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QUASICLASSICAL ANALYSIS OF HYPOELLIPTIC OPERATORS

Abstract. In this paper we take into account the hypoelliptic operators introduced by Hörmander in [3] and we develop their quasiclassical analysis, obtaining an asymptotic formula for their counting function, as the Planck constant ϵ goes to 0, under somewhat more general hypotheses than Hörmander's ones.

1. Introduction

At the beginning of our work we give a brief sketch of the physical motivations underlying the mathematical branch known as *quasiclassical analysis*, without having the pretension of being totally exhaustive. For further details we refer to [7], [8] and [9] among others.

In Classical Mechanics it is well known that a particle of mass m can be described by a curve $t \to (x(t), \xi(t))$ (called the *classical trajectory*) such that, in the case the particle is subject to a force field $F = -\operatorname{grad} V$, the so called *Hamiltonian equations* are fulfilled:

$$\begin{cases} \dot{\xi}(t) = -\operatorname{grad} V(x(t)) \\ \dot{x}(t) = \frac{1}{m}\xi(t), \end{cases}$$

where $x(t)$ is the *position* of the particle at any time t and $\xi(t) = m\dot{x}(t)$ is the *momentum* of the particle.

A *classical observable* is any real smooth function $h(x, \xi)$ defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$: its value at the point $(x(t), \xi(t))$ gives an information on the position of the particle at time t. An example of classical observable is the *total energy* $\frac{1}{2m}\xi^2 + V(x)$.

In Quantum Mechanics a particle is instead described by a function $\psi(t, x)$ (called the *wave function*) such that for all $t \in \mathbb{R}$

$$\int_{\mathbb{R}^n} |\psi(t, x)|^2 \, dx = 1,$$

that is $\psi_t : x \mapsto \psi(t, x)$ belongs to $L^2(\mathbb{R}^n)$ and $\|\psi_t\|_{L^2(\mathbb{R}^n)} = 1$. The function ψ_t is called the *state* of the particle at time t.

The classical quantities of position x and momentum ξ can be investigated in Quantum Mechanics by considering the following two operators:

1. the *multiplication operator*
$$x : \psi \to x\psi$$
for the position;
2. the *derivation operator*
$$\epsilon D_x : \psi \to -i\epsilon \operatorname{grad} \psi$$
for the momentum,

where ϵ is the *Planck constant*. *

Note that these operators are selfadjoint with respect to the L^2 scalar product. More generally, any selfadjoint operator on L^2 is called a *quantum observable*.

The question about the correspondence between classical observable $h(x, \xi)$ and quantum observable $h(x, \epsilon D_x)$ is called the problem of *quantization* and one of the purposes of pseudodifferential calculus is just to provide a correspondence between the space of all classical observables endowed with the usual multiplication and the space of quantum observables endowed with the composition of operators.

For example, to the classical total energy $\frac{1}{2m}\xi^2 + V(x)$ we associate the operator

$$-\frac{\epsilon^2}{2m}\Delta + V(x)$$

(where Δ is the standard Laplacian) which is the *Schrödinger operator*.

Actually, it is convenient to consider the so called *Weyl quantization*:

$$h^w(x, \epsilon^2\xi)u(x) = \int e^{i(x-y)\cdot\xi} h\left(\frac{x+y}{2}, \epsilon^2\xi\right) u(y)\, dy\, đ\xi \tag{1}$$

or the unitarily equivalent form

$$h^w(\epsilon x, \epsilon\xi)u(x) = \int e^{i(x-y)\cdot\xi} h\left(\epsilon\frac{x+y}{2}, \epsilon\xi\right) u(y)\, dy\, đ\xi. \tag{2}$$

(for suitable functions u and h) since these operators are symplectically invariant, as remarked at the end of the paper, and are selfadjoint if and only if the symbol h is real valued.

Since Classical Mechanics (which is much simpler than Quantum Mechanics) describes quite well most of common elementary physical phenomena, we expect that Quantum Mechanics is a kind of generalization of Classical Mechanics, in the sense that one could recapture the classical properties of a system by making some approximation of its quantum properties. This is the so called *Bohr correspondence principle*: Classical Mechanics is nothing but the limits of Quantum Mechanics as the Planck constant ϵ tends to 0. This motivates an interest in the asymptotic properties of Weyl

*In order to be consistent with the notations used in [13] here we denote the Planck constant by ϵ and not by h, since we use h to denote our operators.

operators (1) or (2) when $\epsilon \to 0$. The corresponding asymptotic analysis is called the *quasiclassical analysis*.

