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F. Messina

LOCAL SOLVABILITY FOR SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF CONSTANT STRENGTH

Abstract. The main goal of the present paper is to study the local solvability of semilnear partial differential operators of the form

$$F(u) = P(D)u + f(x, Q_1(D)u, ..., Q_M(D)u),$$

where $P(D), Q_1(D), ..., Q_M(D)$ are linear partial differntial operators of constant coefficients and $f(x, v)$ is a C^∞ function with respect to x and an entire function with respect to v.
Under suitable assumptions on the nonlinear function f and on $P, Q_1, ..., Q_M$, we will solve locally near every point $x^0 \in \mathbb{R}^n$ the next equation

$$F(u) = g, \;\; g \in B_{p,k},$$

where $B_{p,k}$ is a wieghted Sobolev space as in Hörmander [13].

1. Introduction

During the last years the attention in the literature has been mainly addressed to the semilinear case:

$$P(x, D)u + f(x, D^\alpha u)_{|\alpha|\leq m-1} = g(x) \tag{1}$$

where the nonlinear function $f(x, v)$, $x \in \mathbb{R}^n$, $v \in \mathbb{C}^M$, is in $C^\infty(\mathbb{R}^n, \mathcal{H}(\mathbb{C}^M))$ with $\mathcal{H}(\mathbb{C}^M)$ the set of the holomorphic functions in $\mathbb{C}^M$ and where the local solvability of the linear term $P(x, D)$ is assumed to be already known.
See Gramchev-Popivanov[10] and Dehman[4] where, exploiting the fact that the non-linear part of the equation (1) involves derivatives of order $\leq m - 1$, one is reduced to applications of the classical contraction principle and Brower's fixed point Theorem, provided the linear part is invertible in some sense. The general case of $P(x, D)$ satisfying the $(\mathcal{P})$ condition of Nirenberg and Trèves [21] has been settled in Hounie-Santiago[12], by combining the contraction principle with compactness arguments.
Corcerning the case of linear part with multiple characteristics, we mention the recent results of Gramchev-Rodino[11], Garello[6], Garello[5], Garello-Gramchev-Popivanov-Rodino[7], Garello-Rodino[8], Garello-Rodino[9], De Donno-Oliaro[3], Marcolongo[17], Marcolongo-Oliaro[18], Oliaro[22].

The main goal of the present paper is to study the local solvability of semilinear partial differential operators of the form

(2) $$F(u) = P(D)u + f(x, Q_1(D)u, ..., Q_M(D)u)$$

where $P(D)$, $Q_1(D)$, ..., $Q_M(D)$ are linear partial differential operators with constant coefficients and $f(x, v)$ is as before a C^∞ function with respect to x and an entire function with respect to v.
We will introduce suitable assumptions on the nonlinear function f and on $P, Q_1, ..., Q_M$ in order to solve locally near a point $x^0 \in \mathbb{R}^n$ the next equation

(3) $$F(u) = g, \quad g \in B_{p,k},$$

where $B_{p,k}$ is a weighted Sobolev space as in Hörmander [13], with $1 \le p \le \infty$ and k temperate weight function. Hörmander introduced these spaces exactly in connection with the problem of the solvability of linear partial differential operators with constant coefficients, namely one can find a fundamental solution T of $P(D)$ belonging locally to $B_{\infty,\widetilde{P}}$, see [13].
There are suitable assumptions on the temperate weight function k under which the space $B_{p,k}$ forms an algebra, see [20], [19]. In [20] we have also proved, under the same conditions, invariance after composition with analytic functions. These results allow us to look for a solution u, in a related $B_{p,k}$, giving meaning to the nonlinear term of (2).
More precisely, from basic properties of these spaces (see [13], [24] and the next Section 2 for notations and results), we know that if $u \in B_{p,k\widetilde{P}}$ then $P(D)u \in B_{p,k}$ and $Q_i(D)u \in B_{p,k\widetilde{P}/\widetilde{Q_i}}$ for $i = 1, .., M$. Assuming that $P \succ Q_i$ for every i, we get

(4) $$B_{p,k\widetilde{P}/\widetilde{Q_i}} \hookrightarrow B_{p,k},$$

then, one should require k satisfying the hypotheses which grant $f(x, Q_1(D)u, ..., Q_M(D)u) \in B_{p,k}$. Under these conditions the equation (3) will be well defined in the classical sense, for $g \in B_{p,k}$ and $u \in B_{p,k\widetilde{P}}$.
In this paper we will prove two theorems about the local solvabilty of (3) under different assumptions on the non-linear function f and the linear terms $P, Q_1, ..., Q_M$, completing the results of [20].
In the first theorem, Theorem 11, we assume $\frac{\widetilde{Q_i}(\xi)}{\widetilde{P}(\xi)} \to 0$ when $|\xi| \to \infty$ for $i = 1, ..., M$. From this hypothesis it follows that the inclusion $B_{p,k\widetilde{P}/\widetilde{Q_i}} \hookrightarrow B_{p,k}$ is compact (see [13]).
However here we assume on the nonlinearity $f(x, 0) = 0$, corresponding to the standard setting in literature; this is essentially weaker than the hypotheses in [20] $f(x_0, v) = 0$ for every $v \in \mathbb{C}^M$.
In the proof, we will generalize a well-known property of the standard Sobolev spaces to the case of the weighted spaces $B_{p,k}$, namely, if k_1 and k_2 are temperate weight functions, $k_2(\xi) \ge N > 0$ for all $\xi \in \mathbb{R}^n$, x^0 is a point in $\mathbb{R}^n$ and

(5) $$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty$$

then

$$\|u\|_{p,k_2} \leq C(\epsilon)\|u\|_{p,k_1} \quad \forall u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}) \tag{6}$$

where $\overline{B_\epsilon(x^0)} = \overline{\{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon\}}$ and $C(\epsilon) \to 0$ for $\epsilon \to 0$.
Applying the previous property and the Schauder Fixed Point Theorem, we will conclude that the equation (3) is locally solvable at every $x^0 \in \mathbb{R}^n$, and the solution belongs to $B_{p,k\widetilde{P}}$.
In the second result, Theorem 12, the assumptions are weaker than those in the first theorem, but the functional frame is now limited to suitable spaces $H_k := B_{2,k}$.
More precisely, we suppose $g \in H_k$ having the algebra property, and k of the particular form $k = \psi^r$, with $r \in \mathbb{R}$, $\psi \in C^\infty$ satisfying the "slowly varying" estimates, stronger than the temperate condition. These estimates were introduced by Beals [1], cf. Hörmander [14], in connection with the pseudodifferential calculus. Namely, as a particular case of the classes of Beals [1], we may consider the classes of symbols S^ν_ψ, whose definition is obtained by replacing $(1 + |\xi|)$ in the classic $S^\nu_{1,0}$ bounds with the weight $\psi(\xi)$. The corresponding pseudodifferential operators OPS^ν_ψ have a natural action on the weighted Sobolev spaces H_{ψ^r}. For most of the related properties and definitions recalled in the following let us refer to Rodino [23].
After having transformed the equation (3) into an equivalent fixed point problem, we will deduce, using the assumption $Q_i \prec\prec P$ and applying the properties of the pseudodifferential calculus, that there exists $C(\epsilon)$ such that $C(\epsilon) \to_{\epsilon \to 0} 0$ and

$$\|Q_i(D)u\|_{H_k} \leq C(\epsilon)\|P(D)u\|_{H_k}$$

for $i = 1, ..., M$ and every $u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0))$, x^0 any point in $\mathbb{R}^n$.
The Contraction Principle will allow us to conclude again that the equation (3) is locally sovable, and the solution belongs to $H_{k\widetilde{P}}$, at every point of $\mathbb{R}^n$.
We end this introduction by giving a simple example of an operator with multiple characteristics to which our Theorems 11 and 12 apply. Namely consider

$$D^4_{x_1}u - L^2u + f(u, (D^2_{x_1} - L)u, (D^2_{x_2} + L)u) = g. \tag{7}$$

where L is a constant vector field in $\mathbb{R}^n$. Our results will provide for (7) solutions in different scales of functional spaces $B_{p,k}$, cf. Section 2, exibiting novelty in comparison with the papers mentioned at the beginning. We also point out that even for some semilinear partial differential equations with simple characteristics Theorem 11 and Theorem 12 imply new results for the local solvability in more general functional spaces.

