinian graded k-algebra then:

$$A \text{ is a level algebra}$$
$$\Leftrightarrow$$
$$\text{there is exactly one shift in the last free module in an MFR of } A.$$

We can take advantage of this alternate definition of a level Artinian algebra to define level algebras of any dimension. We do that now.

Let $R = k[x_0, \ldots, x_n]$, $A = R/I$ with I homogeneous (so A is a standard graded k-algebra). Suppose that the Krull dimension of A is d (written $K - dim(A) = d$). It follows from Hilbert's Syzygy Theorem that a MFR of A (as R-module) has the form

$$0 \to \mathcal{F}_\ell \to \cdots \to \mathcal{F}_0 \to R \to A \to 0$$

where $\ell \leq n$.

THEOREM 1 (AUSLANDER-BUCHSBAUM).

$$\ell = n - d \quad \Leftrightarrow \quad \textit{A is a Cohen-Macaulay ring (C-M ring).}$$

Since Artinian rings are always C-M, this seems the right start for a generalization of level rings to rings of arbitrary dimension.

Moreover, when A is a C-M ring, and $|\ k\ | = \infty$, we can choose linear forms $\overline{L_1}, \ldots, \overline{L_d} \in A_1$, which are a regular sequence on A. The ring $B = A/(\overline{L_1}, \ldots, \overline{L_d})$ is called an *Artinian Reduction of A*.

If we write $B = R/(L_1, \ldots, L_d J) = R/J$ and denote by $\overline{R}$ the ring

$$\overline{R} = k[x_0, \ldots, x_{n-d}] \simeq R/(L_1, \ldots, L_d),$$

then B is an $\overline{R}$-module and:

1) the Hilbert series of A is $\frac{h(t)}{(1-t)^d}$ where $h(t) \in \mathbb{Z}[t]$ and the coefficients of $h(t)$ are precisely the entries of the h-vector, $h(B)$;

2) the graded Betti numbers of A and of B are the same.

Thus, it makes sense to say:

$$A \text{ is level } \Leftrightarrow B \text{ is level.}$$

and it makes sense to speak of the h-vector of A as that of B.

EXAMPLES 1. 1) Any Gorenstein algebra is a level algebra of type 1 (and conversely). It follows that any complete intersection variety has level homogeneous coordinate ring.

2) Let $I = (y^3, x^3y, x^5) \subset k[x, y]$ and let $A = k[x, y]/I$. Then

$$A = \langle \overline{1} \rangle \oplus \langle \overline{x}, \overline{y} \rangle \oplus \left\langle \begin{array}{c} \overline{x}^2 \\ \overline{xy} \\ \overline{y}^2 \end{array} \right\rangle \oplus \left\langle \begin{array}{c} \overline{x}^2\overline{y} \\ \overline{xy}^2 \\ \overline{x}^3 \end{array} \right\rangle \oplus \left\langle \begin{array}{c} \overline{x}^2\overline{y}^2 \\ \overline{x}^4 \end{array} \right\rangle.$$

Thus, $h(A) = (1, 2, 3, 3, 2)$.

It's easy to see that $soc(A)$ is a monomial ideal and that only $\overline{x}^2\overline{y}^2$ and $\overline{x}^4$ annihilate both $\overline{x}$ and $\overline{y}$. Thus, A is a level algebra (of type 2).

3) Consider s generic points in $\mathbb{P}^n$ where, for some integer d, we have

$$\binom{d+n}{n} \leq s \leq \binom{d+1+n}{n} - \frac{1}{n}\binom{d+n}{n-1}.$$

Then, for such an s, the homogeneous coordinate ring of the s points is always a level algebra. This was proved in [43] and was a partial positive response to the Minimal Resolution Conjecture of Lorenzini [44].

2. Problems Being Considered

What are the kinds of problems concerning level algebras that are being studied?

Question 1) To fix the notation, let us suppose that A is a standard graded k-algebra of embedding dimension c, i.e. $\dim_k A_1 = c$.

