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ON THE SOLUTIONS OF THE DISSIPATION INEQUALITY

Abstract.
We present some recent results on the existence of solutions to the Dissipation Inequality.

1. Introduction

In this review paper we outline recent results on the properties of the **Dissipation Inequality**, shortly **(DI)**. The **(DI)** is the following inequality in the unknown operator P:

$$(\mathbf{DI}) \qquad 2\Re e \langle Ax, P(x+Du)\rangle + F(x+Du, u) \geq 0 .$$

Here A is the generator of a C_0-semigroup e^{At} on a Hilbert space X and $D \in \mathcal{L}(U, X)$ where U is a second Hilbert space; $F(x, u)$ is a continuous quadratic form on $X \times U$,

$$F(x, u) = \langle x, Qx\rangle + 2\Re e \langle Sx, u\rangle + \langle u, Ru\rangle .$$

Positivity of $F(x, u)$ is not assumed.

We require that $P = P^* \in \mathcal{L}(X)$.

We note that the unknown P appears linearly in the **(DI)**, which is also called **Linear Operator Inequality** for this reason.

The **(DI)** has a central role in control theory. We shortly outline the reason by noting the following special cases:

- The case $D = 0$, $S = 0$, $R = 0$. In this case, **(DI)** takes the form of a Lyapunov type inequality,
$$2\Re e \langle Ax, Px\rangle \geq -\langle x, Qx\rangle .$$
- If $Q = 0$ and $R = 0$ (but $S \neq 0$) and if $B = -AD \in \mathcal{L}(U, X)$ we get the problem
$$(1) \qquad 2\Re e \langle Ax, Px\rangle \geq 0 \qquad B^*P = -S .$$
This problem is known as *Lur'e Problem* and it is important for example in stability theory, network theory and operator theory.
- The case $S = 0$, $R = I$ and $Q = -I$ is encountered in scattering theory while the case $S = 0$, $Q \geq 0$ and coercive R corresponds to the **standard** regulator problem of control theory.

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We associate to **(DI)** the following quadratic regulator problem "with stability": we consider the control system

$$\dot{x} = A(x - Du)\,. \tag{2}$$

We call a pair $(x(\cdot), u(\cdot))$ an *evolution of system (2) with initial datum* x_0 when $x(\cdot)$ is a (mild) solution to (2) with input $u(\cdot)$ and $x(0) = x_0$.

We associate to control system (2) the quadratic cost

$$J(x_0; u) = \int_0^{+\infty} F(x(t), u(t))\,\mathrm{d}t\,. \tag{3}$$

The relevant problem is the following one: we want to characterize the condition $V(x_0) > -\infty$ for each x_0 where $V(x_0)$ **is the infimum of (3) over the class of those square integrable evolutions which have initial datum** x_0. (The term "with stability" refers to the fact that we only consider the square integrable evolutions of the system).

Of course, Eq. (2) has no meaning in general. One case in which it makes sense is the case that $B = -AD$ is a bounded operator (*distributed* control action). In this case the problem has been essentially studied in [7] but for one crucial aspect that we describe below.

More in general, large classes of boundary control systems can be put in the form (2), as shown in [6], where two main classes have been singled out, the first one which corresponds to "hyperbolic" systems and the second one which corresponds to "parabolic" systems.

We illustrate the two classes introduced in [6]:

- The class that models in particular most control problems for the heat equation: the semigroup e^{At} is holomorphic (we assume exponentially stable for simplicity) and $\mathrm{im} D = \mathrm{im}[-A^{-1}B] \subseteq \mathrm{dom}(-A)^{\tilde{\gamma}}$, $\tilde{\gamma} < 1$.
- The class that models in particular most control problems for string and membrane equations: e^{At} is a C_0-semigroup, $A^{-1}B \in \mathcal{L}(X)$ and

  $$\int_0^T \|B^* e^{A^* t} x\|^2\,\mathrm{d}t \le k_T \|x\|^2\,. \tag{4}$$

  It is sufficient to assume that the previous inequality holds for one value of T since then it holds for every T.