2. Preliminary results

We begin with a short survey on $B_{p,k}$ spaces.

DEFINITION 1. *A positive function k defined in $\mathbb{R}^n$ is called a temperate weight function if there exist positive constants C and N such that*

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta)\,; \quad \xi,\ \eta \in \mathbb{R}^n. \tag{8}$$

The set of all such functions will be denoted by K.

EXAMPLE 1. The basic example of a function in K, is the function $\widetilde{P}$ defined by

$$\widetilde{P}^2(\xi) = \sum_{|\alpha|\geq 0} |P^{(\alpha)}(\xi)|^2 \tag{9}$$

where P is a polynomial and so that the sum is finite. Here $P^{(\alpha)} = \partial^\alpha P$. It follows immediately from Taylor's formula that

$$\widetilde{P}(\xi + \eta) \leq (1 + C|\xi|)^m \widetilde{P}(\eta)$$

where m is the degree of P and C is a constant depending only on m and the dimension n.

The proofs of the following results are omitted for shortness; let us refer for them to [13].

DEFINITION 2. *If $k \in K$ and $1 \leq p \leq +\infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that the Fourier Transform $\widehat{u}$ is a function and*

$$\|u\|_{p,k} = \left((2\pi)^{-n} \int |k(\xi)\widehat{u}(\xi)|^p d\xi\right)^{1/p} < +\infty. \tag{10}$$

When p=$+\infty$ we shall interpret $\|u\|_{p,k}$ as $ess.sup|k(\xi)\widehat{u}(\xi)|$. We shall also write $H_k := B_{2,k}$, endowed with the natural Hilbert structure.

EXAMPLE 2. The usual Sobolev spaces $H_{(s)}$ correspond to the temperate weight function $k_s(\xi) = (1 + |\xi|^2)^{s/2}$, with $p = 2$.

$B_{p,k}$ is a Banach space with the norm (10). We have

$$\mathcal{S} \subset B_{p,k} \subset \mathcal{S}' \tag{11}$$

also in topological sense.

THEOREM 1. *If k_1 and k_2 belong to K and there exists $C > 0$ such that*

$$k_2(\xi) \leq Ck_1(\xi),\ \xi \in \mathbb{R}^n,$$

it follows that $B_{p,k_1} \hookrightarrow B_{p,k_2}$.

REMARK 1. Let $k \in K$, in view of estimate (8) with $\eta = 0$ and Theorem 1, one can find $s \in \mathbb{R}$ such that $B_{p,(1+|\xi|^2)^{s/2}} \hookrightarrow B_{p,k}$.

We shall now study how differential operators with constant coefficients act in the spaces $B_{p,k}$. Recall that if $P(\xi)$ is a polynomial in n variables $\xi_1, ..., \xi_n$ with complex coefficients, then a differential operator $P(D)$ is defined by replacing ξ_j by $D_j = -i\partial/\partial_{x_j}$ and the function $\tilde{P}$ is defined by (9).

THEOREM 2. *If $u \in B_{p,k}$ it follows that $P(D)u \in B_{p,k/\tilde{P}}$.*

THEOREM 3. *If $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ and $u_2 \in B_{\infty,k_2}$, it follows that $u_1 * u_2$ belongs to B_{p,k_1k_2}, and we have the estimate*

$$\|u_1 * u_2\|_{p,k_1k_2} \leq \|u_1\|_{p,k_1}\|u_2\|_{\infty,k_2}. \tag{12}$$

THEOREM 4. *If $u \in B_{p,k}$ and $\phi \in \mathcal{S}$, it follows that $u\phi \in B_{p,k}$ and that*

$$\|\phi u\|_{p,k} \leq \|\phi\|_{1,M_k}\|u\|_{p,k}, \tag{13}$$

where $M_k \in K$ is defined by $M_k(\xi) = \sup_\eta \frac{k(\xi+\eta)}{k(\eta)}$.

LEMMA 1. *Let $k \in K$. For every $\phi \in \mathcal{S}$ there exists an equivalent weight function h (i.e. $C^{-1}k(\xi) \leq h(\xi) \leq Ck(\xi)$) such that*

$$\|\phi u\|_{p,h} \leq 2\|\phi\|_{1,1}\|u\|_{p,h}. \tag{14}$$

We stress that the weight h depends on ϕ. We will use Lemma 1 later.

THEOREM 5. *If k_1 and k_2 belong to K and* H *is a compact set in $\mathbb{R}^n$, the inclusion mapping of $B_{p,k_1} \cap \mathcal{E}'(\mathrm{H})$ into B_{p,k_2} is compact if*

$$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty. \tag{15}$$

Conversely, if the mapping is compact for one set H *with interior points, it follows that* (15) *is valid.*

The Frèchet space $B^{\mathrm{loc}}_{p,k}$ is defined in the standard way and corresponding properties are valid for it, in particular we have the following variant of Theorem 3.

THEOREM 6. *Let $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ and $u_2 \in B^{loc}_{\infty,k_2}$, it follows that $u_1 * u_2$ belongs to B^{loc}_{p,k_1k_2}.*

Under suitable conditions regarding a temperate weight function k the corresponding space $B_{p,k}$ is an algebra; for the proof of the next theorems, let us refer to [20].

THEOREM 7. *Let $1 < p < +\infty$, $1/p + 1/q = 1$, $u, v \in B_{p,k}$ and $K(\xi,\eta) = \frac{k(\xi)}{k(\xi-\eta)k(\eta)}$ satisfy*

$$\sup_\xi \int |K(\xi,\eta)|^q d\eta \leq C_0 < +\infty. \tag{16}$$

then $uv \in B_{p,k}$ and $\|uv\|_{p,k} \leq C\|u\|_{p,k}\|v\|_{p,k}$.

The previous theorem can be generalized to the invariance of the spaces $B_{p,k}$ under the composition with entire functions.

DEFINITION 3. *We write* $f(x,v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$, *where* $\mathcal{H}(\mathbb{C}^M)$ *is the set of the entire functions in* $\mathbb{C}^M$, *if* $f(x,v) = \sum_{|\alpha|\geq 0} c_\alpha(x) v^\alpha$ *where* $c_\alpha(x) \in C^\infty(\mathbb{R}^n)$ *and for every compact subset* $\mathrm{H} \subset \mathbb{R}^n$, *there exist* $C_\beta(\mathrm{K}), \lambda_\alpha(\mathrm{K}) > 0$ *such that* $\sup_{x\in\mathrm{K}} |\partial^\beta c_\alpha(x)| < C_\beta \lambda_\alpha$, $\sum_{|\alpha|\geq 0} \lambda_\alpha v^\alpha$ *being an entire function of* $v \in \mathbb{C}^M$.

THEOREM 8. *Let* $f(x,v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$, $u_1, ..., u_M \in B_{p,k}$ *with k satisfying the hypotheses of Theorem 7, then* $f(x, u_1(x), ..., u_M(x)) \in B^{loc}_{p,k}$.

Now we recall two particular definitions of comparison between differential polynomials and related theorems of characterization (see Hörmander [13], Trèves [24] for the proofs).

Let Ω be an open subset of $\mathbb{R}^n$, P, Q two differential operators with C^∞ coefficients in Ω.

DEFINITION 4. *We say that P is stronger than Q in* Ω *if to every relatively compact open subset* Ω' *of* Ω, *there exists a constant* $C(P,Q,\Omega')$ *such that, for all functions* $\psi \in C^\infty_0(\Omega')$,

$$\int |Q(x,D)\psi(x)|^2 dx \leq C(P,Q,\Omega') \int |P(x,D)\psi(x)|^2 dx. \tag{17}$$

DEFINITION 5. *Let* x^0 *be a point of* Ω. *We say that* P *is infinitely stronger than* Q *at* x^0 *if to every* $\epsilon > 0$ *there exists* $\eta > 0$ *such that, for all functions* $\psi \in C^\infty_0(\Omega)$ *having their support in the open ball centered at* x^0, *with radius* η,

$$\int |Q(x,D)\psi(x)|^2 dx \leq \epsilon \int |P(x,D)\psi(x)|^2 dx. \tag{18}$$

In other words, P is infinitely stronger than Q at x^0 if estimate (17) holds for some open neighborhoord Ω' of x^0, and if we may choose the constant $C(P,Q,\Omega')$ so that it converges to zero when Ω' converges to the set $\{x^0\}$. If P is stronger than Q and Q stronger than P in Ω, we say that P and Q are equally strong or equivalent in Ω.
If P and Q have constant coefficients, the validity of Definition 4 does not depend on Ω. In this situation, we simply say that $P(D)$ is stronger than $Q(D)$ and we shall write $P(D) \succ Q(D)$. Similarly, the translation invariance of $P(D)$ and $Q(D)$ implies that if $P(D)$ is infinitely stronger than $Q(D)$ at some point of $\mathbb{R}^n$, this is also true at any other point of $\mathbb{R}^n$. Thus we shall say that $P(D)$ is infinitely stronger than $Q(D)$, and write $P(D) \succ\succ Q(D)$.