In our work we consider the Weyl operators already considered by Hörmander in [3] for which he obtained an asymptotic formula for their counting function $\mathcal{N}(\tau)$ as $\tau \to +\infty$ (see Section 2). Then we consider the corresponding quantum observables $h^w(\epsilon x, \epsilon\xi)$ and we obtain a quasiclassical asymptotic formula for the associated counting function $\mathcal{N}_\epsilon(\tau)$ as $\epsilon \to 0$, under somewhat more general hypotheses than Hörmander's ones.

We employ the following notation: given two functions $f, g : X \to \mathbb{R}$, and a subset $A \subset X$, we write

$$f(x) \prec g(x), \quad \forall x \in A,$$

if there exists a constant C such that

$$f(x) \leq Cg(x), \quad \forall x \in A.$$

I would like to acknowledge professor Buzano for his precious suggestions while writing this paper.

2. Basic definitions and results

First we recall some definitions and results from [2], [4] and [13].

Let us begin with a brief review of Weyl-Hörmander calculus.

Let

$$\phi(x, \xi; y, \eta) = \sum_{j=1}^{n} \xi_j y_j - x_j \eta_j$$

be the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$.

DEFINITION 1. *A Riemannian metric $g_{x,\xi}(y, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is* slowly varying *if there exist a positive real number r such that*

$$g_{x,\xi}(y, \eta) \prec g_{t,\tau}(y, \eta) \prec g_{x,\xi}(y, \eta),$$

for all $(x, \xi), (t, \tau), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$g_{x,\xi}(x - t, \xi - \tau) \leq r.$$

DEFINITION 2. *A positive function $m(x, \xi)$ is said to be g* continuous *if there is a positive real number r such that*

$$m(y, \eta) \prec m(x, \xi) \prec m(y, \eta),$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$g_{x,\xi}(x - y, \xi - \eta) \leq r.$$

DEFINITION 3. *A Riemannian metric* $g_{x,\xi}(y,\eta)$ *on* $\mathbb{R}^n \times \mathbb{R}^n$ *is* locally ϕ temperate *if it is slowly varying and there exist two positive real numbers* r *and* N *and a slowly varying metric* $G_x(y)$ *on* $\mathbb{R}^n$ *such that*

$$G_x(y) \le g_{x,\xi}(y,\eta),$$

for all $(x,\xi), (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$*, and*

$$g_{x,\xi}(z,\zeta) \prec g_{y,\eta}(z,\zeta)(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N, \tag{3}$$

for all $(x,\xi), (y,\eta), (z,\zeta) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$G_x(x-y) \le r.$$

The quadratic form $g^{\phi}_{x,\xi}(y,\eta)$ appearing in (3) is the *dual metric*

$$g^{\phi}_{x,\xi}(y,\eta) = \sup\{(\phi(y,\eta;z,\zeta))^2 : g_{x,\xi}(z,\zeta) = 1\}.$$

DEFINITION 4. *A positive function* $m(x,\xi)$ *is* locally ϕ, g temperate *if it is* g *continuous and there exist two positive real numbers* r *and* N *such that*

$$m(x,\xi) \prec m(y,\eta)(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,$$

for all $(x,\xi), (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$G_x(x-y) \le r.$$

Now we recall the definition of the class of symbols of Weyl-Hörmander $S(m,g)$.

DEFINITION 5. *Let* g *be a slowly varying Riemannian metric. Let* m *be a* g *continuous function. Then we say that a function* $h \in S(m,g)$ *if it is smooth and*

$$\sup_{x,\xi} \frac{|h|^g_k(x,\xi)}{m(x,\xi)} < +\infty, \quad \forall k \in \mathbb{N},$$

where

$$|h|^g_k(x,\xi) = \sup_{y_j,\eta_j} \frac{|h^{(k)}((x,\xi);(y_1,\eta_1),\ldots,(y_k,\eta_k))|}{\prod_{j=1}^k g_{x,\xi}(y_j,\eta_j)^{1/2}}$$

and $h^{(k)}((x,\xi);(y_1,\eta_1),\ldots,(y_k,\eta_k))$ *is the k-th differential of* h *at* (x,ξ).

We can introduce here the operators we deal with in the next section.

