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$\mathfrak{P}$-SPACES OF IWASAWA TYPE AND ALGEBRAIC RANK ONE

Abstract. A Riemannian manifold is a $\mathfrak{P}$-space if the Jacobi operators along the geodesics are diagonalizable by a parallel orthonormal basis. We show that a solvable Lie group of Iwasawa type and algebraic rank one which is a $\mathfrak{P}$-space is a symmetric space of noncompact type and rank one. In particular, irreducible, non-flat homogeneous Einstein $\mathfrak{P}$-spaces with nonpositive curvature and algebraic rank one, are rank one symmetric spaces of noncompact type.

Let M be a Riemannian manifold, R its curvature tensor and R_X the Jacobi operator defined by $R_X Y = R(Y, X)X$, where X is a unit tangent vector. Following [3], we say that M is a $\mathfrak{P}$-space if for every geodesic γ in M the associated Jacobi operators $R_{\gamma'(t)}$ are diagonalizable by a parallel orthonormal basis along γ, condition that is satisfied for symmetric spaces.

In this paper we study the homogeneous $\mathfrak{P}$-spaces of Iwasawa type and algebraic rank one and in particular, those with nonpositive curvature which are Einstein, since irreducible, non-flat homogeneous Einstein spaces with nonpositive curvature are represented as Lie groups of Iwasawa type (see [6]). The geometry of Lie groups of Iwasawa type and algebraic rank one which at a first glance seems to be complicated, becomes very simple when they are $\mathfrak{P}$-spaces: they are Damek–Ricci spaces whose geometric structure is well known (Damek-Ricci spaces are defined in Section 1, following [2], Chapter 4).

An outline of the paper is as follows. In Sections 1 and 2, we give the basic results concerning the Lie algebras of Iwasawa type and algebraic rank one, its geodesics in various directions and the expression of the Jacobi operator along them. Properties of its eigenvalues in the special case of parallel eigenvectors are obtained in Section 3. The geometric hypothesis involving condition (P) about the eigenvalues is strongly used in Section 4 to obtain algebraic properties of the solvable Lie algebra. This information allows us to show that an Iwasawa type Lie group of algebraic rank one satisfying condition (P) is a Damek-Ricci space. By applying a result from [2], we obtain the following:

THEOREM. *If S is a solvable Lie group of Iwasawa type and algebraic rank one which is an $\mathfrak{P}$-space, then S is a rank one symmetric space of noncompact type.*

The class of Riemannian manifolds obtained by considering Lie groups of Iwasawa type contains as a subclass the Damek-Ricci spaces, and more generally the irreducible,

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non-flat homogeneous Einstein spaces with nonpositive curvature. As a consequence, we have

COROLLARY. *An irreducible, non-flat homogeneous Einstein $\mathfrak{P}$-space with non-positive curvature and algebraic rank one is a symmetric space of noncompact type and rank one.*

1. Lie algebras of Iwasawa type and algebraic rank one

A solvable Lie algebra $\mathfrak{s}$ with inner product $\langle , \rangle$ is a metric Lie algebra of Iwasawa type if it satisfies the conditions

(i) $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a}$, the orthogonal complement of $\mathfrak{n}$, is abelian.

(ii) The operators ad_H are symmetric for all $H \in \mathfrak{a}$.

(iii) For some $H_0 \in \mathfrak{a}$, $\mathrm{ad}_{H_0}|_{\mathfrak{n}}$ has positive eigenvalues.

The simply connected Lie group S with Lie algebra $\mathfrak{s}$ and left invariant metric induced by the inner product $\langle , \rangle$ will be called of Iwasawa type. The Levi Civita connection and the curvature tensor associated to the metric can be computed by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle , \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

for any X, Y, Z in $\mathfrak{s}$.

For each unit vector X in $\mathfrak{s}$, R_X, the Jacobi operator associated to X, is the symmetric endomorphism of $\mathfrak{s}$ defined by $R_X Y = R(Y, X)X$. We will say that either the metric Lie algebra $\mathfrak{s}$ satisfies condition (P) or S is a $\mathfrak{P}$-space if for every geodesic γ in S the associated Jacobi operator $R_{\gamma'(t)}$ can be diagonalized by a parallel orthonormal basis of $T_{\gamma(t)}S$. We note that condition (P) is equivalent to the fact $R_X \circ R'_X = R'_X \circ R_X$ for all $X \in \mathfrak{s}$ (see Corollary 5 of [3] and note that S is a real analytic C^∞-manifold). We recall that S is an Einstein space if

$$\mathrm{Ric}(X) = \mathrm{tr} R_X = c\,|X|^2 , \ c \text{ constant, for all } X \in \mathfrak{s}.$$

If $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ is a metric Lie algebra of Iwasawa type, let $\mathfrak{z}$ denote the center of $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and let $\mathfrak{v}$ be the orthogonal complement of $\mathfrak{z}$ with respect to the metric $\langle , \rangle$ restricted to $\mathfrak{n}$. Thus $\mathfrak{n}$ decomposes as $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, and for all $H \in \mathfrak{a}$ $\mathrm{ad}_H : \mathfrak{z} \to \mathfrak{z}$ and hence, $\mathrm{ad}_H : \mathfrak{v} \to \mathfrak{v}$ since ad_H is symmetric. We recall that $\mathfrak{n}$ is said to be 2-step nilpotent if $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$.

For any $Z \in \mathfrak{z}$ the skew-symmetric linear operator $j_Z : \mathfrak{v} \to \mathfrak{v}$ is defined by

$$\langle j_Z X, Y\rangle = \langle [X, Y], Z\rangle \ \text{ for all } X, \ Y \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}.$$

Equivalently, $j_Z X = (\mathrm{ad}_X)^* Z$ for all $X \in \mathfrak{v}$, where $(\mathrm{ad}_X)^*$ denotes the adjoint of ad_X. The operators j_Z coincide with the usual one in the case of a 2-step nilpotent $\mathfrak{n}$ (see [5]) and their properties determine the geometry of $\mathfrak{n}$ and $\mathfrak{s}$, as we will see.

We now assume that $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ is a metric Lie algebra of Iwasawa type and algebraic rank one; that is, $\mathfrak{a} = \mathbf{R}H$ where H, $|H| = 1$, is chosen such that all eigenvalues of

$\mathrm{ad}_H|_{\mathfrak{n}}$ are positive. S is called a Damek-Ricci space in the special case that $\mathrm{ad}_H|_{\mathfrak{z}} = \mathrm{Id}$, $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\,\mathrm{Id}$ and $j_Z^2 = -\,|Z|^2\mathrm{Id}$ for all $Z \in \mathfrak{z}$ (see [2], p. 78).

We recall that since $\mathrm{ad}_H|_{\mathfrak{n}}$ is a symmetric operator, $\mathfrak{n}$ has an orthogonal direct sum decomposition into eigenspaces $\mathfrak{n}_\mu$, for all eigenvalues μ of $\mathrm{ad}_H|_{\mathfrak{n}}$, which are invariant by ad_H with the property $[\mathfrak{n}_\mu, \mathfrak{n}_{\mu'}] \subset \mathfrak{n}_{\mu+\mu'}$ (by the Jacobi identity), whenever $\mu+\mu'$ is an eigenvalue of $\mathrm{ad}_H|_{\mathfrak{n}}$ (see [7]). Moreover, since $\mathfrak{z}$ and $\mathfrak{v}$ are ad_H-invariant, by the same argument they also have decompositions into their eigenspaces as $\mathfrak{z} = \sum_\lambda \mathfrak{z}_\lambda$, and $\mathfrak{v} = \sum_\mu \mathfrak{v}_\mu$.

1.1. Algebraic structure of the Lie algebra $\mathfrak{s}$

The definition of the Lie algebra structure on $\mathfrak{s}$ implies that, as a Lie algebra, $\mathfrak{s}$ is the semidirect sum $\mathfrak{s} = \mathfrak{n} +_\sigma \mathfrak{a}$ of $\mathfrak{n}$ and $\mathfrak{a} = \mathbf{R}H$, by considering the $\mathbf{R}$-algebra homomorphism $\sigma = \mathrm{ad}\colon \mathfrak{a} \to \mathrm{der}\,\mathfrak{n}$, $H \to (\mathrm{ad}_H : \mathfrak{n} \to \mathfrak{n})$. Carrying this over to the group level means that $S = N \times_\tau A$ is a semidirect product of N and $A = \mathbf{R}$ (considered in the canonical way), where

$$\tau : A \to \mathrm{Aut}N,\ \ \tau_a : x \to axa^{-1}, (d\tau_a)_e = \mathrm{Ad}(a),$$

is given by $a \exp Xa^{-1} = \exp_{\mathfrak{n}}(\mathrm{Exp}(t\mathrm{ad}_H)X)$ for all $X \in \mathfrak{n}$, $a = t$, and Exp denotes the exponential map of matrices. Note that S is diffeomorphic to $\mathfrak{s}$ under the map $(X, r) \to \big(\exp_{\mathfrak{n}} X, r\big)$ since $\exp_{\mathfrak{n}} : \mathfrak{n} \to N$, the exponential map of N, is a diffeomorphism.

