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SOLITARY WAVES: PHYSICAL ASPECTS AND MATHEMATICAL RESULTS

Abstract. This paper originates from a series of lectures given by the second author at the University of Torino, during the Third Fortnight on Nonlinear Analysis (September 24 - October 6, 2001) and is divided into two parts, almost independent. Part 1 proposes a set of ideas concerning the introduction of nonlinear field equations in the theory of particles; indeed, moving from very general mathematical assumptions, we present a deterministic model which exhibits relativistic phenomena. Part 2 contains a survey on some questions concerning solitary waves; we start with old, even historical, matters and touch some more recent developements.

1. Some remarks on variational principles for Lorentz invariant field equations

1.1. Introduction

In this introduction we make some simple basic assumptions, which are shared today by every fundamental theory in physics and yield, as it will be shown in the next sections, very deep consequences.

Throughout these notes, we will not be completely rigorous. For example, we will generally not specify the functional spaces to which the functions we use belong, simply assuming that all the functions involved are 'good enough' to allow the computations; in particular, we assume that all the integrals we introduce are finite.

(α) **The universe is variational**, that is *we suppose that all the physical phenomena are governed by differential equations which admit a variational formulation.*

This variational principle is at once reasonable, if we think that all the fundamental equations of physics (†) can be seen as the Euler-Lagrange equations of a suitable action functional. For instance

$$J(x) = \int \left[\frac{1}{2} \sum_{j=1}^{k} m_j \left| \dot{x}_j \right|^2 - V(t, x) \right] dt \qquad (Newton)$$

gives the equations of motion of k particles having masses m_j and whose positions at time t is given by $x_j(t) \in \mathbb{R}^3$, where V is the potential energy of the system; more

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†Diffusion equations, which are deeply related to stochastic processes, are not variational; nevertheless they are not fundamental, by which we mean that they can be derived from the Newton equation via the Boltzmann theory.

generally, using generalized coordinates $q(t) = (q_1(t), \ldots, q_k(t))$, the dynamics of a finite dimensional system is determined by the functional

$$J(q) = \int L(t, q, \dot{q})\, dt \qquad (\textit{Lagrange})$$

where L is the Lagrangian of the system, or by the functional

$$J(q, p) = \int [\dot{q} \cdot p - H(t, q, p)]\, dt \qquad (\textit{Hamilton})$$

where H is the Hamiltonian of the system and $p(t) = (p_1(t), \ldots, p_k(t))$ the generalized momenta.

Also the dynamics of fields can be related to variational principles. From a mathematical point of view, a field is a function

$$\psi : (x, t) = (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R} \longmapsto \psi(x, t) \in \mathbb{R}^k$$

from the *physical space-time* into the *internal parameters space*. From a physical point of view, a field can be regarded as a modification of the 'ether' (or 'vacuum'), an ipotetic entity whose *internal state* at point x and time t is described by the value $\psi(x, t)$. On the subject, we have for example the following variational formulations:

$$J(\mathbf{A}, \varphi) = \frac{1}{2} \int \int \left[E^2 - B^2\right] dx dt \qquad (\textit{Maxwell})$$

where $E = |\mathbf{A}_t + \nabla\varphi|$ and $B = |\nabla \times \mathbf{A}|$ (see Section 1.3 below),

$$J(\psi) = \int \int \left[\frac{\hbar^2}{2m} |\nabla\psi|^2 - \hbar i \psi_t \overline{\psi} + V(x) |\psi|^2\right] dx dt \quad (\textit{Schrödinger})$$

where m is the particle mass, $\hbar$ the Planck constant and V the potential,

$$J\left(g_{ij}\right) = \int \int R\left(g_{ij}\right) dx dt \qquad (\textit{Einstein})$$

where R is the scalar curvature of the metric tensor g_{ij}.

REMARK 1. *The variational principle has a very long history and can be traced back even to the thought of Aristotle. However, its very discovery has been attributed to P.L.M. de Maupertuis (1698-1759), after he was engaged in polemics with the followers of G.W. Leibniz; in his work* "Examen philosophique de la preuve de l'existence de Dieu" *(1756), he stated the principle of minimal action as an evidence of rationality in the divine creation. It is well known that these metaphysical ideas were then formalized by L. Euler and G.L. Lagrange in the eighteenth century, but the ultimate reason for which the variational principle holds true in nature is still today a mystery. On the subject, we refer to the essay* "Le Meilleur des mondes possibles. Mathématiques et destinée" *by I. Ekeland [46].*

(β) **The universe is invariant for the Poincaré group**, that is *we suppose that all the equations of the universe are invariant with respect to the group generated by the following transformations:*

- *time translations*, i.e. transformations depending on one parameter having the form
$$\begin{cases} x \longmapsto x \\ t \longmapsto t + t_0 \end{cases}$$
- *space translations*, i.e. transformations depending on three parameters having the form
$$\begin{cases} x \longmapsto x + x^0 \\ t \longmapsto t \end{cases}$$
- *space rotations*, i.e. transformations depending on three parameters having the form
$$\begin{cases} x \longmapsto Rx \quad \text{with } R \in O(3) \\ t \longmapsto t \end{cases}$$
- *Lorentz transformations*, i.e. space-time rotations depending on one parameter v having the form
$$\begin{cases} x_1 \longmapsto \gamma\,(x_1 - vt) \\ x_2 \longmapsto x_2 \\ x_3 \longmapsto x_3 \\ t \longmapsto \gamma\left(t - \frac{v}{c^2}x_1\right) \end{cases} \qquad \begin{cases} x_1 \longmapsto x_1 \\ x_2 \longmapsto \gamma\,(x_2 - vt) \\ x_3 \longmapsto x_3 \\ t \longmapsto \gamma\left(t - \frac{v}{c^2}x_2\right) \end{cases} \qquad \begin{cases} x_1 \longmapsto x_1 \\ x_2 \longmapsto x_2 \\ x_3 \longmapsto \gamma\,(x_3 - vt) \\ t \longmapsto \gamma\left(t - \frac{v}{c^2}x_3\right) \end{cases}$$
where $\gamma = \frac{c}{\sqrt{c^2 - v^2}}$, $|v| < c$ and c is a constant (dimensionally a velocity).

Indeed, the Poincaré group $\mathcal{P}$ is the ten parameters Lie group generated by the above transformations together with the time and parity inversions $t \longmapsto -t$ and $x \longmapsto -x$.

The assumption of the first three invariances cannot be omitted if we want to make a physical theory, for they express the possibility of repeating experiments. More precisely, translational invariances ask for time and space to be homogeneuos (i.e., whenever and wherever an experiment is performed, it gives the same results) and rotational invariance requires that the space is isotropic (i.e., there is no privileged directions in the universe). Finally, the Lorentz invariance is an empirical fact and we will see that it is the very responsible of relativistic effects; meanwhile, we can already state some of its first consequences, as follows.

• Speaking about rigid bodies, we need lenght to be an invariant; because of the fact that the transformations involving velocity mingle space coordinates with time, we have to state on the contrary that euclidean distance cannot be invariant. So we conclude that *rigid bodies do not exist*.

• If we want to describe the physical phenomena by means of differential equations, the link between space and time mentioned above forces us to use *partial differential equations*.

REMARK 2. *In the spirit of the so-called "Erlangen program" by F.C. Klein, every theory can be regarded as the set of those properties which are invariant with respect to a certain group of transformations; for example, euclidean geometry is classified as the set of properties which are invariant with respect to rotations, translations and congruences, as well as affine geometry is invariant for the linear group plus translations, as well as topology is invariant under continuous transformations, and so on. In this respect, if we assume the invariance for the* Galileo transformations

$$\begin{cases} x_1 \longmapsto x_1 - vt \\ x_2 \longmapsto x_2 \\ x_3 \longmapsto x_3 \\ t \longmapsto t \end{cases} \qquad \begin{cases} x_1 \longmapsto x_1 \\ x_2 \longmapsto x_2 - vt \\ x_3 \longmapsto x_3 \\ t \longmapsto t \end{cases} \qquad \begin{cases} x_1 \longmapsto x_1 \\ x_2 \longmapsto x_2 \\ x_3 \longmapsto x_3 - vt \\ t \longmapsto t \end{cases}$$

instead of the ones of Lorentz, we obtain classical physics instead of relativity.

The invariance with respect to the Poincaré group $\mathcal{P}$ and the variational principle can be joined together as follows. If a physical phenomenon is described by the critical points of a functional J on a proper functional linear space V, i.e.

$$\delta J(u) = 0 \quad \text{with } u \in V,$$

then invariance means that there is a representation

$$T : \mathcal{P} \to \mathrm{GL}(V)$$

such that

$$\forall u \in V \quad \forall g \in \mathcal{P} \quad J(u) = J\left(T_g u\right). \tag{1}$$

This implies that u is a solution if and only if $T_g u$ is a solution.

1.2. Scalar fields

Let us consider the action of the Poincaré group $\mathcal{P}$ on a scalar field $\psi \in \mathbb{C}$:

$$\forall g \in \mathcal{P} : \quad T_g \psi(x,t) = \psi\left(g^{-1}(x,t)\right) \tag{2}$$

where $(x,t) = (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R}$.

Probably, the simplest Lagrangian which is invariant with respect to this action is

$$\mathcal{L}_\psi = |\psi_t|^2 - c^2 |\nabla \psi|^2 \tag{3}$$

and the Euler-Lagrange equation for the functional

$$\mathcal{S}_0(\psi) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}} \mathcal{L}_\psi(x,t)\, dx dt \tag{4}$$

is

$$\Box \psi = 0 \tag{5}$$

where $\Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta$ is the d'Alembert operator. Indeed, it's just a matter of substitution to verify that $\mathcal{L}_{T_g \psi}(x,t) = \mathcal{L}_\psi\left(g^{-1}(x,t)\right)$ for every $g \in \mathcal{P}$ and then the invariance (1) obviously holds for the functional (4), if we notice that $|Jac(g)| = 1$ for every $g \in \mathcal{P}$. Note that the Poincaré invariance (1) is reflected by the fact that if $\psi(x,t)$ is a solution of (5) then also all the functions defined by (2) are solutions; in particular we will consider the function

$$\psi\left(\gamma\left[x_1 - vt\right], x_2, x_3, \gamma\left[t - \frac{v}{c^2}x_1\right]\right).$$

The main focus of this paper is however the more realistic (from a physical point of view) semilinear wave equation

$$\boxed{\Box \psi + W'(\psi) = 0} \tag{6}$$

which is still the simplest, but now *nonlinear*, equation which satisfies the above assumptions. Here

$$W' = \frac{\partial W}{\partial \psi^1} + i\frac{\partial W}{\partial \psi^2}$$

is the gradient of a C^1 function $W : \mathbb{C} \to \mathbb{R}$ under the usual identification $\psi \equiv \left(\psi^1, \psi^2\right)$ between $\mathbb{C}$ and $\mathbb{R}^2$. Other useful and rather natural assumptions on W will be made in the following. The action functional corresponding to equation (6) is then

$$\mathcal{S}(\psi) = \frac{1}{2}\int_{\mathbb{R}^3 \times \mathbb{R}} \left[|\psi_t|^2 - c^2 |\nabla \psi|^2\right] dx dt - \int_{\mathbb{R}^3 \times \mathbb{R}} W(\psi)\, dx dt. \tag{7}$$

1.2.1. Conservation laws

By Emmy Noether's theorem, the invariance of a variational integral under the action of a group of transformations smoothly dependent on one parameter yields a conservation law for any of its extremals. In the present case (*), we have the following integrals of the motion, provided that W is of class C^2 and having suitable decay properties on the field $\psi(x,t)$ at 'spatial' infinity.

- Time translation invariance implies the conservation of *energy*, which takes the form

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\psi_t|^2 + c^2 |\nabla \psi|^2\right] dx + \int_{\mathbb{R}^3} W(\psi)\, dx. \tag{8}$$

*see Subsection 1.5.1.

- Space translations invariance implies the conservation of *momentum*, which takes the form

$$\mathbf{P}(\psi) = -\mathrm{Re}\int_{\mathbb{R}^3} \overline{\psi}_t \nabla\psi \, dx = -\int_{\mathbb{R}^3} \left[u_t \nabla u + u^2 S_t \nabla S \right] dx \tag{9}$$

  where we have used the polar form $\psi(x,t) = u(x,t)\, e^{iS(x,t)}$ with $u, S \in \mathbb{R}$.

- Space rotations invariance implies the conservation of *angolar momentum*, which takes the form

$$\mathbf{M}(\psi) = \mathrm{Re}\int_{\mathbb{R}^3} \overline{\psi}_t \,(\mathbf{x} \times \nabla\psi)\, dx. \tag{10}$$

- Considering transformations in the target space, not in the physical one, the invariance under the *gauge transformation*

$$\psi(x,t) \longmapsto \psi(x,t)\, e^{i\theta} \qquad (\theta \in \mathbb{R})$$

  which clearly holds when W depends only on $|\psi|$, yields another constant of the motion, called *charge*. Again with the use of a polar expression, it has the form

$$\mathcal{C}(\psi) = \mathrm{Im}\int_{\mathbb{R}^3} \psi_t \overline{\psi}\, dx = \int_{\mathbb{R}^3} u^2 S_t \, dx.$$

  Note that the phase translations invariance $\psi \mapsto \psi e^{i\theta}$ is the simplest one we can imagine in the target space (in fact, $\mathbb{S}^1$ is the simplest Lie group).

REMARK 3. *It is easy to see that the right-hand sides of (8)-(1.2) change in the 'natural' way by a change of reference frame. For instance, considering the change of coordinates* $x' = Rx + x^0$ *with* $R \in O(3)$ *and* $\psi'(x', t) = \psi\left(R^{-1}\left(x' - x^0\right), t\right)$*, we get*

$$E\left(\psi'\right) = E(\psi) \quad \textit{and} \quad P\left(\psi'\right) = RP(\psi)$$

(P *denotes the column of the components of* $\mathbf{P}$*).*

1.2.2. Solitary waves

As a consequence of the assumptions we have made, *point* particles are inadequate to describe interactions. For instance, the problem arises considering the Maxwell equation

$$\nabla \cdot \mathbf{E} = \delta$$

(where the Dirac function δ represents a unit-charge point particle) and introducing the gauge potentials associated to the electromagnetic field (see below): in the electrostatic case, one is led to consider the equation

$$-\triangle\varphi = \delta$$

which has no finite-energy solution (because the fundamental solution $\varphi(x) = \frac{1}{4\pi |x|}$ of the Laplacian bears $\int |\nabla \varphi|^2 = +\infty$) and this makes classical electrodynamics inconsistent (see also [52]).

Hence, turning back to our equation (6), we are interested in the case in which the nonlinearity causes the formation of stable structures which can be considered, owing to their particle-like behaviour, as *extended* particles, namely *solitary waves* and *solitons* (see Part 2); roughly speaking, a solitary wave is a solution of a field equation whose energy travels as a localized packet and a soliton is a solitary wave which asymptotically preserves its localization property under interactions with other localized disturbances.