As we said, for simplicity of exposition, we assume exponential stability. The simplification which is obtained when the semigroup is exponentially stable is that the class of the controls is $L^2(0, +\infty; U)$, independent of x_0. However, this condition can be removed.

The crucial result in the case of *distributed control action* is as follows (see [14] for the finite dimensional theory and [7] for distributed systems with distributed control action):

THEOREM 1. *If $AD \in \mathcal{L}(U, X)$, then $V(x_0)$ is finite for every x_0 if and only if there exists a solution to* **(DI)** *and in this case $V(x_0)$ is a continuous quadratic form on X: $V(x_0) = \langle x_0, Px_0\rangle$. The operator P of the quadratic form is the* maximal *solution to* **(DI)**.

The result just quoted can be extended to both the classes of boundary control systems introduced in [6], see [9, 11]. Rather than repeating the very long proof, it is possible to use a device, introduced in [10, 8], which associates to the boundary control system an "augmented" system, with distributed control action. From this distributed system it is possible to derive many properties of the **(DI)** of the original boundary control system. This device is illustrated in sect. 2.

With the same method it is possible to extend the next result:

THEOREM 2. *If $V(x_0) > -\infty$, i.e. if* **(DI)** *is solvable, then*

$$\Pi(i\omega) = F(-i\omega(i\omega I - A)^{-1}Du + Du, u) \geq 0 \qquad \forall \omega \in \mathbb{R}\,. \tag{5}$$

The function $\Pi(i\omega)$ was introduced in [12] and it is called the **Popov function**.

As the number $i\omega$ are considered "frequencies", condition (5) is a special "frequency domain condition".

At the level of the frequency domain condition we encounter a crucial difference between the class of "parabolic" and "hyperbolic" systems:

THEOREM 3. *In the parabolic case if $V(x_0) > -\infty$, then $R \geq 0$. Instead, in the "hyperbolic" case, we can have $V(x_0) > -\infty$ even if $R = -\alpha I$, $\alpha > 0$.*

Proof. It is clear that

$$\Pi(i\omega) = F((i\omega I - A)^{-1}Bu, u)$$

($B = -AD$) and $\lim_{|\omega|\to+\infty}(i\omega I - A)^{-1}Bu = 0$ because $\mathrm{im}D = \mathrm{im}[-A^{-1}B] \subseteq \mathrm{dom}(-A)^{\gamma}$ (here we use exponential stability, but the proof can be adapted to the unstable case.) Hence, $0 \leq \lim_{|\omega|\to+\infty}\Pi(i\omega) = \langle u, Ru\rangle$ for each $u \in U$. This proves that $R \geq 0$.

Clearly an analogous proof cannot be repeated in the "hyperbolic" case; and the analogous result does not hold, as the following example shows:

the system is described by

$$x_t = -x_\theta \qquad 0 < \theta < 1, \qquad t > 0 \qquad x(t, 0) = u(t)$$

(this system is exponentially stable since the free evolution is zero for $t > 1$).

The functional $F(x, u)$ is

$$F(x, u) = \|x(\cdot)\|^2_{L^2(0,1)} - \alpha|u|^2$$

so that

$$J(x_0; u) = \int_0^{+\infty}\{\|x(t,\cdot)\|^2_{L^2(0,1)} - \alpha|u(t)|^2\}\,\mathrm{d}t\,.$$

If $x(0, \theta) \equiv 0$ then

$$\hat{x}(z, \theta) = e^{-z\theta}\hat{u}(z)$$

so that

$$\langle u, \Pi(i\omega)u\rangle = [1 - \alpha]|u|^2\,.$$

This is nonnegative for each $\alpha \leq 1$ in spite of the fact that $R = -\alpha I$ can be negative. Hence, in the hyperbolic boundary control case, **the condition $R \geq 0$ does not follows from the positivity of the Popov function**.

