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# UNIFORM ERROR BOUNDS FOR CAUCHY PRINCIPAL VALUE INTEGRALS

**Abstract.** We prove a uniform convergence theorem for the numerical evaluation of certain weighted Cauchy principal value integrals which also gives the order of magnitude of the error. This theorem is then applied to derive uniform error bounds in the numerical integration of these integrals based on several spline approximations. These bounds are given in terms of the norms of the underlying spline spaces.

## 1. Introduction

In this paper we derive a uniform error bound, with respect to the parameter $\lambda$, in the numerical evaluation of weighted Cauchy principal value (CPV) integrals of the form

$$I(\omega_{\alpha\beta} f;\lambda) := \fint_J \omega_{\alpha\beta}(x)\frac{f(x)}{x-\lambda}dx, \tag{1}$$

where $J := [-1,1]$, $\quad \lambda \in \mathring{J} := (-1,1)$ and $\omega_{\alpha\beta}$ is the Jacobi weight function

$$\omega_{\alpha\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1. \tag{2}$$

A sufficient condition on $f$, ensuring that $I(\omega_{\alpha\beta} f;\lambda)$ exists, is

$$f \in L_\mu(A), \quad A > 0, \quad 0 < \mu \leq 1, \tag{3}$$

where

$$L_\mu(A) := \{g \in C(J) : \omega(g;t;J) \leq At^{\mu}\}$$

and, for $g \in C(J)$, $\omega(g;t;J)$ is the modulus of continuity

$$\omega(g;t;J) := \max_{\substack{x,x+h\in J\\ 0<|h|\leq t}} |g(x+h)-g(x)|.$$

A standard way to evaluate $I(\omega_{\alpha\beta} f;\lambda)$ numerically is to replace $f$ by an approximation $f_n$ from a sequence $\{f_n\}$ such that $I(\omega_{\alpha\beta} f_n;\lambda)$ can be evaluated analytically or easily approximated numerically for all $n$ [24].

In many cases, $f_n$ is given in the form

$$f_n := \sum_{i=0}^{n} f(x_{in})\varphi_{in}$$

so that $I(\omega_{\alpha\beta} f; \lambda)$ is approximated by the quadrature sum

$$I(\omega_{\alpha\beta} f_n; \lambda) = \sum_{i=0}^{n} f(x_{in}) I(\omega_{\alpha\beta}\varphi_{in}; \lambda)\rangle. \tag{4}$$

We consider the approximation error

$$r_n(x) := f(x) - f_n(x) \tag{5}$$

and the quadrature error $I(\omega_{\alpha\beta} r_n; \lambda)$.

A general uniform convergence theorem, providing sufficient conditions for $I(\omega_{\alpha\beta} r_n; \lambda) \to 0$ as $n \to \infty$, uniformly with respect to $\lambda \in \overset{\circ}{J}$, is proved by Rabinowitz in [20], whereas in [24] he generalized this result by proving a uniform convergence theorem for Hadamard finite part (HFP) integrals which are derivatives of CPV integrals. In [11] examples of splines sequences for the evaluation of such HFP integrals are provided.

The previous uniform convergence results [20,24] did not provide uniform error bounds. In [16] a uniform bound for $|I(\omega_{\alpha\beta} r_n; \lambda)|$ is given in the case $\alpha = \beta = 0$, where $f \in C^1(J)$ is approximated by splines with not-a-knot end conditions [5]. In [17,18] this result has been generalized to HFP integrals approximated by different types of spline-based quadratures, for which error bounds are provided uniformly with respect to $\lambda$ in $[-1+\delta, 1-\delta]$, for every $\delta > 0$.

Here we continue Rabinowitz's work [20] by proving a uniform convergence theorem for the numerical evaluation of the CPV integral (1) with Jacobi weight (2) and $f$ satisfying (3), which also gives a bound for $|I(\omega_{\alpha\beta} r_n; \lambda)|$ uniformly with respect to $\lambda \in \overset{\circ}{J}$.

We shall consider sequences of approximations $\{f_n\}$ with $f_n \in S^m_{Y_n}$, where $S^m_{Y_n}$ is a space of polynomial splines of order $m$ based on a sequence of partitions $\{\Pi_N : t_{0N} := -1 < t_{1N} < \cdots < t_{NN} := 1\}$, from which the sequence of spline knots $\{Y_n\}$ and quadrature nodes $\{X_n\}$ are derived. Setting $h_{iN} := t_{i+1,N} - t_{iN}$, we say that $\{\Pi_N\}$ is locally uniform (l.u.) if there exists a constant $L \geq 1$, so that $L^{-1} \leq h_{i-1,N}/h_{iN} \leq L\rangle, \quad 1 \leq i \leq N-1, \rangle\text{all } N\rangle$. We define the norm of $\Pi_N$, $H_N = H_n := \max_{0 \leq i \leq N-1} h_{iN}$ and we shall assume that $H_n \to 0$ as $n \to \infty$.