With a view to investigate the properties of such waves, it is convenient to consider first the so-called *standing* solitary waves, which have the form

$$\psi_0(x,t) = u_0(x)\, e^{-i\omega_0 t}, \; u_0 \in \mathbb{R} \tag{11}$$

where $\omega_0 \in \mathbb{R}$ can be considered as the frequence of an internal clock. It is then natural to assume that W only depends on $|\psi|$, i.e.

$$W(\psi) = f(|\psi|) \tag{12}$$

where we may take $f \in C^1(\mathbb{R};\mathbb{R})$, f even, $f'(0) = 0$. So, the function ψ_0 is a solution of (6) if and only if (†)

$$-\triangle u_0 + \frac{1}{c^2} f'(u_0) = \frac{\omega_0^2}{c^2} u_0 \tag{13}$$

which means that u_0 is a critical point of the *reduced action* functional

$$F(u_0) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u_0|^2 - \frac{\omega_0^2}{c^2}u_0^2\right]dx + \frac{1}{c^2}\int_{\mathbb{R}^3} f(u_0)\,dx. \tag{14}$$

The existence of a solution to this problem has been first proved in the 70's by W.A. Strauss [79] and H. Berestycki and P.L. Lions [27], [28], [29] for example with the choice

$$W(\psi) = \tfrac{1}{2}|\psi|^2 - \tfrac{1}{p}|\psi|^p \quad \text{and} \quad 1 - \omega_0^2 > 0, \; p \in \,]2, 6[\,.$$

Finally, observing that to any standing wave we can associate a family of travelling waves by applying a Lorentz transformation, we will denote by $\psi_{\mathbf{v}}$ the solitary wave travelling with vector velocity $\mathbf{v}$ ($|\mathbf{v}| < c$) which is obtained from the (11) standing one. For sake of simplicity, we consider $\mathbf{v} = (v, 0, 0)$ and hence

$$\psi_{\mathbf{v}}(x,t) = u_0(\gamma[x_1 - vt], x_2, x_3)\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{15}$$

where $\mathbf{k} = \frac{\omega}{c^2}\mathbf{v}$ and $\omega = \gamma\omega_0$.

† see the beginning of Subsection 1.5.2

1.2.3. Space contraction and time dilation

Taking ψ_0 as in (11) and moving it, we have obtained the expression (15): looking at the first argument of the space function u_0, we can see that the size shrinks by a factor $\gamma > 1$ in the direction of the movement; this is the *contraction of lenght* phenomenon.

The position of the travelling object (15) at the time t can be defined as the barycenter

$$q(t) = \frac{\int |\psi_{\mathbf{v}}(x,t)|^2 \, x \, dx}{\int |\psi_{\mathbf{v}}(x,t)|^2 \, dx}.$$

By a change of variables one easily deduces that $q(t) = q(0) + t\mathbf{v}$; let us assume, for sake of simplicity, $q(0) = (0,0,0)$. If we follow this center during the movement and look at what happens in the position $q(t)$, we have

$$\begin{aligned} \psi_{\mathbf{v}}(q(t), t) &= u_0(\gamma [tv - vt], 0, 0) \, e^{i\left(\frac{\omega}{c^2} vtv - \omega t\right)} \\ &= u_0(0,0,0) \, e^{i\omega t\left(\frac{v^2}{c^2} - 1\right)} \\ &= u_0(0,0,0) \, e^{-i\gamma\omega_0 t \gamma^{-2}} \\ &= u_0(0,0,0) \, e^{-i\frac{\omega_0}{\gamma} t} \end{aligned}$$

i.e., the internal frequence is now $\frac{\omega_0}{\gamma} < \omega_0$. Time slows down: the internal clock oscillates slower and the movement makes the structure life to last more; this is the *dilation of time* phenomenon.

1.2.4. The dynamics

The conservation laws stated in the previous Subsection 1.2.1 hold true for any solution of our equation (6). As we are interested in solitary waves, we now compute those constants of the motion referred to these particular solutions.

Let us begin with considering standing solitary waves (11). Under the assumption (12) with $W(0) = 0$, we can use (‡) Pohožaev identity

$$\int |\nabla u_0|^2 \, dx = \frac{3\omega_0^2}{c^2} \int u_0^2 dx - \frac{6}{c^2} \int W(u_0) \, dx \tag{16}$$

for the equation (13), to eliminate W from (8) and get

$$E(\psi_0) = \frac{c^2}{3} \int_{\mathbb{R}^3} |\nabla u_0|^2 \, dx + \omega_0^2 \int_{\mathbb{R}^3} u_0^2 dx. \tag{17}$$

Turning then to the associated travelling waves (15), we also get (§)

$$E(\psi_{\mathbf{v}}) = \gamma E(\psi_0) \tag{18}$$

‡see Subsection 1.5.2

§see Subsection 1.5.3 and Subsection 1.5.4

and

$$\mathbf{P}(\psi_{\mathbf{v}}) = \frac{\gamma E(\psi_0)}{c^2}\mathbf{v} \tag{19}$$

Since $\gamma > 1$ depends on the velocity v, we conclude that the energy of a travelling solitary wave grows with velocity by a factor γ and his momentum is proportional both to $\mathbf{v}$ and γ.

Since as a matter of fact we have regarded physical phenomena as solutions of partial differential equations, so far it wouldn't have made sense to speak about the 'mass' of a solitary wave; right now it is possible instead to define the *mass* $m(\psi_{\mathbf{v}})$ of any travelling object. Recalling that in classical mechanics we have $\mathbf{p} = m\mathbf{v}$, the parallelism obtained in (19) allows us to define the mass still as the ratio between momentum and velocity, namely

$$m(\psi_{\mathbf{v}}) = \gamma \frac{E(\psi_0)}{c^2} \tag{20}$$

and then we have that mass is not constant, but, as the energy, increases with velocity by a factor γ.

Moreover, from (18) and (20), we surprisingly deduce the *Einstein equation*

$$E(\psi_{\mathbf{v}}) = m(\psi_{\mathbf{v}})\, c^2$$

for mass and energy.

1.2.5. Conclusions

At the beginning of this section, assuming variational principle and Poincaré invariance we were led to consider the equation (6), as the simplest one satisfying these requirements; in the next subsections, we have seen that all the peculiar facts of special relativity

- space contraction and time dilation
- increase of mass with velocity
- Einstein equation $E = mc^2$

can be easily deduced from that equation. If we think that Poincaré group was born from electromagnetism as the exact group for which Maxwell equations are invariant, this could appear natural; nevertheless, it is somewhat surprising that relativistic features can be derived from a single equation without extra assumptions. Moreover, if we recall that the operator $\Box$ was born a century before Maxwell theory (with J.B. d'Alembert, 1717-1783) and that, interpreting c as the speed of sound, it occurs in the Newtonian mechanics of an elastic membrane, the whole discussion can be seen under a different perspective.

First of all, the theory of relativity is consistent with the ideas of *ether theory*, namely of a universe filled up with an elastic medium having c as 'sound speed'.

Moreover, relativistic effects turn out to be related to mathematical principles and they are also exhibited by a classical model; in this respect, the theory of relativity resembles noneuclidean geometries: it can be seen as 'embedded' into classical mechanics and its validity should be considered not so absolute as many believe.

Finally, the idea of an *absolute time* proves to be consistent with relativity.

1.3. Vector fields

So far we have seen the consequences of variational principle and Poincaré invariance on scalar fields; now we turn to consider the way in which the Poincaré group $\mathcal{P}$ acts on a vector field $U = (\mathbf{A}, \varphi) \in \mathbb{R}^3 \times \mathbb{R}$:

$$\forall g \in \mathcal{P} : \quad T_g U(x,t) = gU\left(g^{-1}(x,t)\right)$$

where $(x,t) = (x_1, x_2, x_3, t) \in \mathbb{R}^4$.

Assuming hereafter

$$\boxed{c = 1}$$

(which amounts to a suitable choice of measure units), the simplest Lagrangian which is invariant under this action is

$$\mathcal{L}_{(\mathbf{A},\varphi)} = |\nabla\varphi|^2 - \varphi_t^2 - |\nabla\mathbf{A}|^2 + |\mathbf{A}_t|^2$$

and the Euler-Lagrange equations for the functional

$$\mathcal{V}(\mathbf{A},\varphi) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}} \mathcal{L}_{(\mathbf{A},\varphi)}(x,t)\, dxdt$$

are

$$\left\{ \begin{array}{l} \Box\mathbf{A} = 0 \\ \Box\varphi = 0. \end{array} \right. \tag{21}$$

Assuming, as usual in these notes, that $\mathbf{A}$ and its derivatives have all the needed integrability properties, integrations by parts yield

$$\mathcal{V}(\mathbf{A},\varphi) = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}} \left[|\nabla\varphi|^2 - \varphi_t^2 - |\nabla\times\mathbf{A}|^2 - (\nabla\cdot\mathbf{A})^2 + |\mathbf{A}_t|^2\right] dxdt$$

where $\nabla\cdot$ and $\nabla\times$ denote the divergence and curl operators. In deriving this expression, recall the algebraic identity $-\triangle\mathbf{V} = \nabla\times(\nabla\times\mathbf{V}) - \nabla(\nabla\cdot\mathbf{V})$.

Besides Poincaré invariance, there are other transformations for which $\mathcal{V}$ is invariant. For instance, we can change φ and $\mathbf{A}$ in such a way that

$$\varphi_t + \nabla\cdot\mathbf{A} = 0 \tag{22}$$

and $\mathcal{V}$ is not changed; this implies that, for every critical point of $\mathcal{V}$ we can find a gauge transformation that yields a solution to (21) for which (22) is satisfied. Thus we may assume the extra equation (22), which is the *gauge of Lorentz*, just as a mathematical trick in order to reduce the number of solutions and, as a matter of fact, this means that we are minimizing $\mathcal{V}$ over the manifold given by this equation, which, being it linear, represents a hyperplane in a suitable space of functions.

By some computations and integrations by parts (again assuming all the necessary integrability) using (22) we obtain

$$\mathcal{V}(\mathbf{A},\varphi)=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}}\left[|\mathbf{A}_t+\nabla\varphi|^2-|\nabla\times\mathbf{A}|^2\right]dxdt \tag{23}$$

and, when $\mathcal{V}$ has this form, its Euler-Lagrange equations become

$$\begin{cases}\dfrac{\partial}{\partial t}(\mathbf{A}_t+\nabla\varphi)=-\nabla\times(\nabla\times\mathbf{A})\\ \nabla\cdot(\mathbf{A}_t+\nabla\varphi)=0.\end{cases} \tag{24}$$

Then, setting

$$\begin{aligned}\mathbf{E}&=-(\mathbf{A}_t+\nabla\varphi)\\ \mathbf{B}&=\nabla\times\mathbf{A}\end{aligned} \tag{25}$$

from (23) and (24) we get

$$\mathcal{V}(\mathbf{A},\varphi)=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}}\left[|\mathbf{E}|^2-|\mathbf{B}|^2\right]dxdt \tag{26}$$

and

$$\begin{cases}\nabla\times\mathbf{B}=\mathbf{E}_t\\ \nabla\cdot\mathbf{E}=0\\ \nabla\times\mathbf{E}=-\mathbf{B}_t\\ \nabla\cdot\mathbf{B}=0.\end{cases} \tag{27}$$

For this, recall the algebraic identities $\nabla\times(\nabla f)\equiv 0$ and $\nabla\cdot(\nabla\times\mathbf{V})\equiv 0$, and observe that $\nabla\times(\mathbf{A}_t+\nabla\varphi)=\nabla\times\mathbf{A}_t+\nabla\times(\nabla\varphi)=\frac{\partial}{\partial t}[\nabla\times\mathbf{A}]$. System (27) is the system of *Maxwell equations in empty space*. Note again that there is no physics in our discussion and even the Maxwell equations turn out to be a consequence of variational principle and Poincaré invariance.

1.4. Maxwell equations and matter

Since in this theory matter is supposed to be a field, satisfying the very simple equation (6) we have examined, by gauge invariance arguments there is a standard way to couple Maxwell equations and matter (¶) and we are led to describe the interaction between the electromagnetic field $(\mathbf{E},\mathbf{B})$ and the matter field ψ by the functional

$$\begin{aligned}\mathcal{I}(\psi,\mathbf{A},\varphi)\;=\;&\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}}\left[|D_t\psi|^2-|\mathbf{D}_x\psi|^2\right]dxdt-\int_{\mathbb{R}^3\times\mathbb{R}}W(\psi)\,dxdt\\ &+\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}}\left[|\mathbf{E}|^2-|\mathbf{B}|^2\right]dxdt\end{aligned} \tag{28}$$

¶see [50] or also [16], [19], [18]

which is the total action, formally obtained by addictioning (7) and (26) and substituting the usual derivatives with the so-called *gauge covariant derivatives*

$$D_t = \partial_t + i\varphi \quad \text{and} \quad \mathbf{D}_x = \nabla - i\mathbf{A}$$

where $(\mathbf{A}, \varphi)$ is the pair of gauge potentials associated to $(\mathbf{E}, \mathbf{B})$ by the relations (25).

By means of the usual polar form $\psi = ue^{iS}$, under the assumption (12) we can write $\mathcal{I}$ explicitly as (‖)

$$\mathcal{I}(u, S, \mathbf{A}, \varphi) = \int_{\mathbb{R}^3 \times \mathbb{R}} \mathcal{L}_{u,S,\mathbf{A},\varphi}(x, t)\, dxdt$$

where

$$\begin{aligned} \mathcal{L}_{u,S,\mathbf{A},\varphi} &= \frac{1}{2}\left[u_t^2 - |\nabla u|^2 - u^2\left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right)\right] + \\ &\quad - f(u) + \frac{1}{2}\left[|\mathbf{A}_t + \nabla\varphi|^2 - |\nabla \times \mathbf{A}|^2\right] \end{aligned} \tag{29}$$

and then, making its variations with respect to u, S, $\mathbf{A}$ and φ, we get the following equations:

$$\begin{gathered} \Box u + \left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right) u + f'(u) = 0 \\ \frac{\partial}{\partial t}\left[(S_t + \varphi)\, u^2\right] - \nabla \cdot \left[u^2 (\nabla S - \mathbf{A})\right] = 0 \\ \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}[\mathbf{A}_t + \nabla\varphi] - u^2 (\nabla S - \mathbf{A}) = 0 \\ \nabla \cdot (\mathbf{A}_t + \nabla\varphi) - (S_t + \varphi)\, u^2 = 0. \end{gathered}$$

These equations can be simplyfied if we set

$$\begin{aligned} \rho &= (S_t + \varphi)\, u^2 \\ \mathbf{j} &= u^2 (\nabla S - \mathbf{A}) \end{aligned}$$

so to obtain

$$\begin{cases} \Box u + \left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right) u + f'(u) = 0 \\ \rho_t - \nabla \cdot \mathbf{j} = 0 \\ \nabla \times \mathbf{B} - \mathbf{E}_t = \mathbf{j} \\ \nabla \cdot \mathbf{E} = -\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\mathbf{B}_t \end{cases} \tag{30}$$

where the second is a *continuity equation* and the last four are *Maxwell equations* with respect to a matter distribution having ρ and $\mathbf{j}$ respectively as the *charge* and *current* densities. On the subject, we quote from [17] the following existence result, which concerns the electrostatic case with the matter field given by a standing wave.

‖The computations concerning the present Section 1.4 are collected in Subsection 1.5.5.

THEOREM 1. *If* $W(\psi) = \frac{1}{2}|\psi|^2 - \frac{1}{p}|\psi|^p$ *with* $4 < p < 6$ *and* $1 - \omega_0^2 > 0$*, then the system* (30) *has infinitely many solutions* $(u, S, \mathbf{A}, \varphi)$ *with* $u \in H^1(\mathbb{R}^3)$*,* $S(x,t) = -\omega_0 t$*,* $\mathbf{A} = 0$ *and* $\varphi \in D^{1,2}(\mathbb{R}^3)$.