We assume that $\mathfrak{n}$ is 2-step nilpotent. In this case we have that for any $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$, if Z^* and Y are eigenvectors of ad_H restricted to $\mathfrak{z}$ and $\mathfrak{v}$, with associated eigenvalues λ and μ, respectively, then the product in S yields

$$\begin{aligned} & (\exp_{\mathfrak{n}}(Z+X), r)\cdot(\exp_{\mathfrak{n}}(Z^*+Y), s) \\ = \ & (\exp_{\mathfrak{n}}(Z + e^{r\lambda}Z^* + \frac{1}{2}e^{r\mu}[X,Y] + X + e^{r\mu}Y), r+s). \end{aligned}$$

In fact, note that by the definition of the product in S we have

$$\begin{aligned} & (\exp_{\mathfrak{n}}(Z+X), r)\cdot(\exp_{\mathfrak{n}}(Z^*+Y), s) \\ = \ & \big(\exp_{\mathfrak{n}}(Z+X)\tau_r(\exp_{\mathfrak{n}}(Z^*+Y)), r+s\big) \\ = \ & \big(\exp_{\mathfrak{n}}(Z+X)\exp_{\mathfrak{n}}\big(\mathrm{Exp}(r\mathrm{ad}_H)Z^* + \mathrm{Exp}(r\mathrm{ad}_H)Y\big), r+s\big) \\ = \ & \big(\exp_{\mathfrak{n}}(Z+X)\exp_{\mathfrak{n}}(e^{r\lambda}Z^* + e^{r\mu}Y), r+s\big), \end{aligned}$$

since $\exp_{\mathfrak{n}} X \exp_{\mathfrak{n}} Y = \exp_{\mathfrak{n}}(X + Y + \frac{1}{2}[X,Y])$ gives the multiplication law in N (see the Campbell-Hausdorff formula in [7]).

1.2. Global coordinates in S

We introduce global coordinates in S given by $\varphi = (x_1, \ldots, x_k, y_1, \ldots, y_m, r)$, defined as follows. If $\{Z_1, \ldots, Z_k\}$ and $\{X_1, \ldots, X_m\}$ ($k = \dim\mathfrak{z}$, $m = \dim\mathfrak{v}$) are orthonormal bases of eigenvectors of ad_H in $\mathfrak{z}$ and $\mathfrak{v}$, with associated eigenvalues $\{\lambda_1, ..., \lambda_k\}$ and $\{\mu_1, ..., \mu_m\}$ respectively, then

$$\varphi(x_1, ..., x_k, y_1, ..., y_m, r) = \left(\exp_{\mathfrak{n}}(x_1 Z_1 + ... + x_k Z_k + y_1 X_1 + ... + y_m X_m), r\right).$$

Following the same argument as the given in [2], p. 82 for Damek-Ricci spaces, we see in the case of 2-step nilpotent $\mathfrak{n}$ that

$$\begin{aligned}
\left.\frac{\partial}{\partial x_i}\right|_{\varphi(x_1,...,x_k,y_1,...,y_m,r)} &= e^{-r\lambda_i} Z_i(\varphi(x_1, ..., x_k, y_1, ..., y_m, r)),\\
\left.\frac{\partial}{\partial y_i}\right|_{\varphi(x_1,...,x_k,y_1,...,y_m,r)} &= e^{-r\mu_i} X_i(\varphi(x_1, ..., x_k, y_1, ..., y_m, r))\\
&\quad + \frac{1}{2}\sum_{j,s} e^{-r\lambda_s} y_j \langle j_{Z_s} X_i, X_j\rangle Z_s(\varphi(x_1, ..., x_k, y_1, ..., y_m, r)),\\
\left.\frac{\partial}{\partial r}\right|_{\varphi(x_1,...,x_k,y_1,...,y_m,r)} &= H(\varphi(x_1, ..., x_k, y_1, ..., y_m, r)),
\end{aligned}$$

where Z_i, X_i and H on the right-hand side denote the left invariant vector fields on S associated to the corresponding vectors in $\mathfrak{s}$.

1.3. Curvature formulas

By applying the connection formula given at the beginning of this section, one obtains $\nabla_H = 0$ and if $Z, Z^* \in \mathfrak{z}$, $X, Y \in \mathfrak{v}$ then $\nabla_Z Z^* = \nabla_{Z^*} Z = \langle [H, Z], Z^*\rangle H$, $\nabla_X Z = \nabla_Z X = -\frac{1}{2} j_Z X$ and
$\nabla_X Y = \frac{1}{2}[X, Y] + \langle [H, X], Y\rangle H$, in case of 2-step nilpotent $\mathfrak{n}$. Consequently, by a direct computation, we obtain the following formulas (see [4], Section 2):

(i) $R_H = -\mathrm{ad}_H^2$.

(ii) If either $Z \in \mathfrak{z}_\lambda$, $|Z| = 1$, or $X \in \mathfrak{v}_\mu$, $|X| = 1$,

$$R_Z H = -\lambda^2 H \text{ and } R_X H = -\mu^2 H.$$

(iii) If $Z \in \mathfrak{z}_\lambda$, $|Z| = 1$, for any $Z^* \in \mathfrak{z}$ and $X \in \mathfrak{v}$, we have

$$R_Z Z^* = \lambda\left(\langle Z, \mathrm{ad}_H Z^*\rangle Z - \mathrm{ad}_H Z^*\right) \text{ and } R_Z X = -\frac{1}{4} j_Z^2(X) - \lambda \mathrm{ad}_H X.$$

In the case that $\mathfrak{n}$ is 2-step nilpotent, we obtain
(iv) If $X \in \mathfrak{v}_\mu$, $|X| = 1$, for any $Z \in \mathfrak{z}$, $Y \in \mathfrak{v}$

$$R_X Z = \frac{1}{4}[X, j_Z X] - \mu \mathrm{ad}_H Z \text{ and } R_X Y = -\frac{3}{4} j_{[X,Y]} X - \mu \mathrm{ad}_H Y.$$

2. Geodesics and associated Jacobi operators

Throughout this section and the following ones, $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ will denote a metric Lie algebra of Iwasawa type and algebraic rank one, where $\mathfrak{a} = \mathbf{R}H$, $|H| = 1$, is chosen such that all eigenvalues of $\mathrm{ad}_H|_{\mathfrak{n}}$ are positive, and $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ is expressed as in the previous section. Let γ_Y denote the geodesic in S satisfying $\gamma_Y(0) = e$ (the identity of S) and $\gamma_Y'(0) = Y$. For any $X \in \mathfrak{s}$, the associated left invariant field along the geodesic γ_Y will be denoted by $X(t) = X(\gamma_Y(t)) = (\mathrm{dL}_{\gamma_Y(t)})_e X$.

Next, we compute the geodesic γ_Y with $Y \in \mathfrak{n}$, an eigenvector of $\mathrm{ad}_H|_{\mathfrak{n}}$.

LEMMA 1. *If $Y \in \mathfrak{n}$ is an eigenvector of $ad_H|_{\mathfrak{n}}$ with eigenvalue α, then*

$$\gamma_Y(t) = \left(\exp_{\mathfrak{n}}\left(\frac{\tanh \alpha t}{\alpha}\right) Y, -\frac{1}{\alpha}\ln(\cosh \alpha t)\right)$$

with associated tangent vector field

$$\gamma_Y'(t) = \frac{1}{\cosh \alpha t} Y(t) - \tanh \alpha t H(t).$$

Proof. Let S_0 be the simply connected Lie group associated to $\mathfrak{s}_0$, the Lie algebra spanned by $\{Y, H\}$. Note that S_0 has global coordinates $\varphi(x, r) = (\exp_{\mathfrak{n}} xY, r)$ and it is a totally geodesic subgroup of S with connection $\nabla^{S_0} = \nabla|_{S_0}$ satisfying

$$\nabla_Y Y = \alpha H,\ \nabla_Y H = -[H, Y] = -\alpha Y,\ \nabla_H = 0.$$

Since the coordinate fields associated to φ are given by

$$\left.\frac{\partial}{\partial x}\right|_{\varphi(x,r)} = e^{-r\alpha} Y(\varphi(x, r)), \quad \left.\frac{\partial}{\partial r}\right|_{\varphi(x,r)} = H(\varphi(x, r)),$$

the Christoffel symbols are easily computed by the formulas

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x}} \left.\frac{\partial}{\partial x}\right|_{\varphi(x,r)} &= \alpha e^{-2r\alpha} H(\varphi(x, r)),\ \nabla_{\frac{\partial}{\partial x}} \left.\frac{\partial}{\partial r}\right|_{\varphi(x,r)} = -\alpha \left.\frac{\partial}{\partial x}\right|_{\varphi(x,r)}, \\ \nabla_{\frac{\partial}{\partial r}} \left.\frac{\partial}{\partial r}\right|_{\varphi(x,r)} &= 0 = \nabla_{\frac{\partial}{\partial r}} \left.\frac{\partial}{\partial x}\right|_{\varphi(x,r)}. \end{aligned}$$