The first question one can consider is the following: fix c and then describe ALL the h-vectors of level algebras having embedding dimension c.

This question can be divided into several sub-questions:

a) Answer the question in the case that A is Artinian;

b) Answer the question in the case where A is the coordinate ring of a reduced (non-generate) set of points in $\mathbb{P}^c$.

c) Answer the question when A is CM and the coordinate ring of an irreducible curve in $\mathbb{P}^{c+1}$ (an irreducible surface in $\mathbb{P}^{c+2}$, etc.).

Question 2) This question may seem a bit technical, but it has been very important in the study of Gorenstein Artinian algebras (level algebras of type 1) so I thought to introduce the notion and the questions concerning it right away.

DEFINITION 2. *A standard graded Artinian algebra $A = \oplus_{i=0}^{s} A_i$ is said to have the Weak Lefschetz Property (WLP), if there is a linear form $L \in A_1$ such that*

$$A_i \xrightarrow{\times L} A_{i+1}$$

has maximal rank for $i = 0, \ldots, s-1$.

REMARKS 1. 1) If A has the WLP and $h(A) = (1, h_1, \ldots, h_s)$ then the h_i satisfy

$$1 \leq h_1 \ldots \leq h_t \geq h_{t+1} \geq \ldots \geq h_s$$

i.e. $h(A)$ is unimodal.

2) Furthermore, if A has the WLP and $h_i = h_{i+1}$ for any i, then

$$A_{i+r} \overset{\times L}{\to} A_{i+r+1}$$

is surjective for $r \geq 0$. (i.e. after a "plateau" the entries in an h-vector cannot increase, something which is not excluded simply by the unimodality of the h-vector.)

Why study this property?

1) Every algebra with codimension 2 has this property (see [35]).

2) If h is the h-vector of a Gorenstein algebra of codimension 3 then there is at least one Artinian k-algebra A with the property that

i) A is Gorenstein with $h(A) = h$;
ii) A has the WLP.

Moreover, EVERY complete intersection in codimension 3 has the WLP (this is a difficult theorem in [35]). In fact, there is NO known example of a codimension three Gorenstein algebra which does not have the WLP.

I think it will be important to establish the limits of the WLP for level algebras.

3. What's known?

In this section I would like to consider the two questions mentioned above in the context of level algebras and report on what is known and hence what is still left to study.

Codimension 2. Thanks to a general result of Iarrobino [39] (and an independent argument by Valla (private communication) in the case of level algebras) we have, using the Hilbert-Burch theorem, very complete information concerning the relationship between socle vectors and h-vectors.

In the case of level algebras we have the following results:

a) on the Artinian level, the Hilbert functions are completely characterized (Iarrobino, Valla). In fact, if $h = (1, 2, h_2, \ldots, h_s)$ then there is a level Artinian algebra A with $h(A) = h \Leftrightarrow 2h_i \leq h_{i-1} + h_{i+1}, i = 1, \ldots, s-1$.

b) all algebras have the WLP (as noted above).

c) the Hilbert functions of level sets of points in $\mathbb{P}^2$ are characterized ([24]). They are precisely the same Hilbert functions arising from a).

d) the Hilbert functions of reduced, irreducible level curves in $\mathbb{P}^3$ are characterized [15]. They are precisely the Hilbert functions that arise from a).

e) in recent work, the parameter spaces for level algebras are discussed in some detail [14], although some questions remain.

Codimension 3. In passing to this next codimension, things immediately get quite a lot more cloudy! R. Stanley [55] has given us the first general (i.e. any codimension) results (in the Artinian case) which can be applied to this problem.

Facts. If $h = (1, h_1, \ldots, h_s)$ is a level sequence then it follows that:

i) for every $t < s$, $(1, h_1, \ldots, h_t)$ is also a level sequence;

ii) for every $1 \leq i, j \leq s$ such that $i + j \leq s$, we have
$$h_i \leq h_j h_{i+j} \ .$$

iii) if $h_s = 1$ then $h_{s-i} = h_i$ for all i.

