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A SIMPLIFIED ALGORITHM FOR DIGITAL PLANES RECOGNITION

Abstract. Debled proposed an efficient algorithm for the recognition of rectangular digital plane pieces based on some geometric conjectures. The aim of this paper is to propose an analytic approach of the recognition problem of such pieces. Our results allow us to well understand the geometrical constructions given by Debled and to make a mathematical foundation to her conjectures. Moreover, we propose a simplified form of Debled algorithm. Our method give us new way to treat the recognition problem of non rectangular pieces.
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1. Introduction

The aim of *digital geometry* is to give a discrete alternative to euclidean geometry. The ambient space is usually $\mathbb{Z}^n$ rather than $\mathbb{R}^n$. The classical results of euclidean geometry fail generally in the discrete environment. Already the *uniqueness* of geodesic segments connecting two points in $\mathbb{R}^n$ is not satisfied in the discrete case. Moreover, the basic notions as limits, neighboring, continuity, connectness, $\cdots$ etc are lost and need to be redefined accurately. Then, discrete theories can be built or adapted from the continuous case. Discrete planes and lines play a crucial role in digital geometry. Indeed, the discretisation of a continuous curve or a surface need to be approximated (locally) by a collection of segments or plane pieces. As a simple example, let us consider the soccer ball. Its discretisation can be easily assimilated to a collection of pentagons and hexagons that are, alternatively, glued together a long their boundary edges. Conversely, the decomposition of discrete objects into facets is a very important problem in image processing (parametrization, compression, $\cdots$). Therefore, a rigorous mathematical definition of discrete planes is necessary. *Pixels* and *voxels* (i.e., unit squares and cubes) are generally used to represent respectively points of $\mathbb{Z}^2$ and $\mathbb{Z}^3$. The intersection of a real plane in $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ is a disconnected set of scattered voxels. To get a set that looks connected, see Figure 1, we have to consider the intersection of a sufficient number of parallel real planes with $\mathbb{Z}^3$. The connected set obtained by the smallest number of a such real planes is called a *naïve* plane. This intuitive idea has been used in the eighties by many authors. The arithmetic definition of digital planes was introduced for the first time by Reveilles in 1991, [12]. In 1995, Debled proposed, in her PhD thesis [3], an efficient algorithm for the recognition of rectangular pieces of digital planes. Debled's algorithm has many advantages among other existing algorithms. It has a quadratic complexity, it is incremental and is based on some intuitive geometric properties and criteria. Many of these properties have been conjectured by Debled and almost of them have been proved in [2] and [11]. In this

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paper, we address the problem of digital plane pieces recognition following Debled's approach. The results that we prove here allow us to simplify Debled's algorithm. Furthermore, our approach is independent from the rectangular forms of plane pieces considered by Debled. Therefore, it can be developed to recognize plane pieces of various forms. The remaining of this paper is organized as follows. In section 2, we describe the existing related work to the recognition problem of plane pieces. In Section 3, we resume the fundamental notions needed to understand the remaining of the material. In Section 4, we summarize Debled recognition algorithm while in Section 5, we present our recognition theorems that allow us to simplify the above algorithm. We present in the last Section some examples to illustrate our approach.

2. Related Work

There is a wide literature on the problem of recognizing digital plane pieces. In [8, 9], Kim and Rosenfeld showed that a digital surface is a piece of a naïve digital plane if and only if there exists a face of the convex hull of the surface such that the distance between the points of the surface and the plane that supports the face is less than 1. They proposed an algorithm based on this property of complexity $O(p^4)$, where p is the number of points on the surface. In 1991, Kim and Stojmenović [10] improved this algorithm to obtain another algorithm of complexity $O(p^2 logp)$. In the same year, Stojmenović and Tošić [13] presented two other algorithms, the first with complexity $O(p\, logp)$ based on the construction of two convex hulls and the second with complexity $O(p)$ and based on linear programming in $3D$. These algorithms have a low complexity but they are not incremental which is a drawback in application. Furthermore, the construction of the convex hull in $3D$ is a delicate and expensive operation. In [14, 15], Veelaert, relying on a generalization of a regularity property of digital straight lines introduced by Hung [7], developed a simple algorithm of complexity $O(p^2)$ which is satisfactory for small sets ($p \leq 100$).

In 1991, J. P. Reveilles introduced [12] for the first time the arithmetic definition of digital planes led by relation (1) below. Indeed, a bounded subset ν of $\mathbb{Z}^3$ is said to be a *digital* plane piece if there exist four integers a, b, c, ω in $\mathbb{Z}$ and an entire number μ such that the set ν is a solution of the double Diophantine inequality

$$(1) \qquad \begin{cases} \mu \leq ax + by + cz < \mu + \omega \\ (x, y) \in \Pi \end{cases},$$

where Π is the projection of ν on the Oxy coordinate plane. Numbers a, b, c and the lower bound μ are called the *characteristics* of the plane that we denote by $P(a, b, c, \mu, \omega)$. Number ω is called the arithmetic thickness of the plane. When $\omega = sup(|a|, |b|, |c|)$, the plane $P(a, b, c, \mu, \omega)$ is called a *naïve* plane and denoted by $P(a, b, c, \mu)$ [1]. Figure 1 shows the geometric shape of a digital plane.

The above definition translates the fact that a digital plane is the intersection of $\mathbb{Z}^3$ with a family $\{\nu_t\}_{t\in[\mu,\mu+\omega-1]}$ of real parallel planes ν_t defined by $ax + by + cz = t$. Let us suppose that the greatest common divisor of a, b, c is 1. We say that a piece ν is recognized if it contains sufficiently points to compute all the characteristics

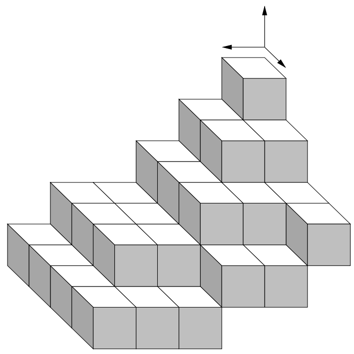


Figure 1: The shape of a digital plane piece

a, b, c, μ, ω. To this aim, it is sufficient that ν possesses four affinely independent points that generate the bounding real planes ν_μ and $\nu_{\mu+\omega-1}$. In particular, these points are not all co-planar. In 1995, Debled proposed, in her PhD thesis [3], an efficient algorithm for the recognition of rectangular pieces of digital planes. This algorithm has more advantages than the other algorithms quoted above. It uses a simple and intuitive geometric properties of digital planes. It has a quadratic complexity and is incremental. Other authors addressed the problem of recognition in higher dimensions. We cite, for instance, the work of Françon *et al.* [5] who used Fourier-Motzkin algorithm to resolve relation (1). They obtained an algorithm whose complexity can reach, in the worst case, $O(p^{2^n})$, where p is the number of elements of ν and n is the dimension. More recently two algorithms were proposed. The first is due to Gerard [6]. The algorithm, relying on geometrical properties of strings and convex hulls, recognizes hyper-planes sets with a complexity of $O(p^{2(n-1)})$. The second algorithm, due to Vittone [16], is based on Farey sequences associated to a given set of voxels. It is rapid and does not need too much memory but it does not consider all the points of the given set. In [11] we gave a proof the main conjecture of debled that allows the recognition of the 1-exterior case, see Theorem 1 below. In [2] proofs of almost all the other conjectures of Debled related to the 1-exterior case are presented.