DEFINITION 6. *A differential operator* h^w *is* formally hypoelliptic† *if its Weyl symbol* $h(x,\xi)$ *satisfies the following conditions:*

†This definition is not to be confused with Definition 2.3 of Chapter III of [10]

1. *h is a smooth function such that*

$$h(x,\xi) \neq 0, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n;$$

2. *there exists a locally ϕ temperate metric $g_{x,\xi}(y,\eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $|h|$ is locally ϕ, g temperate and*

$$h \in S(|h|, g).$$

In the proof of our main theorem (see Theorem 5) we will make use of the following two results.

THEOREM 1. *Consider a positive and formally hypoelliptic symbol $h \in S(h, g)$ and assume that there exists a positive real number γ such that*

$$g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta),$$

for all (x,ξ), $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, then h^w is semi-bounded from below and essentially self-adjoint in $L^2(\mathbb{R}^n)$.

Moreover, if

$$h(x,\xi) \to +\infty, \qquad \text{as } |x| + |\xi| \to +\infty, \tag{4}$$

then the closure H of h^w in $L^2(\mathbb{R}^n)$ has discrete spectrum diverging to $+\infty$.

Proof. This is Theorem 1 of [13]. Actually we must observe that in the proof of Theorem 1 of [13] we did not assume that the metric g should satisfy the so called *principle of indetermination*, that is

$$\sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq 1.$$

In our case, from the fact that there exists $\gamma > 0$ such that

$$g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta)$$

and that $h(x,\xi) \to +\infty$ as $|x| + |\xi| \to +\infty$, it follows that

$$\sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq C$$

for a suitable constant $C > 0$. If $0 < C \leq 1$ then the principle of indetermination is trivially satisfied. If $C > 1$ it is sufficient to replace g with the following new metric

$$\tilde{g} = \frac{g}{\sqrt{C}}.$$

It is easy to show that $\tilde{g}^{\phi} = \sqrt{C} g^{\phi}$. Therefore

$$\sup_{x,\xi} \frac{\tilde{g}_{x,\xi}}{\tilde{g}^{\phi}_{x,\xi}} = \sup_{x,\xi} \frac{\frac{g_{x,\xi}}{\sqrt{C}}}{\sqrt{C} g^{\phi}_{x,\xi}} = \frac{1}{C} \sup_{x,\xi} \frac{g_{x,\xi}}{g^{\phi}_{x,\xi}} \leq 1.$$

□

Thanks to Theorem 1, we can define the *counting function* of the operator H:

(5) $\mathcal{N}(\tau) =$ number of eigenfunctions of H, corresponding to eigenvalues less than or equal to τ.

THEOREM 2. *Under the hypotheses of Theorem 1 assume there exists* $\kappa > 0$ *such that*

$$h^{-\kappa} \in L^1. \tag{6}$$

Then for all $0 < \delta < \frac{\gamma}{3}$ *we have*

$$\mathcal{N}(\tau) = \mathcal{W}(\tau)\left\{1 + O\big(\mathcal{R}(\tau)\big)\right\}, \qquad as\ \tau \to +\infty, \tag{7}$$

where

$$\mathcal{W}(\tau) = (2\pi)^{-n} \int\limits_{h \leq \tau} dx\, d\xi, \tag{8}$$

and

$$\mathcal{R}(\tau) = \frac{\mathcal{W}\left(\tau + \tau^{1-\delta}\right) - \mathcal{W}\left(\tau - \tau^{1-\delta}\right)}{\mathcal{W}(\tau)}. \tag{9}$$

REMARK 1. Estimate (7) is known as *Weyl formula.*

Proof. This is Theorem 2 of [13].

□

3. Quasiclassical Analysis of Hypoelliptic Operators

Consider a formally hypoelliptic operator h^w.

Let us introduce the operator h_ϵ^w whose Weyl symbol is

$$h_\epsilon(x, \xi) = h(\epsilon x, \epsilon \xi),$$

where ϵ is a real parameter, such that $0 < \epsilon \leq 1$.

Then we define a new Riemannian metric ${}^\epsilon g_{x,\xi}(y, \eta)$ in this way:

$${}^\epsilon g_{x,\xi}(y, \eta) = g_{\epsilon x, \epsilon \xi}(\epsilon y, \epsilon \eta) = \epsilon^2 g_{\epsilon x, \epsilon \xi}(y, \eta),$$

where $g_{x,\xi}(y, \eta)$ is the one appearing in Definition 6. We start by giving some results concerning this new metric ${}^\epsilon g$.

PROPOSITION 1. *The metric* ${}^\epsilon g_{x,\xi}(y, \eta)$ *is slowly varying and locally* ϕ *temperate, for all* $0 < \epsilon \leq 1$.