THEOREM 9. *Let* $P(D)$, $Q(D)$ *be two differential polynomials in* $\mathbb{R}^n$. *The following properties are equivalent:*

(a) $P(D)$ *is stronger than* $Q(D)$;

(b) the function $\frac{\tilde{Q}(\xi)}{\tilde{P}(\xi)}$ *is bounded in* $\mathbb{R}^n$;

(c) the function $\frac{|Q(\xi)|}{\tilde{P}(\xi)}$ *is bounded in* $\mathbb{R}^n$.

REMARK 2. With reference to operator (2), let us observe the following. If we assume that $Q_i \prec P$ for every $i = 1, ..., M$, in view of Theorem 9, we have $\frac{\widetilde{Q}_i(\xi)}{\widetilde{P}(\xi)} \le C$; therefore for every $k \in K$

$$k(\xi) \le C\frac{k(\xi)\widetilde{P}(\xi)}{\widetilde{Q}_i(\xi)}. \tag{19}$$

In view of Theorem 1 and (19) we obtain

$$B_{p,\frac{k\widetilde{P}}{\widetilde{Q}_i}} \hookrightarrow B_{p,k}, \quad \forall i = 1, ..., M. \tag{20}$$

We conclude this section by recalling the definition and the main properties of the pseudodifferential operators with symbols in the classes S_ψ, particular case of the classes of Beals [1].

We say that a positive continuous function $\psi(\xi)$ in $\mathbb{R}^n$ is a basic weight function if there are positive constants c, C such that

$$\begin{aligned} &(1) \quad c(1+|\xi|)^c \le \psi(\xi) \le C(1+|\xi|), \\ &(2) \quad c \le \psi(\xi+\theta)\psi(\xi)^{-1} \le C, \quad \text{if} \quad |\theta|\psi(\xi)^{-1} \le c. \end{aligned} \tag{21}$$

For $\nu \in \mathbb{R}$ we define S^ν_ψ to be the set of all $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \le c_{\alpha\beta}\psi(\xi)^{\nu-|\beta|}, \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n. \tag{22}$$

Let

$$Af(x) = a(x,D)f(x) = (2\pi)^{-n}\int [ix\xi]a(x,\xi)\hat{f}(\xi)d\xi, \quad f \in \mathcal{C}^\infty_0(\mathbb{R}^n), \tag{23}$$

with $a(x,\xi) \in S^\nu_\psi$. The standard rules of the calculus of the pseudo-differential operators hold for operators of the form (23). Let us review shortly the properties which we shall use in the following.

Recall first that for every basic weight function $\psi(\xi)$ we may find a smooth basic weight function $\psi_0(\xi)$, which is equivalent to $\psi(\xi)$ (i.e. $\psi_0(\xi)\psi(\xi)^{-1}$ and $\psi_0(\xi)^{-1}\psi(\xi)$ are bounded in $\mathbb{R}^n$), such that

$$|D^\beta_\xi \psi_0(\xi)| \le c_\beta \psi_0(\xi)\psi_0(\xi)^{-|\beta|}, \quad \xi \in \mathbb{R}^n. \tag{24}$$

Note that, let ψ be a smooth function satisfying (24) and (21), then ψ is a temperate weight function.

Equivalent basic weight functions define the same class of symbols; therefore we may assume in the following that $\psi(\xi)$ satisfies (24).

Moreover from (24) it follows that $\psi \in S^1_\psi$ and so, for every $r \in \mathbb{R}$, $\psi^r \in S^r_\psi$.

The operator A in (23) maps continuously $\mathcal{C}^\infty_0(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$ and it extends to a linear continuous operator from $\mathcal{E}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.

According to the previous notations, write H_{ψ^ν} for the Hilbert space of the distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ which satisfy

$$\|f\|^2_{H_{\psi^\nu}} = \int \psi(\xi)^{2\nu} |\hat{f}(\xi)|^2 d\xi < \infty. \tag{25}$$

If A has symbol in S^ν_ψ, then for every $s \in \mathbb{R}$:

$$A : H_{\psi^{\nu+s}} \to H_{\psi^s} \quad \text{continuously.} \tag{26}$$

A map $A : \mathcal{C}^\infty_0(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is said to be smoothing if it has a continuous extension mapping $\mathcal{E}'(\mathbb{R}^n)$ into $\mathcal{C}^\infty(\mathbb{R}^n)$; for given operators $A_1, A_2 : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ we shall write $A_1 \sim A_2$ if the difference $A_1 - A_2$ is smoothing.
If $a(x, \xi)$ is in $\cap_\nu S^\nu_\psi$, then $a(x, D)$ is smoothing.

THEOREM 10. *Let $a_1(x, \xi)$ be in $S^{\nu_1}_\psi$, let $a_2(x, \xi)$ be in $S^{\nu_2}_\psi$. Then the product $a_1(x, D)a_2(x, D)$ is in $S^{\nu_1+\nu_2}_\psi$ with symbol*

$$a(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} \partial^\alpha_\xi a_1(x, \xi) D^\alpha_x a_2(x, \xi). \tag{27}$$

3. Statement of the main results

We will study the following semilinear partial differential operator (2) where

(1) $P(D)$ is a linear partial differential operator with constant coefficients;

(2) $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$ and $f(x, 0) = 0$ for every $x \in \mathbb{R}^n$;

(3) $Q_1(D), ..., Q_M(D)$ are linear partial differential operators with constant coefficients such that $Q_i(D) \prec P(D)$ for $i = 1, ..., M$.

We want to solve locally near every point $x^0 \in \mathbb{R}^n$ the equation (3) under the following stronger assumptions on $Q_i(D)$, cf. Introduction.

THEOREM 11. *Let $g \in B_{p,k}$, with k satisfying the assumptions of Theorem 7 and $k(\xi) \geq N > 0$, $\forall \xi \in \mathbb{R}^n$, $1 < p < \infty$. Consider the operator F defined by* (2) *where, for $i = 1, ..., M$,*

$$\frac{\widetilde{Q}_i(\xi)}{\widetilde{P}(\xi)} \to 0, \ \textit{when } |\xi| \to \infty; \tag{28}$$

then for every $x^0 \in \mathbb{R}^n$ one can find a constant $\epsilon_0 > 0$ and $u^0 \in B_{p,k\tilde{P}}$ such that

$$F(u^0)(x) = g(x), \ \forall x \in \Omega \tag{29}$$

where $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

THEOREM 12. *Let $g \in H_k$, k satisfying the assumptions of Theorem 7 with $p = 2$ be such that $k = \psi^r$ with $r \in \mathbb{R}$, $\psi^r(\xi) \geq N > 0$ for every $\xi \in \mathbb{R}^n$ and assume that* (21) *holds. Consider the operator F defined by* (2) *where, for $i = 1, \dots, M$,*

$$Q_i \prec\prec P; \tag{30}$$

then for every $x^0 \in \mathbb{R}^n$ one can find a constant $\epsilon_0 > 0$ and $u^0 \in H_{k\widetilde{P}}$ such that

$$F(u^0)(x) = g(x), \quad \forall x \in \Omega \tag{31}$$

where $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

4. Proof of the results

To prove Theorem 11 we will apply a property of the Sobolev spaces true also for the weighted Sobolev spaces (see [16]).