Hence, we obtain the geodesic $\gamma_Y(t) = \varphi(x(t), r(t))$, where $x(t)$ and $r(t)$ are solutions of the differential equations

$$\begin{aligned} x'' - 2\alpha x' r' &= 0, \\ r'' + \alpha e^{-2r\alpha}(x')^2 &= 0. \end{aligned}$$

Using that $\gamma_Y'(t) = x'(t) \frac{\partial}{\partial x}\big|_{\gamma_Y(t)} + r'(t) \frac{\partial}{\partial r}\big|_{\gamma_Y(t)}$, $\left|\gamma_Y'(t)\right| = 1$, with

$$x'(t) = e^{2\alpha r(t)},\ \left.\frac{\partial}{\partial x}\right|_{\gamma_Y(t)} = e^{-\alpha r(t)} Y(t) \text{ and } \left.\frac{\partial}{\partial r}\right|_{\gamma_Y(t)} = H(t),$$

we have the equivalent equations

$$\begin{aligned} x'(t) &= e^{2\alpha r(t)}, \\ r''(t) + \alpha(1 - (r'(t))^2 &= 0 \end{aligned}$$

whose solutions satisfy

$$x(t) = \int_0^t e^{2\alpha r(u)} du \text{ and } r'(t) = -\tanh \alpha t.$$

Therefore, we get

$$r(t) = -\frac{1}{\alpha} \ln(\cosh \alpha t), \quad x(t) = \frac{1}{\alpha} \tanh \alpha t,$$

and the expression of γ_Y and γ_Y' follows as claimed.

□

PROPOSITION 1. *If $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ are eigenvectors of ad_H with associated eigenvalues λ and μ, respectively, then for any $Y \in \mathfrak{v}$, we have*

$$\begin{aligned} (i) \;\; R_{\gamma_Z'(t)} Y(t) &= \frac{1}{\cosh^2 \lambda t} (dL_{\gamma_Z(t)})_e \\ &\cdot \left(R_Z Y - \sinh^2 \lambda t \; ad_H^2 Y - \sinh \lambda t j_Z \left(\frac{1}{2} \lambda \, Id - ad_H \right) Y \right) \\ (ii) \;\; R_{\gamma_X'(t)} Z(t) &= \frac{1}{\cosh^2 \mu t} (dL_{\gamma_X(t)})_e \\ &\cdot \left(R_X Z - \lambda^2 \sinh^2 \mu t Z + (\lambda - \frac{1}{2} \mu) \sinh \mu t \; j_Z X \right) \\ (iii) \;\; R_{\gamma_X'(t)} Y(t) &= \frac{1}{\cosh^2 \mu t} (dL_{\gamma_X(t)})_e \\ &\cdot \left(R_X Y - \sinh^2 \mu t \; ad_H^2 Y - \sinh \mu t \left(\frac{1}{2} \mu Id - ad_H \right) [X, Y] \right) \end{aligned}$$

in the case of 2*-step nilpotent* $\mathfrak{n}$*, with* $Y \perp X$ *in* $\mathfrak{v}$*.*

Proof. (i) Let $Z \in \mathfrak{z}$ and $\gamma_Z(t)$ be the associated geodesic. Since

$$\gamma_Z'(t) = \frac{1}{\cosh \lambda t} Z(t) - \tanh \lambda t H(t), \text{ we have that}$$

$$\begin{aligned} R(Y(t), \gamma_Z'(t)) \gamma_Z'(t) &= \frac{1}{\cosh^2 \lambda t} (\mathrm{d}L_{\gamma_Z(t)})_e \\ &\cdot \left(R_Z Y + \sinh^2 \lambda t R_H Y - \sinh \lambda t \, (R(Y, Z) H + R(Y, H) Z) \right). \end{aligned}$$

Using the Bianchi identity and the connection formulas we compute

$$\begin{aligned} R(Y,Z)H + R(Y,H)Z &= 2R(Y,H)Z - R(Z,H)Y \\ &= 2\nabla_{[H,Y]}Z - \nabla_{[H,Z]}Y \\ &= -j_Z(\mathrm{ad}_H Y) + \frac{1}{2}\lambda j_Z Y, \end{aligned}$$

that substituted in the above expression, gives (i) as stated since $R_H = -\mathrm{ad}_H^2$.

(ii)-(iii) Assume that $X \in \mathfrak{v}$ is an eigenvector of ad_H with eigenvalue μ, and let $Y \perp X$ in $\mathfrak{v}$. Using the expression of $\gamma_X'(t)$, in the same way as (i) we get

$$\begin{aligned} R(Z(t),\gamma_X'(t))\gamma_X'(t) &= \frac{1}{\cosh^2 \mu t}(\mathrm{d}L_{\gamma_X(t)})_e \\ &\cdot \Big(R_X Z + \sinh^2 \mu t R_H Z - \sinh \mu t \,(R(Z,X)H + R(Z,H)X)\Big). \end{aligned}$$

Hence, the expression of $R_{\gamma_X'(t)}Z(t)$ follows as claimed since

$$\begin{aligned} R(Z,X)H + R(Z,H)X &= 2\nabla_{[H,Z]}X - \nabla_{[H,X]}Z = 2\lambda\nabla_Z X - \mu\nabla_X Z \\ &= -(\lambda - \frac{1}{2}\mu) j_Z X. \end{aligned}$$

Finally, we have

$$\begin{aligned} R(Y(t),\gamma_X'(t))\gamma_X'(t) &= \frac{1}{\cosh^2 \mu t}(\mathrm{d}L_{\gamma_X(t)})_e \\ &\cdot \Big(R_X Y + \sinh^2 \mu t R_H Y - \sinh \mu t \,(R(Y,X)H + R(Y,H)X)\Big). \end{aligned}$$

In the same way as above, in the case of 2-step nilpotent $\mathfrak{n}$, we compute

$$\begin{aligned} R(Y,X)H + R(Y,H)X &= 2\nabla_{[H,Y]}X - \nabla_{[H,X]}Y \\ &= -[H,[X,Y]] + \frac{1}{2}\mu[X,Y] \,(\langle X,Y\rangle = 0), \end{aligned}$$

which completes the proof of the proposition.

□

3. Eigenvectors and eigenvalues along Jacobi operators

In this section we assume that $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ is non-abelian and j_Z is non-singular on $\mathfrak{v}$ for all $Z \in \mathfrak{z}$. Note that if λ is an eigenvalue of $\mathrm{ad}_H|_{\mathfrak{z}}$ and $Z \in \mathfrak{z}_\lambda$ then $j_Z|_{\mathfrak{v}_\mu} : \mathfrak{v}_\mu \to \mathfrak{v}_{\lambda-\mu}$ for any eigenvalue μ of $\mathrm{ad}_H|_{\mathfrak{v}}$. In fact, for $X \in \mathfrak{v}_\mu$ and $Y \in \mathfrak{v}_{\mu'}$

$$\langle j_Z X, Y\rangle = \langle [X,Y], Z\rangle \neq 0 \Rightarrow [X,Y] \neq 0,$$

and thus $\mu + \mu' = \lambda$ since $[X,Y] \in \mathfrak{n}_{\mu+\mu'}$ and has non-zero component in $\mathfrak{z}_\lambda$.

Hence, for any eigenvalue μ of $\mathrm{ad}_H|_{\mathfrak{v}}$, $\lambda-\mu$ is also an eigenvalue of $\mathrm{ad}_H|_{\mathfrak{v}}$, $\lambda > \mu$ and the symmetric operator j_Z^2 preserves $\mathfrak{v}_\mu$. Moreover, the map $Z \to [X, j_Z X]$ defines a symmetric operator on $\mathfrak{z}$ ($\langle [X, j_Z X], Z^* \rangle = \langle j_{Z^*}X, j_Z X\rangle = \langle [X, j_{Z^*}X], Z\rangle$) such that $[X, j_Z X] \in \mathfrak{z}_\lambda$ for all $X \in \mathfrak{v}_\mu$ since $\langle [X, j_Z X], Z\rangle = |j_Z X|^2 \neq 0$, $[X, j_Z X] \in \mathfrak{n}_\lambda$ (the Jacobi identity) and $\lambda > \mu$.

Next, we describe the eigenvalues of the operators $R_{\gamma_Z'(t)}$, $R_{\gamma_X'(t)}$ and the parallel vector fields along the geodesics $\gamma_Z(t)$ and $\gamma_X(t)$ for some $Z \in \mathfrak{z}_\lambda$ and $X \in \mathfrak{v}_\mu$.

LEMMA 2. *Let $Z \in \mathfrak{z}_\lambda$ and $X \in \mathfrak{v}_\mu$ be unit vectors. We set $Y = \frac{j_Z X}{|j_Z X|}$.*

(i) If X is an eigenvector of j_Z^2, then $R_{\gamma_Z'(t)}$ has an eigenvector $U(t) = x(t)X(t) + y(t)Y(t)$ with $x(t)^2 + y(t)^2 = 1$ and associated eigenvalue $a_Z(t)$ satisfying

$$a_Z(t)\cosh^2\lambda t = \langle R_Z Y, Y\rangle - (\lambda-\mu)^2\sinh^2\lambda t - (\frac{1}{2}\lambda - \mu)\,|j_Z X|\,\frac{x(t)}{y(t)}\sinh\lambda t$$

whenever $y(t) \neq 0$.