We shall consider two different spline spaces $S^m_{Y_N}$ for the construction of a sequence of quadrature sums $\{I(\omega_{\alpha\beta} f_n; \lambda)\}$ of the form (4) such that $|I(\omega_{\alpha\beta} r_n; \lambda)|$ satisfy the obtained error bounds.

## 2. Uniform quadrature error bound

The following theorem provides a bound, uniform with respect to $\lambda \in \overset{\circ}{J}$, for the quadrature error $|I(\omega_{\alpha\beta} r_n; \lambda)|$, with $r_n$ given in (5).

THEOREM 1. *Consider the CPV integral (1) with Jacobi weight (2) and $f$ satisfying (3). Assume that we are given a sequence of approximations $\{f_n\}$ and a sequence of positive quantities $H_n$ such that $f_n \in C(J)$ and*

*(i)* $r_n(1) = 0 \quad$ *if* $\rangle\alpha \leq 0$
$r_n(-1) = 0 \quad$ *if* $\rangle\beta \leq 0$

*(ii)* $\max\limits_{x \in J} |r_n(x)| \leq c H_n^{\mu}\rangle, \quad c > 0$

*(iii)* $r_n \in L_\mu(A_1)\rangle, \quad$ *for some* $A_1 > 0$ *and all* $n$.

*If* $\mu + \gamma > 0$, *where* $\gamma := \min\rangle(\alpha, \beta)$, *then*

$$\left|I(\omega_{\alpha\beta} r_n; \lambda)\right| = \begin{cases} O(H_n^{\mu} |\log H_n|)\rangle, & \gamma \geq 0 \\ O(H_n^{\mu+\gamma}), & \gamma < 0\rangle. \end{cases}$$

*In particular, the $O$-term holds uniformly with respect to $\lambda$ in $\mathring{J}$.*

*Proof.* We set

$$g_n(x) := \frac{r_n(x) - r_n(\lambda)}{x - \lambda}$$

and we write

$$I(\omega_{\alpha\beta} r_n; \lambda) = \int_{-1}^{1} \omega_{\alpha\beta}(x) g_n(x) dx + r_n(\lambda) I(\omega_{\alpha\beta}; \lambda) := T_1 + T_2.$$

By the symmetry of the problem, we can assume that $\lambda \geq 0$.

In order to bound $|T_2|$ we consider that in a neighbourhood of $\lambda = 1$ [20]

$$I(\omega_{\alpha\beta}; \lambda) = \begin{cases} O\big((1-\lambda)^{\alpha}\big) + C & \text{if } \alpha \text{ is not an integer} \\ O\big((1-\lambda)^{\alpha} |\log(1-\lambda)| + C\big) & \text{if } \alpha \text{ is an integer.} \end{cases} \tag{6}$$

We can find $s > 0$ sufficiently small so that, for all $\lambda$ in $(1-s, 1))\rangle, \rangle$ (6) holds. Assume that $n$ is sufficiently large so that $H_n < s$. In the sequel we shall denote by $k_j\rangle, \rangle j = 1, \ldots, 7\rangle, \rangle$ all positive constants independent of $\lambda$ and $n$.

We consider at first the case $\lambda \in [1 - H_n, 1)$, i.e. $1 - \lambda \leq H_n$. If $\alpha < 0$ then, since $r_n(1) = 0$, we have that [20]

$$|r_n(\lambda)| = |r_n(1) - r_n(\lambda)| \leq \omega(r_n; 1-\lambda; J) = O\big((1-\lambda)^{\mu}\big) \tag{7}$$

and, using (6) and (7), for some $k_1 > 0$

$$|T_2| \leq k_1 (1-\lambda)^{\mu+\alpha} \leq k_1 H_n^{\mu+\alpha}.$$

If $\alpha = 0$, by using (6) and (7), we consider that the function $x^{\mu} |\log x|$ increases monotonically in $\Big(0, (1/e)^{\frac{1}{\mu}}\Big]$, so that, for $n$ such that $H_n \leq (1/e)^{\frac{1}{\mu}}\rangle, \rangle$ we have for some $k_2 > 0$

$$|T_2| \leq k_2 (1-\lambda)^{\mu} |\log(1-\lambda)| \leq k_2 H_n^{\mu} |\log H_n|\rangle.$$