The proofs for these facts are actually quite simple. For $i)$, simply take a level algebra A for which $h(A) = h$ and let $B = \oplus_{i \leq t} A_i$. Then B is a level algebra with the required h-vector.

As for $ii)$, it's enough to show that the multiplication in A, which induces a map $A_i \to Hom_k(A_j, A_{i+j})$, is injective when A is level.

As for $iii)$, this follows from the fact that the bilinear map
$$A_i \times A_{s-i} \to A_s \simeq k$$
(derived from the multiplication in A) is non-singular with the hypothesis of $iii)$.

If we restrict our attention, for a moment, only to level algebras of codimension 3 and also of type 1 (i.e. the Gorenstein algebras) then much more is known, thanks to the structure theorem of Buchsbaum and Eisenbud [11]. For example,

iv) there is a complete characterization of the h-vectors in the Artinian case. ([56]);

v) there is a complete characterization of the h-vectors of Gorenstein sets of points in $\mathbb{P}^3$ [29]. They are the same h-vectors as in $iv)$.

vi) there is a complete characterization of the h-vectors of reduced and irreducible codimension 3 Gorenstein varieties [36]. In this case the h-vectors are strictly fewer than those of $iv)$.

vii) for any Gorenstein h-vector there is at least one Gorenstein algebra with that h-vector and with the WLP. [32]

viii) there are extensive results about parameter spaces in the book of Iarrobino and Kanev [42], although they indicate many interesting open problems.

As soon as we try to say anything about level algebras of type > 1 (still in codimension 3) we discover that almost nothing is known.

A. Lorenzini and I have found a very simple (but effective) combinatorial method to help us determine which h-vectors CANNOT be the h-vector of a level algebra. Our results, which apply in any codimension, concern the concept of *cancelability*.

This concept is based on two extremely useful theorems (in fact, only one when the codimension is exactly three). These theorems are:

A. the theorem of Bigatti-Hulett-Pardue ([3, 38, 52]) concerning the extremal property of the minimal free resolution of the lex-segment ideals. I won't recall the full theorem here but just the special case of it which we will use in this exposition. Roughly speaking, the theorem says the following:

given an h-vector $h = (1, n, h_2, \ldots, h_s)$ of some Artinian algebra, then the lex-segment ideal $I \subset P = k[x_1, \ldots, x_n]$ for which $h(P/I) = h$ has a very beautiful property. Namely, if we write down the MFR of $A = P/I$ as P-module, then it has the form,

$$0 \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_0 \to P \to A \to 0$$

where

$$\mathcal{F}_j = \oplus_{i=j+2}^{j+2+r_j} P(-i)^{\beta_{j,i}},\ r_j \geq 0$$

and the graded Betti numbers enjoy the following extremal property: if J is any other ideal of P for which $h(P/J) = h$ and the graded Betti numbers of P/J are $\{\tau_{j,i}\}$ then

$$\tau_{j,i} \leq \beta_{j,i}$$

for every j and every i.

The Theorem of Peeva [53] roughly says that any cancellation from the Bigatti-Hullet-Pardue resolution can be done in adjacent free module The following example will make clear how we can use this result.

EXAMPLE 1. Consider the h-vector $h = (1, 3, 5, 7, 6, 6, 2)$. The lex-segment ideal, I, with this h-vector has the following MFR,

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to P \to A = P/I \to 0$$

where

$$\mathcal{F}_2 = P(-6)^2 \oplus P(-7)^1 \oplus P(-8)^4 \oplus P(-9)^2.$$

(Note: 9 = 6+3 where 6 is the socle degree and 3 is the codimension of I. This relationship, between the socle degree, the codimension and the maximum shift in the last place of a MFR of an Artinian algebra, is always there.)