Proof. We know that g is slowly varying. Then there exists a positive real number r such that

$$g_{x,\xi}(t,\tau) \prec g_{y,\eta}(t,\tau) \prec g_{x,\xi}(t,\tau)$$

for all (x,ξ), (y,η), $(t,\tau) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $g_{x,\xi}(x-y,\xi-\eta) \leq r$. Therefore

$$g_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau) \prec g_{\epsilon y,\epsilon\eta}(\epsilon t,\epsilon\tau) \prec g_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau)$$

when $g_{\epsilon x,\epsilon\xi}(\epsilon(x-y),\epsilon(\xi-\eta)) \leq r$, that is

$${}^{\epsilon}g_{x,\xi}(t,\tau) \prec {}^{\epsilon}g_{y,\eta}(t,\tau) \prec {}^{\epsilon}g_{x,\xi}(t,\tau)$$

when ${}^{\epsilon}g_{x,\xi}(x-y,\xi-\eta) \leq r$ and so we have proved that ${}^{\epsilon}g$ is slowly varying, for all $0 < \epsilon \leq 1$.

Let us define a new metric

$${}^{\epsilon}G_x(y) = G_{\epsilon x}(\epsilon y).$$

Therefore it is obvious that

$${}^{\epsilon}G_x \leq {}^{\epsilon}g_{x,\xi}.$$

By hypothesis we know that the metric g is locally ϕ temperate. Therefore there exist two positive real numbers $r > 0$ and N such that

$$g_{x,\xi}(t,\tau) \prec g_{y,\eta}(t,\tau)(1 + g^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,$$

when $G_x(x-y) \leq r$. Therefore

$$g_{\epsilon x,\epsilon\xi}(t,\tau) \prec g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon(x-y),\epsilon(\xi-\eta)))^N,$$

when $G_{\epsilon x}(\epsilon(x-y)) \leq r$, for all $0 < \epsilon \leq 1$, that is when ${}^{\epsilon}G_x(x-y) \leq r$. Now, we have that

$$\begin{aligned}
g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon t,\epsilon\tau) &= \sup_{(y,\eta)\neq 0} \frac{\phi(\epsilon t,\epsilon\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \sup_{(y,\eta)\neq 0} \frac{\epsilon^2\phi(t,\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \epsilon^4 \sup_{(y,\eta)\neq 0} \frac{\phi(t,\tau;y,\eta)^2}{\epsilon^2 g_{\epsilon x,\epsilon\xi}(y,\eta)} = \\
&= \epsilon^4 \sup_{(y,\eta)\neq 0} \frac{\phi(t,\tau;y,\eta)^2}{g_{\epsilon x,\epsilon\xi}(\epsilon y,\epsilon\eta)} = \\
&= \epsilon^4 ({}^{\epsilon}g)^{\phi}_{x,\xi}(t,\tau).
\end{aligned}$$

Thus we can conclude that

$$\begin{aligned}
\epsilon^2 g_{\epsilon x,\epsilon\xi}(t,\tau) &\prec \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon x-\epsilon y,\epsilon\xi-\epsilon\eta))^N \prec \\
&\prec \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + \frac{1}{\epsilon^4}\cdot g^{\phi}_{\epsilon x,\epsilon\xi}(\epsilon x-\epsilon y,\epsilon\xi-\epsilon\eta))^N = \\
&= \epsilon^2 g_{\epsilon y,\epsilon\eta}(t,\tau)(1 + \frac{1}{\epsilon^4}\cdot \epsilon^4 ({}^{\epsilon}g)^{\phi}_{x,\xi}(x-y,\xi-\eta))^N,
\end{aligned}$$

when ${}^{\epsilon}G_x(x-y) \le r$, that is

$$ {}^{\epsilon}g_{x,\xi}(t,\tau) \prec {}^{\epsilon}g_{y,\eta}(t,\tau)(1+({}^{\epsilon}g)^{\phi}_{x,\xi}(x-y,\xi-\eta))^N, $$

when ${}^{\epsilon}G_x(x-y) \le r$, for all $0<\epsilon\le 1$.

□

PROPOSITION 2. *We have that*

$$ h_\epsilon \in S(|h_\epsilon|, {}^{\epsilon}g), $$

uniformly with respect to $0<\epsilon\le 1$.