1.5. Computations and proofs

1.5.1. Deduction of conservation laws

Let (**)

$$J(v) = \int_{\mathbb{R}^4} \mathcal{L}\left(\underline{x}, v\left(\underline{x}\right), Dv\left(\underline{x}\right)\right) d\underline{x} \qquad \left(v \in \mathbb{R}^N,\ D = \left(\partial_{x_1}, \ldots, \partial_{x_4}\right)\right) \tag{31}$$

be a functional whose Lagrangian $\mathcal{L}\left(\underline{x}, v, Dv\right)$ is defined all over $\mathbb{R}^{4+N+4N}$ and at least of class C^2, and let us consider a smooth transformation

$$T_\varepsilon : \begin{cases} \underline{x}^* = X\left(\underline{x}, y; \varepsilon\right) \in \mathbb{R}^4 \\ y^* = Y\left(\underline{x}, y; \varepsilon\right) \in \mathbb{R}^N \end{cases}$$

from $\mathbb{R}^4 \times \mathbb{R}^N$ into itself, smoothly dependent on a parameter $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$ and such that T_0 is the identity map. For any smooth function $v : \overline{\Omega} \to \mathbb{R}^N$ of some bounded domain $\Omega \subset \mathbb{R}^4$, we set

$$\forall \underline{x} \in \overline{\Omega} \quad \eta_\varepsilon\left(\underline{x}\right) = X\left(\underline{x}, v\left(\underline{x}\right); \varepsilon\right)$$

$$\Omega_\varepsilon^* = \eta_\varepsilon(\Omega)$$

$$\forall \underline{x}^* \in \overline{\Omega_\varepsilon^*} \quad v_\varepsilon^*\left(\underline{x}^*\right) = Y\left(\eta_\varepsilon^{-1}\left(\underline{x}^*\right), v\left(\eta_\varepsilon^{-1}\left(\underline{x}^*\right)\right); \varepsilon\right)$$

where it can be shown that $\eta_\varepsilon : \overline{\Omega} \to \overline{\Omega_\varepsilon^*}$ is a diffeomorphism, provided that ε_0 is sufficiently small. By Noether theorem (see for example [56] or [55]), if for any domain $\Omega \subset \mathbb{R}^4$ and for any smooth mapping $v : \overline{\Omega} \to \mathbb{R}^N$ we have

$$\forall \varepsilon \in]-\varepsilon_0, \varepsilon_0[\quad \int_\Omega \mathcal{L}\left(\underline{x}, v\left(\underline{x}\right), Dv\left(\underline{x}\right)\right) d\underline{x} = \int_{\Omega_\varepsilon^*} \mathcal{L}\left(\underline{x}, v_\varepsilon^*\left(\underline{x}\right), Dv_\varepsilon^*\left(\underline{x}\right)\right) d\underline{x},$$

then, for every extremal $v \in C^2$, we have

$$\operatorname{div} G = \sum_{\alpha=1}^{4} \frac{\partial G_\alpha}{\partial x_\alpha} = 0$$

where

$$G_\alpha = \sum_{\beta=1}^{4} \left(\sum_{k=1}^{N} \frac{\partial \mathcal{L}}{\partial v_{x_\alpha}^k} \frac{\partial v^k}{\partial x_\beta} - \delta_{\alpha\beta} \mathcal{L} \right) \mu_\beta - \sum_{k=1}^{N} \frac{\partial \mathcal{L}}{\partial v_{x_\alpha}^k} \omega_k$$

**we will denote $\underline{x} \in \mathbb{R}^4$ either by (x, t) or by (x_1, x_2, x_3, x_4)

and

$$\mu\left(\underline{x}\right)=\frac{\partial X}{\partial\varepsilon}\left(\underline{x},v\left(\underline{x}\right);0\right)\quad\text{and}\quad\omega\left(\underline{x}\right)=\frac{\partial Y}{\partial\varepsilon}\left(\underline{x},v\left(\underline{x}\right);0\right).$$

Then, Gauss integration theorem yields

$$\int_{\partial B_R\times[t_1,t_2]}G\cdot\nu\,d\sigma-\int_{B_R}G_4\left(x,t_1\right)dx+\int_{B_R}G_4\left(x,t_2\right)dx=0$$

and hence, provided that $\int_{\partial B_R\times[t_1,t_2]}G\cdot\nu\,d\sigma\to0$ as $R\to+\infty$, we obtain that the integral

$$\int_{\mathbb{R}^3}\left[\sum_{\beta=1}^{4}\left(\sum_{k=1}^{N}\frac{\partial\mathcal{L}}{\partial v_t^k}\frac{\partial v^k}{\partial x_\beta}-\delta_{4\beta}\mathcal{L}\right)\mu_\beta-\sum_{k=1}^{N}\frac{\partial\mathcal{L}}{\partial v_t^k}\omega_k\right]dx$$

does not depend on t.

We turn now to the proof of the assertions of Subsection 1.2.1. Let us consider the action functional (7); its Lagrangian density is

$$\mathcal{L}_\psi=\mathcal{L}\left(\psi,D\psi\right)=\frac{1}{2}\left(\left(\psi_t^1\right)^2+\left(\psi_t^2\right)^2\right)-\frac{c^2}{2}\sum_{\substack{k=1,2\\ j=1,2,3}}^{n}\left(\psi_{x_j}^k\right)^2-W\left(\psi^1,\psi^2\right)$$

which is of class C^2. Assuming suitable conditions on the decay properties of its critical points, the above argument can be applied, using the invariance properties of $\mathcal{S}$, with respect to the following transformations. Recall that, in what follows, $N=2$.

- *Time translations.* In this case

$$\begin{cases}\underline{x}^*=\underline{x}+(0,0,0,t_0)\\ y^*=y.\end{cases}$$

  Here $\varepsilon=t_0$, $\mu\left(\underline{x}\right)\equiv(0,0,0,1)$, $\omega\left(\underline{x}\right)\equiv0$, and the integral

$$E\left(\psi\right):=\int_{\mathbb{R}^3}\left[\sum_{k=1}^{2}\frac{\partial\mathcal{L}}{\partial\psi_t^k}\frac{\partial\psi^k}{\partial t}-\mathcal{L}_\psi\right]dx$$

  is constant in time. Being $\frac{\partial\mathcal{L}}{\partial\psi_t^k}=\frac{\partial\psi^k}{\partial t}$, we get the formula (8).

- *Space translations.* These transformations are given by

$$\begin{cases}\underline{x}^*=\underline{x}+\left(x_1^0,0,0,0\right)\\ y^*=y\end{cases}\quad\begin{cases}\underline{x}^*=\underline{x}+\left(0,x_2^0,0,0\right)\\ y^*=y\end{cases}\quad\begin{cases}\underline{x}^*=\underline{x}+\left(0,0,x_3^0,0\right)\\ y^*=y\end{cases}$$

  where $\varepsilon=x_1^0$ (or $\varepsilon=x_2^0$ or $\varepsilon=x_3^0$). Then $\mu\left(\underline{x}\right)\equiv(1,0,0,0)$ (or $\mu\left(\underline{x}\right)\equiv(0,1,0,0)$ or $\mu\left(\underline{x}\right)\equiv(0,0,1,0)$), $\omega\left(\underline{x}\right)\equiv0$ and we obtain that the integrals

$$-P_j\left(\psi\right):=\int_{\mathbb{R}^3}\sum_{k=1}^{2}\frac{\partial\mathcal{L}}{\partial\psi_t^k}\frac{\partial\psi^k}{\partial x_j}dx=\int_{\mathbb{R}^3}\left(\psi_t^1\psi_{x_j}^1+\psi_t^2\psi_{x_j}^2\right)dx\quad\text{for } j=1,2,3$$

do not depend on t. Being $\nabla\psi = \nabla\psi^1 + i\nabla\psi^2$ and $\overline{\psi}_t = \psi_t^1 - i\psi_t^2$, we get

$$\begin{aligned}\overline{\psi}_t \nabla\psi &= \left(\overline{\psi}_t \psi_{x_1}, \overline{\psi}_t \psi_{x_2}, \overline{\psi}_t \psi_{x_3}\right) = \\ &= \left(\psi_t^1\psi_{x_j}^1 + \psi_t^2\psi_{x_j}^2 + i\left(\psi_t^1\psi_{x_j}^2 - \psi_t^2\psi_{x_j}^1\right)\right)_{j=1,2,3}\end{aligned}$$

and hence formula (9).

- *Space rotations.* In this case

$$\begin{cases} \underline{x}^* = (R_1 x, t) \\ y^* = y \end{cases} \qquad \text{with} \quad R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $\varepsilon = \theta$. We have $\mu_1\left(\underline{x}\right) = (0, -x_3, x_2, 0)$, $\omega_1\left(\underline{x}\right) \equiv 0$ and the integral

$$\begin{aligned} M_1(\psi) \quad &:= \int_{\mathbb{R}^3} \left[\sum_{k=1}^{2} \frac{\partial\mathcal{L}}{\partial\psi_t^k}\frac{\partial\psi^k}{\partial x_2}(-x_3) + \sum_{k=1}^{2} \frac{\partial\mathcal{L}}{\partial\psi_t^k}\frac{\partial\psi^k}{\partial x_3}x_2\right]dx \\ &= \int_{\mathbb{R}^3}\left[-x_3\left(\psi_t^1\psi_{x_2}^1 + \psi_t^2\psi_{x_2}^2\right) + x_2\left(\psi_t^1\psi_{x_3}^1 + \psi_t^2\psi_{x_3}^2\right)\right]dx \end{aligned}$$

is constant in time. Similarly, with

$$R_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad \text{and} \quad R_3 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we obtain $\mu_2\left(\underline{x}\right) = (x_3, 0, -x_1, 0)$, $\mu_3\left(\underline{x}\right) = (-x_2, x_1, 0, 0)$ and $\omega_2\left(\underline{x}\right) \equiv 0 \equiv \omega_3\left(\underline{x}\right)$, and the correspondent constant integrals

$$\begin{aligned} M_2(\psi) &:= \int_{\mathbb{R}^3}\left[x_3\left(\psi_t^1\psi_{x_1}^1 + \psi_t^2\psi_{x_1}^2\right) - x_1\left(\psi_t^1\psi_{x_3}^1 + \psi_t^2\psi_{x_3}^2\right)\right]dx \\ M_3(\psi) &:= \int_{\mathbb{R}^3}\left[-x_2\left(\psi_t^1\psi_{x_1}^1 + \psi_t^2\psi_{x_1}^2\right) + x_1\left(\psi_t^1\psi_{x_2}^1 + \psi_t^2\psi_{x_2}^2\right)\right]dx. \end{aligned}$$

Explicit computations of $\overline{\psi}_t(\mathbf{x} \times \nabla\psi)$ yield

$$\begin{aligned} \mathrm{Re}\overline{\psi}_t(\mathbf{x}\times\nabla\psi) \quad &= \quad \mathrm{Re}\left(\overline{\psi}_t\left[x_2\psi_{x_3} - x_3\psi_{x_2}\right], \overline{\psi}_t\left[x_3\psi_{x_1} - x_1\psi_{x_3}\right],\right. \\ &\qquad\qquad \left.\overline{\psi}_t\left[x_1\psi_{x_2} - x_2\psi_{x_1}\right]\right) \\ &= \quad \left(x_2\left[\psi_t^1\psi_{x_3}^1 + \psi_t^2\psi_{x_3}^2\right] - x_3\left[\psi_t^1\psi_{x_2}^1 + \psi_t^2\psi_{x_2}^2\right],\right. \\ &\qquad\quad x_3\left[\psi_t^1\psi_{x_1}^1 + \psi_t^2\psi_{x_1}^2\right] - x_1\left[\psi_t^1\psi_{x_3}^1 + \psi_t^2\psi_{x_3}^2\right], \\ &\qquad\quad \left. x_1\left[\psi_t^1\psi_{x_2}^1 + \psi_t^2\psi_{x_2}^2\right] - x_2\left[\psi_t^1\psi_{x_1}^1 + \psi_t^2\psi_{x_1}^2\right]\right) \end{aligned}$$

and we obtain (10).

- To get formula (1.2), we notice that

$$\psi e^{i\theta} = \psi^1 \cos\theta - \psi^2 \sin\theta + i\left(\psi^1 \sin\theta + \psi^2 \cos\theta\right).$$

Considering the transformation

$$\begin{cases} \underline{x}^* = \underline{x} \\ y^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} y \end{cases}$$

we have $\mu\left(\underline{x}\right) \equiv 0$ and $\omega\left(\underline{x}\right) = \left(-\psi^2\left(\underline{x}\right), \psi^1\left(\underline{x}\right)\right)$, and the integral

$$\mathcal{C}(\psi) := \int_{\mathbb{R}^3} \left[\frac{\partial \mathcal{L}}{\partial \psi_t^1}\left(-\psi^2\right) + \frac{\partial \mathcal{L}}{\partial \psi_t^2}\psi^1\right] dx = \int_{\mathbb{R}^3} \left(-\psi_t^1 \psi^2 + \psi_t^2 \psi^1\right) dx$$

is constant with respect to t. On the other hand

$$\psi_t \overline{\psi} = \left(\psi_t^1 \psi^1 + \psi_t^2 \psi^2\right) + i\left(\psi_t^2 \psi^1 - \psi_t^1 \psi^2\right)$$

and we get (1.2).

1.5.2. Pohožaev identity

Setting $\psi_0(x,t) = u_0(x)\, e^{-i\omega_0 t}$, we have

$$\begin{array}{l} \frac{\partial \psi_0}{\partial t} = -i\omega_0 u_0 e^{-i\omega_0 t}\ , \quad \frac{\partial^2 \psi_0}{\partial t^2} = -\omega_0^2 u_0 e^{-i\omega_0 t} \\ \frac{\partial^2 \psi_0}{\partial x_\beta^2} = \frac{\partial^2 u_0}{\partial x_\beta^2} e^{-i\omega_0 t} \quad \text{for } \beta = 1,2,3 \end{array}$$

and hence

$$\Box \psi_0 = \left(-\omega_0^2 u_0 - c^2 \triangle u_0\right) e^{-i\omega_0 t}.$$

Under the assumption $W(\psi) = f(|\psi|)$ with $f \in C^1(\mathbb{R};\mathbb{R})$ even and such that $f'(0) = 0$, it is easy to check that $W(u) = f(u)$ for $u \in \mathbb{R}$ and $W'(\psi) = f'(|\psi|)\frac{\psi}{|\psi|}$ with $W'(0) = 0$, so that $W'(u) = f'(u)$ for $u \in \mathbb{R}$ and

$$W'\left(e^{i\theta}\psi\right) = e^{i\theta} W'(\psi) \quad \text{for } \theta \in \mathbb{R}.$$

Hence the function ψ_0 satisfies (6) if and only if

$$-\triangle u_0 = \frac{\omega_0^2}{c^2} u_0 - \frac{1}{c^2} f'(u_0) =: g(u_0).$$

Applying Pohožaev identity in $\mathbb{R}^3$ (for which see [29] or [62]), we have that solutions of (13) satisfies

$$\int_{\mathbb{R}^3} |\nabla u_0|^2\, dx = 6 \int_{\mathbb{R}^3} G(u_0)\, dx \tag{32}$$

where $G(u_0) := \int_0^{u_0} g(v)\,dv = \frac{\omega_0^2}{2c^2}u_0^2 - \frac{1}{c^2}f(u_0)$, and (32) gives (16).