3. Basic Notions

Generally, in digital topology, the *grid distance* D_1 and the *lattice-point distance* D_2 are used rather than the euclidean distance. These two metrics are defined by the following relations

$$D_1(P, P') = \sum_{i=1}^{n} \mid x_i - x_i' \mid \quad \text{and} \quad D_2(P, P') = max\left(\mid x_i - x_i' \mid, i = 1, \cdots, n\right);$$

where x_i and x_i' are respectively the coordinates of points P and P'. The unit balls of D_1 and D_2 are depicted in Figures 2, 3(a) for two and three dimensions. However, a third distance D_3 combining D_1 and D_2 is usually used too. This distance is defined by

$$D_3(P, P') = max_{j=1,\cdots,n} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \mid x_i - x'_i \mid \right)$$

The unit ball of this distance is depicted in Figure 3(b) for the 3D case.

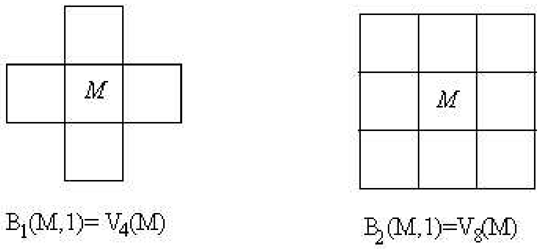
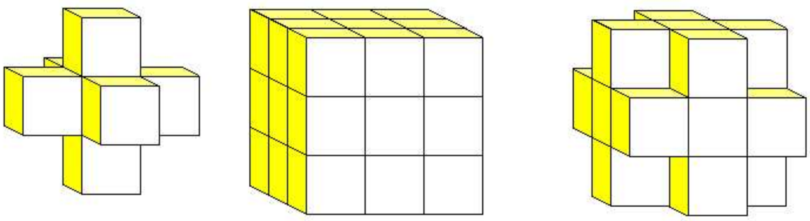


Figure 2: The unit ball of D_1 and D_2 in two dimension, the center has four neighbors for D_1 and eight neighbors for D_2.



$B_1(M, 1) = V_6(M)$ $B_2(M, 1) = V_{26}(M)$ $V_{18} = B_2(M, 1) \cap B_1(M, 2)$

(a) (b)

Figure 3: The unit balls of distances D_1, D_2 and D_3 in three dimension, the center has six neighbors for D_1, 26 neighbors for D_2 and 18 neighbors for D_3.

In each case, the unit ball around a point M is the smallest neighborhood of M. In $\mathbb{Z}^2$, this neighborhood contains 4 neighbors of M for the distance D_1 and 8 neighbors for the distance D_2. Following the case, it is denoted by $V_4(M)$ and $V_8(M)$. In $\mathbb{Z}^3$, we have respectively 6, 26 and 18 neighbors for the distances D_1, D_2 and D_3. We denote the corresponding neighborhoods by $V_6(M)$, V_{26} and V_{18}. Points of $V_k(M)$ are said to be *k-adjacent* to M. A *k-connected* path is a sequence of points $\{P_0, \cdots, P_n\}$ such that each two successive points P_i and P_{i+1} are k-adjacent. A discrete set of points Ω is said to be *k-connected* if and only if any couple of its points can be linked by a k-connected path in Ω. When a k-connected path goes through a set E without intersecting it, we say that E has a k-connected *hole*. In Figure 4 we illustrate the different holes that may exist in $\mathbb{Z}^3$. In the remaining of this paper we will consider only naïve plane pieces

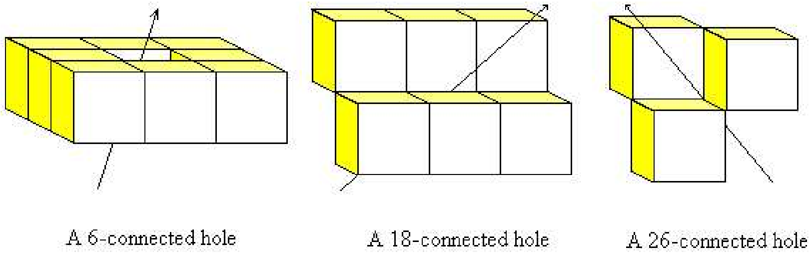


Figure 4: The different existing kind of holes in $\mathbb{Z}^3$.

ν that are 18-connected without 6-connected holes. We recall that a such planes are defined by the following relation

$$(2) \qquad \begin{cases} \mu \le ax + by + cz < \mu + sup(\mid a \mid, \mid b \mid, \mid c \mid) \\ (x, y) \in \Pi \end{cases},$$

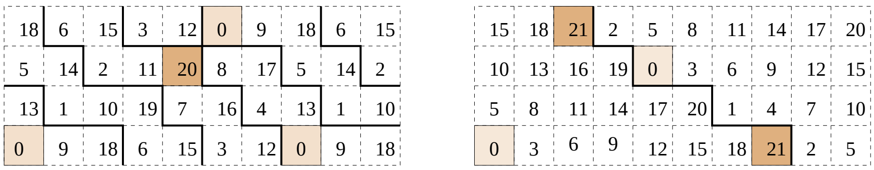
where Π is the projection of ν on the Oxy coordinate plane. The non-existence of 6-connected holes, implies that the projection Π is simply connected (i.e., without holes in its interior). The problem of studying pieces of digital naïve planes can be reduced, via rotations and symmetries, to the case $0 \le a \le b \le c$. Indeed, the simplex $0 \le x \le y \le z \le 1$ is a fundamental domain, [3], of the isometry group of the unit cube of $\mathbb{R}^3$. Thus relation (2) can be reduced to

$$(3) \qquad \mu \le ax + by + cz < \mu + c \ \ \forall\ (x, y) \in \Pi.$$

Let $M(x, y, z)$ be a point of a naïve digital plane $P(a, b, c, \mu)$, the quantity $r(M) = ax + by + cz$ is called the *remainder* of M with respect to P. The Diophantine inequality(3) allows us to define $P(a, b, c, \mu)$ as the set of points $(x, y, z) \in \mathbb{Z}^3$ such that $\frac{\mu - ax - by}{c} \le z < \frac{\mu - ax - by}{c} + 1$. Since the plane projects injectively on Oxy, we can represent $P(a, b, c, \mu)$ in Oxy by level lines corresponding to values of z. We can also represent the plane $P(a, b, c, \mu)$ by the remainder of its points. In Figure 5, we combine the remainder and level lines representations to represent $P(9, 13, 21, 0)$ on plane Oxy. The real plane defined by $r(a, b, c)(M) = k$ is called the plane of *index* k. The plane of index μ is called *lower leaning plane* of P and the plane of index $\mu + c - 1$ is called *upper leaning plane* of P. We denote them by (Pi) and (Ps) respectively.
A piece of a digital plane is said to be *recognized* if it possesses four leaning points that satisfy one of the following two cases:

- Three upper (resp. lower) leaning points and one lower (resp. upper) leaning point. This configuration is referred to as CAS3.1, see figure 5(a).

- Two upper leaning points and two lower leaning points. This configuration is refereed to as CAS2.2, see figure 5(b).