THEOREM 13. *If k_1 and k_2 belong to K, $k_2 \geq N > 0$, x^0 is a point in $\mathbb{R}^n$ and*

$$\frac{k_2(\xi)}{k_1(\xi)} \to 0 \quad \text{for} \quad |\xi| \to \infty \tag{32}$$

then

$$\|u\|_{p,k_2} \leq C(\epsilon)\|u\|_{p,k_1} \quad \forall u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}) \tag{33}$$

where $\overline{B_\epsilon(x^0)} = \overline{\{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon\}}$ and $C(\epsilon) \to 0$ for $\epsilon \to 0$.

Proof. First of all note that from the Theorem 5 it follows that the injection of $B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into B_{p,k_2} is compact.
To prove the Theorem we have to verify that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})} \frac{\|u\|_{p,k_2}}{\|u\|_{p,k_1}} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0 \tag{34}$$

that is equivalent to prove that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}),\, \|u\|_{p,k_1} = 1} \|u\|_{p,k_2} = C(\epsilon) \quad \text{where} \quad C(\epsilon) \to 0. \tag{35}$$

We suppose, ab absurdo, that $C(\epsilon)$ does not tend to 0 when ϵ tends to 0, then there exists a sequence $\{\epsilon_\nu\}_{\nu \in \mathbb{N}}$ such that $\epsilon_\nu \to 0$ when $\nu \to \infty$ and $C(\epsilon_\nu) \nrightarrow 0$ when $\nu \to \infty$. We can deduce that there exists a positive constant r and a subsequence $\{\epsilon_{\nu_j}\}_{j \in \mathbb{N}}$ such that $\epsilon_{\nu_j} \to 0$ when $j \to \infty$ and $C(\epsilon_{\nu_j}) > r$ for every j. Then we obtain that

$$\sup_{u \in B_{p,k_1} \cap \mathcal{E}'(\overline{B_{\epsilon_{\nu_j}}(x^0)}),\, \|u\|_{p,k_1} = 1} \|u\|_{p,k_2} = C(\epsilon_{\nu_j}) > r \quad \text{for every } j. \tag{36}$$

From the definition of supremum there exists a sequence $u_{\nu_j} \in B_{p,k_1}$ with support contained in the ball of fixed center x^0 with radius ϵ_{ν_j} such that

$$\|u_{\nu_j}\|_{p,k_1} = 1 \quad \text{and} \quad \|u_{\nu_j}\|_{p,k_2} \geq r. \tag{37}$$

The sequence $\{u_{\nu_j}\}_{j\in\mathbb{N}}$ is bounded in B_{p,k_1}, then, according to the compactness of the injection of $B_{p,k_1} \cap \mathcal{E}'(\overline{B_{\epsilon_{\nu_j}}(x^0)})$ into B_{p,k_2}, we may assume that there exists a subsequence, still denoted by u_{ν_j}, and a distribution $u \in B_{p,k_2}$ such that

$$\|u_{\nu_j} - u\|_{p,k_2} \to 0 \quad \text{when} \quad j \to \infty. \tag{38}$$

But from the properties of the topology of B_{p,k_2}, see 11, we obtain that

$$\|u_{\nu_j} - u\|_{\mathcal{S}'} \to 0 \quad \text{when} \quad \mathrm{j} \to \infty. \tag{39}$$

Since u necessarily has support contained in $\{x^0\}$, we have $u = \sum_{0\leq|\alpha|\leq m} c_\alpha \delta_{x^0}^{(\alpha)}$. We have two possibilities:

1. $u \equiv 0$, which is absurd, indeed $\|u_{\nu_j}\|_{p,k_2} \to \|u\|_{p,k_2}$ when $j \to \infty$ and $\|u_{\nu_j}\|_{p,k_2} \geq r > 0$ for every $j \in \mathbb{N}$
2. $u \not\equiv 0$, which is absurd, indeed a nontrivial linear combination of derivatives of the distribution δ_{x^0} does not belong to $B_{p,k}$ if $k \geq N > 0$.

□

Proof of Theorem 11. Fix a point $x^0 \in \mathbb{R}^n$, choose

$$\varphi \in C_0^\infty(\mathbb{R}^n),\ \mathrm{supp}\varphi \subset \overline{B_1(0)}, \varphi \equiv 1 \text{ in } \overline{B_{1/2}(0)} \tag{40}$$

and define

$$\varphi_\epsilon(x) = \varphi\left(\frac{x - x^0}{\epsilon}\right). \tag{41}$$

We also introduce the function $\psi \in C_0^\infty(\mathbb{R}^n)$ defined as $\psi(x) = \varphi(x - x^0)$. We observe that from the general theory of the linear partial differential operators with constant coefficients (see [13]) it follows that there exists a fundamental solution $T \in B^{\mathrm{loc}}_{\infty,\tilde{P}}$ of the linear part $P(D)$ of the semilinear operator F.

In order to obtain our result we shall replace the weight function k by an equivalent function $h \in K$; obviously we have $B_{p,k} = B_{p,h}$. The function h will be determined later; Lemma 1 will play a crucial role in this connection (see Hörmander [14] Vol. II Theorem 13.3.3 for a similar argument). Let us define

$$B_{p,h;R} = \{u \in B_{p,h} / \|u\|_{p,h} \leq R\}.$$

We can consider the operator

$$\tilde{F}_\epsilon : B_{p,h;R} \to B_{p,h} \tag{42}$$

with $R \geq 2\|g\|_{p,h}$ given by the following espression

(43) $$\tilde{F}_\epsilon[v] = g - \psi f(x, Q_1(D)(\varphi_\epsilon \psi L \psi v), ..., Q_M(D)(\varphi_\epsilon \psi L \psi v))$$

where $L := T*$, T the fundamental solution of the operator $P(D)$.
Note that, from the hypothesis (28),

(44) $$\frac{h(\xi)}{\frac{h(\xi)\tilde{P}(\xi)}{\tilde{Q}_i(\xi)}} \to 0 \quad \text{when} \quad |\xi| \to \infty$$

and so, for every $i = 1, .., M$, the injection of $B_{p,\frac{h\tilde{P}}{\tilde{Q}_i}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$ is compact. In Theorem 11 we have also supposed that $k(\xi) \geq N > 0$, then we obtain that

(45) $$\frac{h(\xi)\tilde{P}(\xi)}{\tilde{Q}_i(\xi)} \geq C > 0 \quad \text{for every } \xi.$$

From (44) and (45) it follows that the temperate weight functions $\frac{h\tilde{P}}{\tilde{Q}_i}$ and h satisfy the hypothesis of the Theorem 13. Note that, if $v \in B_{p,h;R}$ then $\psi v \in B_{p,h} \cap \mathcal{E}'$; therefore $L\psi v \in B^{loc}_{p,h\tilde{P}}$ and $\psi L \psi v \in B_{p,h\tilde{P}}$ so we obtain $\tilde{F}_\epsilon[v] \in B_{p,h}$ and, let x^0 be a fixed point in $\mathbb{R}^n$, for every $i = 1, .., M$, there exists a function C_i of the variable ϵ such that $C_i(\epsilon) \to 0$ when $\epsilon \to \infty$ and

(46) $$\|u\|_{p,\frac{h\tilde{P}}{\tilde{Q}_i}} \leq C_i(\epsilon)\|u\|_{p,h} \quad \forall u \in B_{p,h} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)}).$$

To prove Theorem 11 we will verify that there exists $\epsilon_0 > 0$ such that the corresponding operator defined by

(47) $$\begin{gathered} \tilde{F}_{\epsilon_0} : B_{p,h;R} \to B_{p,h} \\ \tilde{F}_{\epsilon_0}[v] = g - \psi f(x, Q_1(D)(\varphi_{\epsilon_0} \psi L \psi v), ..., Q_M(D)(\varphi_{\epsilon_0} \psi L \psi v)) \end{gathered}$$

verifies the hypotheses of the Schauder Fixed Point Theorem, namely it is continuous, it is defined from $B_{p,h;R}$ to itself and it maps bounded sets into relatively compact sets. First of all we will prove that $\tilde{F}_\epsilon(B_{p,h;R})$ is relatively compact for every $\epsilon > 0$; to this end we will consider a sequence $\{v_j\}_{j\in\mathbb{N}}$ of distributions in $B_{p,h;R}$ and we will verify that there exists a convergent subsequence in $B_{p,k}$ of the sequence $\{\tilde{F}_\epsilon[v_j]\}_{j\in\mathbb{N}}$.
Set $\{u_j\}_{j\in\mathbb{N}} := \{\varphi_\epsilon \psi L \psi v_j\}_{j\in\mathbb{N}}$, then $\|u_j\|_{p,h\tilde{P}} \leq R_1(\epsilon)$. Indeed, by Theorem 3 and Theorem 4,