1.5.3. Energy of travelling solitary waves

We need the following lemma.

LEMMA 1. *For every critical point u of the functional F defined in* (14)*, we have*

$$\int_{\mathbb{R}^3} u_{x_\beta}^2 dx = \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^2\,dx \quad \textit{for } \beta = 1, 2, 3.$$

Proof. We set $u_\lambda(x) := u\left(\lambda^{-1}x_1, x_2, x_3\right)$ and

$$F(u_\lambda) = \frac{1}{2\lambda}\int u_{x_1}^2 dx + \frac{\lambda}{2}\int\left(u_{x_2}^2 + u_{x_3}^2\right)dx - \lambda\int\left(\frac{\omega_0^2}{2c^2}u^2 - f(u)\right)dx.$$

As u is a critical point, we obtain

$$\left.\frac{d}{d\lambda}F(u_\lambda)\right|_{\lambda=1} = 0,$$

hence

$$\frac{1}{2}\int u_{x_1}^2 dx = \frac{1}{2}\int\left(u_{x_2}^2 + u_{x_3}^2\right)dx - \int\left(\frac{\omega_0^2}{2c^2}u^2 - f(u)\right)dx.$$

Repeating the argument, we also obtain

$$\begin{aligned}\frac{1}{2}\int u_{x_2}^2 dx &= \frac{1}{2}\int\left(u_{x_1}^2 + u_{x_3}^2\right)dx - \int\left(\frac{\omega_0^2}{2c^2}u^2 - f(u)\right)dx\\ \frac{1}{2}\int u_{x_3}^2 dx &= \frac{1}{2}\int\left(u_{x_1}^2 + u_{x_2}^2\right)dx - \int\left(\frac{\omega_0^2}{2c^2}u^2 - f(u)\right)dx\end{aligned}$$

and hence

$$\int u_{x_\beta}^2 dx = \int u_{x_\alpha}^2 dx \quad \text{for } \alpha, \beta = 1, 2, 3.$$

□

Let us denote by

$$\psi_0(y, t) = u_0(y_1, y_2, y_3)\,e^{-i\omega_0 t}$$

the standing wave and by

$$\psi(x, t) = u_0(\gamma[x_1 - vt], x_2, x_3)\,e^{i(kx_1 - \omega t)}, \quad k = \frac{\omega}{c^2}v,\ \omega = \gamma\omega_0$$

the travelling one, obtained as

$$\psi(x, t) = \psi_0(g(x, t))$$

where $g : \mathbb{R}^4 \to \mathbb{R}^4$ is a Lorentz transformation. Since

$$\psi_t = \left(-\gamma v \tfrac{\partial u_0}{\partial y_1} - i\omega_0 \gamma u_0\right) e^{i(kx_1-\omega t)}, \quad \psi_{x_2} = \tfrac{\partial u_0}{\partial y_2} e^{i(kx_1-\omega t)}$$
$$\psi_{x_1} = \left(\gamma \tfrac{\partial u_0}{\partial y_1} + i\omega_0 \gamma \tfrac{v}{c^2} u_0\right) e^{i(kx_1-\omega t)}, \quad \psi_{x_3} = \tfrac{\partial u_0}{\partial y_3} e^{i(kx_1-\omega t)}$$
$$|\nabla \psi|^2 = \gamma^2 \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 + \omega^2 \tfrac{v^2}{c^4} u_0^2 + \left(\tfrac{\partial u_0}{\partial y_2}\right)^2 + \left(\tfrac{\partial u_0}{\partial y_3}\right)^2$$
$$|\psi_t|^2 = \gamma^2 v^2 \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 + \omega^2 u_0^2$$

we get

$$\begin{aligned}
&\tfrac{1}{2}\int \left[|\psi_t|^2 + c^2 |\nabla\psi|^2\right](x,t)\,dx = \\
&= \tfrac{\gamma^2(v^2+c^2)}{2} \int \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 (\gamma[x_1 - vt], x_2, x_3)\,dx + \\
&\quad + \tfrac{c^2}{2} \int \left[\left(\tfrac{\partial u_0}{\partial y_2}\right)^2 + \left(\tfrac{\partial u_0}{\partial y_3}\right)^2\right] (\gamma[x_1 - vt], x_2, x_3)\,dx + \\
&\quad + \tfrac{\omega_0^2}{2}\gamma^2 \tfrac{c^2+v^2}{c^2} \int u_0^2 (\gamma[x_1 - vt], x_2, x_3)\,dx \\
&= \tfrac{\gamma(v^2+c^2)}{2} \int \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 dy + \tfrac{c^2}{2}\tfrac{1}{\gamma} \int \left[\left(\tfrac{\partial u_0}{\partial y_2}\right)^2 + \left(\tfrac{\partial u_0}{\partial y_3}\right)^2\right] dy + \\
&\quad + \tfrac{\omega_0^2}{2} \tfrac{c^2+v^2}{c^2} \gamma \int u_0^2 dy \\
&= \left(\tfrac{\gamma(v^2+c^2)}{6} + \tfrac{1}{\gamma}\tfrac{c^2}{3}\right) \int \left|\nabla_y u_0\right|^2 dy + \tfrac{\omega_0^2}{2}\tfrac{c^2+v^2}{c^2}\gamma \int u_0^2 dy
\end{aligned}$$

where we have used Lemma 1 after the change $(y_1, y_2, y_3) = (\gamma[x_1 - vt], x_2, x_3)$. On the other hand, using assumption (12), the same change of variables and identity (16), we have

$$\begin{aligned}
\int W(\psi(x,t))\,dx &= \int W\left(u_0(\gamma[x_1 - vt], x_2, x_3)\, e^{i(kx_1-\omega t)}\right) dx \\
&= \int W(u_0(\gamma[x_1 - vt], x_2, x_3))\,dx \\
&= \tfrac{1}{\gamma} \int W(u_0(y_1, y_2, y_3))\,dy \\
&= \tfrac{1}{\gamma}\left(-\tfrac{c^2}{6} \int \left|\nabla_y u_0\right|^2 dy + \tfrac{\omega_0^2}{2} \int u_0^2 dy\right).
\end{aligned}$$

Recalling (8), (17) and $\gamma^2 = \frac{c^2}{c^2-v^2}$, we obtain

$$\begin{aligned}
E(\psi) &= \gamma \left(\tfrac{(v^2+c^2)}{6} + \tfrac{1}{\gamma^2}\tfrac{c^2}{6}\right) \int \left|\nabla_y u_0\right|^2 dy + \gamma \tfrac{\omega_0^2}{2} \left(\tfrac{c^2+v^2}{c^2} + \tfrac{1}{\gamma^2}\right) \int u_0^2 dy \\
&= \gamma \left(\tfrac{c^2}{3} \int \left|\nabla_y u_0\right|^2 dy + \omega_0^2 \int u_0^2 dy\right) = \gamma E(\psi_0).
\end{aligned}$$

1.5.4. Momentum of travelling solitary waves

As before, we denote the standing wave by

$$\psi_0(y,t) = u_0(y_1, y_2, y_3)\, e^{-i\omega_0 t}$$

and consider the travelling wave

$$\psi_{\mathbf{v}_1}(x,t) = u_0(\gamma[x_1 - vt], x_2, x_3)\, e^{i(kx_1 - \omega t)} =: u(x,t)\, e^{iS(x,t)}$$

where $k = v\omega/c^2$, $\omega = \gamma\omega_0$ and $\mathbf{v}_1 = (v, 0, 0)$. Recall from Subsection 1.2.1 that

$$\mathbf{P}(\psi_{\mathbf{v}_1}) = -\int \left[u_t \nabla u + u^2 S_t \nabla S\right] dx =: -\int p\, dx.$$

We have

$$u_t = -\gamma v \tfrac{\partial u_0}{\partial y_1}, \quad u_{x_1} = \gamma \tfrac{\partial u_0}{\partial y_1}, \quad S_t = -\omega, \quad S_{x_1} = k.$$

Therefore, it is $p(x,t) = (p_1(x,t), p_2(x,t), p_3(x,t))$ with

$$\begin{aligned}
p_1(x,t) &= -\left[\gamma^2 v \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 + k\omega u_0^2\right](\gamma[x_1 - vt], x_2, x_3)\\
p_2(x,t) &= -\gamma v \left[\tfrac{\partial u_0}{\partial y_1}\tfrac{\partial u_0}{\partial y_2}\right](\gamma[x_1 - vt], x_2, x_3)\\
p_3(x,t) &= -\gamma v \left[\tfrac{\partial u_0}{\partial y_1}\tfrac{\partial u_0}{\partial y_3}\right](\gamma[x_1 - vt], x_2, x_3).
\end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned}
P_1(\psi_{\mathbf{v}_1}) &= \gamma^2 v \int \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 (\gamma[x_1 - vt], x_2, x_3)\, dx + \\
&\quad + \gamma^2\omega_0^2 \tfrac{v}{c^2} \int u_0^2(\gamma[x_1 - vt], x_2, x_3)\, dx\\
&= \gamma v \int \left(\tfrac{\partial u_0}{\partial y_1}\right)^2 dy + \gamma\omega_0^2 \tfrac{v}{c^2}\int u_0^2 dy\\
&= \gamma v \left(\tfrac{1}{3}\int \left|\nabla_y u_0\right|^2 dy + \tfrac{\omega_0^2}{c^2}\int u_0^2 dy\right) = \gamma v \frac{E(\psi_0)}{c^2}
\end{aligned} \tag{33}$$

and

$$P_j(\psi_{\mathbf{v}_1}) = v \int \tfrac{\partial u_0}{\partial y_1}\tfrac{\partial u_0}{\partial y_j} dy \quad \text{for } j \neq 1. \tag{34}$$

Similarly, if we had considered $\mathbf{v}_2 = (0, v, 0)$ and $\mathbf{v}_3 = (0, 0, v)$, we would have obtained

$$P_2(\psi_{\mathbf{v}_2}) = \gamma v \tfrac{E(\psi_0)}{c^2} \quad \text{and} \quad P_j(\psi_{\mathbf{v}_2}) = v\int \tfrac{\partial u_0}{\partial y_2}\tfrac{\partial u_0}{\partial y_j} dy \ \text{ for } j \neq 2 \tag{35}$$

$$P_3\left(\psi_{\mathbf{v}_3}\right)=\gamma v\frac{E(\psi_0)}{c^2}\quad\text{and}\quad P_j\left(\psi_{\mathbf{v}_3}\right)=v\int\frac{\partial u_0}{\partial y_3}\frac{\partial u_0}{\partial y_j}dy\ \text{ for } j\neq 3. \tag{36}$$

Since $\mathbf{P}_\alpha=\mathbf{P}\left(\psi_{\mathbf{v}_\alpha}\right)$ and $\mathbf{v}_\alpha$ are invariants (see Remark 3), there exists a unique linear map m such that

$$\mathbf{P}_\alpha=m\left(\mathbf{v}_\alpha\right)\quad\text{for } \alpha=1,2,3 \tag{37}$$

and from (33)-(37) we deduce that in the reference frame with coordinates $x=(x_1,x_2,x_3)$ it is represented by the matrix

$$M=\begin{pmatrix}\frac{\gamma E(\psi_0)}{c^2} & \int\frac{\partial u_0}{\partial y_2}\frac{\partial u_0}{\partial y_1}dy & \int\frac{\partial u_0}{\partial y_3}\frac{\partial u_0}{\partial y_1}dy\\ \int\frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_2}dy & \frac{\gamma E(\psi_0)}{c^2} & \int\frac{\partial u_0}{\partial y_3}\frac{\partial u_0}{\partial y_2}dy\\ \int\frac{\partial u_0}{\partial y_1}\frac{\partial u_0}{\partial y_3}dy & \int\frac{\partial u_0}{\partial y_2}\frac{\partial u_0}{\partial y_3}dy & \frac{\gamma E(\psi_0)}{c^2}\end{pmatrix}.$$

Since M is real symmetric, there is a reference frame $x'=Rx$ with $R\in O(3)$ in which the matrix M' of m is diagonal and, exploiting the invariance of $E(\psi_0)$ (see Remark 3), the above computations can be carried over again to obtain

$$M'=RMR^{-1}=\frac{\gamma E(\psi_0)}{c^2}I.$$

Hence, M itself is $\gamma E(\psi_0)/c^2$ times the identity matrix and from (37) we get

$$\mathbf{P}\left(\psi_{\mathbf{v}_\alpha}\right)=\frac{\gamma E(\psi_0)}{c^2}\mathbf{v}_\alpha\quad\text{for } \alpha=1,2,3.$$

1.5.5. Euler-Lagrange equations for $\mathcal{L}_{u,S,\mathbf{A},\varphi}$

Let us first derive the expression (29) of the Lagrangian density $\mathcal{L}_{u,S,\mathbf{A},\varphi}$, which gives the action functional $\mathcal{I}$ by means of the polar form

$$\psi(x,t)=u(x,t)\,e^{iS(x,t)}\quad\text{with } u,S\in\mathbb{R}.$$

Letting $(\mathbf{A},\varphi)=\left(A^1,A^2,A^3,\varphi\right)$, we have

$$\begin{aligned}|D_t\psi|^2=\ &|\psi_t+i\varphi\psi|^2=\left|u_te^{iS}+uiS_te^{iS}+i\varphi ue^{iS}\right|^2=|u_t+i\left(uS_t+\varphi u\right)|^2\\ =\ &u_t^2+u^2\left(S_t+\varphi\right)^2\end{aligned}$$

and

$$\begin{aligned}|\mathbf{D}_x\psi|^2 &= |\nabla\psi-i\psi\mathbf{A}|^2=\left|e^{iS}\left(u_{x_\alpha}+iu\left(S_{x_\alpha}-A^\alpha\right)\right)_{\alpha=1,2,3}\right|^2\\ &= \sum_{\alpha=1}^3\left|u_{x_\alpha}+iu\left(S_{x_\alpha}-A^\alpha\right)\right|^2=\sum_{\alpha=1}^3\left[u_{x_\alpha}^2+u^2\left(S_{x_\alpha}-A^\alpha\right)^2\right]\\ &= |\nabla u|^2+u^2\left|\nabla S-\mathbf{A}\right|^2\end{aligned}$$

and then, using the assumption (12), from (28) we get

$$\mathcal{I}(u, S, \mathbf{A}, \varphi) = \int_{\mathbb{R}^3 \times \mathbb{R}} \mathcal{L}_{u,S,\mathbf{A},\varphi}(x,t)\, dxdt$$

where the Lagrangian is given by the formula (29), i.e. (setting $D = (\nabla, \partial_t)$)

$$\begin{aligned}
\mathcal{L}_{u,S,\mathbf{A},\varphi} = {} & \mathcal{L}(u, \mathbf{A}, \varphi, Du, D\mathbf{A}, D\varphi) = \\
= {} & \tfrac{1}{2}\left[u_t^2 - |\nabla u|^2 - u^2\left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right) + |\mathbf{A}_t + \nabla\varphi|^2 + \right. \\
& \left. - |\nabla \times \mathbf{A}|^2 + \right] - f(u) \\
= {} & \tfrac{1}{2}\left[u^2 (S_t + \varphi)^2 - \textstyle\sum_{\alpha=1}^3 \left(u_{x_\alpha}^2 + u^2\left(S_{x_\alpha} - A^\alpha\right)^2 - \left(A_t^\alpha + \varphi_{x_\alpha}\right)^2\right) + \right. \\
& \left. - \left(A_{x_2}^3 - A_{x_3}^2\right)^2 - \left(A_{x_3}^1 - A_{x_1}^3\right)^2 - \left(A_{x_1}^2 - A_{x_2}^1\right)^2 + u_t^2\right] - W(u).
\end{aligned}$$

Recalling the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial v^k} - \sum_{\alpha=1}^{4} \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial v_{x_\alpha}^k}\left(\underline{x}, v\left(\underline{x}\right), Dv\left(\underline{x}\right)\right)\right] = 0 \qquad \text{for } k = 1, ..., N \tag{38}$$

for the generic action functional (31), if we make the variations of $\mathcal{I}$ with respect to u, S, $\mathbf{A}$ and φ, we get the following equations.