(a) (b)

Figure 5: (a) CAS3.1. The remainder and level lines representation of a piece of $P(9, 13, 21, 0)$ which contains three lower leaning points and one upper leaning point. (b) CAS2.2. The remainder and level lines representation of a piece of $P(3, 5, 22, 0)$ that contains two lower leaning points and two upper leaning points

When the added point M is k-exterior, many geometric constructions are built and key voxels are extracted. These geometric constructions depend on *leaning polygons* and their positions with respect to M. These leaning polygons are defined as follows:

- If $r(M) < \mu$, we call (PS) the convex hull of the upper leaning points in the piece and we call it the *upper leaning polygon*. We define, in this case, the *convex polygon of pivots* CVP to be the upper leaning polygon PS.
  In the same way, we call PI the *lower leaning polygon*. In this case, we define the *convex polygon of antipodes* CVA to be the lower leaning polygon PI. Thus, we have $CVP = PS$ and $CVA = PI$.

- If $r(M) \geq \mu + c$, the upper leaning polygon PS is called the convex polygon of antipodes CVA and the lower leaning polygon PI is called the convex polygon of pivots CVP. We have $CVA = PS$ and $CVP = PI$.

In figure 6(a) we give an example of the convex polygons of pivots and antipodes for a recognized piece in the plane $P(5, 6, 7, -1)$.

The *polygonal line of pivot vectors* L is constructed depending on the added k-exterior point and its associated convex polygon of pivots. All constructions are realized on the projections in the plane Oxy. Four cases are possible:

1. The CVP is reduced to one point. In this case, the polygonal line of pivot vectors L is reduced to this point.

2. The CVP is formed by points that are not all collinear. In this case, the polygonal line L of pivot vectors is composed by points of the CVP such that their projections in the plane Oxy are located on the part of the boundary of the convex hull of the CVP projection that disappears when the point M is added, see Figure 6(b).

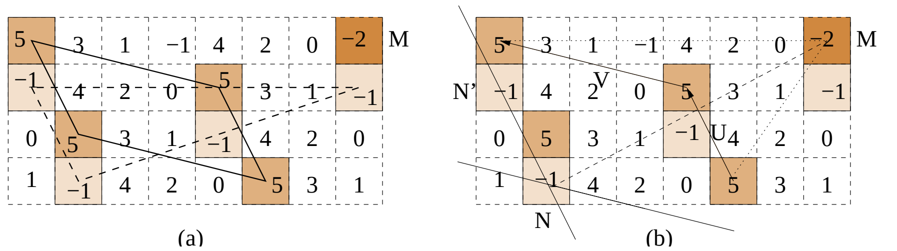


Figure 6: Point M is 1-exterior of remainder -2. In (a) the parallelogram in bold is the convex of pivot points CVP and the dashed triangle is the convex polygon of antipodes CVA. In (b) $U \cup V$ form the polygonal line of pivot vectors L; point N is an antipode for both pivot vectors U and V and it is a separating for vector U. Point N' is an antipode for the vector U only.

3. The CVP is composed by collinear points in the plane Oxy:

   – If the projection of M is collinear with the projected points of the CVP, then the polygonal line L of pivot vectors is reduced to the nearest point of the CVP to M, see Figure 7(a).

   – If the projection M is not collinear with the projected points of the CVP, then L is equal to all points of the CVP, see Figure 7(b).

An *antipode A of a pivot vector V* associated to a k-exterior point M is a summit of the CVA hat has the maximal distance, among all points of the CVA, from the line directed by V and containing M. Furthermore, if the end points of V are separated by the line MA, then A is called a *separating antipode*.

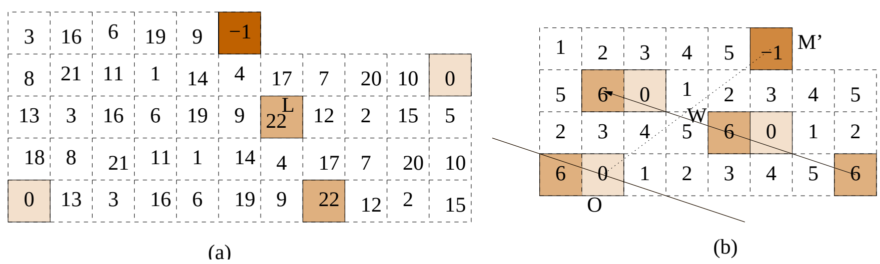


Figure 7: In (a) point M is 1-exterior to a piece of plane $P(13, 18, 23, 0)$, the polygonal line of pivot vectors is reduced to the point L. In (b) point M is 1-exterior to a piece of $P(1, 3, 7, 0)$; the polygonal line of pivot vectors is formed by the vector W. The point O is a separating antipode for the pivot vector W.

4. Debled's Algorithm

Debled's algorithm sweeps a rectangular piece by sections that are parallel to a coordinate plane. At the beginning, one fixes, for instance, $y = 0$ and let x vary. In this case, Debled's algorithm tries to recognize digital straight lines until x reaches its maximum value in Π. We recall that digital straight lines are defined by similar Diophantine inequalities of 2 variables. Then, y is incremented by 1 and x varies again. In this case, Debled's algorithm tries to recognize pieces of digital planes. At each step the algorithm computes the characteristics of the new plane. Three cases may occur

- If the new point $M(x, y)$, added to the last recognized plane piece, satisfies relation (1), then the same characteristics are kept.
- If the added point M satisfies one of the following relations

  $$ax + by + cz = \mu - 1 \qquad \text{or} \qquad ax + by + cz = \mu + c$$

  where (a, b, c) is the normal vector of the last recognized piece, then point M is said to be *1-exterior* to the last piece. Debled conjectured that the new piece is recognized in a new plane. The new characteristics $A \leq B \leq C$ are constructed by means of other conjectures that define the vector basis following the position of the polygonal lines of pivots and antipodes with respect to the added point M. Furthermore, these conjectures insure that the new characteristic C is the smallest one among all possible planes that contain the piece. All of these conjectures have been checked by numerous examples. The complexity of Debled's algorithm corresponding to this step is at most linear in the number of points on the piece [3] p.180.
- The third possibility is that M satisfies one of the following inequalities

  $$ax + by + cz < \mu - 1 \qquad \text{or} \qquad ax + by + cz > \mu + c$$

  Point M is said to be *strongly k-exterior*, where k is, respectively, equal to $\mu - (ax + by + cz)$ or $(ax + by + cz) - \mu - c + 1$. The new piece belongs to a digital plane if M is not too distant from the piece. Debled gave three validity criteria to be checked for the new piece to conclude its flatness. These criteria are sufficient but not necessary [3] p.172.