(48) $$\begin{aligned} \|u_j\|_{p,h\tilde{P}} &= \|\varphi_\epsilon \psi(T * \psi v_j)\|_{p,h\tilde{P}} = \|\varphi_\epsilon \psi(\tilde{\psi} T * \psi v_j)\|_{p,h\tilde{P}} \leq C_\epsilon \|\tilde{\psi} T * \psi v_j\|_{p,h\tilde{P}} \\ &\leq C_{1,\epsilon}\|\tilde{\psi} T\|_{\infty,\tilde{P}}\|\psi v_j\|_{p,h} \leq C_{2,\epsilon}\|\tilde{\psi} T\|_{\infty,\tilde{P}}\|v_j\|_{p,h} \leq C_{3,\epsilon} R \end{aligned}$$

where $\widetilde{\psi} \in C_0^\infty$ and $\widetilde{\psi} \equiv 1$ in K= $\{x \in \mathbb{R}^n/(x + \mathrm{supp}\psi v_j) \cap \mathrm{supp}\varphi_\epsilon \psi \neq \emptyset\}$.
Set

$$\{E_i v_j\}_{i=1,\dots,M}=\{(E_1 v_j, ..., E_M v_j)\}:=\{Q(D)u_j\}=\{(Q_1(D)u_j, ..., Q_M(D)u_j)\};$$

by Theorem 2, we also obtain $\|E_i v_j\|_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \leq R_2(\epsilon)$ for every $i = 1, ..., M$ and $j \in \mathbb{N}$.
The sequences $\{E_i v_j\}_{j\in\mathbb{N}}$ are bounded in $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}}$ and belong to $\mathcal{E}'(\overline{B_\epsilon(x^0)})$, indeed $\{u_j\} \in \mathcal{E}'(\overline{B_\epsilon(x^0)})$ and the differential operators $Q_i(D)$ do not increase the support.
Then, according to the compactness of the injection of $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$, we may suppose that there exists a subsequence $\{v_{j_\nu}\}_{\nu\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and a distribution z_1 such that

$$\|E_1 v_{j_\nu} - z_1\|_{p,h} \to 0 \quad \text{when} \quad \nu \to \infty.$$

The sequence $\{E_2 v_{j_\nu}\}_{\nu\in\mathbb{N}}$ is a subsequence of $\{E_2 v_j\}_{j\in\mathbb{N}}$ and so is still bounded in $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_2}}}$, then there exists $\{v_{j_{\nu_l}}\}_{l\in\mathbb{N}}$ and a distribution z_2 such that

$$\|E_2 v_{j_{\nu_l}} - z_2\|_{p,h} \to 0 \quad \text{when} \quad l \to \infty.$$

Iterating this process for every $i = 1, .., M$ we will find a subsequence $\{v_{j_m}\}_{m\in\mathbb{N}}$ of $\{v_j\}_{j\in\mathbb{N}}$ and M distributions $z_1, .., z_M$ such that

$$\|E_i v_{j_m} - z_i\|_{p,h} \to 0 \quad \text{when} \quad m \to \infty.$$

From the continuity of the injection of $B_{p,\frac{h\widetilde{P}}{\widetilde{Q_i}}} \cap \mathcal{E}'(\overline{B_\epsilon(x^0)})$ into $B_{p,h}$, for every i, we may assume that the sequences $\{E_i v_{j_m}\}_{m\in\mathbb{N}}$ are bounded in $B_{p,h}$ and so $\|E_i v_{j_m}\|_{p,h} \leq R^*(\epsilon)$ for every $1 \leq i \leq M$ and $m \in \mathbb{N}$.
To complete the proof we have to verify that $\{\psi f(x, E_1 v_{j_m}, .., E_M v_{j_m})\}_{m\in\mathbb{N}}$ is convergent in $B_{p,h}$.
We will prove that the operator

$$\overline{F} : B_{p,h;R^*} \times B_{p,h;R^*} \times ... \times B_{p,h;R^*} \to B_{p,h} \tag{49}$$

defined as

$$\overline{F}[w^1, .., w^M] = \psi f(x, w^1, .., w^M) \tag{50}$$

is sequentially continuous, then

$$\|\widetilde{F}_\epsilon[v_{j_n}] + \overline{F}[z_1, .., z_M] - g\|_{p,h} =$$
$$= \|g - \overline{F}[E_1 v_{j_n}, .., E_M v_{j_n}] + \overline{F}[z_1, .., z_M] - g\|_{p,h}$$

will tend to zero if $\|\overline{F}[E_1 v_{j_n}, .., E_M v_{j_n}] - \overline{F}[z_1, .., z_M]\|_{p,h}$ tends to zero.
Let $\{w_n\}_{n\in\mathbb{N}} := \{(w_n^1, ..., w_n^M)\}_{n\in\mathbb{N}} \in B_{p,h;R^*} \times \dots \times B_{p,h;R^*}$ and $w := (w^1, .., w^M) \in$

$B_{p,h;R^*}\times\ldots\times B_{p,h;R^*}$ be such that $(w_n^1,\ldots w_n^M)\to(w^1,\ldots,w^M)$ for $n\to\infty$, we must show that

$$\overline{F}[w_n^1,\ldots,w_n^M]\to\overline{F}[w^1,\ldots,w^M]. \tag{51}$$

Reminding the Cavalieri Lagrange Formula, we have to estimate

$$\begin{aligned}&\overline{F}[w_n^1,\ldots w_n^M]-\overline{F}[w^1,\ldots w^M]\|_{p,h}=\\ &\|\psi f(x,w_n^1,..,w_n^M)-\psi f(x,w^1,..,w^M)\|_{p,h}=\\ &\|\psi\sum_{i=1}^M(w_n^i-w^i)\int_0^1\partial_{v_i}f(x,w^1+t(w_n^1-w^1),..,w_n^M+t(w_n^M-w^M))dt\|_{p,h}.\end{aligned} \tag{52}$$

Set

$$G_i(x,y,z):=\int_0^1\partial_{v_i}f(x,y^1+t(z^1-y^1),..,y^M+t(z^M-y^M))dt \tag{53}$$

and note that $G_i(x,y,z)\in C^\infty(\mathbb{R}_x^n,\mathcal{H}(\mathbb{C}^{2M}))$ for every $i=1,..,M$.
Then $G_i(x,y,z)=\sum_{|\alpha|\ge 0}a_{i,\alpha}(x)(y,z)^\alpha$ where $a_{i,\alpha}(x)\in C^\infty(\mathbb{R}^n)$ and for every compact subset $\mathrm{H}\subset\subset\mathbb{R}^n$, there exist $C_\beta(\mathrm{H}),\lambda_\alpha(\mathrm{H})>0$ such that $\sup_{x\in\mathrm{H}}|\partial^\beta a_{i,\alpha}(x)|<C_\beta\lambda_\alpha$, being $\sum_{|\alpha|\ge 0}\lambda_\alpha(y,z)^\alpha$ an entire function.
Applying Theorem 7, we may further estimate (52) by

$$C\sum_{i=1}^M\|(w_n^i-w^i)\|_{p,h}\|\psi G_i(x,w_n,w)\|_{p,h}. \tag{54}$$

Set $\mathrm{H}:=\operatorname{supp}\psi$, from the Fourier Transform properties we can deduce that there exist $s\in\mathbb{N}$ and $l\in\mathbb{N}$ such that

$$\|\psi a_{i,\alpha}\|_{p,h}\le A(\mathrm{H})\|\psi a_{i,\alpha}\|^l_{C^{(s)}},\ \forall\alpha\in\mathbb{N}. \tag{55}$$