□

It is clear that the frequency domain condition may hold even if the **(DI)** is not solvable, as the following example shows:

EXAMPLE 1. The example is an example of a scalar system,

$$\dot{x} = -x + 0u \qquad y = x\,.$$

It is clear that $\Pi(i\omega) \geq 0$, is nonnegative; but $PB = C$, i.e. $P0 = 1$, is not solvable.

A problem that has been studied in a great deal of papers is the problem of finding additional conditions which imply solvability of the **(DI)** in the case that the frequency domain condition (5) holds. A special instance of this problem is the important Lur'e problem of stability theory.

This problem is a difficult problem which is not completely solved even for finite dimensional systems. Perhaps, the most complete result is in [2]: if a system is finite dimensional and $\Pi(i\omega) \geq 0$, then a sufficient condition for solvability of **(DI)** is the existence of a number ω_0 such that $\det \Pi(i\omega_0) \neq 0$.

It is easy to construct examples which show that this condition is far from sufficient.

In the context of hyperbolic systems, the following result is proved in [11].

THEOREM 4. *Let condition (4) hold and let the system be* exactly controllable. *Under these conditions, if the Popov function is nonnegative then there exists a solution to* **(DI)** *and, moreover, the* maximal *solution P of* **(DI)** *is the strong limit of the decreasing sequence $\{P_n\}$, where P_n is the maximal solution of the* **(DI)**

$$2\Re e\, \langle Ax,\, P(x + Du)\rangle + F(x + Du, u) + \frac{1}{n}\{\|u\|^2 + \|x\|^2\} \geq 0\,. \tag{6}$$

The last statement is important because it turns out that P_n solves a Riccati equation, while there is no equation solved by P in general.

The proof of Theorem 4 essentially reproduces the finite dimensional proof in [14]. Hence, the "hyperbolic" case is "easy" since the finite dimensional proof can be adapted. In contrast with this, the "parabolic" case requires new ideas and it is "difficult". Consistent with this, only very partial results are available in this "parabolic" case, and under quite restrictive conditions. These results are outlined in sect. 3.

Before doing this we present, in the next section, the key idea that can be used in order to pass from a boundary control system to an *"augmented"* but *distributed* control system.

2. The augmented system

A general model for the analysis of boundary control systems was proposed by Fattorini ([4]). Let X be a Hilbert space and σ a linear closed densely defined operator, $\sigma : X \to X$. A second operator τ is linear from X to a Hilbert space U.

We assume:

Assumption We have: $\operatorname{dom}\sigma \subseteq \operatorname{dom}\tau$ and τ is continuous on the Hilbert space $\operatorname{dom}\sigma$ with the graph norm.

The "boundary control system" is described by:

$$\begin{cases} \dot{x} = \sigma x \\ \tau x = u \end{cases} \qquad x(0) = x_0 \tag{7}$$

where $u(\cdot) \in L^2_{\mathrm{loc}}(0, +\infty; U)$.

We must define the "strong solutions" $x(\cdot; x_0, u)$ to system (7). Following [3] the function $x(\cdot) = x(\cdot; x_0, u)$ is a strong solution if there exists a sequence $\{x_n(\cdot)\}$ of C^1-functions such that

$x_n(t) \in \operatorname{dom}\sigma$ for each $t \geq 0$ and:

$$(8) \qquad \begin{cases} \dot{x}_n(\cdot) - \sigma x_n(\cdot) \to 0 & \text{in } L^2_{\text{loc}}(0, +\infty; X) \\ x_n(0) \to x_0 & \text{in } X \\ \tau x_n(\cdot) \to u(\cdot) & \text{in } L^2_{\text{loc}}(0, +\infty; U) \end{cases}$$

and

- $x_n(\cdot)$ **converges uniformly to** $x(\cdot)$ **on compact intervals in** $[0, +\infty)$.