$$\mathcal{F}_1 = P(-5)^5 \oplus P(-6)^2 \oplus P(-7)^9 \oplus P(-8)^4,$$

and

$$\mathcal{F}_0 = P(-2)^1 \oplus P(-4)^3 \oplus P(-5)^1 \oplus P(-6)^5 \oplus P(-7)^2.$$

We summarize all of this information in the (by now) well known *Macaulay diagram*:

$$\begin{array}{c c}
 & \begin{array}{cccc} & 0 & 1 & 2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & - & - \\ - & 3 & 5 & 2 \\ - & 1 & 2 & 1 \\ - & 5 & 9 & 4 \\ - & 2 & 4 & 2 \end{pmatrix}
\end{array} \tag{1}$$

Now I can explain what we mean by "cancelability". If there were a level algebra B, with h-vector as above, this would imply that the MFR of B would have a Macaulay diagram *below* that of (1), entry by entry, and (moreover) would have last column

$$\begin{pmatrix} - \\ - \\ - \\ - \\ - \\ - \\ 2 \end{pmatrix}$$

Thus, we would have to have *canceled* the entries $\left\{\begin{matrix} 2 \\ 1 \\ 4 \end{matrix}\right\}$ of the last column of (1). One cancels, for example, the $P(-6)^2$ from $\mathcal{F}_2$ with the $P(-6)^2$ of $\mathcal{F}_1$ (canceling can only occur 'down' and to the 'left'). Similarly, we can cancel the $P(-7)^1$ from $\mathcal{F}_2$ from the $P(-7)^9$ of $\mathcal{F}_1$, and so on.

Since, in this case, we can cancel all the unwanted entries from the last column of (1), we say that the h-vector $h = (1, 3, 5, 7, 6, 6, 2)$ is *cancelable*. Notice that this property is completely combinatorial.

Obviously one has:

$$h \text{ is a level sequence } \Rightarrow h \text{ is cancelable.}$$

(In fact, it is not hard to show that, in codimension 2, the converse is also true, see, e.g. [25].)

EXAMPLE 2. If, however, we consider the h-vector, $h = (1, 3, 4, 5, 4, 4, 2)$ then the Macaulay diagram associated to (the lex-segment ideal with h-vector) h is:

$$\begin{array}{cc}
 & \begin{array}{cccc} & 0 & 1 & 2 \end{array} \\
\begin{array}{c} 0\\1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} 1 & - & - & - \\ - & 2 & 1 & - \\ - & - & - & - \\ - & 2 & 3 & 1 \\ - & - & - & - \\ - & 2 & 4 & 2 \\ - & 2 & 4 & 2 \end{pmatrix}
\end{array}$$

and so h is not cancelable.

In order to take advantage of this idea it's important to be able to recognize when an h-vector is, or is not, cancelable. In [25] we give (in codimension 3) necessary and sufficient conditions in order that h be cancelable.

It is possible, in any codimension, to find examples of h-vectors which are not cancelable. For example, we recapture a result of Cho-Iarrobino [13] in this light. For this result we need to use Macaulay's functions $-^{<i>} : \mathbb{N} \to \mathbb{N}$.

PROPOSITION 1. *Let $h = (1, n, h_2, \ldots, h_s)$ be the h-vector of an Artinian algebra, and let h_{i-1}, h_i, h_{i+1} be three consecutive entries of h with the property that*

$$i)\ h_i^{<i>} = h_{i+1} \text{ and } ii)\ (h_{i-1} - b)^{<i-1>} = h_i \text{ for some } b \geq 0$$

If $b > 0$ then h is not cancelable.

Our proof of this result (see [22]), which is different from that of Cho-Iarrobino, uses the fact that in the Macaulay diagram associated to h we have the $(i-1)^{st}$ and i^{th} rows looking like:

$$\begin{array}{rccccc} i-1: & 0 & * & \cdots & * & b \\ i: & 0 & 0 & \cdots & 0 & 0 \end{array}$$

and thus h is not cancelable. (The proof uses the celebrated theorem of Eliahou and Kervaire [20] describing the MFR of stable monomial ideals and a recent result of I. Peeva [53] which says (in our particular case) that one need only look at the next last column to decide if cancelation from the last column is possible.)