If $\alpha > 0$, by using (6) and (ii), for $n$ such that $H_n \leq (1/e)^{\frac{1}{\alpha}}\rangle$, we have for some $k_3, k_4 > 0$

$$|T_2| \leq k_3 H_n^{\mu}\big(H_n^{\alpha}|\log H_n| + C\big) \leq k_4 H_n^{\mu}\rangle.$$

In the case $1 - s \leq \lambda < 1 - H_n$, i.e. $H_n < 1 - \lambda \leq s$, by (6) and (ii), we can write for $\alpha < 0$

$$|T_2| \leq k_5 H_n^{\mu}(1-\lambda)^{\alpha} < k_5 H_n^{\mu+\alpha}, \quad k_5 > 0\rangle,$$

whereas for $\alpha > 0$

$$|T_2| \leq k_6 H_n^{\mu}\rangle, \quad k_6 > 0\rangle.$$

If $\alpha = 0\rangle$ then, for some $k_7 > 0\rangle$,

$$|T_2| \leq k_7 H_n^{\mu}|\log(1-\lambda)| < k_7 H_n^{\mu}|\log H_n|.$$

Finally, in the case $\lambda \in [0, 1-s)\rangle$, $I(\omega_{\alpha\beta};\lambda) = O(1)$ [20] and, consequently, $|T_2| = O\big(H_n^{\mu}\big)$.

In order to bound $|T_1|$ we consider $r \in (0, 1/2)$ and $n$ sufficiently large that $H_n < r$. We write

$$T_1 = \int_{-1}^{-1+r} \omega_{\alpha\beta}(x) g_n(x) + \int_{-1+r}^{1} \omega_{\alpha\beta}(x) g_n(x) dx := I_1 + I_2\rangle.$$

We consider that for $\lambda > 0$ and $0 < r < 1/2$

$$1 + x < |x - \lambda| \quad , \quad -1 \leq x \leq -1 + r \tag{8}$$

and

$$(1-x)\begin{cases} > |x-\lambda|, & -1 + r \leq x < (1+\lambda)/2 \\ = |x-\lambda|, & x = (1+\lambda)/2 \\ < |x-\lambda|, & (1+\lambda)/2 < x \leq 1\rangle. \end{cases} \tag{9}$$

In order to bound $|I_1|$ we set

$$\eta_1 := \max_{x\in[-1,-1+r]} (1-x)^{\alpha}$$

and write

$$|I_1| \leq \eta_1 \int_{-1}^{-1+r} (1+x)^{\beta}|g_n(x)|dx = \eta_1\Big[\int_{-1}^{-1+H_n} (1+x)^{\beta}|g_n(x)|dx$$

$$+ \int_{-1+H_n}^{-1+r} (1+x)^{\beta}|g_n(x)|dx\Big] := \eta_1[R_1 + R_2]$$

Since by (8) $|x-\lambda|^{\mu-1} < (1+x)^{\mu-1}$, we obtain

$$R_1 \leq A_1 \int_{-1}^{-1+H_n} (1+x)^{\beta+\mu-1} dx = \frac{A_1 H_n^{\mu+\beta}}{\mu+\beta}\rangle.$$

In order to bound $R_2$ we consider the following cases. If $\beta < 0$, using (8) and (ii), we have that

$$R_2 \leq 2cH_n^\mu \int_{-1+H_n}^{-1+r} (1+x)^{\beta-1} dx \leq \frac{2c}{|\beta|} H_n^\mu (r^\beta + H_n^\beta) < \frac{4c}{|\beta|} H_n^{\mu+\beta} \rangle.$$

If $\beta \geq 0$ then, setting $\eta_2 := \max\limits_{x\in[-1,-1+r]} (1+x)^\beta$, using (ii) and noting that $|-1+H_n-\lambda| > H_n$ and that

$$|-1+r-\lambda| > H_n \rangle, \tag{10}$$

we obtain

$$R_2 \leq 2\eta_2 c H_n^\mu \int_{-1+H_n}^{-1+r} |x-\lambda|^{-1} dx < 4\eta_2 c H_n^\mu |\log H_n|.$$