In the special case that the sequence $x_n(\cdot)$ is stationary, $x_n(\cdot) = x(\cdot)$, we shall say that $x(\cdot)$ is a *classical* solution to problem (7).

Assumption 1. Let us consider the "elliptic" problem $\sigma x = u$. We assume that it is "well posed", i.e. that there exists an operator $D \in \mathcal{L}(U, X)$ such that

$$x = Du \text{ iff } \{\sigma x = 0 \text{ and } \tau x = u\}\,.$$

Moreover we assume that the operator A defined by

$$\operatorname{dom} A = \operatorname{dom}\sigma \cap \ker\tau \qquad Ax = \sigma x$$

generates a strongly continuous semigroup on X.

As we said already, for simplicity of exposition, we assume that the semigroup e^{At} is exponentially stable.

Now we recall the following arguments from [1]. Classical solutions to Eq. (7) solve

$$(9) \qquad \dot{x} = A(x - Du) \qquad x(0) = x_0\,.$$

Let $u(\cdot)$ be an absolutely continuous control and $\xi(t) = x(t) - Du(t)$. Then, $\xi(\cdot)$ is a classical solution to

$$(10) \qquad \dot{\xi} = A\xi - D\dot{u} \qquad \xi_0 = \xi(0) = x(0) - Du(0)$$

and conversely.

As the operator A generates a C_0-semigroup, it is possible to write a "variation of constants" formula for the solution ξ. "Integration by parts" produces a variation of constants formula, which contains unbounded operators, for the function $x(\cdot)$. This is the usual starting point for the study of large classes of boundary control systems. Instead, we "augment" system (9) and we consider the system:

$$(11) \qquad \begin{cases} \dot{\xi} = A\xi - Dv \\ \dot{u} = v \end{cases}$$

Here we consider formally $v(\cdot)$ as a new "input", see [10, 8].

Moreover, we note that it is possible to stabilize the previous system with the simple feedback $v = -u$, since e^{At} is exponentially stable.

The cost that we associate to (11) is the cost

$$(12) \qquad J(x_0; u) = \int_0^{+\infty} F(\xi(t) + Du(t), u(t))\,\mathrm{d}t\,.$$

This cost does not depend explicitly on the new input $v(\cdot)$: it is a quadratic form of the state, which is now $\Xi = [\xi, u]$.

It is proved in [9] that the value function $\mathcal{V}(\xi_0, u_0)$ of the augmented system has the following property:

$$\mathcal{V}(\xi_0 + Du_0, u_0) = V(x_0)\,.$$

We apply the stabilizing feedback $v = -u$ and we write down the **(DI)** and the Popov function for the stabilized augmented system. The **(DI)** is

$$2\Re e\,\langle \mathcal{A}\Xi, W\Xi\rangle + \langle \Xi, \mathcal{Q}\Xi\rangle \geq 0 \qquad \forall \Xi \in \mathrm{dom}\mathcal{A}\,, \qquad W\mathcal{D} = 0\,. \tag{13}$$

where

$$\mathcal{A} = \begin{bmatrix} A & -D \\ 0 & I \end{bmatrix}, \qquad \Xi = \begin{bmatrix} \xi \\ u \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} Q & S^* + QD \\ D^*Q + S & R + D^*S^* + SD + D^*QD \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} -D \\ I \end{bmatrix}.$$

The Popov function is:

$$P(i\omega) = \frac{\Pi(i\omega)}{1+\omega^2} \tag{14}$$

It is clear that the transformations outlined above from the original to the augmented system do not affect the positivity of the Popov function and that if $\omega^s\Pi(i\omega)$ is bounded from below, then $\omega^{s+2}P(i\omega)$ is bounded from below.