- $\delta_u \mathcal{I} = 0$. In this case, we apply (38) with $N = 1$ and $v = u$.

  Since

$$\frac{\partial \mathcal{L}}{\partial u} = -u\left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right) - f'(u)\ , \qquad \frac{\partial \mathcal{L}}{\partial u_{x_\alpha}} = -u_{x_\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial u_t} = u_t\ , \quad \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial u_{x_\alpha}}\right] = -\sum_{\alpha=1}^{3} u_{x_\alpha x_\alpha} = -\triangle u$$

  we get

$$-u\left(|\nabla S - \mathbf{A}|^2 - (S_t + \varphi)^2\right) - f'(u) - u_{tt} + \triangle u = 0.$$

- $\delta_S \mathcal{I} = 0$. In this case, $N = 1$ and $v = S$.

  Since

$$\frac{\partial \mathcal{L}}{\partial S} = 0, \quad \frac{\partial \mathcal{L}}{\partial S_{x_\alpha}} = -u^2\left(S_{x_\alpha} - A^\alpha\right), \quad \frac{\partial \mathcal{L}}{\partial S_t} = u^2 (S_t + \varphi)$$

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial S_{x_\alpha}}\right] = -\sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left[u^2\left(S_{x_\alpha} - A^\alpha\right)\right] = -\operatorname{div}\left[u^2(\nabla S - \mathbf{A})\right]$$

  we get

$$\operatorname{div}\left[u^2(\nabla S - \mathbf{A})\right] - \frac{\partial}{\partial t}\left[u^2(S_t + \varphi)\right] = 0.$$

- $\delta_{\mathbf{A}}\mathcal{I} = 0$. In this case, $N = 3$ and $v = \left(A^1, A^2, A^3\right)$.

  We have

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = u^2 \left(S_{x_\alpha} - A^\alpha\right) \ , \quad \frac{\partial \mathcal{L}}{\partial A^\alpha_t} = A^\alpha_t + \varphi_{x_\alpha}$$

  while the matrix $\left(\frac{\partial \mathcal{L}}{\partial A^\alpha_{x_\beta}}\right)_{\alpha,\beta=1,2,3}$ is given by

$$\left(\frac{\partial \mathcal{L}}{\partial A^\alpha_{x_\beta}}\right) = \begin{pmatrix} 0 & A^2_{x_1} - A^1_{x_2} & A^3_{x_1} - A^1_{x_3} \\ A^1_{x_2} - A^2_{x_1} & 0 & A^3_{x_2} - A^2_{x_3} \\ A^1_{x_3} - A^3_{x_1} & A^2_{x_3} - A^3_{x_2} & 0 \end{pmatrix}$$

  and hence

$$\sum_{\beta=1}^3 \frac{\partial}{\partial x_\beta}\left[\frac{\partial \mathcal{L}}{\partial A^\alpha_{x_\beta}}\right] = \sum_{\beta\neq\alpha}\left(A^\beta_{x_\alpha x_\beta} - A^\alpha_{x_\beta x_\beta}\right).$$

  If we compute

$$\nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A^3_{x_2} - A^2_{x_3} & A^1_{x_3} - A^3_{x_1} & A^2_{x_1} - A^1_{x_2} \end{vmatrix}$$

$$= \begin{pmatrix} A^2_{x_1x_2} - A^1_{x_2x_2} - A^1_{x_3x_3} + A^3_{x_1x_3} \\ A^3_{x_2x_3} - A^2_{x_3x_3} - A^2_{x_1x_1} + A^1_{x_2x_1} \\ A^1_{x_3x_1} - A^3_{x_1x_1} - A^3_{x_2x_2} + A^2_{x_3x_2} \end{pmatrix}$$

  we get, from (38), the following vector equation

$$u^2\left(\nabla S - \mathbf{A}\right) - \nabla \times (\nabla \times \mathbf{A}) - \frac{\partial}{\partial t}\left[\mathbf{A}_t + \nabla\varphi\right] = 0.$$

- $\delta_\varphi\mathcal{I} = 0$. In this case, $N = 1$ and $v = \varphi$.

  Since

$$\frac{\partial \mathcal{L}}{\partial \varphi} = u^2\left(S_t + \varphi\right) \ , \quad \frac{\partial \mathcal{L}}{\partial \varphi_{x_\alpha}} = A^\alpha_t + \varphi_{x_\alpha} \ , \quad \frac{\partial \mathcal{L}}{\partial \varphi_t} = 0$$
$$\sum_{\alpha=1}^3 \frac{\partial}{\partial x_\alpha}\left[\frac{\partial \mathcal{L}}{\partial \varphi_{x_\alpha}}\right] = \sum_{\alpha=1}^3 \frac{\partial}{\partial x_\alpha}\left[A^\alpha_t + \varphi_{x_\alpha}\right] = \operatorname{div}\left[\mathbf{A}_t + \nabla\varphi\right]$$

  we get

$$u^2\left(S_t + \varphi\right) - \operatorname{div}\left[\mathbf{A}_t + \nabla\varphi\right] = 0.$$

2. Concentration phenomena and solitary waves

2.1. Introduction

A definition of solitary waves and solitons as solutions to field equations will be discussed in Section 2.2. In this introductory section, we want to touch the reasons for which the interest in these notions arises in nonlinear equations, rather than to expound the analysis rigorously.

Accordingly, we have to concern ourselves with the fact that *the most wave phenomena are dispersive*, as we are going to see through some sample equations in 1+1 dimensions.

Let us consider the well known D'Alembert equation

$$\psi_{tt} - c^2 \psi_{xx} = 0 \qquad \psi \in \mathbb{C} \tag{39}$$

which is the first and simplest of all wave equations. This equation presents two features which are of relevance to our discussion:

(I) *there are solutions representing a wave packet travelling at uniform velocity and with no distorsion in shape*; we can construct such a solution taking a localized (††) function f on $\mathbb{R}$ and considering $f(x \pm ct)$, which moves undistorted at velocity $\pm c$

(II) *there are solutions representing separated wave packets approaching each other essentially undistorted and asymptotically retaining their original shapes and velocities after collision*; this is the situation we get taking for instance f and g compactly supported and constructing the solution $\psi(x,t) = f(x-ct) + g(x+ct)$, which consists of two wave packets clearly not interacting at all when $|t|$ is large.

A further comprehension of feature **(I)** relies on the Fourier analysis of the motion. If we look for well-behaved localized solutions (i.e. decaying sufficiently rapidly as $|x| \to \infty$ with fixed t and such to allow the interchange of time derivative and spatial integral), by taking the Fourier transform of (39) we obtain the ordinary differential equation

$$\widehat{\psi}_{tt} = -c^2 k^2 \widehat{\psi}$$

which is solved by $\widehat{\psi}(k,t) = A(k)\, e^{-i\omega t} + B(k)\, e^{i\omega t}$ where $A, B \in \mathbb{C}$ are constant with respect to variable t and $\omega = ck$. Then, provided that A and B decay sufficiently rapidly as $|k| \to \infty$, we can recover

$$\psi(x,t) = \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \left[A(k)\, e^{i(kx-\omega t)} + B(k)\, e^{i(kx+\omega t)} \right] dk \tag{40}$$

††We shall use the adjective 'localized' for those functions of space variables which are finite in some finite region and decay to zero at infinity, as rapidly as the discussion requires; otherwise, referring to functions of space-time variables, we mean that the localization property is required at any time t.

by the inverse transform, and (40) gives the general solution of (39) by the sum of two wave packets written each as a superposition of plane-wave components. Note that the arbitrary functions A and B are related to the intial conditions $\psi(x,0) = f(x)$ and $\psi_t(x,0) = g(x)$ by $A = \frac{1}{2}\left(\widehat{f} + i\frac{\widehat{g}}{\omega}\right)$ and $B = \frac{1}{2}\left(\widehat{f} - i\frac{\widehat{g}}{\omega}\right)$.

The relation $\omega = \omega(k)$, which gives the frequency of the normal mode $e^{i(kx+\omega t)}$ (or, up to a sign, of $e^{i(kx-\omega t)}$) as a function of the wavenumber k, is the *dispersion relation* of the equation (39). From it we deduce the *phase velocity*

$$v_F = \omega(k)/k \qquad (\text{or } v_F = -\omega(k)/k)$$

which gives the velocity of the wave fronts of a single normal mode, and also the *group velocity*

$$v_G = \omega'(k) \qquad (\text{or } v_G = -\omega'(k)\,)$$

which gives the velocity of a superposition of normal modes with nearly the same wavelenght $2\pi/k$ (see [68] or [82] for further readings on group velocity).

As a result, once the solutions

$$f(x \pm ct) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(k)\, e^{i(kx \pm \omega t)} dk$$

are written as superpositions of Fourier components, we see that the normal modes travel at the same phase velocity $v_F = \pm c$ and with constant group velocity $v_G = \pm c$, and thereby they won't spread as time goes on.

So, features **(I)** and **(II)** hold for the equation (39) because it is both *linear* and *dispersionless* ($\omega'' = 0$). However, the most wave phenomena occur in a dispersive medium and the related equations are more complicated. It's easy to see that the addiction of also the simplest kinds of terms to (39) tends to destroy even feature **(I)**.

Consider for instance the (linear) Klein-Gordon equation (*)

$$\psi_{tt} - c^2\psi_{xx} + \omega_0^2\psi = 0 \qquad \psi \in \mathbb{C}. \tag{41}$$

By the usual Fourier transform method, formula (40) still gives the general solution, but the dispersion relation we get is

$$\omega^2 = c^2k^2 + \omega_0^2. \tag{42}$$

Since groups of waves with nearly the same length propagate at the group velocity $v_G = \pm c^2k/\sqrt{c^2k^2 + \omega_0^2}$, while individual components move through the group with their phase velocity $v_F = \pm\sqrt{c^2k^2 + \omega_0^2}/k$, we see that short-wave groups travel faster than the long-wave ones and thereby the components disperse. Hence we conclude that *the linear theory predicts the dispersal of any localized disturbance*, and it obviously comes from the term ω_0^2 in (42) and thus from the *dispersive term* $\omega_0^2\psi$ in (41).

*It was derived by O. Klein [60] and W. Gordon [58] for a charged free particle in an electromagnetic field, using the ideas of quantum theory.

Similarly, the loss of feature **(I)** may be caused by the addiction to (39) of a simple *nonlinear term*. For instance, though not all the solutions of the equation

$$\psi_{tt} - c^2\psi_{xx} + \psi^3 = 0 \qquad \psi \in \mathbb{C}$$

are known, a numerical approach shows that the term ψ^3 is still dispersive, for it causes the spreading of an arbitrary wave packet (see [72]).

Dispersion phenomena occur in several types of equations. For instance, the well known (linear) Schrödinger equation

$$i\psi_t = -\tfrac{\hbar}{2m}\psi_{xx} \qquad \psi \in \mathbb{C}$$

has the general localized solution

$$\psi(x,t) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(k)\, e^{i(kx-\omega t)} dk \qquad (\text{where } f(x) = \psi(x,0))$$

with dispersion relation $\omega = \frac{\hbar}{2m}k^2$ and then velocities $v_F = \frac{\hbar}{2m}k$ and $v_G = \frac{\hbar}{m}k$.

In contrast to dispersion, a nonlinearity may also lead to the *concentration* of a disturbance. For example to study the equation (†)

$$u_t + uu_x = 0 \qquad u \in \mathbb{R},$$

the method of characteristics applies and the dependence of the propagation velocity of solutions on their height tends to steepen a disturbance until the formation of a shock wave (see [65], [82], [66] and [67] for a detailed discussion).

Actually, for both nonlinear and dispersive equations it is possible that concentration and dispersion effects are just in balance, in such a way that features **(I)** and **(II)** are exhibited. The attractive question is then whether a *nonlinear field equation* admits special solutions which enjoy the essential spirit of feature **(I)** and also eventually **(II)**; when the answer is positive, such solutions are referred to as *solitary waves* and *solitons*, respectively.

2.2. Solitary waves

Solitary waves were first observed by J. Scott Russell on the canal from Edimburgh to Glasgow in 1834. Reporting to the British Association, he wrote [75]:

> "I believe I shall best introduce this phænomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation,

† This equation is an approximation to the more accurate model $u_t + uu_x = \mu u_{xx}$, derived by J.M. Burgers [34] for turbulent flows in a channel (see also [59] and [36]).

> then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminuition of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phænomenon which I have called the Wave of Translation [...]."

Doing laboratory experiments, Russell also deduced empirically that the steady velocity of a solitary wave in a channel of uniform depth h is given by

$$v^2 = g(h + a) \tag{43}$$

where a is the amplitude of the wave above the undisturbed water surface and g is the gravitational constant.

J. Boussinesq [33] in 1871 and Lord Rayleigh [73] in 1876 independently recovered this formula through the inviscid equations of water waves and, developing the theory, D.J. Korteweg and G. de Vries [61] derived in 1895 a nonlinear 1+1 dimensional equation governing the motion of long, small amplitude waves propagating in a shallow water channel

$$u_t = \frac{3}{2}\sqrt{\frac{g}{h}}\left(uu_x + \frac{2}{3}\alpha u_x + \frac{1}{3}\sigma u_{xxx}\right) \quad \text{with } \sigma = \frac{1}{3}h^3 - \frac{T}{\rho}\frac{h}{g} \tag{44}$$

where u is the wave height above the undisturbed water surface, α a small but otherwise arbitrary constant related to the uniform motion of the fluid, T the surface tension and ρ the density (for a derivation of equation (44) see also [5]). The equation (44) has been extensively studied and ever since 1960 [54] it has been rediscovered in many physical contexts, such as hydromagnetic waves, plasma physics, lattice dynamics (see [45] and [3] for details and further references).

This equation admits solitary wave solutions

$$u(x,t) = 4k^2 \operatorname{sech}^2\left(\frac{k}{\sqrt{\sigma}}\left(x - x_0 + 2k^2\sqrt{\frac{g}{h}}t\right)\right) - \frac{2}{3}\alpha \tag{45}$$

with k and x_0 arbitrary constant, and they were known even to Korteweg and de Vries. Nevertheless, their remarkable properties were discovered only in 1965 through the numerical approach of N.J. Zabusky and M.D. Kruskal [83], who considered an equation like (44) with periodic boundary conditions (suitable for numerical computations) in $x = 0$ and $x = 2$. They found that the initial condition $u(x, 0) = \cos \pi x$, after steepening and nearly forming a shock wave owing to the prevailing nonlinear term uu_x, then develops in a train of eight distinct waves, for the dispersive term u_{xxx} becomes relevant and comes to balance the nonlinear concentration effect. These waves, each like (45) but with different speeds and amplitudes, move through one another in the same direction and when a taller wave overtakes a smaller one they interact nonlinearly, regaining nevertheless their original velocities and amplitudes after collision. So, they behave almost as if the principle of superposition were valid and the only interaction result is a phase shift. Finally, after a long time, something very close to the initial configuration recurs.