Indeed, let us consider $B_{p,(1+|\xi|^2)^{s'/2}}$ such that $s'\in\mathbb{N}$ and $B_{p,(1+|\xi|^2)^{s'/2}}\hookrightarrow B_{p,h}$; set $1/p+1/q=1$, if $p\ge 2$ then $1\le q\le 2$ and the Sobolev space $W^{q,s'}=\{u\in\mathcal{S}'/\mathcal{F}^{-1}((1+|\xi|^2)^{s'/2}\widehat{u})\in L^q\}$ is included with continuity in $B_{p,(1+|\xi|^2)^{s'/2}}$. Then in this case

$$\begin{aligned}\|\psi a_{i,\alpha}\|_{p,h}\le\|\psi a_{i,\alpha}\|_{W^{q,s'}}&=\sum_{|\beta|\le s'}\|\partial^\beta(\psi a_{i,\alpha})\|_{L^q}\\ &\le\sum_{|\beta|\le s'}\left(\int_{\mathrm{K}}(\sup(|\partial^\beta(\psi a_{i,\alpha})|)^q)dx\right)^{1/q}\\ &=(\mathrm{meas H})^{1/q}\sum_{|\beta|\le s'}\|\partial^\beta(\psi a_{i,\alpha})\|_{C^0}=A\|\psi a_{i,\alpha}\|_{C^{(s')}}\end{aligned}$$

where $A := (\mathrm{measH})^{1/\mathrm{q}}$.
If $1 < p < 2$ we apply the Hölder inequality with $r = 2/p$:

$$\begin{aligned}\|\psi a_{i,\alpha}\|^p_{p,h} &= \int |h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|^p d\xi\\ &= \int ((1+|\xi|^2)^{s''}|h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|)^p \left(\frac{1}{(1+|\xi|^2)^{s''}}\right)^p d\xi\\ &\le \left(\int ((1+|\xi|^2)^{s''}|h(\xi)\widehat{\psi a_{i,\alpha}}(\xi)|)^2 d\xi\right)^{1/r}\left(\int \frac{1}{(1+|\xi|^2)^{s''pr'}}\right)^{1/r'}\\ &\le D\left(\int ((1+|\xi|^2)^{s^*}|\widehat{\psi a_{i,\alpha}}(\xi)|)^2 d\xi\right)^{1/r} = D\|\psi a_{i,\alpha}\|^p_{H_{(s^*)}},\end{aligned}$$

where r' is such that $1/r + 1/r' = 1$, s'' is such that $s''pr' > n/2$ and $s^* = s' + s'' \in \mathbb{N}$. But we have

$$\|\psi a_{i,\alpha}\|_{H_{(s^*)}} \le A'\|\psi a_{i,\alpha}\|_{C^{(s^*)}},$$

where $A' := \sqrt{\mathrm{measH}}$, so in this case we have get (55) with $l = p$.
Therefore applying the algebra property of $B_{p,h}$, the estimate (55) and the algebra property of $C^{(s)}$ we obtain

$$\begin{aligned}\|\psi G_i(x, w_n, w)\|_{p,h} &\le A\sum_{\alpha\ge 0}\|\psi a_{i,\alpha}\|^l_{C^{(s)}}\|(y,z)^\alpha\|_{p,h}\\ &\le A'\|\psi\|^l_{C^{(s)}}\sum_{\alpha\ge 0}\|a_{i,\alpha}|_{\mathrm{K}}\|^l_{C^{(s)}}\left(C^{|\alpha|}(\|y\|_{p,h}, \|z\|_{p,h})^\alpha\right)\\ &\le A''(\mathrm{K})\sum_{\alpha\ge 0}\max_{|\beta|\le s}\{C^l_\beta\}\lambda_\alpha(CR^*)^{|\alpha|}\\ &= A'''(\mathrm{K})\sum_{\alpha\ge 0}\lambda_\alpha(CR^*)^{|\alpha|} < D.\end{aligned}\tag{56}$$

Replacing (56) in (54) we have

$$\|\overline{F}[w^1_n, ..., w^M_n] - \overline{F}[w^1, ..., w^M]\|_{p,h} \le D\sum_{i=1}^M \|w^i_n - w^i\|_{p,h}$$

that tends to zero when $\|w^i_n - w^i\|_{p,h} \to 0$ for every $i = 1, ..M$.
As second step of the proof, now we will prove that there exists $\epsilon_0 > 0$ such that the corresponding operator $\widetilde{F}_{\epsilon_0}$ is defined from $B_{p,h;R}$ to itself, namely let $v \in B_{p,h}$ such that $\|v\|_{p,h} \le R$ then $\|\widetilde{F}_{\epsilon_0}[v]\|_{p,h} \le R$.
We have to estimate

$$\|\widetilde{F}_\epsilon[v]\|_{p,h} \le \|\psi f(x, Q_1\varphi_\epsilon\psi L\psi v, .., Q_M\varphi_\epsilon\psi L\psi v)\|_{p,h} + \|g\|_{p,h}.\tag{57}$$

We now determine our choice of the equivalent function h (see the beginning of the proof). Namely we take, as a new equivalent weight, h such that $h\tilde{P}$ is equivalent to

$k\tilde{P}$, with k the original weight, and satisfies the estimate (14) in Lemma 1 with $\phi = \varphi_\epsilon$. We obtain

$$\|\varphi_\epsilon \psi L \psi v\|_{p,h\widetilde{P}} \le 2\|\varphi_\epsilon\|_{1,1}\|\psi L \psi v\|_{p,h\widetilde{P}} \le 2A\|\varphi_\epsilon\|_{1,1}\|v\|_{p,h\widetilde{P}}. \tag{58}$$

We introduce the function, Φ_ϵ defined as $\Phi_\epsilon(x) := \varphi(x/\epsilon)$ so from the Fourier Transform properties, we have that $\|\varphi_\epsilon\|_{1,1} = \|\Phi_\epsilon\|_{1,1}$, with φ,φ_ϵ as in (40) and (41). Noting that

$$\widehat{\Phi_\epsilon}(\xi) = \epsilon^n \widehat{\varphi}(\epsilon\xi) \tag{59}$$

we obtain

$$\begin{aligned} \|\varphi_\epsilon\|_{1,1} = \|\Phi_\epsilon\|_{1,1} = \|\widehat{\Phi}_\epsilon\|_{L^1} = \int |\epsilon^n \widehat{\varphi}(\epsilon\xi)| d\xi \\ = \int |\widehat{\varphi}(z)| dz, \end{aligned} \tag{60}$$

so we have proved that $\|\varphi_\epsilon\|_{1,1}$ does not depend on ϵ.
Replacing (60) in (58), we get $\|\varphi_\epsilon \psi L \psi v\|_{p,h\widetilde{P}} \le A'\|v\|_{p,h} = R_3$ and so $\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,\frac{h\widetilde{P}}{Q_i}} \le R_4$.
Applying Theorem 13 we obtain

$$\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,h} \le C_i(\epsilon)\|Q_i \varphi_\epsilon \psi L \psi v\|_{p,\frac{h\widetilde{P}}{Q_i}} \le C_i(\epsilon) R_4$$

with $C_i(\epsilon) \to 0$ when $\epsilon \to 0$, for every $i = 1, .., M$.
Set $Q_i \varphi_\epsilon \psi L \psi v := E_{i,\epsilon} v$ and $C(\epsilon) := \max_{i=1,..,M}\{C_i(\epsilon)\}$, then $C(\epsilon) \to 0$ when $\epsilon \to 0$ and $\|E_{i,\epsilon} v\|_{p,h} \le C(\epsilon) R_4$. But $f(x, v) \in C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^M))$ with $f(x, 0) = 0$, then, by the same arguments used to obtain (56)

$$\begin{aligned} &\|\psi f(x, Q_1\varphi_\epsilon \psi L \psi v, .., Q_M \varphi_\epsilon \psi L \psi v)\|_{p,h} = \|\psi f(x, E_{1,\epsilon} v, .., E_{M,\epsilon} v)\|_{p,h} \\ &= \|\psi \sum_{|\alpha|>0} c_\alpha(x)(E_{1,\epsilon} v, .., E_{M,\epsilon} v)^\alpha\|_{p,h} \\ &\le \sum_{|\alpha|>0} C_1 \|\psi c_\alpha\|_{p,h} (\|(E_{1,\epsilon} v\|_{p,h}, .., \|E_{M,\epsilon} v\|_{p,h})^\alpha C^{|\alpha|} \\ &\le \sum_{|\alpha|>0} C_2 \lambda_\alpha (R_4 C(\epsilon), .., R_4 C(\epsilon))^\alpha C^{|\alpha|} = \sum_{|\alpha|>0} C_2 \lambda_\alpha (\widetilde{C}(\epsilon))^{|\alpha|} \\ &= \sum_{l>0} C_2 \mu_l (\widetilde{C}(\epsilon))^l \end{aligned} \tag{61}$$

where $\widetilde{C}(\epsilon) \to 0$ when $\epsilon \to 0$ and $\sum_{l>0} \mu_l v^l < +\infty$ for every $v \in \mathbb{C}$.
If we have chosen ϵ sufficiently small, then $\widetilde{C}(\epsilon) < 1$ and we may further estimate

$$\begin{aligned} \sum_{l>0} C_2 \mu_l (\widetilde{C}(\epsilon))^l = C_2 \widetilde{C}(\epsilon) \sum_{l\ge 0} \mu_{l+1} (\widetilde{C}(\epsilon))^l \\ \le C_2 \widetilde{C}(\epsilon) \sum_{l\ge 0} \mu_{l+1} = C_4 \widetilde{C}(\epsilon). \end{aligned} \tag{62}$$