Proof. Using Definition 5 we have that, for all $k\in\mathbb{N}$,

$$ \begin{aligned} \sup_{x,\xi} \frac{|h_\epsilon|_k^{{}^{\epsilon}g}(x,\xi)}{|h_\epsilon(x,\xi)|} &= \sup_{x,\xi} \frac{\sup_{y_j,\eta_j} \frac{|h_\epsilon^{(k)}((x,\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\prod_{j=1}^k {}^{\epsilon}g_{x,\xi}(y_j,\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi} \frac{\sup_{y_j,\eta_j} \frac{|h^{(k)}((\epsilon x,\epsilon\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\prod_{j=1}^k g_{\epsilon x,\epsilon\xi}(\epsilon y_j,\epsilon\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi} \frac{\sup_{y_j,\eta_j} \frac{\epsilon^k|h^{(k)}((\epsilon x,\epsilon\xi);(y_1,\eta_1),\dots,(y_k,\eta_k))|}{\epsilon^k\prod_{j=1}^k g_{\epsilon x,\epsilon\xi}(y_j,\eta_j)^{1/2}}}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi} \frac{|h|_k^g(\epsilon x,\epsilon\xi)}{|h_\epsilon(x,\xi)|} = \\ &= \sup_{x,\xi} \frac{|h|_k^g(\epsilon x,\epsilon\xi)}{|h(\epsilon x,\epsilon\xi)|} = \sup_{x,\xi} \frac{|h|_k^g(x,\xi)}{|h(x,\xi)|} < +\infty, \end{aligned} $$

uniformly with respect to $0<\epsilon\le 1$, since $h\in S(|h|,g)$ by hypothesis.

□

We can now state the following proposition:

PROPOSITION 3. *Let us suppose that there exists a real number* $\gamma>0$ *such that*

$$ g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta), \tag{10} $$

for all $(x,\xi),(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$*. Then we have that*

$$ {}^{\epsilon}g_{x,\xi}(y,\eta) \prec (h_\epsilon(x,\xi)\epsilon^{-\frac{4}{\gamma}})^{-\gamma}({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta), $$

for all $(x,\xi),(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$*, for all* $0<\epsilon\le 1$.

Proof. Recalling that

$$({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta) = \epsilon^{-2} g^{\phi}_{\epsilon x,\epsilon\xi}(y,\eta),$$

and using (10) we have:

$$\begin{aligned}\frac{{}^{\epsilon}g_{x,\xi}(y,\eta)}{({}^{\epsilon}g)^{\phi}_{x,\xi}(y,\eta)} &= \frac{\epsilon^2 g_{\epsilon x,\epsilon\xi}(y,\eta)}{\epsilon^{-2} g^{\phi}_{\epsilon x,\epsilon\xi}(y,\eta)} \prec \\ &\prec h(\epsilon x,\epsilon\xi)^{-\gamma}\epsilon^4 = \\ &= (h_\epsilon(x,\xi)\cdot\epsilon^{-\frac{4}{\gamma}})^{-\gamma}.\end{aligned}$$

□

Due to this proposition, from now on we will work with the following symbol:

$$H_\epsilon(x,\xi) = \epsilon^{-\frac{4}{\gamma}} h_\epsilon(x,\xi).$$

Now we formulate Theorem 1 in this new context:

THEOREM 3. *Consider a positive and formally hypoelliptic symbol* $h \in S(h, g)$ *and assume that there exists a positive real number* γ *such that* (10) *is satisfied and* $h(x,\xi) \to +\infty$ *as* $|x| + |\xi| \to +\infty$.

Then the operator $H^w_\epsilon(x,\xi)$*, corresponding to the new symbol* $H_\epsilon(x,\xi)$*, is semi-bounded from below and essentially self-adjoint in* $L^2(\mathbb{R}^n)$ *and its closure has discrete spectrum diverging to* $+\infty$.

Proof. This is an immediate consequence of Theorem 1 .

□

REMARK 2. Thanks to Theorem 3 we can define the *counting function* of the closure of the operator H^w_ϵ:

$\mathcal{N}_{H_\epsilon}(\tau) =$ number of eigenfunctions of the closure of H^w_ϵ, corresponding to eigenvalues less than or equal to τ.

Before claiming our results, we have to state the following theorem, which is a direct consequence of Theorem 2.