From [22] one can find other cases of h-vectors, $h = (1, n, h_2, \ldots, h_s)$ which are not cancelable. For example,

i) if $h_d = h_{d+1} = d$ and $h_{d-1} > d$; or

ii) if $h_d = H_{d+1} = d + 1$ and $h_{d-1} > d + 1$; or

iii) if $h_d = h_{d+1} = P < 2d$ and $h_{d-1} \geq p + n$ and $d \geq n + 2$; or

iv) if $h_{d-1}^{<d-1>} - 1 = h_d$ and $(h_{d-2} - \epsilon)^{<d-2>} = h_{d-1}$ and $\epsilon \geq n$.

REMARK 2. Unfortunately, the example we gave above, $h = (1, 3, 5, 7, 6, 6, 2)$, which is cancelable, is not the h-vector of a level algebra (see [22] for the argument). Thus, immediately in codimension 3, we see that cancelability is only a necessary condition for being a level sequence.

4. Existence methods

Up to this point we have seen only conditions which can be used to say that certain h-vectors CANNOT be the h-vector of a level algebra. We now turn to a more 'positive' point of view and discuss methods to construct level algebras.

There are, in fact, many different methods for constructing level algebras. Many of these can be seen in our monograph [22]. In this exposition I will just discuss one method of constructing level algebras. It is a method which we have baptized the *linked sum* method. The method is interesting because it uses non-Artinian level algebras in a fixed codimension to construct Artinian level algebras in one higher codimension.

The method is based on the following observations: suppose Z is a finite set of points in $\mathbb{P}^n$ whose homogeneous coordinate ring is a level algebra (we'll call such a set of points a *level set of points*). In whatever way we partition Z into two disjoint subsets X and Y, the ring $A = k[x_0, \ldots, x_n]/(I_X + I_Y)$ is an Artinian level algebra.

To prove that, use the exact sequence

$$0 \to I_Z \to I_X \oplus I_Y \to I_X + I_Y \to 0 \tag{2}$$

the first map being $f \to (f, -f)$ and the second being $(f, g) \to f + g$. Since I_Z is a level algebra, its MFR finishes in a free module with only one shift. The mapping cone construction for the resolution of $I_X + I_Y$ then has only one shift in the last term of a free resolution and hence only one shift in the last term of its MFR.

By varying the Hilbert function of X and Y we vary that of the ring A. Thus, the method is most useful when Z is a set of points which has subsets with lots of different Hilbert functions. I.e. Z should not be a very *uniform* set of points. This means we should take Z to be a very special set of points to get the maximum use of the construction.

Fortunately, there is a very simple way to construct level sets of points in $\mathbb{P}^2$ which are very special. The method was first used by Harima in [33] to construct examples of Gorenstein algebras. We adapt the method of Harima to our situation.

We consider *rectangles* of points – sets of points which Harima called a *Basic Configuration*. More precisely, if d and e are positive integers, then a Basic Configuration $B(d, e)$ consists of de points carefully positioned on e rows and d columns to form a rectangular grid of points.

E.g. a $B(2, 4)$ is a set of 8 points positioned as follows:

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

A *pure configuration* is a collection of basic configurations which are put together in a very careful way. More precisely, choose a set of coordinate axes in $\mathbb{P}^2$ such that all our points are in the affine plane $z \neq 0$, then

a *pure configuration* is a finite union $\cup_{i=1}^{r} B(d_i, e_i)$ where $e_1 > e_2 > \cdots > e_r$ such that the y-coordinates of any point in $B(d_i, e_i)$ is the y-coordinate of a point in $B(d_{i-1}, e_{i-1})$.