(ii) Assume that $\mathfrak{n}$ is 2-step nilpotent. If X satisfies $[X, j_Z X] = |j_Z X|^2 Z$, then $R_{\gamma_X'(t)}$ has an eigenvector $U(t) = x(t)Z(t) + y(t)Y(t)$, $x(t)^2 + y(t)^2 = 1$, whose associated eigenvalue $a_X(t)$ is given by

$$\begin{aligned} & a_X(t)\cosh^2\mu t \\ = \; & \frac{1}{4}|j_Z X|^2 - \lambda\mu - \lambda^2\sinh^2\mu t + (\lambda - \frac{1}{2}\mu)\,|j_Z X|\,\frac{y(t)}{x(t)}\sinh\mu t \quad \textit{or} \\ & a_X(t)\cosh^2\mu t \\ = \; & -\frac{3}{4}|j_Z X|^2 - (\lambda-\mu)(\mu + (\lambda-\mu)\sinh^2\mu t\,) + (\lambda - \frac{1}{2}\mu)\,|j_Z X|\,\frac{x(t)}{y(t)}\sinh\mu t \end{aligned}$$

according to $x(t) \neq 0$ or $y(t) \neq 0$, respectively.

Proof. Note that by the properties of Z and X, the spaces $\mathrm{span}\{X(t), Y(t)\}$ and $\mathrm{span}\{Z(t), Y(t)\}$ are invariant under the symmetric operators $R_{\gamma_Z'(t)}$ and $R_{\gamma_X'(t)}$, respectively. The assertion of the lemma follows from the expression of $R_{\gamma_Z'(t)}$ and $R_{\gamma_X'(t)}$ given in Proposition 1 and using in each case the equalities

$$R_{\gamma_Z'(t)}(U(t)) = a_Z(t)(U(t)) \text{ and } R_{\gamma_X'(t)}(U(t)) = a_X(t)U(t).$$

Note that in the last case,

$$R_X Z = \frac{1}{4}|j_Z X|^2 Z - \lambda\mu Z \quad\text{and}\quad R_X Y = -\frac{3}{4}|j_Z X|^2 Y - \mu(\lambda-\mu)Y$$

since $\mathfrak{n}$ is 2-step nilpotent.

□

PROPOSITION 2. *Let $Z \in \mathfrak{z}_\lambda$ be a unit vector. If $X \in \mathfrak{v}_\mu$, $|X| = 1$, is an eigenvector of j_Z^2, then the parallel vector field U along the geodesic γ_Z with $U(0) = x_0 X + y_0 Y$*

($Y = \frac{j_Z X}{|j_Z X|}$), $x_0^2 + y_0^2 = 1$, is given by $U(t) = x(t)X(t) + y(t)Y(t)$ where

$$x(t) = x_0 \cos s(t) - y_0 \sin s(t)\ ,\ \ y(t) = x_0 \sin s(t) + y_0 \cos s(t),\ \textit{and}$$

$$s(t) = \frac{1}{2}\,|j_Z X| \int_0^t \frac{du}{\cosh \lambda u}.$$

Proof. We first note that $(\mathrm{d}L_{\gamma_Z(t)})_e(\mathrm{span}\{X, j_Z X\})$ is invariant under $\nabla_{\gamma_Z'(t)}$ since $\gamma_Z'(t) = \frac{1}{\cosh \lambda t} Z(t) - \tanh \lambda t H(t)$, $\nabla_Z X = -\frac{1}{2} j_Z X$, $\nabla_Z j_Z X = \frac{1}{2}\,|j_Z X|^2\, X$ and $\nabla_H = 0$.
Hence the parallel vector field U along γ_Z with $U(0) = x_0 X + y_0 Y$ is given by $U(t) = x(t)X(t) + y(t)Y(t)$ satisfying the equation $\nabla_{\gamma_Z'(t)} U(t) = 0$, which gives

$$x'(t)X + x(t)\frac{1}{\cosh \lambda t}\nabla_Z X + y'(t)Y + y(t)\frac{1}{\cosh \lambda t}\nabla_Z Y = 0 \text{ for all real } t$$

since X and Y are left invariant. Thus,

$$x'(t)X - x(t)\frac{1}{2\cosh \lambda t} j_Z X + y'(t)Y - y(t)\frac{1}{2\cosh \lambda t} j_Z Y = 0,$$

and $x(t), y(t)$ are solutions of the differential equations

$$x'(t) + \frac{|j_Z X|}{2\cosh \lambda t} y(t) = 0,\ \ y'(t) - \frac{|j_Z X|}{2\cosh \lambda t} x(t) = 0$$

since $j_Z Y = -\,|j_Z X|\, X$. By expressing these equations in the matrix form as

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \frac{1}{2}\frac{|j_Z X|}{\cosh \lambda t}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

the solutions are given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathrm{Exp} s(t) J \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},\ \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$s(t) = \frac{1}{2}\,|j_Z X| \int_0^t \frac{du}{\cosh \lambda u}$ (Exp denotes the exponential map of matrices). The assertion of the proposition follows since $\mathrm{Exp} s J = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}$.

□

PROPOSITION 3. *Assume that $\mathfrak{n}$ is 2-step nilpotent. Let $X \in \mathfrak{v}_\mu$ and $Z \in \mathfrak{z}_\lambda$ be unit vectors satisfying $[X, j_Z X] = |j_Z X|^2\, Z$. Then the parallel vector field U along the geodesic γ_X with $U(0) = x_0 Z + y_0 Y$ ($Y = \frac{j_Z X}{|j_Z X|}$), $x_0^2 + y_0^2 = 1$, is given by $U(t) = x(t)Z(t) + y(t)Y(t)$ where*

$$x(t) = x_0 \cos s(t) - y_0 \sin s(t)\ ,\ \ y(t) = x_0 \sin s(t) + y_0 \cos s(t)\ \textit{and}$$

$$s(t) = \frac{1}{2}\,|j_Z X| \int_0^t \frac{du}{\cosh \mu u}.$$

Proof. Note that the parallel displacement $U(t)$ of $U(0)$ along $\gamma_X(t)$ is expressed as $U(t) = x(t)Z(t)+y(t)Y(t)$, since $(\mathrm{d}L_{\gamma_X(t)})_e(\mathrm{span}\{Z, j_Z X\})$ is invariant under $\nabla_{\gamma'_X(t)}$. In the same way as in Proposition 2, the equation $\nabla_{\gamma'_X(t)}U(t) = 0$ gives

$$x'(t)Z - x(t)\frac{|j_Z X|}{2\cosh \mu t}Y + y'(t)Y + y(t)\frac{|j_Z X|}{2\cosh \mu t}Z = 0 \text{ for all real } t.$$

Hence, $x(t)$ and $y(t)$ are solutions of the differential equations

$$x'(t) + \frac{|j_Z X|}{2\cosh \mu t}y(t) = 0, \quad y'(t) - \frac{|j_Z X|}{2\cosh \mu t}x(t) = 0,$$

which are given by

$$x(t) = x_0 \cos s(t) - y_0 \sin s(t), \quad y(t) = x_0 \sin s(t) + y_0 \cos s(t)$$

with $s(t) = \frac{1}{2}\,|j_Z X| \int_0^t \frac{du}{\cosh \mu u}$.

□

4. Condition (P) on Lie algebras of Iwasawa type and rank one

In this section we will assume that $\mathfrak{s}$ is a metric Lie algebra of Iwasawa type with non-abelian $\mathfrak{n}$ and algebraic rank one satisfying condition (P). We summarize in the proposition below some properties of the algebraic structure of the Lie algebra $\mathfrak{n} = \mathfrak{z}\oplus\mathfrak{v}$ (Similar ones were obtained in [4], Proposition 1.3). The following lemma will be useful in what follows.