THEOREM 4. *Under the hypotheses of Theorem 1, assume that there exists* $k > 0$ *such that*

$$h^{-k} \in L^1(\mathbb{R}^{2n}).$$

Then for all $0 < \delta < \frac{\gamma}{3}$ *we have*

$$\mathcal{N}_{H_\epsilon}(\tau) = \mathcal{W}_\epsilon(\tau)\{1 + O(\mathcal{R}_\epsilon(\tau))\},$$

as $\tau \to +\infty$, *uniformly with respect to* $0 < \epsilon \leq 1$, *where*

$$\mathcal{W}_\epsilon(\tau) = (2\pi)^{-n} \iint_{H_\epsilon(x,\xi)\leq\tau} dx\, d\xi$$

and

$$\mathcal{R}_\epsilon(\tau) = \frac{\mathcal{W}_\epsilon(\tau + \tau^{1-\delta}) - \mathcal{W}_\epsilon(\tau - \tau^{1-\delta})}{\mathcal{W}_\epsilon(\tau)}.$$

Proof. By means of a change of coordinates we immediately obtain that

$$\begin{aligned} \|H_\epsilon^{-k}\|_{L^1} &= \epsilon^{\frac{4k}{\gamma}} \int |h_\epsilon(x,\xi)|^{-k}\, dx\, d\xi = \\ &= \epsilon^{\frac{4k}{\gamma}} \int h(\epsilon x, \epsilon\xi)^{-k}\, dx\, d\xi = \\ &= \epsilon^{\frac{4k}{\gamma}-2n} \int h(x,\xi)^{-k}\, dx\, d\xi \leq \\ &\leq \int h(x,\xi)^{-k}\, dx\, d\xi = \|h^{-k}\|_{L^1}, \end{aligned}$$

if we take $\frac{4k}{\gamma} - 2n \geq 0$, that is

$$\gamma \leq \frac{2k}{n}.$$

Therefore we obtain that the integrability of the symbol $h(x,\xi)$ implies the integrability of the new symbol $H_\epsilon(x,\xi)$ and that the L^1 norm of H_ϵ^{-k} is uniformly bounded with respect to $0 < \epsilon \leq 1$. The rest of the proof is an immediate consequence of Theorem 2, Proposition 1 and Proposition 2.

□

Now we can state and prove our main result.

THEOREM 5. *Let* $\mathcal{N}_\epsilon(\tau)$ *be the counting function associated to the operator* h_ϵ^w. *Let*

$$\mathcal{W}(\tau) = (2\pi)^{-n} \iint_{h(x,\xi)\leq\tau} dx\, d\xi.$$

Let τ *be not a critical value of* $h(x,\xi)$, *that is* grad $h(x,\xi) \neq 0$ *on the surface* $\{(x,\xi) : h(x,\xi) = \tau\}$. *Under the hypotheses of Theorem 4, we have the following asymptotic formula for* $\mathcal{N}_\epsilon(\tau)$*:*

$$\mathcal{N}_\epsilon(\tau) = \epsilon^{-2n}(\mathcal{W}(\tau) + O(\epsilon^\theta)), \tag{11}$$

as $\epsilon \to 0$, *for all* $0 < \theta < \frac{4}{3}$.

COROLLARY 1. *The asymptotic formula* (11) *is valid when* τ *belongs to the complementary of a set of zero measure.*

Proof of Corollary 1. We know from Theorem 5 that in the asymptotic formula (11) we must exclude the critical values of $h(x,\xi)$. Corollary 1 is an immediate consequence of Sard Theorem, according to which the set of the critical values of $h(x,\xi)$ has zero measure.

□

Proof of Theorem 5. Since $\mathcal{N}_\epsilon(\tau)$ is the counting function associated to the operator h^w_ϵ, then it is clear that H^w_ϵ has exactly $\mathcal{N}_\epsilon(\epsilon^{\frac{4}{\gamma}}\tau)$ eigenvalues less than or equal to τ and that

$$(2\pi)^{-n} \iint\limits_{h(\epsilon x,\epsilon\xi)\epsilon^{-\frac{4}{\gamma}}\le\tau} dx\, d\xi = \epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{4}{\gamma}}\tau).$$

Thanks to Theorem 4, we obtain that for all $0<\delta<\frac{\gamma}{3}$ there exists a real number $C_\delta>0$ such that

$$\begin{aligned}|\mathcal{N}_\epsilon(\epsilon^{\frac{4}{\gamma}}\tau)-\epsilon^{-2n}\mathcal{W}(\epsilon^{\frac{4}{\gamma}}\tau)| &\le \\ &\le C_\delta\epsilon^{-2n}(\mathcal{W}(\epsilon^{\frac{4}{\gamma}}(\tau+\tau^{1-\delta})-\mathcal{W}(\epsilon^{\frac{4}{\gamma}}(\tau-\tau^{1-\delta})),\end{aligned}$$

as $\tau\to+\infty$, for all $0<\epsilon\le 1$.