5. Recognition Theorems

Let S be a recognized piece of the plane $P(a, b, c, \mu)$ and M an added point to S. If the remainder of M with respect to $P(a, b, c, \mu)$ is between μ and $\mu + c - 1$, then $S' = S \cup \{M\}$ is still recognized in the same plane. If the point M is 1-exterior, Debled's conjectures [3, 4] provide the existence of a new plane in which S' is recognized. We proved in [11] a key theorem that insures the existence of planes that contain S'. We gave also a geometric method to construct the smallest characteristics. This result confirms the main conjecture of Debled and is expressed by the following theorem

THEOREM 1 ([11]). *Let S be a recognized piece of the plane $P(a, b, c, \mu)$ and $M(x_0, y_0, z_0) \in \mathbb{Z}^3$ be a point such that $S' = S \cup \{M\}$ is convex. If M is 1-exterior to S, then there exists $(A, B, C, \mu') \in \mathbb{Z}^4$ with $A \wedge B \wedge C = 1$ such that $S' = S \cup \{M\}$ is a piece of the plane $P(A, B, C, \mu')$.*

In the proof of this theorem we gave two necessary and sufficient conditions on the choice of points that allow the construction of the new base. These conditions are:

$$\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0) = c, \tag{4}$$

$$\begin{aligned} r_m - (\mu + c - 1) + \frac{c}{C} \quad &\leq \quad \frac{\beta_2(x - x_0) - \alpha_2(y - y_0)}{\beta_2(x_1 - x_0) - \alpha_2(y_1 - y_0)} \\ &\leq \quad r_m - (\mu - 1) \;\; \forall m \in S, \end{aligned} \tag{5}$$

where (x_2, y_2), (x_1, y_1) and (α_2, β_2) are respectively the projections on Oxy of a pivot point M_2, an antipode M_1 and a vector V_2 formed by M_2 and another pivot point associated to the polygonal line of pivots or the polygonal line of antipodes. Relation (5) can be expressed under the following geometric form:

$$r_m - (\mu + c - 1) + \frac{h_2}{H} \leq \frac{\epsilon_m h_m}{H} \leq r_m - (\mu - 1) \;\; \forall m \in S, \tag{6}$$

where r_m is the remainder of a point m; ϵ_m is the sign of $\beta_2(x - x_0) - \alpha_2(y - y_0)$; numbers H, h_2 et h_m are respectively the euclidean distances between points M_1, M_2 and m and the straight real line directed by V_2 and containing M, see Figure 8. We will see below that all these results and relations can be found again as a particular case of the strongly extrior results, see Remarks 1 and 2.

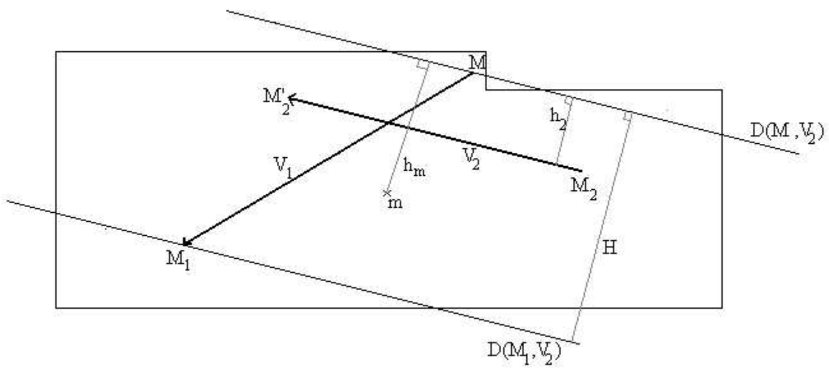


Figure 8: The projection of a piece on Oxy coordinate plane where the added point M is exterior. Vector V_2 is defined by pivot points M_2 and M_2'; M_1 is an antipode. Numbers H, h_2 and h_m are the respective distances of M_1, M_2 and point m to the straight real line $D(M, V_2)$ directed by vector V_2.

Now let us suppose that M is strongly exterior to the piece $S \subset P(a, b, c, \mu)$. This implies that there exists an integer $P \geq 2$ such that M satisfies one of the following equalities:

$$ax_0 + by_0 + cz_0 = \mu - P \quad \text{(i.e., } M \text{ is located under the piece } S\text{)}$$

or

$$ax_0 + by_0 + cz_0 = \mu + c - 1 + P \quad \text{(i.e., } M \text{ is located over the piece } S\text{).}$$

An axial rotation of the plane by π reverses the positions of its points. Points located under the plane get over it and vice versa. The characteristics of the plane remain unchanged. This fact allows us to reduce our study to only one case, let say M being located under the plane. All relations that we will obtain are still valid in the case where M is located over the plane. The following proposition is a direct consequence of the piece convexity.

PROPOSITION 1. *Let S be a piece of a discrete plane $P(a, b, c, \mu)$. If $M(x_0, y_0, z_0)$ is a strongly P-exterior point such that $S' = S \cup \{M\}$ is convex, then $P \leq c$.*

Proof. Suppose that the remainder of M is $r(M) = \mu - P$ and let consider the point $M'(x_0, y_0, z_0 + 1)$. The remainder of M' is $r(M') = \mu - P + c$. If $r(M') < \mu$, then M is disconnected from S and S' is not convex. Hence, $r(M') \geq \mu$ which gives $P \leq c$. In the same way, if $r(M) = \mu + c - 1 + P$, then the remainder $\mu - 1 + P$ of point M"$(x_0, y_0, z_0 - 1)$ must be smaller than $\mu + c - 1$, otherwise we lose the convexity of S'. This also implies $P \leq c$. □

Since 1-exterior case is recognizable, then our idea for the strong exterior case is to apply some suitable rotations that reduce the "exteriority" degree and get, when it is possible, the 1-exterior case in a new discrete plane. Then Debled's algorithm can be applied again to recognize the transformed piece.

PROPOSITION 2. *Let S be a recognized piece of a digital plane $P(a, b, c, \mu)$ and $M \in \mathbb{Z}^3$ be a point such that $S' = S \cup \{M\}$ is convex. Suppose that M is P-exterior ($P \geq 1$) to M. Then there exist three integers $0 \leq A \leq B \leq C$, a rational number $\omega \geq C$ and a rational function $\mu(x, y)$ such that for all point $m(x, y, z) \in S'$ we have*

$$\mu(x, y) \leq Ax + By + Cz \leq \mu(x, y) + \omega \tag{7}$$

Proof. Let us suppose that the added point $M(x_0, y_0, z_0)$ is located under plane $P(a, b, c, \mu)$. Point M satisfies the relation $ax_0 + by_0 + cz_0 = \mu - P$. Let $M_2(x_2, y_2, z_2)$ be a point on the polygonal line of pivots and $V_2(\alpha_2, \beta_2, \gamma_2)$ a vector based on M_2 and located either on the polygonal line of pivots if this one is not reduced to a point or parallel to the polygonal line of antipodes. Vector V_2 satisfies the relation $a\alpha_2 + b\beta_2 + c\gamma_2 = 0$. Let $V_1(\alpha_1, \beta_1, \gamma_1)$ be a vector linking M to an antipode M_1 sufficiently distant from M to exceed at least the polygonal line of antipodes. Vector

V_1 satisfies the relation $a\alpha_1 + b\beta_1 + c\gamma_1 = P$. Let V_1' be the vector of coordinates $V_1'(\alpha_1, \beta_1, \gamma_1 - \frac{P}{c})$. Vector V_1' satisfies the relation $a\alpha_1 + b\beta_1 + c(\gamma_1 - \frac{P}{c}) = 0$. The vectorial product $V_1' \wedge V_2 = (A', B', C')$ is a rational multiple of $N(a, b, c)$. Let λ be this multiplicity coefficient. Numbers A', B', C' are given by:

$$A' = (\beta_1\gamma_2 - \beta_2\gamma_1) + \beta_2\frac{P}{c} = \lambda a, \quad B' = (\alpha_2\gamma_1 - \alpha_1\gamma_2) - \alpha_2\frac{P}{c} = \lambda b,$$

$$C' = \alpha_1\beta_2 - \alpha_2\beta_1 = \lambda c.$$

Let A, B, C be the numbers defined by :

$$A = \beta_1\gamma_2 - \beta_2\gamma_1, \qquad B = \alpha_2\gamma_1 - \alpha_1\gamma_2, \qquad C = \alpha_1\beta_2 - \alpha_2\beta_1$$