In order to bound $|I_2|$ we set

$$\eta_3 := \max_{x\in[-1+r,1]} (1+x)^\beta \rangle, \quad \bar{g}_n(x) := (1-x)^\alpha \frac{r_n(x) - r_n(\lambda)}{x-\lambda}$$

and write

$$|I_2| \leq \eta_3 \Big[ \int_{-1+r}^{\lambda-H_n} |\bar{g}_n(x)| dx + \int_{\lambda-H_n}^{1} |\bar{g}_n(x)| dx \Big] := \eta_3 [R_3 + R_4]$$

In order to bound $R_3$ we have to consider the following cases.
If $\lambda - H_n < 1 - r$ then, by setting $\eta_4 := \max\limits_{x\in[-1+r,1-r]} (1-x)^\alpha$ and using (ii) and (10) we obtain

$$R_3 \leq 2\eta_4 c H_n^\mu |\log H_n| \rangle. \tag{11}$$

In the case $\lambda - H_n \geq 1 - r$, if $\alpha \geq 0$ then (11) holds with $\eta_4 := \max\limits_{x\in[-1+r,1]} (1-x)^\alpha$. If $\alpha < 0$, then using (9), (10) and (ii), we obtain

$$R_3 < 2cH_n^\mu \int_{-1+r}^{\lambda-H_n} |x-\lambda|^{\alpha-1} dx \leq \frac{4c}{|\alpha|} H_n^{\mu+\alpha} \rangle.$$

In order to bound $R_4$ we have to consider the following cases:

**a)** $\lambda + H_n \leq 1 - r \rangle$.
In this case we write

$$\begin{aligned} R_4 &= \int_{\lambda-H_n}^{\lambda+H_n} |\bar{g}_n(x)| dx + \int_{\lambda+H_n}^{1-r} |\bar{g}_n(x)| dx \\ &+ \int_{1-r}^{1-H_n} |\bar{g}_n(x)| dx + \int_{1-H_n}^{1} |\bar{g}_n(x)| dx := P_1 + P_2 + P_3 + P_4 \rangle. \end{aligned}$$

Setting $\eta_5 := \max_{x\in[-1,1-r]}(1-x)^\alpha$, by using (iii) we have that

$$P_1 \leq \eta_5 A_1 \int_{\lambda-H_n}^{\lambda+H_n} |x-\lambda|^{\mu-1} = \frac{2A_1}{\mu}\eta_5 H_n^\mu \rangle.$$

Using (ii) and a)

$$P_2 \leq 2c\eta_5 H_n^\mu \int_{\lambda+H_n}^{1-r} \frac{1}{|x-\lambda|}dx \leq 4c\eta_5 H_n^\mu |\log H_n|\rangle.$$

Moreover we have that

$$P_3 \leq 2cH_n^\mu \int_{1-r}^{1-H_n} (1-x)^\alpha \frac{1}{|x-\lambda|}dx\rangle.$$

Then in order to bound $P_3$ we consider at first the case $\alpha \geq 0$. Setting $\eta_6 := \max_{x\in[1-r,1]}(1-x)^\alpha$ and, using a), we have that

$$P_3 \leq 2c\eta_6 H_n^\mu \int_{1-r}^{1-H_n} \frac{1}{|x-\lambda|}dx \leq 4c\eta_6 H_n^\mu |\log H_n|\rangle.$$

If $\alpha < 0$ and $(1+\lambda)/2 \leq 1-r$ then, using (9),

$$P_3 \leq 2cH_n^\mu \int_{1-r}^{1-H_n} (1-x)^\alpha |x-\lambda|^{-1}dx \leq 2cH_n^\mu \int_{1-r}^{1-H_n} (1-x)^{\alpha-1}dx \leq \frac{4c}{|\alpha|}H_n^{\mu+\alpha}\rangle.$$

If $\alpha < 0$ and $1-r < (1+\lambda)/2 < 1-H_n$, then

$$\begin{aligned} P_3 \quad &\leq 2cH_n^\mu \Big[ \int_{1-r}^{(1+\lambda)/2} (1-x)^\alpha |x-\lambda|^{-1}dx \\ &+ \int_{(1+\lambda)/2}^{1-H_n} (1-x)^\alpha |x-\lambda|^{-1}dx \Big] \leq \frac{4c}{|\alpha|}\left(\frac{1}{2^\alpha}+1\right)H_n^{\mu+\alpha}\rangle, \end{aligned}$$

since by (9),

$$\int_{1-r}^{(1+\lambda)/2} (1-x)^\alpha |x-\lambda|^{-1}dx \leq \int_{1-r}^{(1+\lambda)/2} |x-\lambda|^{\alpha-1}dx \leq \left(\frac{1}{2^\alpha}+1\right)\frac{1}{|\alpha|}H_n^\alpha$$

and

$$\int_{(1+\lambda)/2}^{1-H_n} (1-x)^\alpha |x-\lambda|^{-1}dx \leq \int_{(1+\lambda)/2}^{1-H_n} (1-x)^{\alpha-1}dx \leq \left(1+\frac{1}{2^\alpha}\right)\frac{1}{|\alpha|}H_n^\alpha.$$