Replace (61) and (62) in (57) and choose ϵ_0 such that $\widetilde{C}(\epsilon_0) < 1$ and $\widetilde{C}(\epsilon_0) \le \frac{R-\|g\|_{p,h}}{C_4}$, then $\|\widetilde{F}_{\epsilon_0}[v]\|_{p,h} \le R$ for every $v \in B_{p,h;R}$.
At the end it is easy to verify that the operator $\widetilde{F}_{\epsilon_0}$ defined from $B_{p,h;R}$ to itself is continuouos. We leave to the reader to check sequential continuity, by using the preceding arguments.
According to the Schauder Fixed Point Theorem we can assume that there exists a fixed point $v^0 \in B_{p,h;R}$ of the operator $\widetilde{F}_{\epsilon_0}$:

$$v^0 = g - \psi f(x, Q(D)(\varphi_{\epsilon_0}\psi L\psi v^0)).$$

Multiply this espession for the function ψ, by convolution properties we then deduce $PL(\psi v^0) = \psi v^0$, therefore

$$PL(\psi v^0) = \psi g - \psi^2 f(x, Q(D)(\varphi_{\epsilon_0}\psi L\psi v^0));$$

we obtain that the distribution $u^0 := L\psi v^0 \in B^{loc}_{p,h\widetilde{P}}$ satisfies the theorem, namely shrinking ϵ_0 we have

$$Pu^0 = g - f(x, Q(D)u^0)$$

in $\Omega = \{x \in \mathbb{R}^n / \|x - x^0\| < \epsilon_0\}$.

□

Proof of Theorem 12. Without loss of generality we suppose $\psi \in C^\infty(\mathbb{R}^n)$ and, for every multi–index β, there exists c_β such that $|\partial^\beta\psi(\xi)| \le c_\beta\psi^{1-|\beta|}(\xi)$, cf. (24).

Fix a point $x^0 \in \mathbb{R}^n$, choose $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset \overline{B_1(x^0)}$, $\varphi \equiv 1$ in $\overline{B_{1/2}(x^0)}$ and $\varphi_\epsilon(x) = \varphi\left(\frac{x-x_0}{\epsilon}\right)$, and consider the operator

$$\widetilde{F}_\epsilon : H_{k;R} \to H_k \tag{63}$$

where $H_{k;R} = \{u \in H_k / \|u\|_{H_k} \le R\}$, given by the following expression

$$\widetilde{F}_\epsilon(v) = g - \varphi f(x, Q_1(D)(\varphi_\epsilon L\varphi v), ..., Q_M(D)(\varphi_\epsilon L\varphi v)) \tag{64}$$

with $L := T*$, T the fundamental solution of the operator $P(D)$.
We will prove that, fixed $R \ge 2\|g\|_{H_k}$, there exists $\epsilon_0 > 0$ such that the corresponding operator $\widetilde{F_{\epsilon_0}}$ is defined from $H_{k;R}$ to itself and is a contraction, then we will apply the Contraction Priciple to the operator $\widetilde{F_{\epsilon_0}}$. ¸Let $v \in H_{k;R}$, set $u := \varphi_\epsilon L\varphi v$ and $w = (w_1, ..., w_M) = Q(D)u$ with $Q(D)u = (Q_1(D)u, ..., Q_M(D)u)$; we already noted in the proof of Theorem 11 that from Cavalieri Lagrange Theorem we get

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \le C\sum_{j=1}^{M} \|(Q_j u^1 - Q_j u^2)\|_{H_k} \|(\varphi G_j(x, Qu^1, Qu^2)\|_{H_k} \tag{65}$$

where $G_j(x, v^1, v^2)$ is defined in (53) and belongs to $C^\infty(\mathbb{R}^n_x, \mathcal{H}(\mathbb{C}^{2M}))$ for $j = 1, ...M$. ¸With respect to the proof of Theorem 11 in which we estimated the second

term of the right-hand part of (65) obtaining $\|(\varphi_\epsilon \varphi G_j(x, Qu^1, Qu^2)\|_{H_k} \leq \epsilon D_3$, in the present case we will fix attention on the first term of the right-hand part of (65) and we expect to get from it the small constant ϵ.
Namely we will estimate the term

$$\|Q_j(D)u^1 - Q_j(D)u^2\|_{H_k} = \|Q_j(D)(u^1 - u^2)\|_{H_k}$$

for $j = 1, ..., M$ and for every $u^1, u^2 \in H_{k\widetilde{P}}$ such that supp u^1, supp u^2 are contained in $B_\epsilon(x^0)$.
From the Fourier trasform properties we get

$$\begin{aligned} \|Q_j(D)u\|_{H_k} &= \|k(\xi)\widehat{Q_j(D)}u(\xi)\|_{L^2} = \|Q_j(\xi)\widehat{k(D)u}(\xi)\|_{L^2} \\ &= \|Q_j(D)k(D)u\|_{L^2} \end{aligned} \tag{66}$$

where $k(D)$ has to be interpreted as pseudodifferential operator corresponding to the symbol $k(\xi)$.
Now let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\quad \chi \equiv 1$ on supp u and supp $\chi \subset B_{2\epsilon}(x^0)$, then

$$\begin{aligned} \|Q_j(D)u\|_{H_k} &= \|Q_j(D)k(D)\chi u\|_{L^2} \\ &\leq \|Q_j(D)\chi k(D)u\|_{L^2} + \|Q_j(D)Au\|_{H_k} \end{aligned} \tag{67}$$

where we have denoted with A the operator $[k(D), \chi] = k(D)\chi - \chi k(D)$.
Remind the assumption $Q_j \prec\prec P$ for $j = 1, .., M$, i.e. there exists $C(\epsilon) \to 0$ for $\epsilon \to 0$ such that

$$\|Q_j(D)v\|_{L^2} \leq C(\epsilon)\|P(D)v\|_{L^2} \quad \forall v \in C_0^\infty(B_\epsilon(x^0))$$

and note that the support of $\chi k(D)u$ is contained in $B_{2\epsilon}(x^0)$, then it follows

(68)

$$\begin{aligned} \|Q_j(D)u\|_{H_k} &\leq C(2\epsilon)\|P(D)\chi k(D)u\|_{L^2} + \|Q_j(D)Au\|_{L^2} \\ &= C(2\epsilon)(\|P(D)k(D)\chi u\|_{L^2} + \|P(D)Au\|_{L^2}) + \|Q_j(D)Au\|_{L^2}. \end{aligned}$$

With the same considerations used to obtain (66), this can be further estimated by

$$C(2\epsilon)\|P(D)u\|_{H_k} + C(2\epsilon)\|P(D)Au\|_{L^2} + \|Q_j(D)Au\|_{L^2}. \tag{69}$$