In the next section we apply the previous arguments to the case that the operator A generates a holomorphic semigroup and $\mathrm{im}D \subseteq (\mathrm{dom}(-A)^\gamma)$, $\gamma < 1$.

3. "Parabolic" case: from the Frequency domain condition to the (DI)

We already said that in the parabolic case only partial results are available. In particular, available results require that the control be scalar so that S is an element of X. This we shall assume in this section. We assume moreover that the operator A has only point spectrum with simple eigenvalues z_k and the eigenvectors v_k form a complete set in X. Just for simplicity we assume that the eigenvalues are real (hence negative). Moreover, we assume that we already wrote the system in the form of a distributed (augmented and stabilized) control system. Hence we look for conditions under which there exists a solution W to (13).

We note that $\mathcal{D} \in X \times U$ and that $P(i\omega)$ is a scalar function: it is the restriction to the imaginary axis of the analytic function

$$P(z) = -\mathcal{D}(zI + \mathcal{A}^*)^{-1}\mathcal{Q}(zI - \mathcal{A})^{-1}\mathcal{D}\,.$$

The function $P(z)$ is analytic in a strip which contains the imaginary axis in its interior.

We assume that $P(i\omega) \geq 0$ and we want to give additional conditions under which (13) is solvable. In fact, we give conditions for the existence of a solution to the following more restricted problem: to find an operator W and a vector $q \in (\mathrm{dom}A)'$ such that

$$2\Re e\,\langle \mathcal{A}\Xi, W\Xi\rangle + \langle \Xi, \mathcal{Q}\Xi\rangle = |\langle\langle \Xi, q\rangle\rangle\|^2 \quad \forall \Xi \in \mathrm{dom}\mathcal{A}\,. \tag{15}$$

The symbol $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the pairing of $(\mathrm{dom}A)'$ and $\mathrm{dom}A$.

The previous equation suggests a form for the solution W:

$$\langle \Xi, W\Xi \rangle = \int_0^{+\infty} \langle e^{\mathcal{A}t}\,\Xi, \mathcal{Q}e^{\mathcal{A}t}\,\Xi\rangle \,\mathrm{d}t - \int_0^{+\infty} |\langle\langle \Xi, e^{\mathcal{A}t}q\rangle\rangle\|^2 \,\mathrm{d}t\,. \tag{16}$$

However, it is clear that in general the operator W so defined will not be continuous, unless q enjoys further regularity. We use known properties of the fractional powers of the generators of holomorphic semigroups and we see that W is bounded if $q \in [\mathrm{dom}(-\mathcal{A}^{\alpha})]'$ with $\alpha < 1/2$.

It is possible to prove that if a solution W to (15) exists then there exists a factorization

$$P(i\omega) = m^*(i\omega)m(i\omega)$$

and $m(i\omega)$ does not have zeros in the right half plane. This observation suggests a method for the solution of Eq. (15), which relies on the computation of a factorization of $P(i\omega)$. The factorization of functions which takes nonnegative values is a classical problem in analysis. The key result is the following one:

LEMMA 1. *If $P(i\omega) \geq 0$ and if $|\ln P(i\omega)|/(1+\omega^2)$ is integrable, then there exists a function $m(z)$ with the following properties:*

- *$m(z)$ is holomorphic and bounded in $\Re e\, z > 0$;*
- *$P(i\omega) = m(-i\omega)m(i\omega)$;*
- *let $z = x + iy$, $x > 0$. The following equality holds:*

$$\ln|m(z)| = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln P(i\omega)\frac{x}{x^2+(\omega-y)^2}\,d\omega \quad \forall z = x+iy, \quad x > 0\,. \tag{17}$$

See [13, p. 121], [5, p. 67].