LEMMA 3. *If S_0 is a totally geodesic subgroup of a Lie group S which is a $\mathfrak{P}$-space, then S_0 is a $\mathfrak{P}$-space. Equivalently, a totally geodesic subalgebra $\mathfrak{s}_0$ of a Lie algebra $\mathfrak{s}$ satisfying condition (P) satisfies condition (P).*

Proof. Let $\mathfrak{s}$ and $\mathfrak{s}_0$ denote the Lie algebras of S and S_0, respectively, with associated curvature tensors R and R_0. Note that for a unit $X \in \mathfrak{s}$ the symmetric operator $R'_X = \nabla_{\gamma'_X(t)}\left(R_{\gamma'_X(t)}\right)\Big|_{t=0}$ is defined along $\gamma_X(t)$ by $R'_{\gamma'_X(t)} = \nabla_{\gamma'_X(t)}\left(R_{\gamma'_X(t)}\right)$ on $\mathfrak{s}(t) = (\mathrm{d}L_{\gamma_X(t)})_e\mathfrak{s}$, and for any orthonormal parallel basis $\{e_i(t)\}$ satisfying $R_{\gamma'_X(t)}e_i(t) = a_i(t)e_i(t)$, we have

$$R'_{\gamma'_X(t)}e_i(t) = \nabla_{\gamma'_X(t)}\left(R_{\gamma'_X(t)}e_i(t)\right) - R_{\gamma'_X(t)}\left(\nabla_{\gamma'_X(t)}e_i(t)\right) = a'_i(t)e_i(t).$$

Let $X \in \mathfrak{s}_0$ be a unit vector, and note that $\mathfrak{s}_0$ and $\mathfrak{s}_0^{\perp}$ are invariant under the symmetric operators R_X and R'_X in $\mathfrak{s}$ since $\mathfrak{s}_0$ is a totally geodesic subalgebra of $\mathfrak{s}$ and $R_0 = R|_{\mathfrak{s}_0}$. Hence, $\mathfrak{s}_0$ and $\mathfrak{s}_0^{\perp}$ are also invariant by the skew-symmetric operator $R_X \circ R'_X - R'_X \circ R_X$. Using that condition (P) is equivalent to $R_X \circ R'_X - R'_X \circ R_X = 0$ for all unit vectors X in $\mathfrak{s}$, it follows that $R_{0X} \circ R'_{0X} - R'_{0X} \circ R_{0X} = 0$ (R_{0X} and R'_{0X} are the restrictions of R_X and R'_X to $\mathfrak{s}_0$, respectively). Thus S_0 is a $\mathfrak{P}$-space.

□

PROPOSITION 4. *Let $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$, $|H| = 1$, be a metric Lie algebra of Iwasawa type that satisfies condition (P). Then*

(i) $ad_H|_{\mathfrak{z}} = \lambda Id$.

(ii) If $\mathfrak{n}$ is non-abelian, $\lambda > \mu_1$, the maximum eigenvalue of $ad_H|_{\mathfrak{v}}$. In particular $\mathfrak{n}_\lambda = \mathfrak{z}$ and $[\mathfrak{n}_{\mu_1}, \mathfrak{v}] \subseteq \mathfrak{z}$.

(iii) For any $Z \in \mathfrak{z}$ the linear map $j_Z : \mathfrak{v} \to \mathfrak{v}$ is an isomorphism with the properties $j_Z|_{\mathfrak{v}_\mu} : \mathfrak{v}_\mu \to \mathfrak{v}_{\lambda-\mu}$ and $j_Z^2\big|_{\mathfrak{v}_\mu} : \mathfrak{v}_\mu \to \mathfrak{v}_\mu$ (isomorphically). In particular, if μ is an eigenvalue of $ad_H|_{\mathfrak{v}}$, then $\lambda - \mu$ is also an eigenvalue of $ad_H|_{\mathfrak{v}}$.

Proof. (i) Let $\mathfrak{s}_0 = \mathfrak{z} \oplus \mathbf{R}H$ be the subalgebra of $\mathfrak{s}$ with associated simply connected Lie group S_0. It follows from the above lemma that S_0 is a $\mathfrak{P}$-space since $\mathfrak{s}_0$ is a totally geodesic subalgebra of $\mathfrak{s}$. Moreover, if $Z \perp Z^*$ are eigenvectors of $\mathrm{ad}_H|_{\mathfrak{z}}$ with associated eigenvalues λ and λ^*, respectively, the Lie algebra spanned by $\{Z, Z^*, H\}$ is a three-dimensional totally geodesic subalgebra of $\mathfrak{s}_0$, whose associated Lie group is a $\mathfrak{P}$-space, by the previous lemma. Then we can assume that $\mathfrak{s} = \mathfrak{s}_0$ and it is spanned by $\{Z, Z^*, H\}$. Using Lemma 7 and the Remark on page 73 of [3], it follows that the condition $R_X \circ R'_X - R'_X \circ R_X = 0$ implies that

$$(\nabla_Z \mathrm{Ric})\,(Z, H) = (\nabla_{Z^*} \mathrm{Ric})\,(Z^*, H),$$

where the Ricci tensor associated to S is defined by $\mathrm{Ric}(X, Y) = \mathrm{tr}(V \to R(V, X)Y)$. We compute

$$\begin{aligned} \mathrm{Ric}(Z, Z^*) &= \mathrm{Ric}(Z, H) = \mathrm{Ric}(Z^*, H) = 0,\ \mathrm{Ric}(Z, Z) = -\lambda(\lambda + \lambda^*), \\ \mathrm{Ric}(Z^*, Z^*) &= -\lambda^*(\lambda + \lambda^*) \text{ and } \mathrm{Ric}(H, H) = -(\lambda^2 + \lambda^{*2}). \end{aligned}$$

As a consequence, it is easy to see that

$$\begin{aligned} (\nabla_Z \mathrm{Ric})\,(Z, H) &= -\mathrm{Ric}(\nabla_Z Z, H) - \mathrm{Ric}(Z, \nabla_Z H) \\ &= \lambda\,(\mathrm{Ric}(Z, Z) - \mathrm{Ric}(H, H)) = \lambda\lambda^*(\lambda^* - \lambda) \end{aligned}$$

and similarly,

$$(\nabla_{Z^*} \mathrm{Ric})\,(Z^*, H) = \lambda\lambda^*(\lambda - \lambda^*).$$

Hence $\lambda^* = \lambda$ and there is a unique eigenvalue of $\mathrm{ad}_H|_{\mathfrak{z}}$.

(ii) Assume that $\mathfrak{n}$ is non-abelian. Then we have that $[\mathfrak{n}_{\mu_1}, \mathfrak{n}_\mu] \neq 0$ for some eigenvalue μ of $\mathrm{ad}_H|_{\mathfrak{v}}$, which implies that $\mu_1 + \mu = \lambda$ since μ_1 is the maximum. Hence, $\lambda > \mu_1 \geq \mu$ for all eigenvalues μ of ad_H in $\mathfrak{v}$, and it also follows that $[\mathfrak{n}_{\mu_1}, \mathfrak{v}] \subseteq \mathfrak{z}$ from the definition of μ_1.

(iii) We recall that $j_Z|_{\mathfrak{n}_\mu} : \mathfrak{n}_\mu \to \mathfrak{n}_{\lambda-\mu}$. Hence j_Z^2 preserves $\mathfrak{n}_\mu$ for all μ (see the beginning of Section 3). Moreover, $\ker j_Z$ is invariant by ad_H since the condition $j_Z X = 0$ implies that

$$\begin{aligned} \langle j_Z([H, X]), Y\rangle &= \langle[[H, X], Y], Z\rangle = -\langle[[X, Y], H], Z\rangle - \langle[[Y, H], X], Z\rangle \\ &= \langle[X, Y], [H, Z]\rangle + \langle j_Z([H, Y]), X\rangle \\ &= \lambda\langle j_Z X, Y\rangle - \langle[H, Y], j_Z X\rangle = 0 \text{ for all } Y \in \mathfrak{v}. \end{aligned}$$

We will show that $j_Z X \neq 0$ for all unit Z in $\mathfrak{z}$ and $X \in \mathfrak{v}$. If $j_Z X = 0$, then by the previous remark we can assume that $X \in \mathfrak{n}_\mu$ and $|X| = 1$. Then $\mathfrak{s}_0$, the Lie algebra spanned by $\{Z, X, H\}$, is a totally geodesic subalgebra of $\mathfrak{s}$ since $\nabla_Z Z = \lambda H$, $\nabla_X X = \mu H$, $\nabla_Z X = \nabla_X Z = 0$ and $\nabla_Z H = -\lambda Z$, $\nabla_X H = -\mu X$, $\nabla_H = 0$. Hence, $\mathfrak{s}_0$ satisfies condition (P) and applying the same argument used to show that $\lambda = \lambda^*$ in (i) above, we obtain that $\lambda = \mu$, contradicting (ii). Consequently, $\lambda - \mu$ is an eigenvalue of $\mathrm{ad}_H|_{\mathfrak{v}}$ whenever μ is an eigenvalue since $j_Z|_{\mathfrak{n}_\mu} : \mathfrak{n}_\mu \to \mathfrak{n}_{\lambda-\mu}$. Assertion (iii) follows since $\dim \mathfrak{v}_\mu \leq \dim \mathfrak{v}_{\lambda-\mu} \leq \dim \mathfrak{v}_\mu$ (j_Z is an isomorphism).

□

Next we will show that under the hypothesis of condition (P), the number of eigenvalues of $\mathrm{ad}_H|_{\mathfrak{n}}$ can be reduced to at most two, namely λ and $\frac{1}{2}\lambda$, attained in $\mathfrak{z}$ and $\mathfrak{v}$, respectively. For any subspace $\mathfrak{u}$ of the Lie algebra $\mathfrak{s}$ and γ_X ($X \in \mathfrak{s}$) a fixed geodesic in S, we will denote by $\mathfrak{u}(t) = (\mathrm{dL}_{\gamma_X(t)})_e \mathfrak{u}$.