Letting $\epsilon^{\frac{4}{\gamma}}\tau=\lambda$ we obtain:

$$\begin{aligned}|\mathcal{N}_\epsilon(\lambda)-\epsilon^{-2n}\mathcal{W}(\lambda)| &\le \\ &\le C_\delta\epsilon^{-2n}(\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))-\mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))\end{aligned} \tag{12}$$

for $0<\delta<\frac{\gamma}{3}$ and $\epsilon\to 0$.

From [9], 28.7, we know that when τ is not a critical value of $h(x,\xi)$ then $\mathcal{W}(\tau)$ is differentiable with

$$\mathcal{W}'(\tau)=(2\pi)^{-n}\int\limits_{V(\tau)}\frac{dS}{|\operatorname{grad} h|}$$

where

$$V(\tau)=\{(x,\xi): h(x,\xi)=\tau\}$$

and dS is the area element of the surface $V(\tau)$. Therefore when λ is not a critical value of h, using Taylor's formula, we can state that

$$\mathcal{W}(\lambda+\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}})=\mathcal{W}(\lambda)+\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1))$$

and

$$\mathcal{W}(\lambda-\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}})=\mathcal{W}(\lambda)-\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1)),$$

as $\epsilon\to 0$.

Thus

$$\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))-\mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta}))=2\lambda^{1-\delta}\epsilon^{\frac{4\delta}{\gamma}}(\mathcal{W}'(\lambda)+o(1)),$$

as $\epsilon \to 0$, and then

$$\mathcal{W}(\lambda(1+\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta})) - \mathcal{W}(\lambda(1-\epsilon^{\frac{4\delta}{\gamma}}\lambda^{-\delta})) = O(\epsilon^{\frac{4\delta}{\gamma}}), \tag{13}$$

as $\epsilon \to 0$.

Since $0 < \delta < \frac{\gamma}{3}$, using (12) and (13) we conclude that

$$\mathcal{N}_\epsilon(\lambda) = \epsilon^{-2n}(\mathcal{W}(\lambda) + O(\epsilon^\theta)),$$

as $\epsilon \to 0$, for all $0 < \theta < \frac{4}{3}$.

□

REMARK 3. As pointed out in the introduction, quasiclassical operators could be also in the form $h^w(x, \epsilon^2\xi)$. In this case they are also called *semiclassical operators* and ϵ^2 is the Planck constant. Everything we obtained about quasiclassical operators is true also for semiclassical operators. Indeed the property of being ϕ temperate is invariant under *symplectic* changes of coordinates and the two transformations $(x, \xi) \to (\epsilon x, \epsilon\xi)$ and $(x, \xi) \to (x, \epsilon^2\xi)$ are symplectically equivalent. It is enough to consider the map

$$\begin{aligned} T : \mathbb{R}^{2n} &\to \mathbb{R}^{2n} \\ (x, \xi) &\mapsto T(x, \xi) = (\epsilon x, \epsilon^{-1}\xi). \end{aligned}$$

For further details see [6], Lemma 18.5.8. In the case of semiclassical operators we can actually keep the old metric: ${}^\epsilon G_x(y) = G_x(y)$.

4. An example

At the end of our paper we give an example of hypoelliptic operator satisfying our hypotheses.

PROPOSITION 4. *Let us consider*

- *a real polynomial $p(\xi)$ vanishing at the origin and hypoelliptic, i. e. such that*

$$\lim_{|\xi| \to +\infty} p(\xi) = +\infty$$

  and

$$\nabla p(\xi) \prec p(\xi)^{1-\rho}, \qquad \textit{for } p(\xi) \geq 1,$$

  with $0 < \rho \leq 1$,

- *a positive smooth function $q(x)$ such that*
  - *$h(x, \xi) = p(\xi) + q(x) > 0, \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$,*
  - *there exists $0 \leq \nu < \rho$ such that for all $\alpha \in \mathbb{N}^n$ we have $D^\alpha q(x) \prec q(x)^{1+\nu|\alpha|}, \forall x \in \mathbb{R}^n$,*

– *there exists $k_1 > 0$ such that $q(x)^{-k_1} \in L^1(\mathbb{R}^n)$.*

Then $h^w_\epsilon(x,\xi) = p^w_\epsilon(\xi) + q(\epsilon x)$ satisfies the hypotheses of Theorem 5.