Hence, we have

$$A' = A + \beta_2\frac{P}{c}, \quad B' = B - \alpha_2\frac{P}{c}, \quad C' = C$$

We can always assume (by considering $-V_2$ in place of V_2 if it is necessary) that $0 \le A'$. Since $0 \le a \le b \le c$, then $0 \le A' \le B' \le C'$ which is equivalent to

$$0 \le A + \beta_2\frac{P}{c} \le B - \alpha_2\frac{P}{c} \le C.$$

If we increase the length of V_1, then the numbers A', B', C' also increase, and so does the distance between each other. Note that the quantities $\beta_2\frac{P}{c}$ and $\alpha_2\frac{P}{c}$ do not change when we increase the length of V_1. Let us take V_1 sufficiently long so that

$$0 \le A \le B \le C$$

For all $m(x, y, z) \in S \subset P(a, b, c, \mu)$ we have:

$$\mu \le ax + by + cz \le \mu + c - 1$$

The added point M satisfies relation $ax_0 + by_0 + cz_0 = \mu - P$. Then

$$\mu - P \le ax + by + cz \le \mu + c - 1,$$

for all points of S'.
Multiplying all members with λ we obtain:

$$\lambda(\mu - P) \le (A + \frac{P\beta_2}{c})x + (B - \frac{P\alpha_2}{c})y + Cz \le \lambda(\mu - 1) + C,$$

which implies that for all points in S' we have:

$$\lambda(\mu - P) - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y) \le Ax + By + Cz$$

$$\le \lambda(\mu - 1) - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y) + C.$$

Let us take

$$\mu(x, y) = \lambda(\mu - P) - P\frac{\beta_2 x - \alpha_2 y}{c},$$

then for all points in S' we have:

$$\mu(x, y) \leq Ax + By + Cz \leq \mu(x, y) + \omega, \tag{7}$$

where $\omega = C + \lambda(P - 1)$. $\square$

Under these conditions, we have

LEMMA 1. *If vector V_2 is chosen so that its coordinates are prime together (i.e., $\alpha_2 \wedge \beta_2 \wedge \gamma_2 = 1$), then the greater common divisor $PGCD(A, B, C) = A \wedge B \wedge C$ divides number P.*

REMARK 1. This result generalizes the 1-exterior case. Indeed, for $P = 1$, numbers A, B, C are prime together.

Proof. Vector $V_1 = \overrightarrow{MM_1} = (\alpha_1, \beta_1, \gamma_1)$ satisfies the relation $a\alpha_1 + b\beta_1 + c\gamma_1 = P$ and vector $V_2 = \overrightarrow{M_2'M_2} = (\alpha_2, \beta_2, \gamma_2)$ satisfies $a\alpha_2 + b\beta_2 + c\gamma_2 = 0$. We call the previous relations characteristic equations of V_1 and V_2. For all points of $P(a, b, c, \mu)$ we have

$$Ax + By + Cz = \lambda r_m - P\frac{\beta_2 x - \alpha_2 y}{c},$$

where $r_m := ax + by + cz$ is the remainder of point m. By replacing λ with its value $\frac{C}{c}$, we get

$$c(Ax + By + Cz) = Cr_m - P(\beta_2 x - \alpha_2 y)$$

Let k_0 be a prime number that divides $PGCD(A, B, C)$. By dividing the previous identity by k_0 we obtain

$$\frac{c(Ax + By + Cz) - Cr_m}{k_0} = -\frac{P(\beta_2 x - \alpha_2 y)}{k_0}$$

Since k_0 divides the left member of this identity, then it divides $P(\beta_2 x - \alpha_2 y)$ for all $m(x, y, z) \in P(a, b, c, \mu)$.
For $x = 0$, $y = 1$ we find that k_0 is a divisor of $P\alpha_2$, and, for $x = 1$, $y = 0$ we find that k_0 is a divisor of $P\beta_2$. Moreover, numbers α_2, β_2 and γ_2 satisfy equality $a\alpha_2 + b\beta_2 = -c\gamma_2$. Therefore, k_0 divides $Pc\gamma_2$. We distinguish two cases

i) k_0 divides $P\gamma_2$. In this case since α_2, β_2 and γ_2 are prime together then, following Bezout theorem, k_0 divides P.

ii) k_0 does not divide $P\gamma_2$. Since k_0 divides numbers $P\alpha_2, P\beta_2$, $A = \beta_1\gamma_2 - \beta_2\gamma_1$ and $B = \alpha_2\gamma_1 - \alpha_1\gamma_2$ then, k_0 divides $P\beta_1\gamma_2$ and $P\alpha_1\gamma_2$. Moreover, by multiplying characteristic equation of V_1 with $P\gamma_2$ we find

$$Pa\gamma_2\alpha_1 + Pb\gamma_2\beta_1 + Pc\gamma_2\gamma_1 = P^2\gamma_2.$$

The first member is divisible by k_0, therefore k_0 divides $P^2\gamma_2$. This implies that k_0 divides P or γ_2 and in both cases, k_0 divides $P\gamma_2$, which contradict the hypothesis of this case.

Therefore, all prime numbers that decompose $PGCD(A, B, C)$ divides the number P. Hence $PGCD(A, B, C)$ divides P. $\square$

Now let suppose that $M(x_0, y_0, z_0)$ is strongly P-exterior ($P \geq 2$) to a recognized piece S of a digital plane $P(a, b, c, \mu)$ such that $S' = S \cup \{M\}$ is convex. Let $M_2(x_2, y_2, z_2)$ be a point on the polygonal line of pivots and $V_2(\alpha_2, \beta_2, \gamma_2)$ a vector based on M_2 and located either on the polygonal line of pivots if it is not reduced to a point or parallel to the polygonal line of antipodes, see Figure 8. Let $V_1(\alpha_1, \beta_1, \gamma_1)$ be a vector linking M to an antipode M_1 sufficiently distant from M to exceed at least the polygonal line of antipodes. Let $0 \leq A \leq B \leq C$ be three integers obtained from Proposition 2 and k be their greater common divisor, then we have

THEOREM 2. *There exists an entire number μ' such that the piece S belongs to the digital plane $P(\frac{A}{k}, \frac{B}{k}, \frac{C}{k}, \mu')$ with M as a leaning point ($q = 0$) or a q-exterior to S with $1 \leq q < P$ if and only if M_1 is arbitrarily chosen on a line $D_1(V_2)$ directed by V_2 such that the following relations hold*

$$P\big(\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)\big) - C(P - 1) = kc(1 - q), \tag{8}$$

$$\frac{r_m - \mu - c + P}{P} + \frac{k(1 - q)c}{|V_2|H} \leq \frac{\epsilon_m h_m}{H} \leq \frac{r_m - \mu + P}{P}\ ;\ \forall m \in S, \tag{9}$$

with H (resp. h_m) being the height of point M_1 (resp. m) to the real line $D(M, V_2)$ passing through M and directed by V_2.

REMARK 2. We note that if we assume that P can take the value 1, then q is necessarily equal to 0 and relations (8) and (9) become equivalent to relations (4) and (6). Hence, Theorem 2 becomes a common theorem for 1-exterior and strongly exterior cases.