Now we have to estimate the terms $\|P(D)Au\|_{L^2}$, $\|Q_j(D)Au\|_{L^2}$.
From the assumption $k = \psi^r$, it follows $k \in S^r_\psi$ and $\chi \in S^0_\psi$, see (22), then, denoting with $a(x, \xi)$ the symbol of the pseudodifferential operator $A = a(x, D)$, from Theorem 10 about the symbolic calculus we get

$$a(x, \xi) = k(\xi)\chi(x) + b(x, \xi) - \chi(x)k(\xi) + c(x, \xi)$$

where $b, c \in S^{r-1}_\psi$, and so $a(x, \xi) \in S^{r-1}_\psi$. We can write

$$\|P(D)Au\|_{L^2} \leq \|AP(D)u\|_{L^2} + \|[P, A]u\|_{L^2},$$

but, from (26) we get

$$\|AP(D)u\|_{L^2} \leq C\|P(D)u\|_{H_{\psi^{r-1}}} \leq C_1\|u\|_{H_{\psi^{r-1}\widetilde{P}}}$$

where we have also applied the properties of the $B_{p,k}$ spaces.
Reminding the condition (1) in (21), we have

$$\psi^{-1}(\xi) \leq c(1+|\xi|)^{-\rho} \quad \text{with} \quad \rho > 0$$

and from the assumption $k = \psi^r$ we obtain

$$\psi^{r-1}(\xi) \leq ck(\xi)(1+|\xi|)^{-\rho}$$

and so

$$\frac{\widetilde{P}(\xi)\psi^{r-1}(\xi)}{\widetilde{P}(\xi)k(\xi)} \to 0 \quad \text{when} \quad |\xi| \to \infty;$$

from Theorem 13 it follows that

$$\|u\|_{H_{\psi^{r-1}\widetilde{P}}} \leq D(\epsilon)\|u\|_{H_{k\widetilde{P}}} \quad \forall u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0)) \tag{70}$$

with $D(\epsilon) \to 0$ for $\epsilon \to 0$, then

$$\|AP(D)u\|_{L^2} \leq C_1 D(\epsilon)\|u\|_{H_{k\widetilde{P}}}.$$

From Theorem 10 it also follows

$$\|[P, A]u\|_{L^2} \leq C_2 \sum_\alpha \|(D_x^\alpha a)(x, D)(D_\xi^\alpha P)(D)\|_{L^2}$$

where the sum is finite because P is a polynomial in the variable ξ. Now we note that

$$(D_x^\alpha a)(x,\xi) \in S_\psi^{r-1} \quad \forall \alpha \in \mathbb{N}^n$$

therefore

$$\begin{aligned} \|(D_x^\alpha a)(x, D)(D_\xi^\alpha P)(D)u\|_{L^2} &\leq C_3\|(D_\xi^\alpha P)(D)u\|_{H_{\psi^{r-1}}} \\ &\leq C_4\|u\|_{H_{\psi^{r-1}\widetilde{D_\xi^\alpha P}}} \quad \forall \alpha \in \mathbb{N}^n. \end{aligned} \tag{71}$$

But $\widetilde{D_\xi^\alpha P}(\xi) \leq \widetilde{P}(\xi)$ for every $\xi \in \mathbb{R}^n$, then

$$\|u\|_{H_{\psi^{r-1}\widetilde{D_\xi^\alpha P}}} \leq C_5\|u\|_{H_{\psi^{r-1}\widetilde{P}}}.$$

From the same considerations used to obtain (70) it follows

$$\|[P, A]u\|_{L^2} \leq C_5 D(\epsilon)\|u\|_{H_{k\widetilde{P}}}$$

and so

$$\|P(D)Au\|_{L^2} \leq C_1(\epsilon)\|u\|_{H_{k\widetilde{P}}}$$

with $C_1(\epsilon) \to 0$ when $\epsilon \to 0$. In analogous way we estimate

$$\|Q_j(D)Au\|_{L^2} \le \|AQ_j(D)u\|_{L^2} + \|[Q_j, A]u\|_{L^2} \le C_2(\epsilon)\|u\|_{H_{k\widetilde{Q_j}}};$$

but, from the assumption $Q \prec\prec P$ it follows $\frac{\widetilde{Q_j}}{\widetilde{P}} < const.$, therefore, applying the properties of the $B_{p,k}$ spaces, we get

$$\|Q_j(D)Au\|_{L^2} \le C_3(\epsilon)\|u\|_{H_{k\widetilde{P}}}. \tag{72}$$

In conclusion we have obtained that there exists $g_1(\epsilon)$ such that $g_1(\epsilon) \to 0$ when $\epsilon \to 0$ and

$$\|Q_j(D)u\|_{H_k} \le g_1(\epsilon)\|u\|_{H_{k\widetilde{P}}} \tag{73}$$

for $j = 1, \ldots, M$ and every $u \in H_{k\widetilde{P}} \cap \mathcal{E}'(B_\epsilon(x^0))$. Replacing (73) in (65) we have

$$\begin{aligned} \|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} &\le g(\epsilon)\sum_{j=1}^{M}\|u^1 - u^2\|_{H_{k\widetilde{P}}}\|G_j(x, Qu^1, Qu^2)\|_{H_k} \\ &\le g_1(\epsilon)\sum_{j=1}^{M}\|\varphi_\epsilon L\varphi v^1 - \varphi_\epsilon L\varphi v^2\|_{H_{k\widetilde{P}}}\|G_j(x, Qu^1, \ldots, Qu^2)\|_{H_k}. \end{aligned} \tag{74}$$

In the proof of Theorem 11 we have already proved, see (58) and (60), that

$$\|\varphi_\epsilon L\varphi v\|_{H_{k\widetilde{P}}} \le A\|v\|_{H_k} \tag{75}$$

with A independent of ϵ, then

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \le g_2(\epsilon)\|v^1 - v^2\|_{H_k}\sum_{j=1}\|G_j(x, Qu^1, Qu^2)\|_{H_k}$$

with $g_2(\epsilon) \to 0$ for $\epsilon \to 0$. But $v^1, v^2 \in H_{k,R}$, then we have that if $\|v\|_{H_k} \le R$ then $\|Q_i u\|_{H_k} \le R_1$ for $i = 1, \ldots, M$ and

$$\|G_j(x, Qu^1, Qu^2)\|_{H_k} \le D \quad \text{for} \quad j = 1, \ldots, M;$$

therefore

$$\|\widetilde{F}_\epsilon(v^1) - \widetilde{F}_\epsilon(v^2)\|_{H_k} \le g_3(\epsilon)\|v^1 - v^2\|_{H_k}$$

with $g_3(\epsilon) \to 0$ for $\epsilon \to 0$. We can choose ϵ_1 such that for $\epsilon \le \epsilon_1$ we have $g_3(\epsilon) < 1$ and $\widetilde{F}_\epsilon$ is then a contraction. Now we will prove that there exists $\epsilon_2 > 0$ such that for $\epsilon \le \epsilon_2$ the corresponding operator $\widetilde{F}_\epsilon$ is defined from $H_{k,R}$ to itself, i.e. let v such that $\|v\|_{H_k} \le R$ for $\epsilon \le \epsilon_2$, then $\|\widetilde{F}_\epsilon v\|_{H_k} \le R$. We have to estimate

$$\begin{aligned} \|\widetilde{F}_\epsilon\|_{H_k} &\le \|\varphi f(x, Q_1(\varphi_\epsilon L\varphi v), \ldots, Q_M(\varphi_\epsilon L\varphi v))\|_{H_k} + \|g\|_{H_k} \\ &\le \left\|\varphi\sum_{|\alpha|>0} c_\alpha(x)(Q_1(\varphi_\epsilon L\varphi v), \ldots, Q_M(\varphi_\epsilon L\varphi v))^\alpha\right\|_{H_k} + \|g\|_{H_k}. \end{aligned} \tag{76}$$

Applying (61) and (62) and with the same considerations used to obtain (73) we get

$$\|\varphi f(x, Q_1(\varphi_\epsilon L\varphi v), ..., Q_M(\varphi_\epsilon L\varphi v))\|_{H_k} \leq \widetilde{D} R_1 g_1(\epsilon)$$

where we have already chosen ϵ such that $R_1 g_1(\epsilon) < 1$. Consider the constant ϵ_2 such that $R_1 g_1(\epsilon) \leq \frac{R}{2D}$ for $\epsilon \leq \epsilon_2$, then $\|\widetilde{F}_\epsilon v\|_{H_k} \leq R$ for every $v \in H_{k;R}$. Choosing $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ we have that the corresponding operator $\widetilde{F_{\epsilon_0}}$ is defined from $H_{k;R}$ to itself and is a contraction. According to the Contraction Principle we can assume that there exists a fixed point $v^0 \in H_{k;R}$ of the operator $\widetilde{F_{\epsilon_0}}$ and we may conclude as in the proof of Theorem 11.

□

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Francesca MESSINA
Dipartimento di Matematica
Università di Milano
Via Saldini 50
20133 Milano, ITALIA
e-mail: `messina@mat.unimi.it`

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