Proof. We give a brief sketch of the proof. First we prove that there exists $k > 0$ such that

$$h^{-k} \in L^1(\mathbb{R}^{2n}). \tag{14}$$

From hypotheses of Proposition 4, it is clear that there exists $R > 0$ such that

$$\sqrt{p(\xi)q(x)} \le p(\xi) + q(x), \quad \forall \xi \notin B(R), \quad \forall x \in \mathbb{R}^n,$$

where $B(R)$ is the open ball with center at the origin and radius R. Therefore, in order to verify (14), it suffices to show that there exists $k_2 > 0$ such that $p^{-k_2} \in L^1(\mathbb{R}^n \setminus B(R))$. We have that $p \to +\infty$, as $|\xi| \to +\infty$. Then, because p is a polynomial, it follows from Tarski-Seidenberg Theorem that there exists $k_0 > 0$ such that

$$p(\xi) \succ |\xi|^{k_0}, \qquad \forall\, |\xi| \ge R. \tag{15}$$

In fact,

$$E = \left\{(s,t,\xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : s = |\xi|^2\,,\ p(\xi) = t\right\}$$

is algebraic in $\mathbb{R}^{2+n}$, and therefore

$$f(s) = \inf_{|\xi|^2 = s} p(\xi) = \inf\left\{t : \exists s\, \exists \xi\, \left((s,t,\xi) \in E\right)\right\} < +\infty$$

is semi-algebraic. Moreover $f(s) \to +\infty$, as $s \to +\infty$; hence (15) follows from Corollary A.2.6 of [5].

It is clear that estimate (15) implies the existence of κ_2.

Then it suffices to verify the hypotheses of Theorem 3. Consider the metric

$$g_{x,\xi}(y,\eta) = h(x,\xi)^{2\nu}|y|^2 + h(x,\xi)^{-2\rho}|\eta|^2.$$

Then one verifies that g and h are locally ϕ temperate, for example see [12]. Moreover one can show that $h \to +\infty$, as $|x| + |\xi| \to +\infty$, $h \in S(h,g)$ and (10) is verified. We omit the details.

□

For example we can take (see [1], section 1.1)

$$p(\xi) = \left(\xi_1^{2k} - \xi_2^{2k-1}\right)^2 + \xi_1^{2k}\xi_2^{2k-2},$$
$$q(x) = \exp\left(x_1^{2m_1} + x_2^{2m_2}\right),$$

where k is an integer greater than 1 and m_1 and m_2 are positive integers.

References

[1] BOGGIATTO P., BUZANO E. AND RODINO L., *Global hypoellipticity and spectral theory*, vol. 92, Akademie Verlag, Berlin 1996.

[2] DENCKER N., *The Weyl calculus with locally temperate metrics and weights,* Archiv för Math. **24** (1986), 59–79.

[3] HÖRMANDER L., *On the asymptotic distribution of the eigenvalues of pseudodifferential operators in* $\mathbb{R}^n$, Archiv för Math. **17** (1979), 297–313.

[4] HÖRMANDER L., *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[5] HÖRMANDER L., *The analysis of linear partial differential operators II*, Grundleheren der mathematischen Wissenschaften, vol. 257, Springer-Verlag, Berlin 1983.

[6] HÖRMANDER L., *The analysis of linear partial differential operators III*, Grundleheren der mathematischen Wissenschaften, vol. 274, Springer-Verlag, Berlin 1985.

[7] MARTINEZ A., *An Introduction to Semiclassical Analysis*, 102 pages, January 1998.

[8] ROBERT D., *Autour de l'Approximation Semi-Classique*, Progress in Mathematics, vol. 68, Birkhäuser, Boston,Basel,Stuttgart 1987.

[9] SHUBIN M.A., *Pseudodifferential operators and spectral theory*, second edition, Springer-Verlag, Berlin 2001.

[10] TAYLOR M.E., *Pseudodifferential Operators*, vol. 34, Princeton University Press, Princeton, New Jersey, 1981.

[11] ZIGGIOTO A., *Comportamento asintotico dello spettro di operatori multi-quasi-ellittici di tipo Schrödinger*, Boll. U.M.I. **IV-A** (2001), no. 8, 567–569.

[12] ZIGGIOTO A. AND BUZANO E., *Weyl formula for multi-quasi-elliptic operators of Schrödinger type*, Annali di Matematica Pura e Applicata (2001), no. 180, 223–243.

[13] ZIGGIOTO A. AND BUZANO E., *Weyl formula for hypoelliptic operators of Schrödinger type*, Proc. Amer. Math. Soc., **131** (1), 265–274.

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