Proof. Let us suppose that point M is located under plane $P(a, b, c, \mu)$, then M satisfies the relation $ax_0 + by_0 + cz_0 = \mu - P$. Integers A, B and C are defined in Proposition 2 by

$$A = \beta_1\gamma_2 - \beta_2\gamma_1, \qquad B = \alpha_2\gamma_1 - \alpha_1\gamma_2, \qquad C = \alpha_1\beta_2 - \alpha_2\beta_1,$$

and antipode M_1 is chosen far away from M so that $0 \leq A \leq B \leq C$. Function $\mu(x, y) = \lambda(\mu - P) - P\frac{\beta_2 x - \alpha_2 y}{c}$ was also defined so that for all points in S' we have:

$$\mu(x, y) \leq Ax + By + Cz \leq \mu(x, y) + C + \lambda(P - 1),$$

where $\lambda = \frac{C}{c}$. Let us divide the previous inequality by the greater common divisor k of A, B and C, we obtain

$$\mu''(x, y) \leq A''x + B''y + C''z \leq \mu''(x, y) + C'' + \lambda''(P - 1),$$

where $\mu"(x,y) = \frac{1}{k}\mu(x,y)$, $A" = \frac{A}{k}$, $B" = \frac{B}{k}$, $C" = \frac{C}{k}$ et $\lambda" = \frac{\lambda}{k} = \frac{C"}{c}$.
Since M is P-exterior and M_2 (one of the end points of the vector V_2) is an upper leaning point of piece S, then we obtain respectively

$$A"x_0 + B"y_0 + C"z_0 = \mu"(x_0, y_0)$$

$$A"x_2 + B"y_2 + C"z_2 = \mu"(x_2, y_2) + C" + \lambda"(P - 1)$$

On the other hand, we want that M becomes q-exterior to $S \subset P(A", B", C", \mu" = \mu"(x_0, y_0) + q)$ and M_2 an upper leaning point of S', then M_2 must satisfy the relation

$$A"x_2 + B"y_2 + C"z_2 = \mu"(x_0, y_0) + q + C" - 1$$

Therefore we obtain

$$\mu"(x_0, y_0) + C" + q - 1 = \mu"(x_2, y_2) + C" + \lambda"(P - 1)$$

which is equivalent to

$$\mu"(x_2, y_2) - \mu"(x_0, y_0) = q - 1 - \lambda"(P - 1)$$

or to

$$P"((\beta_2 x_2 - \alpha_2 y_2) - (\beta_2 x_0 - \alpha_2 y_0)) = c(1 - q) + \frac{C}{k}(P - 1)$$

Thus we obtain the condition

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - C(P - 1) = kc(1 - q) \tag{8}$$

This condition express the position of points M_2, M_2' and M_1 with respect to M.
Now let us look for which region we can choose V_1 so that the piece S is recognized in the digital plane $P(A", B", C", \mu"(x_0, y_0) + q)$ with M a q-exterior, where $q = 0$ or 1. This means that for all point $m(x, y, z) \in S'$ we have:

$$\mu"(x_0, y_0) \leq A"x + B"y + C"z \leq A"x_2 + B"y_2 + C"z_2$$

$$= \mu"(x_0, y_0) + C" - 1 + q$$

which is equivalent to

$$k\mu"(x_0, y_0) \leq Ax + By + Cz \leq k\mu"(x_0, y_0) + C + k(q - 1).$$

We have $Ax + By + Cz = \lambda(ax + by + cz) - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y)$. Let us take $r_m = ax + by + cz$, when the point $m(x, y, z)$ sweeps S', then the number r_m sweeps the interval $[\mu - P, \mu + c - 1]$. Thus we have

$$Ax + By + Cz = \lambda r_m - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y), \quad \text{with } r_m \in [\mu - P, \mu + c - 1]$$

Therefore, we get

$$k\mu"(x_0, y_0) \leq \lambda r_m - P(\frac{\beta_2}{c}x - \frac{\alpha_2}{c}y) \leq k\mu"(x_0, y_0) + C + k(q - 1)$$

or

$$\lambda(\mu - P) - \frac{P}{c}(\beta_2 x_0 - \alpha_2 y_0) \le \lambda r_m - \frac{P}{c}(\beta_2 x - \alpha_2 y)$$
$$\le \lambda(\mu - P) - \frac{P}{c}(\beta_2 x_0 - \alpha_2 y_0) + C + k(q-1)$$

which is equivalent to

$$\lambda(\mu - P - r_m) \le \frac{P}{c}\big[\alpha_2(y - y_0) - \beta_2(x - x_0)\big] \le \lambda(\mu - P - r_m) + C + k(q-1)$$

or

$$C(\mu - P - r_m) \le P\big(\alpha_2(y - y_0) - \beta_2(x - x_0)\big) \le C(\mu + c - P - r_m) + kc(q-1)$$

Dividing the two members by $C = \beta_2\alpha_1 - \alpha_2\beta_1$ and by $-P$, we obtain

$$\frac{r_m - \mu - c + P}{P} + \frac{kc(1-q)}{PC} \le \frac{\beta_2(x - x_0) - \alpha_2(y - y_0)}{\beta_2\alpha_1 - \alpha_2\beta_1} \le \frac{r_m - \mu + P}{P}.$$

This inequality can be expressed in terms of heights by

$$\frac{r_m-\mu-c+P}{P} + \frac{k(1-q)c}{|V_2|H} \le \frac{\epsilon_m h_m}{H} \le \frac{r_m-\mu+P}{P} \tag{9}$$

with $r_m \in [\mu - P, \mu + c - 1]$.
Note that we have

$$\frac{-c}{P} \le \frac{r_m - \mu - c + P}{P} \le \frac{P-1}{P} \quad \forall m \in S'$$

Since $\frac{h_m}{H}$ tends to zero when H tends to infinity, then to get (9) satisfied the value of H should not be very big. □

COROLLARY 1. *Let k be the greater common divisor of characteristics A, B and C. If $q \in \{0, 1\}$, then the number of lines $D_1(V_2)$, that may contain M_1 and needed to recognize the piece, is equal to 1 if $k < P$ and is at most 3 if $k = P$.*

Proof. Indeed, since V_2 is fixed in relation (8), then only C and $k = PGCD(A, B, C)$ depend on M_1. The value C represents the area of the parallelogram generated by the projection of points M, M_1 and V_2 on X, Y coordinates plane. Let suppose that there exists a line $D_{1,q}$ which contain the solution point M_1 of (8). The line $D_{1,q}$ passes generally by the antipode M_1 with respect to the vector V_2 or it comes just after. Value C stay always fixed when M_1 sweeps $D_{1,q}$ since the area of the parallelogram depends only on the the length of V_2 and the height from M to the line $D_{1,q}$. Thus relation (8) depends only on $D_{1,q}$ and the vector V_2.
If for this line, relation (9) is satisfied and $q = 0$ then we had needed only one line for the recognition of S' and if $q = 1$ then M is 1-extérieur to S with respect to the new plane and we apply Debled's algorithm for the case 1-exterior to recognize S'. Thus, in this case, one line $D_{1,1}$ is sufficient.
If relation (9) is not satisfied for the choice of $D_{1,q}$, then we distinguish two cases:

- $k = P$. In this case we can express relation (8) into the following form

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - C(P-1) = Pc(1-q)$$

Since V_2 is a basis vector on S, let $D_{2,q}$ be the first line which comes after $D_{1,q}$ (in the sense that it's more distant from M) and which contains an antipode M_1' of the plane $P(a, b, c, \mu)$. For this new antipode the value C increases by c because M_1', M_1 and V_2 generate a parallelogram of area c. Then (8) becomes