A function which is holomorphic and bounded in the right half plane and which satisfies (17) is called an *outer function.*

The previous arguments show that an outer factor of $P(z)$ exists when $P(i\omega) \geq 0$ and when $P(i\omega)$ decays for $|\omega| \to +\infty$ of the order $1/|\omega|^{\beta}$, $\beta < 1$. Let us assume this condition (which will be strengthened below). Under this condition $P(z)$ can be factorized and, moreover,

$$\begin{aligned}
\ln|m(z)| &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln P(i\omega)\frac{x}{x^2+(\omega-y)^2}\,\mathrm{d}\omega \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \ln\frac{M}{1+\omega^2}\,\frac{x}{x^2+(\omega-y)^2}\,\mathrm{d}\omega \\
&= \ln\left|\frac{1}{1+z^2}\right|.
\end{aligned}$$

This estimates implies in particular that the integrals $\int_{-\infty}^{+\infty} |m(x+iy)|^2\,\mathrm{d}y$ are uniformly bounded in $x > 0$. Paley Wiener theorem (see [5]) implies that

$$m(i\omega) = \int_0^{+\infty} e^{-i\omega t}\check{m}(t)\,\mathrm{d}t, \qquad \check{m}(\cdot) \in L^2(0,+\infty)\,.$$

The function $\check{m}(t)$ being square integrable, we can write the integral

$$\int_0^{+\infty} e^{A^*s} q\check{m}(t)\,\mathrm{d}t$$

and we can try to solve the following equation for q:

$$\int_0^{+\infty} e^{A^* s} q \check{m}(t)\,\mathrm{d}t = -s = \int_0^{+\infty} e^{\mathcal{A}^* t} \mathcal{Q} e^{\mathcal{A} t} \mathcal{D}\,\mathrm{d}t\,. \tag{18}$$

This equation is suggested by certain necessary conditions for the solvability of (1) which are not discussed here.

We note that

$$s \in \mathrm{dom}(-\mathcal{A})^{1-\epsilon} \text{ for each } \epsilon > 0\,. \tag{19}$$

It turns out that equation (18) can always be **formally** solved, a solution being

$$q_k = \langle v_k, q\rangle = -\frac{\langle v_k, s\rangle}{m(-\bar{z}_k)}$$

since $m(z)$ does not have zeros in the right half plane.

Moreover, we can prove that the operator W defined by (16) **formally** satisfies the condition $W\mathcal{D} = 0$. Hence, this operator W will be the required solution of (15) if it is a bounded operator, i.e. if $q \in [\mathrm{dom}(-\mathcal{A}^{\alpha})]'$.

An analysis of formula (17) shows the following result:

THEOREM 5. *The vector q belongs to $(\mathrm{dom}(-\mathcal{A}^*)^{1/2-\epsilon})'$ for some $\epsilon > 0$ if there exist numbers $\gamma < 1$ and $M > 0$ such that*

$$|\omega|^{\gamma}\,\Pi(i\omega) > M$$

for $|\omega|$ large.

Examples in which the condition of the theorem holds exist, see [9].

Let $\zeta_k = -z_k \in \mathbb{R}$. The key observation in the proof of the theorem is the following equality, derived from (17):

$$\begin{aligned}
\log|\zeta|^{\frac{3}{2}-\epsilon} m(\zeta_k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\log|\zeta_k|^{3-2\epsilon} P(i\zeta_k s)]\frac{1}{1+s^2}\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\log \zeta_k^{3-\gamma-2\epsilon}\frac{1}{|s|^{\gamma}}]\frac{1}{1+s^2}\,\mathrm{d}s \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\log \zeta_k |s|^{\gamma} P(i\zeta_k s)]\frac{1}{1+s^2}\,\mathrm{d}s\,.
\end{aligned}$$

The first integral is bounded below if $\gamma \leq 3 - 2\epsilon$ and the second one is bounded below in any case.

We recapitulate: the condition $q \in (\mathrm{dom}(-\mathcal{A}^*)^{1/2-\epsilon})'$ holds if $P(i\omega)$ decays at ∞ of order less than 3. We recall (14) and we get the result.

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