EXAMPLE 3. Here is an example of a pure configuration, $\cup_{i=1}^{4} B(d_i, e_i) = B(1, 7) \cup B(1, 6) \cup B(1, 5) \cup B(1, 4)$.

$$\begin{array}{cccc}
\bullet & & & \\
\bullet & \bullet & & \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{array}$$

Harima [33] gives a compact formula for the Poincare' series of any pure configuration which can be used to prove the following theorem:

THEOREM 2. *Let* $Z = \cup_{i=1}^{r} B(d_i, e_i)$ *be a pure configuration. Then*

$$Z \text{ is level of type } r \ \Leftrightarrow e_i - e_{i+1} = d_{i+1}, i = 1, \ldots, r-1.$$

So, for example, the pure configuration above is a level set of points in $\mathbb{P}^2$.

EXAMPLE 4. The following pure configuration, $Z = B(4, 5) \cup B(2, 3)$ is also a level set of points in $\mathbb{P}^2$. We will divide it into two disjoint subsets – one denoted with $\bullet$'s (which we will call X) and the other denoted with x's (which we will denote Y). We will apply the linked sum method to these three sets to construct a level Artinian algebra.

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & x & x & x & & & \\
\bullet & x & x & x & & x & x \\
x & x & x & x & & x & x \\
x & x & x & x & & x & x
\end{array}$$

Using Harima's theorem we find:

$$H_Z : 1 \ \ 3 \ \ 6 \ \ 10 \ \ 15 \ \ 20 \ \ 24 \ \ 26 \ \ \rightarrow .$$

A simple calculation gives:

$$H_X : 1 \ \ 3 \ \ 5 \ \ 6 \ \ \rightarrow$$

and

$$H_Y : 1 \ \ 3 \ \ 6 \ \ 10 \ \ 14 \ \ 18 \ \ 20 \ \ \rightarrow$$

It now follows from the exact sequence (2) that if $A = k[x, y, z]/(I_X + I_Y)$ then A is level with h-vector

$$h(A) = (1, 3, 5, 6, 5, 4, 2).$$

In our monograph, as mentioned earlier, we give several other construction methods for level algebras. Using these constructions and the theorems which eliminate certain h-vectors as h-vectors of level algebras we have been able to make calculations which give:

1) a list of ALL the h-vectors of Artinian level algebras of codimension 3 having socle degree ≤ 5;

2) a list of all the h-vectors of level algebras of type 2 and socle degree 6 (in codimension 3).

We also make many observations about level algebras with the WLP.

Unfortunately, even with all this data we have been unable to formulate a precise conjecture to characterize the h-vectors of codimension 3 level algebras (even of type 2) and wonder if there is a *pretty* such description. Nevertheless, we do have some conjectures (better – questions) about such objects. These are all taken from our monograph.

I) $h = (1, 3, h_2, \dots, h_s)$ is the h-vector of an Artinian level algebra $\Leftrightarrow$ it is the h-vector of a level set of points in $\mathbb{P}^3$?

II) Let $h = (1, 3, h_2, \dots, h_s)$ be the h-vector of a level Artinian algebra and suppose that $h_i = max\{\, h_j \mid j = 1, \dots, s\}$. Then $(1, 2, h_2 - 3, \dots, h_i - h_{i-1})$ is an O-sequence?

III) $a)$. If $h = (1, 3, h_2, \dots, h_s)$ is the h-vector of a level Artinian algebra then h is the h-vector of a level algebra with the WLP?
$b)$. (less strongly) h is unimodal?

I have decided to include, in this bibliography, not only the papers mentioned above, but also those that I thought might be interesting to someone who wanted to find out more about what I have described. This is a somewhat abbreviated part of the bibliography of our memoir [22] below. A more ample bibliography can be found there.

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AMS Subject Classification: 13D02, 14M05, 14M07, 14M99.

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Lavoro pervenuto in redazione il 06.05.04.