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - (C+c)(P-1) = Pc(1-q) - c(P-1)$$

which is equivalent to

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - (C+c)(P-1) = -c(Pq-1)$$

For $q = 1$ the second member is negative and has not the form $k'c(1-q) > 0$. Even we move the line $D_{2,q}$ the second member of (8) stays always negative. Which implies that S' couldn't be recognized.
For $q = 0$, the second member is equal to c. In this case, let $D_{3,q}$ be the first line after $D_{2,q}$ which contain an antipode and directed by V_2. The point M may become 1-exterior (i.e, $q = 1$) for the new basis computed from $D_{3,q}$. The value $C + c$ increases again by c. Let substitute in (8), we obtain for the new line

$$P[\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)] - (C+2c)(P-1) = c - c(P-1) = c(2-P)$$

If $P \geq 3$, the second member is then negative and the piece is not recognizable.
If $P = 2$, the second member vanishes and will be written under the form $kc(1-q) = 2c(1-1) = 0$.
If relation (9) is satisfied, then the piece S is recognized with M 1-exterior. In this case we apply Debled algorithm to recognize S'.
If (9) is not satisfied then another line $D_{4,q}$ gives a negative second member of relation (8) and the piece S' is not recognized.

- $k < P$. In this case the same work done for $D_{2,q}$ yield the second member of (8) equal to $c\big(k - kq - (P-1)\big)$.
If $q = 0$ then we obtain $k - (P-1)$. Since k divides P then $k < P - 1$ which means that the second member of (8) negative and S' is not recognizable.
If $q = 1$ we obtain $-c(P-1) < 0$ and S' is not recognizable.

Finally, in all cases only three lines are sufficient. $\square$

6. The Simplified Algorithm

The algorithm that we describe in this section directly derives from the discussion and the proofs of the previous results. It uses only the part of Debled's algorithm corresponding to 1-exterior case with small modifications to recognize the general case of rectangular pieces. The complexity of the simplified algorithm decreases to become at

most quadratic in the number of points in the piece.
The algorithm begin by sweeping the piece to be recognized following sections parallel to one coordinate plane, Oxz for instance, by successively adding voxels. At the beginning we initialize $y = 0$ and we let x vary in its interval of definition. At each added voxel the algorithm tries to recognize a piece of a digital straight line. When all values of x are considered we increment y by 1 and we let x sweep again over all its possible values. At each step the algorithm tries to recognize a piece in a digital plane and compute its characteristics. Three cases are possible:

1. If the added point M satisfies the double Diophantine inequality (1) for the plane constructed before adding M, then we keep the same characteristics and the updated piece is still recognized in this plane with the same leaning points with a possible addition of M.

2. If the added point is 1-exterior to the plane, then we apply Debled's algorithm to recognize the new piece. This step consists of computing polygonal pivots and antipodes lines to determine the vector V_2 which satisfies relation (4), and also search an adequate antipode M_1 that satisfies (6)).

3. In the third case, if the added point is strongly exterior, we search the polygonal line of pivots and antipodes and then check relation (8). If this relation is not satisfied for any point M_2 on the polygonal line of pivots with $0 \leq q < P$, then the piece is not recognizable. If point M becomes q-strongly exterior, with $1 < q < P$ then we repeat the process, with $P = q$, until we get $q = 0$ or 1. If relation (8) is satisfied for $q = 0$, then the piece is recognized with M as a leaning point. If relation (8) is satisfied for $q = 1$, then we apply Debled's algorithm for the 1-exterior case at most three times to decide the recognition of the piece. In the first running of the algorithm, point M_1 is taken as in Debled's algorithm. If M becomes 1-exterior, then we apply Debled's algorithm once more and the piece is recognized. At this point, if after one run there are some points which are excluded, then we choose the antipode M_1 on the following line proposed by corollary 1 and we repeat the process. This process is limited to at most three lines fixed by corollary 1.

Examples

EXAMPLE 1. In Figure 9 we represent a recognized piece S of the plane $P(3, 6, 10, 0)$ and the added point $M(0, 2, 0)$ which is 3-exterior of remainder $r(M) = 12$. The polygonal line of pivots contains two points $M_2(0, 0, 0)$ and $M_2'(8, 1, -3)$ which define the vector $V_2(8, 1, -3)$. The point $M_1(3, 0, 0)$ is a separating antipode of the vector V_2. The vector $V_1 = \overrightarrow{MM_1}$ is equal to $(3, -2, 0)$. The vectorial product of V_1 and V_2 is given by $(A, B, C) = (6, 9, 19)$. Here we have $k = PGCD(A, B, C) = 1$. In this case the quantity $P\big(\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)\big) - C(P - 1)$ is equal to $3\big(1(0 - 0) - 8(0 - 2)\big) - 19(3 - 1) = 48 - 38 = 10$. Moreover, for $q = 0$

we obtain $kc(1-q) = 1 * 10 * 1 = 10$. Thus the relation 8 is satisfied for $q = 0$. The value of $\mu(x_0, y_0) + q$ is $-\frac{9}{10}$.So we take $\mu' = [\mu(x_0, y_0)] + 1$. The piece $S' \cup M$ is then recognized into the plane $P(6, 9, 19, 0)$, see Figure 10.

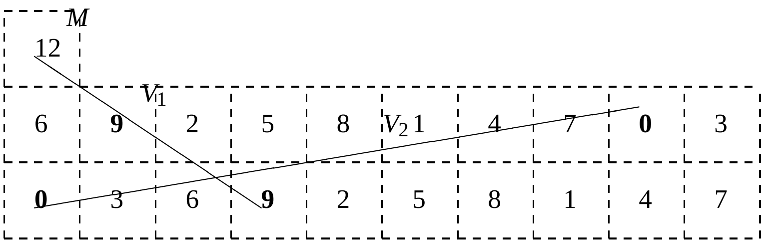


Figure 9: S is a recognized piece in the plane $P(3, 6, 10, 0)$ and M is 3-exterior

18									
9	15	2	8	14	1	7	13	**0**	6
0	6	12	**18**	5	11	17	4	10	16

Figure 10: S' is recognized in the plane $P(6, 9, 19, 0)$ and M is a leaning point

EXAMPLE 2. In Figure 11 we represent a recognized piece S of the plane $P(18, 21, 23, 0)$. The added point $M(3, 5, -7)$ is 2-exterior of remainder $r(M) = 2$. The nearest pivot point to M is $M_2(4, 2, -4)$. The polygonal line of antipodes contains only two points $M_1(0, 0, 0)$ and $M_1'(8, 3, -9)$. The vector V_2 is then $(8, 3, -9)$ and $V_1 = (-3, -5, 7)$. The vectorial product of V_1 et V_2 gives $(A, B, C) = (24, 29, 31)$. With this choice, relation 9 is not satisfied. Take for instance the point $m = (9, 0, -7)$. The quantity $\frac{h_m}{H} = \frac{\beta_2(x-x_0)-\alpha_2(y-y_0)}{C} = \frac{58}{31} = 1,86$

is bigger than $\frac{d_m-\mu+P}{P} = 3/2 = 1,5$. Let take the antipode located on the line directed by V_2 and which comes just after V_1. Let $M"_1(5, -1, -3)$ be a such antipode. The point $M"_1$ is located outside the piece S. In this case the vector V_1 is $\overrightarrow{M"_1M} = (2, -6, 4)$.The vectorial product V_1 with V_2 is $(42, 50, 54) = 2(21, 25, 27)$. Note that $PGCD = k = 2$ divides $P = 2$.