REMARK 1. If $\mathfrak{s}_0 \oplus \mathfrak{u}$ is a direct sum decomposition into subspaces of the Lie algebra $\mathfrak{s}$ such that $\mathfrak{s}_0(t)$ and $\mathfrak{u}(t)$ are invariant under $\nabla_{\gamma'_X(t)}$ and $R_{\gamma'_X(t)}$ for all t, then $e(t) = e_1(t) + e_2(t)$, expressed according to the decomposition $\mathfrak{s}_0(t) \oplus \mathfrak{u}(t)$, is a parallel eigenvector of $R_{\gamma'_X(t)}$ along γ_X if and only if $e_1(t)$ and $e_2(t)$ are also parallel eigenvectors of $R_{\gamma'_X(t)}$.

In the case of an orthogonal direct sum decomposition, if $\mathfrak{s}_0(t)$ is invariant under $\nabla_{\gamma'_X(t)}$ and $R_{\gamma'_X(t)}$, then $\mathfrak{u}(t) = \mathfrak{s}_0(t)^\perp$ is also invariant under $\nabla_{\gamma'_X(t)}$ and $R_{\gamma'_X(t)}$ since $\nabla_{\gamma'_X(t)}$ is skew-symmetric and $R_{\gamma'_X(t)}$ is symmetric.

PROPOSITION 5. *If* $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$, $|H| = 1$, *is a metric Lie algebra of Iwasawa type with non-abelian nilpotent* $\mathfrak{n}$ *satisfying condition* (P), *then the eigenvalues of* $ad_H|_{\mathfrak{n}}$ *are* λ *and* $\frac{1}{2}\lambda$, *when restricted to* $\mathfrak{z}$ *and* $\mathfrak{v}$, *respectively. Then* $\mathfrak{n}$ *is* 2*-step nilpotent.*

Proof. Note that $\mathrm{ad}_H|_{\mathfrak{z}} = \lambda \mathrm{Id}$, by Proposition 4. Next we will show that $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\lambda \mathrm{Id}$. For this purpose, we fix an eigenvalue μ of $\mathrm{ad}_H|_{\mathfrak{v}}$ and assume that $\mu \neq \frac{1}{2}\lambda$. If $Z \in \mathfrak{z}$ is a unit vector, it follows from the definition of $\nabla_{\gamma'_Z(t)}$ and the expression of $R_{\gamma'_Z(t)}$ given by Proposition 1 (i) that for any eigenvalue μ^* of $\mathrm{ad}_H|_{\mathfrak{v}}$, $\mathfrak{v}_{\mu^*}(t) \oplus \mathfrak{v}_{\lambda-\mu^*}(t)$ (or $\mathfrak{v}_{\frac{1}{2}\lambda}(t)$ in case $\mu^* = \frac{1}{2}\lambda$) is invariant under $R_{\gamma'_Z(t)}$ and $\nabla_{\gamma'_Z(t)}$ since $\mathfrak{v}_{\mu^*} \oplus \mathfrak{v}_{\lambda-\mu^*}$ is R_Z-invariant and $\nabla_{\gamma'_Z(t)} X(t) = -\frac{1}{2\cosh \lambda t} j_Z X(t)$.

Assume that the condition (P) is satisfied and let $\{e_i(t) : i = 1, ..., \dim \mathfrak{s}\}$ be an orthonormal parallel basis that diagonalizes $R_{\gamma'_Z(t)}$. Therefore, $e_i(0)$ has a non-zero component $e \in \mathfrak{v}_\mu \oplus \mathfrak{v}_{\lambda-\mu}$ for some $i = 1, ..., \dim \mathfrak{s}$ (otherwise $\{e_i(0) : i = 1, ..., \dim \mathfrak{s}\}$ would be a basis of $\mathfrak{z} \oplus \sum_{\mu^* \neq \mu, \lambda-\mu} \mathfrak{v}_{\mu^*} \oplus \mathbf{R}H$). It follows from the previous remark that $e(t)$, the parallel displacement of e along $\gamma_Z(t)$, is a parallel eigenvector of $R_{\gamma'_Z(t)}$ with $e(t) \in \mathfrak{v}_\mu \oplus \mathfrak{v}_{\lambda-\mu}(t)$.

Now, we choose a basis $\{X_i : i = 1, ..., m = \dim \mathfrak{v}_\mu\}$ of $\mathfrak{v}_\mu$ that diagonalizes the symmetric operator $j_Z^2 : \mathfrak{v}_\mu \to \mathfrak{v}_\mu$. Hence $\{X_i, j_Z X_i : i = 1, ..., m\}$ is an orthogonal basis of $\mathfrak{v}_\mu \oplus \mathfrak{v}_{\lambda-\mu}$ by Proposition 4, and $\mathfrak{v}_\lambda \oplus \mathfrak{v}_{\lambda-\mu} = \oplus_{i=1}^m \mathrm{span}\{X_i, j_Z X_i\}$, where each $\mathrm{span}\{X_i, j_Z X_i\}(t)$ is invariant under $\nabla_{\gamma'_Z(t)}$ and $R_{\gamma'_Z(t)}$ (X_i and $j_Z X_i$ are eigenvectors of R_Z). Applying Remark 1 again, we can choose a unit $X \in \mathfrak{v}_\mu$ such that $j_Z^2 X = -|j_Z X|^2 X$ ($X = X_i$ for some $i = 1, ..., m$ so that $e = \sum e_i$, $e_i \neq 0$) and a parallel eigenvector $U(t)$ of $R_{\gamma'_Z(t)}$ along the geodesic $\gamma_Z(t)$ satisfying $U(t) = x(t)X(t) + y(t)Y(t)$ ($Y = \frac{j_Z X}{|j_Z X|}$), with $x(t)^2 + y(t)^2 = 1$.

We set $U(0) = x_0 X + y_0 Y$ and let $a_Z(t)$ be the associated eigenvalue of $R_{\gamma'_Z(t)}$.

If $x_0 \neq 0$ and $y_0 \neq 0$, we have that $R_Z X = a_Z(0)X$, $R_Z Y = a_Z(0)Y$. Therefore, $\langle R_Z X, X\rangle = a_Z(0) = \langle R_Z Y, Y\rangle$ and consequently $\mu = \frac{1}{2}\lambda$ since

$$\frac{1}{4}|j_Z X|^2 - \lambda\mu = \frac{1}{4}|j_Z X|^2 - \lambda(\lambda - \mu).$$

Assume that $x_0 \neq 0$ and $y_0 = 0$, hence $x_0 = 1$, $y_0 = 0$ and $a_Z(0) = \langle R_Z X, X\rangle$. It follows from Lemma 2 (applied at $t = 0$) and Proposition 2 that

$$a_Z(0) = \langle R_Z Y, Y\rangle - (\frac{1}{2}\lambda - \mu)\,|j_Z X| \lim_{t\to 0} \frac{x(t)}{y(t)} \sinh \lambda t$$

where $x(t) = \cos s(t)$, $y(t) = \sin s(t)$ with $s'(t) = \frac{1}{2}|j_Z X| \frac{1}{\cosh \lambda t}$. We compute

$$\begin{aligned} \lim_{t\to 0}\left(\frac{x(t)}{y(t)}\sinh\lambda t\right) &= \lim_{t\to 0}\left(\frac{(\sinh\lambda t)'}{(\tan s(t))'}\right) = \lambda \lim_{t\to 0}\frac{\cosh^2\lambda t \cosh^2 s(t)}{\frac{1}{2}|j_Z X|} \\ &= \frac{2\lambda}{|j_Z X|} \quad (s(t) \to 0), \end{aligned}$$

which substituted in the above expression gives

$$a_Z(0) = \frac{1}{4}|j_Z X|^2 - \lambda(2\lambda - 3\mu).$$

From the equality $a_Z(0) = \langle R_Z X, X\rangle$, we obtain $\mu = \frac{1}{2}\lambda$ and get a contradiction. The same argument is used in the case $x_0 = 0$, $y_0 = 1$.

Now we observe that the conditions $\mathrm{ad}_H|_{\mathfrak{z}} = \lambda \mathrm{Id}$ and $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\lambda \mathrm{Id}$ imply that the eigenspaces associated to $\mathrm{ad}_H|_{\mathfrak{n}}$ are $\mathfrak{n}_\lambda = \mathfrak{z}$ and $\mathfrak{n}_{\frac{1}{2}\lambda} = \mathfrak{v}$. Thus, $[\mathfrak{v}, \mathfrak{v}] = [\mathfrak{n}_{\frac{1}{2}\lambda}, \mathfrak{n}_{\frac{1}{2}\lambda}] \subseteq \mathfrak{n}_\lambda = \mathfrak{z}$, showing that $\mathfrak{n}$ is 2-step nilpotent.