For the very special waves they have observed and which resemble particles in their way of interacting elastically, Zabusky and Kruskal coined the name of ‘solitons’.

From a mathematical point of view, current literature does not provide a universally accepted definition of solitary waves and solitons. In order to generalize the requirements we have so far described qualitatively as features **(I)** and **(II)** but also to encompass as many cases of interest as possible, different authors offer slightly different definitions (see also [76] and [38]).

Following [72], our working definition of solitary waves will be in terms of the energy rather than the wave-fields themselves.

> Given on the space-time $\mathbb{R}^3 \times \mathbb{R}$ a nonlinear field equation with associated energy functional (see Section 1.2)
>
> $$E(\psi) = \int_{\mathbb{R}^3} \mathcal{E}_\psi(x, t)\, dx,$$
>
> we call *solitary wave* any nonsingular solution whose energy density has a space-time dependence of the form
>
> $$\mathcal{E}_\psi(x, t) = \widetilde{\mathcal{E}}_\psi(x - \mathbf{v}t) \tag{46}$$
>
> where $\widetilde{\mathcal{E}}_\psi$ is a localized function and $\mathbf{v}$ is some velocity vector.

In other words, a solitary wave is a regular solution of a field equation whose energy density is localized and travels undistorted with constant velocity. The definition extends naturally to systems of coupled equations.

The definition of solitons calls for more stringent requirements and a mathematical formulation becomes finer and rather complicated (see again [72] for a working definition). We content ourselves with the intuitive idea discussed so far and guess it will become clearer after the example exposed below in Section 2.3. Besides, we remark that the distinction between these terms is often dropped in the literature, either because all time-dependent solutions should be known before deciding whether a solitary wave is actually a soliton (so to verify that it interacts elastically with other localized solutions), or also because in many discussions solitary waves alone suit the purpose (provided eventually some addictional requirements, such as certain kinds of stability; see for example [20], [21] and [18]).

As a matter of fact, several equations have yielded solitary waves. In contrast, very few of them bear solitons in a strict sense. Although neither a comprehension of the ultimate reasons for which these special solutions persist nor a technique to decide whether a given equation permits solitons have yet been achieved, here is a list of some features which might be involved in the presence of solitary solutions.

(i) *Complete integrability*. In this case the equation behaves as an infinite dimensional completely integrable Hamiltonian system and it can be solved exactly by performing the so-called *Inverse Scattering Transform* (I.S.T.) method, which can be thought of as a nonlinear transformation from physical variables to an infinite set of action-angle variables (see [51], [3], [76]).

This is the case of the Korteweg-deVries equation, which bears infinitely many integrals of motion (see [70]), and also of the sine-Gordon equation (see below)

$$\Box u + \omega_0^2 \sin u = 0 \qquad u \in \mathbb{R}$$

and the nonlinear Schrödinger equation (see [84])

$$i\hbar\psi_t = -\tfrac{\hbar^2}{2m}\triangle\psi - |\psi|^2\,\psi \qquad \psi \in \mathbb{C}$$

in their 1+1 dimensional version.

(ii) *A 'sufficient' number of conservation laws*. For systems which are not completely integrable, the existence of stable waves may be related to the presence of a 'large' number of functions which yield, upon evaluating along the solutions, a vanishing space-time divergence. As discussed in Subsection 1.5.1., this usually provides first integrals of motion.

Structural stability of solutions to nonlinear Klein-Gordon and Schrödinger equations has been proved in [77] and [35].

(iii) *Topological constraints*. In this case, which will be largely discussed in the following, the solutions are characterized by some topological invariant and then we speak of *topological solitary waves*.

This is again the case of the sine-Gordon equation in dimension 1+1 (see below), which provides the simplest example of topological solitons.

2.3. Sine-Gordon equation and Derrick's problem

Hereafter we set the light velocity equal to one:

$$\boxed{c = 1}.$$

In this section, we consider the 1+1 dimensional sine-Gordon equation (*)

$$u_{tt} - u_{xx} + \sin u = 0 \qquad u \in \mathbb{R} \tag{47}$$

*It's name was coined by J. Rubinstein [74] as a pun on 'Klein-Gordon' and it arises in the study of surfaces with constant negative Gaussian curvature in differential geometry and also in many physical applications, such as two-dimensional models of elementary particles, stability of fluid motions, propagation of crystal dislocations (see [3], [4], [44], [45], [72], [7], [32], [37], [53], [57], [71] and [76] for exhaustive discussions and references).

which is probably the simplest equation admitting soliton soutions and can be seen as a pattern for our further discussions (†).

Taking into account the results of Section 1.2, (47) is the Euler-Lagrange equation of the action functional

$$\mathcal{S}(u) = \int_{\mathbb{R}\times\mathbb{R}} \left[\frac{1}{2}\left(u_t^2 - u_x^2\right) - V(u)\right] dx\,dt \tag{48}$$

where we can choose $V(u) = 1 - \cos u$ so to obtain $V \geqslant 0$, and then the related energy functional is

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2 dx + \frac{1}{2}\int_{-\infty}^{+\infty} u_x^2 dx + \int_{-\infty}^{+\infty} (1 - \cos u)\, dx. \tag{49}$$

Note that the potential V has a discrete infinite set of degenerate minima $2\pi\mathbb{Z}$, where it vanishes.

Obviously $u(x,t) \equiv k\pi$ is a trivial solution of (47) for every $k \in \mathbb{Z}$, but of course we are interested in nontrivial solutions. In particular, we will concern ourselves with nonsingular finite-energy solutions (of which solitary waves are special cases).

So, let u be a classical solution with $E(u) < \infty$. By this we mean of course that all the integrals in (49) are finite, and this implies $1 - \cos u(\cdot,t) \in H^1(\mathbb{R})$ for all fixed t, being $[1 - \cos u(\cdot,t)]^2 \leqslant 2[1 - \cos u(\cdot,t)] \in L^1$ and $\left|\frac{d}{dx}\cos u(\cdot,t)\right| \leqslant |u_x(\cdot,t)| \in L^2$. Hence $\lim_{x\to\pm\infty} 1 - \cos u(x,t) = 0$ and from this we deduce that every configuration of u satisfies the asymptotic conditions

$$u(\pm\infty,t) = \lim_{x\to\pm\infty} u(x,t) \in 2\pi\mathbb{Z}. \tag{50}$$

Moreover, if we assume that $u_t \in L^\infty\left(\mathbb{R}^2\right)$ (which is necessarily the case of solitary waves) then the functions of the variable t defined by the left-hand side of (50), being continuous and discrete-valued, must be constant

$$u(\pm\infty,t) \equiv u(\pm\infty) \in 2\pi\mathbb{Z} \tag{51}$$

i.e., u preserves its asymptotic values as t varies. These facts suggest to consider the sets

$$H_{(k_1,k_2)} := \left\{ f \in C^2(\mathbb{R};\mathbb{R}) \mid \lim_{x\to-\infty} f(x) = 2k_1\pi,\ \lim_{x\to+\infty} f(x) = 2k_2\pi \right\}$$

with $k_1, k_2 \in \mathbb{Z}$, and the topological space

$$H = \bigcup_{(k_1,k_2)\in\mathbb{Z}^2} H_{(k_1,k_2)} \subset L^\infty(\mathbb{R}).$$

It is easy to see that $(k_1,k_2) \neq (h_1,h_2) \Rightarrow H_{k_1,k_2} \cap H_{h_1,h_2} = \varnothing$ and that each $H_{(k_1,k_2)}$ is an open path-connected subset of H, called *sector*. The property (51) implies that,

† for other generalizations, see [2] and [81]

for a solution u, the function $x \mapsto u(x,t)$ (which we call a *configuration* of u) stays always in the same connected component $H_{(k_1,k_2)}$ as time evolves. This fact allows a *topological classification* of the finite-energy nonsingular solutions to the equation (47) satisfying (51), each bearing thereby the pair of indices (k_1, k_2). Moreover, consistent with the invariance of the equation (47) (and also of the action (48)) under the change $u \mapsto u + 2\pi k$, we can fix k_1 and relate to such solutions a single integer index given by the difference $k_2 - k_1$, namely

$$Q(u) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} u_x(x,t)\, dx.$$

$Q(u)$ defines a topological index, called *topological charge*. Note that Q is essentially a boundary condition which is constant in time because of the finiteness of energy, in contrast with the other more familiar conserved quantities (see Subsection 1.2.1) coming from the symmetries of the action functional.

Now, we turn on the particular case of *static* (i.e. t-independent) nonsingular finite-energy solutions. They solve the equation

$$-u'' + \sin u = 0 \qquad u : \mathbb{R} \to \mathbb{R} \tag{52}$$

which can be interpreted as a conservative system (or, by a mechanical analogy, as the equation of motion for a unit-mass point particle). The 'mechanical energy'

$$E_M = \frac{1}{2}\left(u'\right)^2 - V(u)$$

(in the analogy, kinetic energy plus potential energy) is constant with respect to x and must equal zero. Indeed $u(\pm\infty) \in 2\pi\mathbb{Z}$ implies $E_M = \lim_{x\to\pm\infty}\left(u'\right)^2/2$, so $E(u) < \infty$ implies $E_M = 0$. Hence in the phase plane we get the zero-'energy' orbits, that is the solutions u for which

- $\forall x \in \mathbb{R} \quad u'(x) = \pm 2 \sin \frac{u(x)}{2}$
- $\exists k \in \mathbb{Z} \quad \forall x \in \mathbb{R} \quad 2\pi k < u(x) < 2\pi(k+1)$
- u is monotone and either $\lim_{x\to-\infty} u(x) = 2k\pi$ and $\lim_{x\to+\infty} u(x) = 2(k+1)\pi$ or $\lim_{x\to-\infty} u(x) = 2(k+1)\pi$ and $\lim_{x\to+\infty} u(x) = 2k\pi$.

This implies that $Q(u) = \pm 1$ for these solutions. Finally, upon integration, we obtain the identities

$$x - x_0 = \pm \int_{u(x_0)}^{u(x)} \frac{du}{2\sin(u/2)} = \pm \ln \frac{\tan[u(x)/4]}{\tan[u(x_0)/4]} \qquad \forall x \in \mathbb{R}.$$

Using again the invariance $u \mapsto u + 2\pi k$, we impose $u(x_0) = \pi$ and we get the explicit solutions

$$u_{\mathrm{K}}(x) = 4 \arctan e^{x-x_0} \quad \text{and} \quad u_{\mathrm{A}}(x) = -4 \arctan e^{x-x_0} \tag{53}$$

which are the so-called *kink* and *antikink*, respectively, and carry $Q(u_{\mathrm{K}}) = 1$ and $Q(u_{\mathrm{A}}) = -1$. Note that the translational invariance of (52) is reflected by the fact that a different choice of the arbitrary constant x_0 only brings the solution to shift in space. The energy density of both kink and antikink is given by the localized function

$$\widetilde{\mathcal{E}}(x) = \frac{16e^{2(x-x_0)}}{\left[1+e^{2(x-x_0)}\right]^2}$$

and hence they are static solitary waves (i.e. corresponding to $v = 0$ in (46)).

By the Lorentz invariance of (48), travelling solitary waves can be trivially obtained on Lorentz-transforming (53) and their energy density turns out to be

$$\mathcal{E}(x,t) = \gamma^2 \widetilde{\mathcal{E}}(\gamma[x - vt])$$

which represents a single bump travelling undistorted with uniform velocity.

The kink and antikink also exhibit a soliton-like behaviour: there exist solutions of (47) representing the interaction of an arbitrary number of kinks and antikinks, which scatter, collide and rise out again essentially unhurt. For instance, the solution

$$u_{\mathrm{KA}}(x,t) = 4\arctan\frac{\sinh(\gamma v t)}{v\cosh(\gamma x)} \tag{54}$$

as time evolves shows first a $(0, -2\pi)$-antikink and a $(-2\pi, 0)$-kink approaching each other, then their vanishing in the collision, and finally a $(2\pi, 0)$-antikink and a $(0, 2\pi)$-kink that re-emerge and go far from each other with the same velocities of the initial pair. Similarly, the configurations of the solutions

$$u_{\mathrm{KK}}(x,t) = 4\arctan\frac{v\sinh(\gamma x)}{\cosh(\gamma v t)} \quad \text{and} \quad u_{\mathrm{AA}}(x,t) = -u_{\mathrm{KK}}(x,t) \tag{55}$$

describe the interaction, respectively, of two kinks (with indices $(-2\pi, 0)$ and $(0, 2\pi)$) and two antikinks, which collide when t is around zero and are far apart at both negative and positive large t, with equal and opposite speeds. Note that $Q(u_{\mathrm{KA}}) = 0$ and $Q(u_{\mathrm{KK}}) = 2 = -Q(u_{\mathrm{AA}})$. Note also that the kinks and antikinks involved in (54) and (55) suffer a vertical shift as a consequence of the collision; but, if we consider $u \in \mathbb{S}^1$ (i.e. we see u as an angle, consistent with the translational symmetry of the equation) such a displacement becomes meaningless and thus we see exactly the same pair of solitary waves before and after the collision (to be precise, up to a time delay, which is the sole residual effect). The existence of more general solutions describing the interaction of an arbitrary number of kinks and antikinks has been proved in the 70's (‡) and is related to the fact that, through the I.S.T. method, all the time-dependent solutions of the equation (47) are known.

In 1963, attempting to find a model for *extended* elementary particles in contrast with *point* particles, U. Enz [47] was led to study an equation like (47). He proved

‡ see [4] and the work of L.D. Faddeev and collaborators ([49] and references therein)

the existence of nonsingular time-independent solutions with energy density localized about a point on the x axis and, under a further request of stability, he found that the energy is bound to assume only certain discrete values, which can be seen as corresponding to the rest energies of elementary particles. Moving from this work, G.H. Derrick proposed, in a celebrated paper [43], the more realistic 3+1 dimensional model given by the nonlinear Klein-Gordon equation

$$\Box\psi + W'(\psi) = 0 \qquad \psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \tag{56}$$

which we have already described and discussed in Section 1.2.

Owing to the relativistic invariance of (56), moving waves can be trivially obtained from static solutions by boosting, i.e. turning to a moving coordinate frame by applying a Lorentz transformation. Thus, we are led to concern ourselves with *finite-energy static solutions* (of which solitary waves are a particular case): they are complex functions of the form

$$\psi(x,t) = u(x) \qquad \text{where} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3$$

with

$$E(u) = \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] dx < \infty \tag{57}$$

where $N = 3$, that solve the equation

$$-\triangle u + W'(u) = 0$$

which is also the Euler-Lagrange equation of the energy functional (57).

In [43] Derrick showed that, if the potential W is nonnegative, any finite-energy static solution of (56) is necessarily trivial, namely it takes a constant value which is a minimum point of W. On the other hand, if the nonnegativity of W is not required, no stable finite-energy static solution is permitted to the equation (56). In fact, the following theorem holds.