In order to bound $P_4$, we obtain using (9) and (iii) that

$$P_4 \leq A_1 \int_{1-H_n}^{1} (1-x)^\alpha |x-\lambda|^{\mu-1}dx \leq A_1 \int_{1-H_n}^{1} (1-x)^{\mu+\alpha-1}dx = \frac{A_1}{\mu+\alpha}H_n^{\mu+\alpha}\rangle.$$

**b)** $\lambda + H_n > 1$.
Using (iii) we write

$$R_4 \leq A_1 \int_{\lambda - H_n}^{1} (1-x)^{\alpha} |x-\lambda|^{\mu-1} dx\rangle.$$

If $\alpha \geq 0$ then we set $\eta_7 := \max_{x \in [-1+r,1]} (1-x)^{\alpha}$ and the following bound holds for $R_4$

$$R_4 \leq \eta_7 A_1 \int_{\lambda - H_n}^{1} |x-\lambda|^{\mu-1} \leq \frac{\eta_7 A_1}{\mu} H_n^{\mu}\rangle,$$

since

$$1 - \lambda < H_n\rangle. \tag{12}$$

If $\alpha < 0$, taking in account that $\lambda < (1+\lambda)/2 < 1$, we set

$$\begin{aligned} \int_{\lambda - H_n}^{1} (1-x)^{\alpha} |x-\lambda|^{\mu-1} dx &= \int_{\lambda - H_n}^{(1+\lambda)/2} (1-x)^{\alpha} |x-\lambda|^{\mu-1} dx \\ &\quad + \int_{(1+\lambda)/2}^{1} (1-x)^{\alpha} |x-\lambda|^{\mu-1} dx := S_1 + S_2\rangle. \end{aligned}$$

Using (9) and (12) we obtain

$$\begin{aligned} S_1 &\leq \int_{\lambda - H_n}^{(1+\lambda)/2} |x-\lambda|^{\mu+\alpha-1} dx \leq \frac{1}{\mu+\alpha} \left[ \left( \frac{1-\lambda}{2} \right)^{\mu+\alpha} + H_n^{\mu+\alpha} \right] \\ &< \frac{H_n^{\mu+\alpha}}{\mu+\alpha} \left( \frac{1}{2^{\alpha+\mu}} + 1 \right) \end{aligned}$$

and, similarly,

$$S_2 \leq \int_{(1+\lambda)/2}^{1} (1-x)^{\mu+\alpha-1} dx = \frac{1}{\mu+\alpha} \left( \frac{1-\lambda}{2} \right)^{\mu+\alpha} \leq \frac{H_n^{\mu+\alpha}}{2^{\mu+\alpha}(\mu+\alpha)}\rangle.$$

□

## 3. Examples of splines for quadratures

In this section we shall give some examples of sequences $\{f_n\}$ which satisfy (i)-(iii) of Theorem 1.

A first example can be derived from the sequence of approximating splines $\{Q_n f\}$, obtained by applying to the function $f$ the quasi interpolating $(q - i)$ operator $Q_n$ defined in [19], and used for numerical integration in [3,4,8].

In order to define $Q_n f$, we start with a polynomial spline space $S^m_{Y_n}$ defined by a positive $m$, the order of the splines in the space, and a partition $\Pi_N$.

The knot set $Y_n := \{y_{0n}, y_{1n}, \dots, y_{n+m-1,n}\}$ is derived from $\Pi_N$ and a set of positive integers $d_i$, $i = 0, 1, \dots, N$ with $d_0 = d_N = m$ and $d_i < m$, $i = 1, \dots, N-1$. The $y_{in}$ are given in nondecreasing order by taking each point of $\Pi_N$ exactly $d_i$ times, so that $n = \sum_{i=0}^{N-1} d_i$.