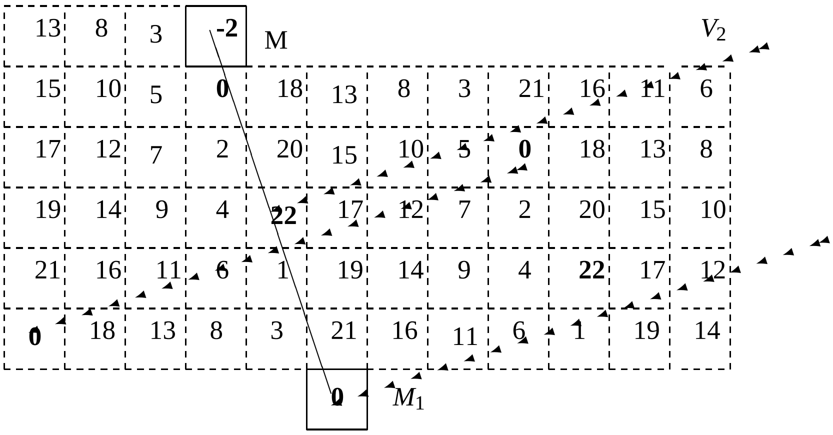


Figure 11:S is a recognized piece in the plane $P(18, 21, 23, 0)$ and M is 2-exterior

The quantity $P\big(\beta_2(x_2 - x_0) - \alpha_2(y_2 - y_0)\big) - C(P - 1)$ is equal to $2\big(3(4 - 3) - 8(2 - 5)\big) - 54(2 - 1) = 54 - 54 = 0$. Moreover, for $q = 1$ we obtain $kc(1 - q) = 0$. Thus relation 8 is satisfied for $q = 1$. We can check that all points of S satisfy the relation 9. For $m(5, 0, -7)$, which did not satisfy (IV') for the first basis, we get $\frac{h_m}{H} = \frac{58}{54} = 1,07$ which is small to $\frac{d_m - \mu + P}{P} = 3/2$ The value of $\mu' = \mu(x_0, y_0) + q$ is $-1 + 1 = 0$. The piece S is then a piece in the plane $P(21, 25, 27, 0)$, see Figure 12.

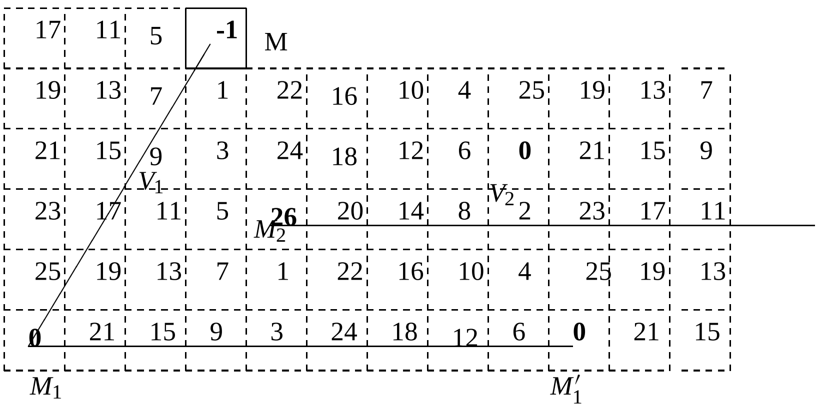


Figure 12: S is recognized in the plane $P(21, 25, 27, 0)$ and M is 1-exterior

Thus we return back to 1-exterior case. The polygonal line of pivots is reduced to the point $M_2(4, 2, -4)$. The polygonal line of antipodes contains two points $M_1(0, 0, 0)$ and $M_1'(9, 0, -7)$. The vector V_2 is then equal to $\overrightarrow{M_1M_1'}(9, 0, -7)$ and $V_1 = \overrightarrow{M_1M} = (-3, -5, 7)$. The vectorial product of V_1 with V_2 gives $(35, 42, 45)$. The piece $S' = S \cup \{M\}$ becomes recognized in the plane $P(35, 42, 45, 0)$ with at least three lower leaning points M, M_1, M_1' and at least one upper leaning point M_2, see Figure 13.

30	20	10	**0**	M							
33	23	13	3	38	28	18	8	43	33	23	13
36	26	16	6	41	31	21	11	1	36	26	16
39	29	19	9	**44**	34	24	14	4	39	29	19
42	32	22	12	2	37	27	17	7	42	32	22
0	35	25	15	5	40	30	20	10	**0**	35	25

Figure 13: S' is recognized in the plane $P(35, 42, 45, 0)$ and M is a leaning point

EXAMPLE 3. In Figure 14 we represent a recognized piece S of the plane $P(4, 8, 13, 0)$. The added point $M(1, 4, -3)$ is 3-exterior of remainder $r(M) = -3$. The nearest pivot point to M is $M_2(1, 1, 0)$. The polygonal line of antipodes contains only two points $M_1(0, 0, 0)$ and $M'_1(7, 3, -4)$. The vector V_2 is then $(7, 3, -4)$ and $V_1 = (-1, -4, 3)$. The vectorial product of V_1 et V_2 gives $(A, B, C) = (16, 35, 52)$ and $\mu' = 0$. With this choice the remainder of the point $m = (0, 3, -1)$ is 53 and therefore it is excluded from by this new plane. The axial rotation on the line L_a containing the antipode $M_1(2, -1, 0)$ and directed by the vector V_2 gives the characteristics $(A, B, C) = (11, 25, 38)$ and $\mu' = 0$ and in this case the remainder of the point $m' = (8, 1, -3)$ is -1. Thus m' is excluded by this rotation. We move the line L_a to another line parallel to its and containing the point $M_1(4, -2, 0)$. The rotation corresponding to this new point gives $(A, B, C) = (5, 11, 17)$. In this case there is no point excluded. The previous points of coordinates $m = (0, 3, -1)$ and $m' = (8, 1, -3)$ become leaning points of remainders 16 and 0 respectively. The added point M becomes 2-exterior with remainder -2, see Figure 15. In this case the nearest upper leaning point to M is m and the new axis L_a of the rotation becomes the line which contains $M_1(0, 0, 0)$ and m'. The corresponding vector V_2 is $(8, 1, -3)$ and $V_1 = (-1, -4, 3)$. Their vectorial product gives $(A, B, C) = (9, 21, 31)$ and $\mu' = 0$. But with this rotation the point m becomes 2-exterior of remainder 2. There are no others rotations which can reach M without excluding m or m'. Thus the piece S' is not recognizable.

6	**-3**							
11	2	6	10	1	5	9	**0**	4
3	7	11	2	6	10	1	5	9
8	**12**	3	7	11	2	6	10	1
0	4	8	**12**	3	7	11	2	6
		0						

Figure 14: S is a recognized piece in the plane $P(4, 8, 13, 0)$and M is 3-exterior

10	**-2**							
16	4	9	14	2	7	12	**0**	5
5	10	15	3	8	13	1	6	11
11	**16**	4	9	14	2	7	12	**0**
0	5	10	15	3	8	13	1	6

Figure 15: The plane $P(5, 11, 17, 0)$ is the extremal plane that contains S with M 2-exterior

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