THEOREM 2. *Let $N \geqslant 3$. The energy functional* (57) *has no nontrivial local minima, i.e.*

$$\delta^2 E(u) \geqslant 0 \text{ with } u \text{ nonconstant} \quad \Longrightarrow \quad \delta E(u) \neq 0.$$

Moreover, if $W \geqslant 0$ then E does not have any nontrivial critical point at all, namely

$$\delta E(u) = 0 \quad \Longrightarrow \quad u(x) \equiv u_0 \text{ with } W(u_0) = 0.$$

Proof. Using Derrick's simple rescaling argument, we set $u_\lambda(x) := u(\lambda x)$ and

$$E(u_\lambda) = \tfrac{1}{2\lambda^{N-2}} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \tfrac{1}{\lambda^N} \int_{\mathbb{R}}^{N} W(u)\, dx =: \tfrac{1}{2\lambda^{N-2}} I_1 + \tfrac{1}{\lambda^N} I_2.$$

If $\delta E(u)\,h = 0$ for any variation h, we have in particular

$$\frac{d}{d\lambda}E(u_\lambda)\Big|_{\lambda=1} = \frac{2-N}{2}I_1 - NI_2 = 0 \tag{58}$$

and therefore

$$\frac{d^2}{d\lambda^2}E(u_\lambda)\Big|_{\lambda=1} = \frac{(2-N)(1-N)}{2}I_1 + N(N+1)\,I_2 = (2-N)\,I_1.$$

Hence the second variation of E at any nonconstant critical point u is negative for a variation corresponding to a uniform stretching of u. Finally, if $W \geqslant 0$ then both I_1 and I_2 are nonnegative and from (58) we deduce $I_1 = I_2 = 0$. □

REMARK 4. *According to Enz's results as well as to our previous discussion on the equation (47), the above argument is not applicable to the 1+1 dimensional case: if* $N = 1$ *we obtain* $E(u_\lambda) = \lambda I_1/2 + I_2/\lambda$ *yielding on differentiation* $I_1 = 2I_2$*, which gives no contraddiction.*

So, even though no restriction is placed on the sign of W, we are forced to seek static solutions with lack of stability (*). In [43], these facts led Derrick to say:

> "We are thus faced with the disconcerting fact that no equation of type" (56) "has any time-independent solutions which could reasonably be interpreted as elementary particles."

At the end of his paper, the author proposed different ways to overcome this difficulty. These ways are briefly described in the following list.

$\boxed{\mathbf{1}}$ We quote from [43]:

> "We could take a Lagrangian in which the derivatives occur in higher powers than the second. For example, with the form" $\left(|\nabla\psi|^2 - |\psi_t|^2\right)^n$ "the nonexistence proof [...] fails for $n > \frac{3}{2}$. Such a Lagrangian, however, leads to a very complicated differential equation."

In this spirit, a considerable amount of work has been done by V. Benci and collaborators (†), and a model equation proposed in [20] will be the topic of the next Section 2.4.

$\boxed{\mathbf{2}}$ Derrick also proposed to use the Dirac operator as linear part instead of the d'Alembert operator, so to obtain first order spinor equations. Unfortunately, for general nonlinearities

> "the condition for stability [...] is now very complicated, and the author has been unable to prove or disprove the existence of stable time-independent solutions [...]."

*see [27], [29] for seeking critical points and [6], [26], [80] for a discussion on stability

†see [10], [11], [14], [15], [17] and [19]-[22]

Investigations on this approach can be found in the work of M.J. Esteban and E. Séré (see [48] and references therein).

$\boxed{3}$ Having in mind the existence results obtained by W. Heisenberg and some other authors, Derrick suggested to make a second quantization of the field equations by

> "replacing the wavefunction by an operator satisfying some postulated commutation relations"

i.e., by considering $u : \mathbb{R}^3 \to H$ where H is a proper operator space.

$\boxed{4}$ We quote again from [43]:

> "Elementary particles might correspond to stable, localized solutions which are *periodic* in time, rather than time-independent. [...] However the condition for stability of solutions is now very complicated, and the author has been unable to demonstrate either the existence or nonexistence of stable solutions [...]."

This point of view has been widely developed by different authors (‡). Here we summarize the basic steps of the strategy adopted under the assumption that W only depends on $|\psi|$, so that $W'\left(e^{i\theta}\psi\right) = e^{i\theta} W'(\psi)$ for $\theta \in \mathbb{R}$ and $W'(u) \in \mathbb{R}$ for $u \in \mathbb{R}$ (see Subsection 1.5.2).

- Make the following ansatz

$$\psi(x,t) = u(x)\, e^{-i\omega_0 t},\ u : \mathbb{R}^3 \to \mathbb{R},\ \omega_0 \in \mathbb{R}$$

namely look for solutions to (56) which are, by definition, *standing* (or *stationary*) *waves*.

- Solve the (elliptic) equation

$$-\triangle u - \omega_0^2 u + W'(u) = 0$$

derived by substitution.

- Use the (Lorentz) group invariance to obtain travelling wave solutions, e.g. $\mathbf{v} = (v, 0, 0)$ and

$$\psi_{\mathbf{v}}(x,t) = u(\gamma[x_1 - vt], x_2, x_3)\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

where $\mathbf{k} = \omega\mathbf{v}$ and $\omega = \gamma\omega_0$.

REMARK 5. *The same strategy can be applied to the nonlinear Schrödinger equation*

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\triangle\psi + W'(\psi) \qquad \psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \tag{59}$$

‡see e.g. [28], [29], [79], [30], [31], [77], [78], [69], [63]

(see [25], [13], [42] and also Section 2.5). Looking for standing waves, one has to solve the elliptic equation

$$-\frac{\hbar^2}{2m}\triangle u - \hbar\omega_0 u + W'(u) = 0.$$

Solutions travelling with vector velocity $\mathbf{v}$ *are obtained by using the (Galilean) invariances of (59):*

$$\psi_{\mathbf{v}}(x,t) = u(x - \mathbf{v}t)\, e^{i[\mathbf{k}\cdot\mathbf{x} - (\omega+\omega_0)t]}$$

where $\mathbf{k} = \frac{1}{\hbar} m\mathbf{v}$ *and* $\omega = \frac{1}{\hbar}\frac{m}{2}|\mathbf{v}|^2$.

2.4. Existence of static solutions

We introduce here an existence result for a 3+1 dimensional model generalizing the one suggested by Derrick in his first proposal.

A first existence result is stated in [20], which also gives a topological classification of static solutions by means of a topological invariant: the topological charge. In order to prove the existence of static solutions with nontrivial charge, a study of the behaviour of sequences of bounded energy is needed, in the spirit of the concentration-compactness principle.

A further generalization is carried out in [14], which developes an existence analysis of the finite-energy static solutions in higher spatial dimension and for a larger class of Lorentz invariant Lagrangian densities, namely

$$\mathcal{L}_\psi = -\frac{1}{2}\alpha(\sigma) - V(\psi) \qquad \text{with} \quad \sigma = |\nabla\psi|^2 - |\psi_t|^2 \tag{60}$$

where $\psi : \mathbb{R}^{N+1} \to \mathbb{R}^k$ with $N \geqslant 2$ and $k \geqslant 1$, $V : \Omega \to \mathbb{R}$ with Ω an open subset of $\mathbb{R}^k$ and

$$\alpha(\sigma) = a\sigma + b|\sigma|^{\frac{p}{2}}$$

with $p > N$, $a \geqslant 0$ and $b > 0$.

The results of [20] were concerned with the case $N = 3$, $a = 1$ and $p = 6$. If $N = 3$ and $a = 0$, (60) is equivalent to the Lagrangian proposed by Derrick. In this section, we will discuss the main results of [14], giving an outline of their proof.

The static solutions $\psi(x,t) = u(x)$ of the Euler-Lagrange equations of the action functional related to $\mathcal{L}_\psi$ solve

$$-a\triangle u - b\frac{p}{2}\triangle_p u + V'(u) = 0 \tag{61}$$

which are the Euler-Lagrange equations of the energy functional

$$E(u) = \int_{\mathbb{R}^N}\left[\frac{a}{2}|\nabla u|^2 + \frac{b}{2}|\nabla u|^p + V(u)\right]dx. \tag{62}$$

Here $\triangle_p u$ denotes the vector whose j-th component is $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u^j\right)$.

We set $k = N + 1$ and assume that V is positive on

$$\Omega = \mathbb{R}^{N+1} \setminus \{\eta\} \quad \text{with } \eta = (1, 0, ..., 0)$$

and singular in η; more precisely, the following assumptions are made on V:

(V1) $V \in C^1(\Omega; \mathbb{R})$

(V2) $V(\xi) \geqslant V(0) = 0$ for every $\xi \in \Omega$

(V3) V is twice differentiable in 0 and the Hessian matrix $V''(0)$ is nondegenerate

(V4) there exist $c, \rho > 0$ such that $|\xi| < \rho \Rightarrow V(\eta + \xi) \geqslant c\,|\xi|^{-\frac{pN}{p-N}}$

(V5) $V(\xi) > 0$ for every $\xi \in \Omega \setminus \{0\}$ and $\nu := \liminf_{|\xi| \to \infty} V(\xi) > 0$.

The presence of $\triangle_p$ in (61) implies that the functions u on which the energy E is finite are continuous and decay to zero at infinity (see below); the presence of the singular term $V'(u)$ implies that such maps u have to take values in $\mathbb{R}^{N+1} \setminus \{\eta\}$, whose N-th homotopy group satisfies $\pi_N\left(\mathbb{R}^{N+1} \setminus \{\eta\}\right) \simeq \mathbb{Z}$. So, the nontrivial topological properties of Ω permit, as in the sine-Gordon equation, to give a topological classification of the static configurations; it is carried out by means of the topological charge, which depends only on the region where the function is concentrated, namely the support.

We are now able to give the statements of the two main existence theorems. In the first one, it is proved the existence of a static solution to the Euler-Lagrange equations related to the Lagrangian (60), which minimizes the energy among the configurations with nontrivial charge. In the second one, under some symmetry assumptions, it is proved the existence of infinitely many solutions, which are constrained minima of the energy.

THEOREM 3. *Let $a \geqslant 0$, $b > 0$ and $p > N \geqslant 2$. If V satisfies* (V1)-(V5) *then there exists a weak solution of equation* (61)*, which is a minimizer of the energy functional* (62) *in the class of those maps whose topological charge is different from zero.*

REMARK 6. *In the above theorem, assumption (V5) can be avoided if $N \geqslant 3$ and $a > 0$.*

THEOREM 4. *Let $a, b > 0$ and $p > N \geqslant 2$. If V satisfies* (V1)-(V5) *and*

(V6) *there exist $c_1, \rho_1 > 0$ and $r > 1$ such that*

$$|\xi| \leqslant \rho_1 \Longrightarrow \left|V'(\xi) - V''(0)\,\xi\right| \leqslant c_1\,|\xi|^r$$

(V7) $V\left(\xi^0, g\tilde{\xi}\right) = V\left(\xi^0, \tilde{\xi}\right)$ *for every* $\xi = \left(\xi^0, \tilde{\xi}\right) \in \mathbb{R} \times \mathbb{R}^N$ *and* $g \in O(N)$

then for every $q \in \mathbb{N} \setminus \{0\}$ there exists a weak solution u_q of equation (61) *such that its topological charge equals q. Moreover*

$$\lim_{q \to \infty} E\left(u_q\right) = +\infty. \tag{63}$$

REMARK 7. *The condition (V7) requires that V is invariant under spatial rotations and the solutions we find in Theorem 4 satisfy a spatial rotation invariance as well, namely*

$$\forall g \in O(N) \quad u_q(x) = \left(u_q^0(gx), g^{-1}\tilde{u}_q(gx)\right) \tag{64}$$

(see Subsection 2.4 below). On the other hand, it is not clear whether, under the addictional assumption (V7), the solution we find in Theorem 3 satisfies the symmetry property (64).

After making some considerations on the functional setting of the variational problem and giving the definition of the topological charge, we will trace an outline of the proof of the above theorems in Subsection 2.4.

2.4.1. Functional setting

Without loss of generality, we can take $b = 1$; so the energy functional becomes

$$E_a(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} V(u)\, dx \quad (a \geqslant 0,\ p > N \geqslant 2)$$

and the natural space where to work is the completion W_a of $C_c^\infty\left(\mathbb{R}^N; \mathbb{R}^{N+1}\right)$ with respect to the norm

$$\|u\|_a := a\|\nabla u\|_{L^2} + \|\nabla u\|_{L^p} + \|u\|_{L^2}. \tag{65}$$

By the Sobolev embedding theorems, since $p > N$, W_a is continuously embedded in $L^\infty\left(\mathbb{R}^N; \mathbb{R}^{N+1}\right)$ and every function u in W_a is Hölder continuous and decays to zero at infinity; indeed

$$W_a \hookrightarrow W^{1,p}\left(\mathbb{R}^N; \mathbb{R}^{N+1}\right) \hookrightarrow L^\infty\left(\mathbb{R}^N; \mathbb{R}^{N+1}\right) \tag{66}$$

$$W_a \subset C^{0,(p-N)/p}\left(\mathbb{R}^N; \mathbb{R}^{N+1}\right) \tag{67}$$

$$\lim_{|x|\to\infty} |u(x)| = 0. \tag{68}$$

Thus, recalling that V is singular in η, it make sense to consider, in the space W_a, the open subset

$$\Lambda_a := \left\{u \in W_a \mid \forall x \in \mathbb{R}^N \quad u(x) \neq \eta\right\}$$

on which the functional E_a turns out to be real-valued, coercive and weakly lower semicontinuous. Moreover, the link between the singularity estimate requested by the assumption (V4) and the Hölder exponent in (67) can be exploited to prove the following lemma.

LEMMA 2. *If* $\{u_n\} \subset \Lambda_a$ *is such that* $u_n \rightharpoonup u \in W_a \backslash \Lambda_a$*, then* $E_a(u_n) \to +\infty$.

2.4.2. Topological charge

We start making some preliminary considerations which lead to the proper definition of the topological charge.

Every $u \in \Lambda_a$, being a continuous function such that $u : \mathbb{R}^N \to \mathbb{R}^{N+1} \backslash \{\eta\}$ and $u(\infty) = 0$, determines a homotopy class

$$[u] \in \pi_N \left(\mathbb{R}^{N+1} \backslash \{\eta\}\right)$$

and thus a topological integer index could be associated to u thanks to the isomorphism $\pi_N \left(\mathbb{R}^{N+1} \backslash \{\eta\}\right) \simeq \mathbb{Z}$. If we set

$$\forall q \in \mathbb{Z} \Lambda_a^q := \{u \in \Lambda_a \mid [u] \simeq q\}$$

we recover $\Lambda_a = \bigcup_{q \in \mathbb{Z}} \Lambda_a^q$ and the natural idea is to minimize E_a on each Λ_a^q. Unfortunately, in this approach the following problems arise:

- Λ_a^q is not weakly closed
- the operator $\triangle_p$ is not weakly continuous and the concentration-compactness methods cannot be applied directly.

These difficulties can be overcome if we are able to 'localize' the charge, so that every bump has its own charge and the charge of any configuration equals the sum of the charge of its bumps.

In order to make this localization possible, the topological charge is defined by means of the Brower topological degree, as follows:

- for every $u \in W_a$, we write $u = \left(u^0, \tilde{u}\right) : \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$
- the open set $K(u) := \left\{x \in \mathbb{R}^N \mid u^0(x) > 1\right\}$ is the *support* of u
- the integer $\mathrm{ch}(u) := \deg(\tilde{u}, K(u), 0)$ is the *topological charge* of u.