A basis for $S^m_{Y_n}$ is provided by the set of normalized B-splines of order $m$ $\{B_i(x); i = 0, \dots, n-1\}$ defined in [25].

We consider, for $l \le m$, a set of points, called $q-i$ points, $\{\tau_{ij}; j = 1, \dots, l; i = 0, 1, \dots, n-1\}$ such that $\tau_{i\nu} \neq \tau_{i\mu}$ for $\nu \neq \mu$.

The spline $Q_n f$ can be represented by [3]

$$Q_n f := \sum_{i=0}^{n-1} \Big( \sum_{j=1}^{l} v_{ij} f(\tau_{ij}) \Big) B_i \rangle, \tag{13}$$

where

$$v_{ij} := \sum_{\nu=j}^{l} \alpha_{i\nu} \Big/ \prod_{\substack{s=1 \\ s\neq j}}^{\nu} (\tau_{ij} - \tau_{is}) \rangle$$

and the coefficients $\alpha_{i\nu}$, defined in [19], are such that $Q_n p = p$ for $p \in P_l$, where $P_l$ is the set of polynomials of order $l$.

Since a $q-i$ point $\tau_{ij}$ may coincide with another $q-i$ point $\tau_{pq}$, the number of distinct function values in (13) can be less than $nl$. Of course, the set of distinct $q-i$ points will coincide with the set of nodes $X_n$ of the quadrature sum obtained in replacing $f$ by $Q_n f$ in (1).

In particular, if we set $l = 2$ in (13) and choose as q-i points the Schoenberg points

$$\{\tau_{i1} := (y_{i+1,n} + \cdots + y_{i+m-1,n}) \big/ (m-1);\ i = 0, 1, \dots, n-1\} \rangle,$$

then $\alpha_{i2} = 0$, $i = 0, \dots, n-1$ [19], leading to the Schoenberg variation diminishing spline. In this case $\{Q_n f\}$ will satisfy (i) and, for l.u. $\{\Pi_N\}\rangle$, $\rangle$ (ii) and (iii). Consequently, $\{Q_n f\}$ will be a viable candidate for $\{f_n\}$ in Theorem 1.

In the case $l \ge 2$ there have been many suggestions for the choice of $q-i$ points [3,8,22,23], which lead to sequences $\{Q_n f\}$ satisfying (ii) and (iii) if $\{\Pi_N\}$ is l.u., but, in general, the condition (i) will be not true. We will obtain a sequence $\{f_n\}$ for Theorem 1 by introducing the following modified splines [10]

$$\hat{Q}_n f := f(-1) B_0 + \sum_{i=1}^{n-2} \Big( \sum_{j=1}^{l} v_{ij} f(\tau_{ij}) \Big) B_i + f(1) B_{n-1} \rangle,$$

for which $\hat{Q}_n f(\pm 1) = f(\pm 1)$ and the conditions (i)-(iii) are true if $\{\Pi_N\}$ is l.u.

A further example is provided by the optimal nodal (o.n.) splines introduced in [13,14], studied in [7,10,12,15] and used for numerical integration in [1,2,6,21].

Let $m > 2$, $N \equiv n \geq m - 1$ and $\Pi_N \equiv X_n$, i.e. the partition $\Pi_N$ coincides with the quadrature nodes set $X_n$. We shall call the points of $X_n$ the primary knots of the spline.

The spline knot set $Y_n$ is obtained by inserting $(m - 2)$ distinct points in each interval $[x_{in}, x_{i+1,n}]$, $i = 0, \ldots, n - 1$, so that $y_{(m-1)i,n} = x_{in}$, $i = 0, \ldots, n$.

Since all knots have multiplicity equal to one, the spline will be $C^{m-2}(J)$. We consider the set of nodal basis functions $\{w_i(x); i = 0, \ldots, n\}$ defined in [12,15] and the spline space $S^m_{Y_n}$ spanned by $\{w_i\}$. The defining formula for $W_n f \in S^m_{Y_n}$ is given by

$$W_n f(x) := \sum_{i=0}^{n} f(x_{in}) w_i(x)\rangle.$$

Since $w_i(x_{jn}) = \delta_{ij}$, $i, j = 0, \ldots, n$, the condition (i) of Theorem 1 is true for $W_n f\rangle.\rangle$ Moreover (ii) holds if $\{X_n\}$ is l.u. and (iii) is true if $\{Y_n\}$ is l.u.. Consequently, $\{W_n f\}$ is a viable candidate for $\{f_n\}$ in Theorem 1.

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