□

EXAMPLE 1. Consider the four-dimensional metric Lie algebra $\mathfrak{s}$ of Iwasawa type and algebraic rank one with nilpotent non-abelian $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. Hence, $\mathfrak{s}$ is spanned by an orthogonal basis $\{Z, X, j_Z X, H\}$ with unit vectors $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$ and Lie bracket

$$\begin{aligned} [Z, X] &= 0 = [Z, j_Z X], \ \ [X, j_Z X] = |j_Z X|^2 Z \\ [H, Z] &= \lambda Z, \ [H, X] = \frac{1}{2}\lambda X, \ [H, j_Z X] = \frac{1}{2}\lambda j_Z X. \end{aligned}$$

Note that $j_Z^2 X = -|j_Z X|^2 X$. We show that the Lie group S associated to $\mathfrak{s}$ is a $\mathfrak{P}$-space if and only if $|j_Z X| = \lambda$. Thus, up to scaling, S is the 2-complex hyperbolic space $\mathbf{C}H^2$.

For this purpose we see that

$$R_{Z+X} \circ R'_{Z+X}(H) - R'_{Z+X} \circ R_{Z+X}(H) = 0 \Leftrightarrow |j_Z X| = \lambda.$$

Note that from the connection formulas,
$\nabla_Z Z = \lambda H,\ \nabla_X X = \frac{1}{2}\lambda H,\ \nabla_{j_Z X} j_Z X = \frac{1}{2}\lambda\, |j_Z X|^2\, H,\ \nabla_Z X = -\frac{1}{2} j_Z X,\ \nabla_H = 0,$
we get

$$\begin{aligned}
R_Z X &= \frac{1}{2}\left(\frac{1}{2}\,|j_Z X|^2 - \lambda^2\right) X,\\
R_Z(j_Z X) &= \frac{1}{2}\left(\frac{1}{2}\,|j_Z X|^2 - \lambda^2\right) j_Z X,\\
R_X Z &= \frac{1}{2}\left(\frac{1}{2}\,|j_Z X|^2 - \lambda^2\right) Z,\\
R_X\, j_Z X &= -\frac{1}{4}\left(3\,|j_Z X|^2 + \lambda^2\right) j_Z X,\\
R_{j_Z X} Z &= \frac{1}{2}\,|j_Z X|^2 \left(\frac{1}{2}\,|j_Z X|^2 - \lambda^2\right) Z,\\
R_{j_Z X} X &= -\frac{1}{4}\,|j_Z X|^2 \left(3\,|j_Z X|^2 + \lambda^2\right) X.
\end{aligned}$$

Hence, taking into account that $R_{Z+X}(\cdot) = R_Z(\cdot) + R_X(\cdot) + R(\cdot, Z)X + R(\cdot, X)Z$, it is a direct computation to see that

(1)
$$\begin{aligned}
R_{Z+X}(H) &= \frac{1}{4}\left(3\lambda j_Z X - 5\lambda^2 H\right),\\
R_{Z+X}(j_Z X) &= -\left(\frac{1}{2}\,|j_Z X|^2 + \frac{3}{4}\lambda^2\right) j_Z X + \frac{3}{4}\lambda\,|j_Z X|^2\, H,\\
R_{Z+X}(Z - X) &= \left(\frac{1}{2}\,|j_Z X|^2 - \lambda^2\right)(Z - X),
\end{aligned}$$

Recall that, by definition, $R'_{Z+X}(\cdot) =$

$$[\nabla_{Z+X}, R_{Z+X}](\cdot) - R(\cdot, \nabla_{Z+X}(Z+X))(Z+X) - R(\cdot, Z+X)\nabla_{Z+X}(Z+X),$$

and by a straightforward computation, using the connection formulas and the definition of $R,$ we obtain the following expressions of R'_{Z+X}

(2)
$$\begin{aligned}
R'_{Z+X}(H) &= \frac{1}{2}\lambda\left(|j_Z X|^2 - \lambda^2\right)(Z - X),\\
R'_{Z+X}(j_Z X) &= -\frac{1}{2}\,|j_Z X|^2\,(|j_Z X|^2 - \lambda^2)(Z - X),
\end{aligned}$$

since

$$[\nabla_{Z+X}, R_{Z+X}](H) = \lambda\left(\frac{1}{2}\,|j_Z X|^2 + \lambda^2\right) Z + \frac{1}{4}\lambda\left(|j_Z X|^2 + \frac{7}{2}\lambda^2\right) X,$$

$$R(H,\nabla_{Z+X}(Z+X))(Z+X)=-\frac{1}{4}\lambda\,|j_ZX|^2\,(Z-X),$$

$$R(H,Z+X)\nabla_{Z+X}(Z+X)=\frac{1}{2}\lambda\left((3\lambda^2+\frac{1}{2}\,|j_ZX|^2)Z+(\frac{3}{4}\lambda^2+|j_ZX|^2)X\right),$$

and

$$[\nabla_{Z+X},R_{Z+X}](j_ZX)=-\frac{1}{4}\,|j_ZX|^2\left((|j_ZX|^2+\frac{9}{2}\lambda^2)Z+(|j_ZX|^2+3\lambda^2)X\right),$$

$$\begin{aligned}R(j_ZX,\nabla_{Z+X}(Z+X))(Z+X) &= -\frac{3}{8}\lambda^2\,|j_ZX|^2\,(Z-X),\\ R(j_ZX,Z+X)\nabla_{Z+X}(Z+X) &= \frac{1}{4}\,|j_ZX|^2\,(|j_ZX|^2-5\lambda^2)Z\\ &\quad -\frac{1}{4}\,|j_ZX|^2\,(|j_ZX|^2\,(3\,|j_ZX|^2+\frac{5}{2}\lambda^2)X.\end{aligned}$$

Finally, we get

$$[R_{Z+X},R'_{Z+X}](H)=\frac{1}{8}\lambda\left(5\,|j_ZX|^2+\lambda^2\right)(|j_ZX|^2-\lambda^2)(Z-X),\quad\text{since}$$

$$\begin{aligned}R_{Z+X}\circ R'_{Z+X}(H) &= \frac{1}{2}\lambda\left(|j_ZX|^2-\lambda^2\right)\left(\frac{1}{2}\,|j_ZX|^2-\lambda^2\right)(Z-X)\ \text{and}\\ R'_{Z+X}\circ R_{Z+X}(H) &= \frac{3}{4}\lambda R'_{Z+X}(j_ZX)-\frac{5}{4}\lambda^2R'_{Z+X}(H)\\ &= -\frac{1}{8}\lambda\left(3\,|j_ZX|^2+5\lambda^2\right)(|j_ZX|^2-\lambda^2)(Z-X),\end{aligned}$$

which are computed using (1) and (2) above. The assertion follows as claimed.

THEOREM 1. *If S is a Lie group of Iwasawa type and algebraic rank one which is a $\mathfrak{P}$-space, then S is a rank one symmetric space of noncompact type.*

Proof. Let $\mathfrak{s}=\mathfrak{n}\oplus\mathbf{R}H$, $|H|=1$, be the metric Lie algebra of Iwasawa type associated to S. If $\mathfrak{n}$ is abelian, then $\mathfrak{s}=\mathfrak{z}\oplus\mathbf{R}H$ with λ as unique eigenvalue of $\mathrm{ad}_H|_{\mathfrak{z}}$; thus S is, up to scaling, the real hyperbolic space.

Assume that $\mathfrak{n}$ is non-abelian, then by Proposition 5 $\mathfrak{n}$ is 2-step nilpotent. We show that $|j_ZX|^2=\lambda^2$ for unit vectors $Z\in\mathfrak{z}$ and $X\in\mathfrak{v}$. For this purpose let $X\neq 0$, $|X|=1$, be a vector in $\mathfrak{v}$. Let $\{Z_1,...,Z_k\}$ be an orthonormal basis of $\mathfrak{z}$ diagonalizing the symmetric operator $Z\to[X,j_ZX]$ on $\mathfrak{z}$. Hence, $\langle j_{Z_i}X,j_{Z_l}X\rangle=\delta_{il}\left|j_{Z_i}X\right|^2$ and $\{j_{Z_1}X,...,j_{Z_k}X\}$ is an orthogonal basis of $j_{\mathfrak{z}}X$. Moreover, since $[X,j_{Z_i}X]=\left|j_{Z_i}X\right|^2Z_i$ $(i=1,...,k)$, it follows that Z_i and $j_{Z_i}X$ are eigenvectors of $R_X|_{\mathfrak{z}}$ and $R_X|_{j_{\mathfrak{z}}X}$, respectively (see Section 1, 1.3), and consequently, $\mathfrak{z}\oplus j_{\mathfrak{z}}X=\oplus_{i=1}^k\mathrm{span}\{Z_i,j_{Z_i}X\}$ where $\mathrm{span}\{Z_i,j_{Z_i}X\}(t)$ is invariant under $R_{\gamma'_X(t)}$ and $\nabla_{\gamma'_X(t)}$ for all $i=1,...,k$.