This definition is well-posed because (68) implies that $K(u)$ is bounded and for every $x \in \partial K(u)$ we have $\tilde{u}(x) \neq 0$, since $u^0(x) = 1$ and $u(x) \neq \eta$. Let us notice that if $K(u)$ consists of m connected components $K_j(u)$ we can define also

$$\mathrm{ch}_j(u) := \deg\left(\tilde{u}, K_j(u), 0\right)$$

so that, by the additivity of the degree, we obtain

$$\mathrm{ch}(u) = \sum_{j=1}^{m} \mathrm{ch}_j(u)$$

and then the topological charge is localized.

Moreover, the continuity property of the degree with respect to the uniform norm and the continuity of the embeddings (66) assure that the topological charge is continuous on Λ_a with respect to the norm (65). Thus the open set Λ_a of W_a splits into *sectors* as follows:

$$\Lambda_a = \bigcup_{q\in\mathbb{Z}} \Lambda_a^q$$

where the sets $\Lambda_a^q := \{u \in \Lambda_a \mid \mathrm{ch}\,(u) = q\}$ are open in W_a.

2.4.3. Existence of minimizers

In order to prove theorem 3, we will show the existence of a minimizer $\overline{u}$ of E_a in the open set

$$\Lambda_a^* := \{u \in \Lambda_a \mid \mathrm{ch}\,(u) \neq 0\}$$

i.e.

$$E_a\,(\overline{u}) = E_a^* := \inf_{\Lambda_a^*} E_a. \tag{69}$$

By the existence property of the degree, $\mathrm{ch}\,(u) \neq 0$ implies $\|u\|_\infty > 1$; hence, E_a is positively bounded from below on Λ_a^*

$$E_a^* \geqslant \Delta_a^* \tag{70}$$

as a consequence of the assertion

$$\exists \Delta_a^* > 0 \quad \forall u \in \Lambda_a \qquad \|u\|_\infty \geqslant 1 \;\Rightarrow\; E_a\,(u) \geqslant \Delta_a^* \tag{71}$$

which immediately follows from the continuous embedding (66).

The key point of the proof is the following lemma.

PROPOSITION 1 (SPLITTING LEMMA). *Let $\{u_n\} \subset \Lambda_a^*$ be such that*

$$E_a\,(u_n) \leqslant M.$$

There exist $l \in \mathbb{N}$ with

$$1 \leqslant l \leqslant M/\Delta_a^* \tag{72}$$

and

$$\overline{u}_1, ..., \overline{u}_l \in \Lambda_a\,, \quad \left\{x_n^1\right\}, ..., \left\{x_n^l\right\} \in \mathbb{R}^N\,, \quad R_1, ..., R_l > 0$$

such that, up to a subsequence,

$$u_n\left(\cdot + x_n^i\right) \rightharpoonup \overline{u}_i \tag{73}$$

$$\|\overline{u}_i\|_\infty \geqslant 1 \tag{74}$$

$$(75) \qquad \left| x_n^i - x_n^j \right| \to \infty \quad \textit{for } i \neq j$$

$$(76) \qquad \sum_{i=1}^{l} E_a(\overline{u}_i) \leqslant \liminf_{n\to\infty} E_a(u_n)$$

$$(77) \qquad \forall x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{l} B_{R_i}\left(x_n^i\right) \qquad |u_n(x)| \leqslant 1.$$

Moreover

$$(78) \quad \mathrm{ch}(u_n) = \sum_{i=1}^{l} \mathrm{ch}(\overline{u}_i) \quad \textit{and} \quad \limsup_{n\to\infty} \left\| u_n - \sum\nolimits_{i=1}^{l} \overline{u}_i\left(\cdot - x_n^i\right) \right\|_\infty \leqslant 1.$$

The Splitting Lemma essentially says that every sequence $\{u_n\}$ of finite energy and nonzero charge can be seen as the superposition of a finite number of diverging bumps $\overline{u}_i\left(\cdot - x_n^i\right)$ and the charge of u_n is equal to the sum of the charge of these bumps.

Proposition 1 is proved in [20] by means of an iterative procedure, which uses the coercivity of E_a together with Lemma 2 to construct a family of functions satisfying (73)-(76) and terminates when even (77) is met; the process must end in a finite number of steps because of the estimate (72), which follows immediately from (74), (76) and (71). Finally, from (73)-(76), (78) is easily deduced.

Now, if $\{u_n\} \subset \Lambda_a^*$ is a minimizing sequence for the minimization problem (69), we can apply Proposition 1. Since $\mathrm{ch}(u_n) \neq 0$, from (78) we deduce that there exists $i_0 \in \{1, ..., l\}$ such that $\mathrm{ch}\left(\overline{u}_{i_0}\right) \neq 0$, i.e. $\overline{u}_{i_0} \in \Lambda_a^*$. Then, using (76), we obtain

$$E_a^* \leqslant E_a\left(\overline{u}_{i_0}\right) \leqslant \sum_{i=1}^{l} E_a(\overline{u}_i) \leqslant \liminf_{n\to\infty} E_a(u_n) = E_a^*$$

and hence $E_a\left(\overline{u}_{i_0}\right) = E_a^*$.

In order to sketch the proof of theorem 4, for sake of simplicity we can assume $a = 1$ and write E, W, $\|\cdot\|$ instead of E_1, W_1, $\|\cdot\|_1$.

Using assumption (V7), simple computations show that the open set Λ and the functional E are invariant under the $O(N)$ action defined in the space W by

$$T_g u(x) = \left(u_0(gx), g^{-1}\tilde{u}(gx)\right).$$

The subspace of fixed points

$$X := \left\{u \in W \mid \forall g \in O(N) \quad T_g u = u\right\}$$

is closed in W and

$$\Lambda_X := \Lambda \cap X$$

turns out to be a *natural* constraint in finding critical points of E; in particular, every local minimum of $E_{|\Lambda_X}$ is also a critical point of E and it is then sufficient to prove the existence of minimizers for the problems

$$E^q := \inf_{\Lambda_X^q} E \qquad \text{with} \quad q \in \mathbb{N} \backslash \{0\}$$

where

$$\Lambda_X^q := \{u \in \Lambda_X \mid \mathrm{ch}\,(u) = q\} = \Lambda^q \cap X$$

is open in X and proves to be not empty.

The main step in the minimization is the deduction of the compactness property given by the following proposition.

PROPOSITION 2. *The functional $E_{|\Lambda_X}$ satisfies the Palais-Smale condition, i.e. any sequence $\{u_n\} \subset \Lambda_X$ such that*

$$E\,(u_n) \ \textit{is bounded} \quad \textit{and} \quad \sup_{v \in X, \|v\|=1} \left\langle E'\,(u_n)\,, v\right\rangle \to 0$$

contains a convergent subsequence.

From Lemma 2 together with the weak lower semicontinuity of E, it easily follows that the sublevels of E

$$E^c = \{u \in \Lambda_X \mid E\,(u) \leqslant c\} \quad \text{for } c \in \mathbb{R}$$

are complete and then, recalling (70), we can use standard minimax arguments to conclude that E^q is attained in Λ_X^q.

Finally, assertion (63) follows from the Splitting Lemma, by contradiction. Indeed, assuming that, up to a subsequence, $E\left(u_q\right)$ is bounded, (78) implies that there exists $Q \in \mathbb{N}$ such that (up to a subsequence) $\mathrm{ch}\left(u_q\right) = Q$, and this contradicts $\mathrm{ch}\left(u_q\right) = q \to +\infty$.

2.5. Some other problems concerned with solitary waves

The results of Section 2.4 bear further developements. In this section we give a nonexhaustive list of some problems concerning solitary waves and brielfy outline the way in which they can be treated exploiting topological solitary waves.

2.5.1. Interaction of solitary waves with the electromagnetic field

Consider the model given by the Lagrangian (60) with $N = 3$: as we have seen in Section 2.4, it admits static solutions $u : \mathbb{R}^3 \to \mathbb{R}^4$ which are classified by means of the topological charge. If we consider travelling waves

$$\psi_v\,(x, t) = u\left(\frac{x_1 - vt}{\sqrt{1-v^2}}, x_2, x_3\right), \quad \mathbf{v} = (v, 0, 0)\,, \ |v| < 1$$

then the fact that the topological charge is a homotopic invariant implies that the function $t \mapsto \operatorname{ch}(\psi_v(\cdot, t))$ is constant, i.e. $\psi_v(\cdot, t)$ does not change sector in Λ_a as t varies; in other words, $\operatorname{ch}(\psi_v(\cdot, t))$ is a topological invariant which can be considered an integral of the motion and interpreted as the electric charge (see [16]). It is then natural to analyze the interaction between the travelling topological solitary wave ψ_v and the electromagnetic field $(\mathbf{E}, \mathbf{B})$ and to try to construct a Lorentz invariant model for the electromagnetic theory, namely a model describing particle-like matter interacting with the electromagnetic field through deterministic differential equations defined in a Newtonian space-time (see also Section 1.4).

This program has been carried out in [19], where a model for the interaction is constructed by using only concepts of classical field theory and the existence of static solutions $(u, \mathbf{A}, \varphi)$ with nontrivial charge ($\operatorname{ch}(u) \neq 0$) to the related system of equations (whose unknowns are the matter field ψ and the gauge potentials $(\mathbf{A}, \varphi)$ associated to the electromagnetic field) is proved. Let us point out that these solutions give rise to solutions $(u_v, \mathbf{A}_v, \varphi_v)$ travelling with velocity $\mathbf{v} = (v, 0, 0)$ by

$$\psi_v(x, t) = u(\gamma[x_1 - vt], x_2, x_3)$$

$$\begin{aligned}
A_v^1(x, t) &= \gamma\left[A^1(\gamma[x_1 - vt], x_2, x_3) - v\varphi(\gamma[x_1 - vt], x_2, x_3)\right] \\
A_v^2(x, t) &= A^2(\gamma[x_1 - vt], x_2, x_3) \\
A_v^3(x, t) &= A^3(\gamma[x_1 - vt], x_2, x_3)
\end{aligned}$$

$$\varphi_v(x, t) = \gamma\left[\varphi(\gamma[x_1 - vt], x_2, x_3) - vA^1(\gamma[x_1 - vt], x_2, x_3)\right]$$

where ψ_v can be seen as a travelling solitary wave 'surrounded' by the electromagnetic field $(\mathbf{A}_v, \varphi_v)$.

2.5.2. Solitary waves and nonlinear Schrödinger equations

The results of Section 2.4, can also be applied to the problem of finding standing waves $\psi(x, t) = u(x)e^{-i\omega_0 t}$ of the nonlinear Schrödinger equation (we set $\hbar = 2m = 1$ for sake of simplicity)

$$i\psi_t = -\triangle\psi + U(x)\psi + \varepsilon^r N\psi \qquad \psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}^N \tag{79}$$

where N is a proper nonlinear differential operator, U a real function, ε a positive parameter and $r \in \mathbb{N}$. In particular, using the same notations of Section 2.4, we consider the operator

$$Nu = -\triangle_p u + V'(u)$$

which can be extended to complex functions in such a way that $N\left(ue^{-i\omega_0 t}\right) = e^{-i\omega_0 t}Nu$. Hence, the standing waves of equation (79) are the solutions of the equation

$$-\triangle u - \omega_0 u + U(x)u + \varepsilon^r\left(-\triangle_p u + V'(u)\right) = 0. \tag{80}$$

If $U(x) \equiv U_0$, choosing $\omega_0 = U_0$ equation (80) reduces to (61) (with $a = 1$ and $b = 2\varepsilon^r/p$) and the solutions u_q found in theorem 4 allow us to construct the family of standing waves

$$\psi_q(x,t) = u_q(x)\, e^{-iV_0 t},\ q \in \mathbb{N}\backslash\{0\}$$

which, by using the (Galileian) invariances of (79), give rise to the travelling solitary wave solutions

$$\psi_{q,\mathbf{k}}(x,t) = u_q(x - 2\mathbf{k}t)\, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)},\ q \in \mathbb{N}\backslash\{0\}$$

where $\mathbf{k} = \mathbf{v}/2$ and $\omega = U_0 + |\mathbf{k}|^2$. In [1] and [12], the properties of these solutions are studied and their orbiltal stability is proved (for suitable values of $\mathbf{k}$).

In the case of $U(x)$ nonconstant, the standing waves of (79) are determined by the solutions of the nonlinear eigenvalue problem

$$-\triangle u + U(x)\, u + +\varepsilon^r \left(-\triangle_p u + V'(u)\right) = \mu u$$

which (with $r > p - N$) has been studied in a perturbative setting in [23] and [24], on $\mathbb{R}^N$ and in a bounded domain respectively (the latter is the case of the so-called *solitary waves in a box*).

2.5.3. Semiclassical limit

One of the basic principles in quantum mechanics is the correspondence principle. According to this principle, the quantum mechanics contains the classical mechanics as limit for $\hbar \to 0$ (see [64]). Let us see how this limit can be formally performed. Considering the Schrödinger equation

$$i\hbar\psi_t = -\tfrac{\hbar^2}{2m}\triangle\psi + V(x)\,\psi \qquad \psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C} \tag{81}$$

we can write

$$\psi(x,t) = u(x,t)\, e^{i\frac{S(x,t)}{\hbar}}, \quad x = (x_1, ..., x_N),\ u, S \in \mathbb{R},\ u > 0$$

so that equation (81) takes the form

$$S_t + \tfrac{1}{2m}|\nabla S|^2 + V(x) - \tfrac{\hbar^2}{2m}\frac{\triangle u}{u} = 0 \tag{82}$$

$$m\frac{\partial}{\partial t}u^2 + \nabla u^2 \cdot \nabla S + u^2 \triangle S = 0. \tag{83}$$

Now we assume that

- $\hbar \to 0$
- $\int |\psi|^2\, dx = \int u^2 dx = 1$

- the probabilty density u^2 concentrates around a point $q(t)$, namely $u(x,t) = v(x - q(t))$ and $v^2(x - q(t)) \to \delta(x - q(t))$.

Thus, from (83) we get

$$-m\nabla v^2 \cdot \dot{q} + \nabla v^2 \cdot \nabla S + v^2 \triangle S = 0$$

and, upon multiplying for a test function ϕ and integrating by parts,

$$0 = \int v^2 (m\dot{q} - \nabla S) \cdot \nabla \phi \, dx \to (m\dot{q} - \nabla S(q,t)) \cdot \nabla \phi(q).$$

Hence, with the above assumptions, equations (82) and (83) formally become

$$S_t + \tfrac{1}{2m} |\nabla S|^2 + V(x) = 0$$

$$m\dot{q} - \nabla S(q,t) = 0$$

which are the Hamilton-Jacobi formulation of the Newton equation

$$m\ddot{q} + \nabla V(q) = 0.$$

However, this formal deduction cannot in general be performed rigorously. One of the main difficulties is related to the fact that equation (81) is dispersive and a wave packet spreads in a short time. For this reason it is interesting to consider the semiclassical limit for nonlinear Schrödinger equations, in which the nonlinear terms compensate the dispersive nature of equation (81) and solitary wave solutions appear.

Pushing further the methods discussed in Section 2.4, this kind of study has been developed in [8], [9] and [39], [40], [41].

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