Assume that condition (P) is satisfied and let $\{e_j(t) : j = 1, ..., \dim\mathfrak{s}\}$ be an orthonormal parallel basis diagonalizing $R_{\gamma'_X(t)}$. Applying the same argument as used in the previous proposition, for each $i = 1, ..., k$, we can choose $1 \le j_i \le \dim\mathfrak{s}$ such that $U_i(t)$, the non-zero component of $e_{j_i}(t)$ in $\mathrm{span}\{Z_i, j_{Z_i}X\}(t)$ is a parallel eigenvector of $R_{\gamma'_X(t)}$ along $\gamma_X(t)$ by Remark 1. We set $Z = Z_i$, $Y = \frac{j_Z X}{|j_Z X|}$, $U = U_i(0) = x_0 Z + y_0 Y$ with $x_0^2 + y_0^2 = 1$, and note that the unit $Z \in \mathfrak{z}$ satisfies $[X, j_Z X] = |j_Z X|^2 Z$. By applying Proposition 3, the parallel eigenvector U along γ_X with $U(0) = U$ is given by $U(t) = x(t)Z(t) + y(t)Y(t)$, with

$$x(t) = x_0 \cos s(t) - y_0 \sin s(t), \quad y(t) = x_0 \sin s(t) + y_0 \cos s(t)$$

and $s(t) = \frac{1}{2}|j_Z X| \frac{1}{\cosh \frac{1}{2}\lambda t}$, whose associated eigenvalue $a_X(t)$ satisfies $a_X(0) = \langle R_X U, U\rangle = x_0^2 \langle R_X Z, Z\rangle + y_0^2 \langle R_X Y, Y\rangle$,

$$a_X(t)\cosh^2\frac{1}{2}\lambda t = \frac{1}{4}|j_Z X|^2 - \frac{1}{2}\lambda^2 - \lambda^2 \sinh^2\frac{1}{2}\lambda t + \frac{3}{4}\lambda |j_Z X| \frac{y(t)}{x(t)} \sinh\frac{1}{2}\lambda t$$

and

$$a_X(t)\cosh^2\frac{1}{2}\lambda t = -\frac{3}{4}|j_Z X|^2 - \frac{1}{4}\lambda^2\cosh^2\frac{1}{2}\lambda t + \frac{3}{4}\lambda |j_Z X| \frac{x(t)}{y(t)} \sinh\frac{1}{2}\lambda t$$

for all real t, by Lemma 2.

If $x_0 \neq 0$ and $y_0 \neq 0$, then $a_X(0) = \langle R_X Z, Z\rangle = \langle R_X Y, Y\rangle$ (Z and Y are eigenvectors of R_X) and we get $|j_Z X|^2 = \frac{1}{4}\lambda^2$ since

$$\frac{1}{4}|j_Z X|^2 - \frac{1}{2}\lambda^2 = -\frac{3}{4}|j_Z X|^2 - \frac{1}{4}\lambda^2.$$

If $x_0 = 1$ and $y_0 = 0$, it follows from Proposition 3 that $x(t) = \cos s(t)$, $y(t) = \sin s(t)$ and in the same way that in the previous proposition,

$$\lim_{t\to 0}\left(\frac{x(t)}{y(t)}\sinh\frac{1}{2}\lambda t\right) = \frac{1}{2}\lambda \lim_{t\to 0}\left(\frac{\cosh\frac{1}{2}\lambda t}{\frac{s'(t)}{\cosh^2 s(t)}}\right) = \frac{\lambda}{|j_Z X|}.$$

Substituting this limit in the last expression of $a_X(t)$ above, we get

$$\begin{aligned} a_X(0) &= -\frac{3}{4}|j_Z X|^2 - \frac{1}{4}\lambda^2 + \frac{3}{4}\lambda|j_Z X| \lim_{t\to 0}\left(\frac{x(t)}{y(t)}\sinh\frac{1}{2}\lambda t\right) \\ &= -\frac{3}{4}|j_Z X|^2 + \frac{1}{2}\lambda^2, \end{aligned}$$

and from the equality $\langle R_X Z, Z\rangle = a_X(0)$ it follows that $|j_Z X|^2 = \lambda^2$.

The same condition is obtained in the case $x_0 = 0$ and $y_0 = 1$, since in this case $x(t) = -\sin s(t)$, $y(t) = \cos s(t)$ and $a_X(0) = \langle R_X Y, Y\rangle$ with $a_X(0)$ computed as

$$a_X(0) = \frac{1}{4}|j_Z X|^2 - \frac{1}{2}\lambda^2 - \frac{3}{4}\lambda|j_Z X| \lim_{t\to 0}\left(\frac{\sinh\frac{1}{2}\lambda t}{\tan s(t)}\right) = \frac{1}{4}|j_Z X|^2 - \frac{5}{4}\lambda^2.$$

Finally, using the same argument on each span$\{Z_i, j_{Z_i}X\}$ $(i = 1, ..., k)$, it follows that for any unit vector $X \in \mathfrak{v}$, an orthonormal basis $\{Z_1, ..., Z_k\}$ can be chosen so that $[X, j_{Z_i}X] = \left|j_{Z_i}X\right|^2 Z_i$ and $\{j_{Z_i}X : i = 1, ..., k\}$ is an orthogonal basis of $j_{\mathfrak{z}}X$ satisfying either $\left|j_{Z_i}X\right|^2 = \frac{1}{4}\lambda^2$ or $\left|j_{Z_i}X\right|^2 = \lambda^2$ for $i = 1, ..., k$. Hence,

$$(*) \quad \frac{1}{4}\lambda^2 |Z|^2 \leq |j_Z X|^2 \leq \lambda^2 |Z|^2 \text{ for all } Z \in \mathfrak{z} \text{ and } X \in \mathfrak{v},\ |X| = 1,$$

since $j_Z X = \sum_{i=1}^k a_i j_{Z_i} X$ is an orthogonal sum for any $Z = \sum_{i=1}^k a_i Z_i$ in $\mathfrak{z}$.

Next we show that the condition $\left|j_{Z_i}X\right|^2 = \frac{1}{4}\lambda^2$ in the above basis is not possible. In fact, for a such Z_i, it follows from $(*)$ above that $\frac{1}{4}\lambda^2 = -\langle j_{Z_i}^2 X, X\rangle$ is the minimum eigenvalue of the symmetric operator $-j_{Z_i}^2$ with X as associated eigenvector. Thus $-j_{Z_i}^2 X = \frac{1}{4}\lambda^2 X$ and $\mathfrak{s}_0$, the Lie algebra spanned by $\{Z_i, X, j_{Z_i}X, H\}$, is a totally geodesic subalgebra of $\mathfrak{s}$. Applying Lemma 3, the associated Lie group to $\mathfrak{s}_0$ is a $\mathfrak{P}$-space, which is not possible by Example 1.

The condition $\left|j_{Z_i}X\right|^2 = \lambda^2$ for all $i = 1, ..., k$, implies that $|j_Z X|^2 = \lambda^2 |Z|^2$ for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$, $|X| = 1$, or equivalently $j_Z^2 = -\lambda^2 |Z|^2 \mathrm{Id}$ for all $Z \in \mathfrak{z}$. Since $\mathrm{ad}_H|_{\mathfrak{z}} = \lambda \mathrm{Id}$ and $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\lambda \mathrm{Id}$ it follows that S is, up to scaling of the metric, a Damek-Ricci space. Theorem 2 of [2], Section 4.3 implies that S is a rank one symmetric space of noncompact type.

□

COROLLARY 1. *If M is an irreducible, non-flat homogeneous Einstein $\mathfrak{P}$-space with nonpositive curvature and algebraic rank one, then M is a symmetric space of noncompact type and rank one.*

Proof. Since M is irreducible and non-flat, by applying Corollary 1 of [8] it follows that M is a simply connected homogeneous space with nonpositive curvature. Hence, M can be represented as a solvable Lie group S of algebraic rank one with left invariant metric of nonpositive curvature, whose associated metric Lie algebra $\mathfrak{s}$ decomposes as an orthogonal direct sum $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$ and $\mathfrak{a}$ is one-dimensional (see [1]). By applying Proposition 4.9 and Theorem 4.10 of [6], S is isometric to a Lie group of Iwasawa type and algebraic rank one. It follows from Theorem 1 that M is a symmetric space of noncompact type and rank one.

□

References

[1] AZENCOTT R. AND WILSON E., *Homogeneous manifolds with negative curvature I*, Trans. Amer.Math. Soc. **215** (1976), 323–362.

[2] BERNDT J., TRICERRI F. AND VANHECKE L.,*Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Math. **1598**, Springer, Berlin 1995.

[3] BERNDT J., VANHECKE L., *Two natural generalizations of locally symmetric spaces*, Differential Geometry and its Applications **2** (1992), 57–80.

[4] DOTTI I. AND DRUETTA M.J., *Negatively curved homogeneous Osserman spaces*, Differential Geometry and Applications **11** (1999), 163–178.

[5] EBERLEIN P., *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. Ecole. Norm. Sup. **27** (1994), 611–660.

[6] HEBER J., *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), 279–352.

[7] HELGASON S., *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York 1978.

[8] WOLF J., *Homogeneity and bounded isometries in manifolds of negative curvature*, Illinois J. Math. **8** (1964), 14–18.

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